Today, the first computer task. In addition, some exercises on Gaussian random fields.

The computer task will be on the expansion of an isolated spherical underdensity. To this end, exactly the same equations can be used as for a spherical overdensity, but then for a density fluctuation $\delta < 0$.

The intention is for you to compute numerically the solutions for a range of evolving voids (density profiles, velocity profiles, expansion factors, etc.) and write a short report (few pages) on the results. In this report you describe how you made the computations, what you observe in the resulting plots, discuss this within the context of general structure evolution, etc.

Part I: Generalities.

In this computer task we restrict ourselves exclusively to the development of voids in an Einstein-de Sitter Universe.

For an Einstein-de Sitter Universe, we have

$$\Omega(t) \,=\, \Omega_i \,=\, 1\,. \tag{1}$$

Recall that the expansion factor $a(t)$ for an EdS universe is

$$a(t) \,=\, \left(\frac{t}{t_0}\right)^{2/3}\,, \tag{2}$$

while the cosmic time t is related to the Hubble parameter $H(t)$ through

$$t = \frac{2}{3}\frac{1}{H(t)} \tag{3}$$

Last week, we inferred the equations for an unbound spherical density fluctuation. In this computer task we will restrict ourselves exclusively to unbound shells around voids. The expansion equations for an unbound spherical shell are:

$$\begin{aligned}
\mathcal{R}(\Phi_r) &= \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})}\,\left(\cosh\Phi_r - 1\right) \\
H_i(t - t_{r0}) &= \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}}\,\left(\sinh\Phi_r - \Phi_r\right)
\end{aligned} \tag{4}$$

where t_{r0} is a time integration constant, specific for each shell (do not confuse this with the universal present time, t_0).

The significance of the parameters Δ_i and α_i have been provided in the previous set of work sets. The parameter Δ_i is the mean density deficit at initial time t_i, i.e. the integral over the initial density profile δ,

$$\begin{aligned}
\Delta(r,t) &= \frac{3}{r^3}\int_0^r \left[\frac{\rho(y,t)}{\rho_u(t)} - 1\right]\,y^2\,dy \\
&= \frac{3}{r^3}\int_0^r \delta(y,t)\,y^2\,dy\,.
\end{aligned} \tag{5}$$

Important is the value of $\Delta(r,t)$ at the initial time, in particular also the Δ with respect to an Einstein-de Sitter Universe, Δ_{ci}. For an Einstein-de Sitter Universe, of course

$$\Delta_{ci} = \Delta_i = \Delta(r_i, t_i)\,. \tag{6}$$

To determine the specific density profile $\delta(r,t)$ of a void, we also need to have the spatial gradient of the initial density profile,

$$\Delta'_{ci} \,=\, \frac{d\Delta_{ci}}{dx_i} \,=\, \frac{3}{x_i}\,\left(\delta_i(x_i) - \Delta_i(x_i)\right)\,, \tag{7}$$

where $x_i = r_i/a_i$ is the comoving radius of the shell r_i.

The parameter α quantifies the velocity perturbation

$$\alpha(r,t) \,=\, \left(\frac{v}{Hr}\right)^2\,. \tag{8}$$

At the initial time t_i, the void still resides in the linear growth phase, for which we found that:

$$\alpha_i \,=\, -\frac{2}{3} f(\Omega_i)\Delta(r_i,t_i) \,=\, -\frac{2}{3}\Delta(r_i,t_i)\,, \tag{9}$$

where $f(\Omega_i)$ is the well-known Peebles factor at time t_i, when $\Omega(t) = \Omega_i$. Also of α_i we need the spatial gradient,

$$\alpha'_i \,=\, -\frac{2}{3}\,\Delta'_{ci}\,. \tag{10}$$

Note that this expression is strictly valid only for an Einstein-de Sitter Universe.

Important for this computer exercise is that for each time t, you may determine the development angle $\Phi_{r,t}$ for a shell by inverting the time equation, ie. determining the Φ_r for which

$$Q(\Phi_{r,t}) \,=\, \sinh\Phi_{r,t} - \Phi_{r,t} = \frac{2(\alpha_i - \Delta_{ci})^{3/2}}{1+\Delta_{ci}}\,H_i\,(t - t_{r0}) \tag{11}$$

which you can solve numerically for any time t with the help of the Newton-Raphson method.

Voids form around underdensities in the mass distribution. The mass deficit generates a net repulsive peculiar gravity. In other words, in comoving space the void shells start to grow in size. The expansion of a spherical void is described by the spherical model equations above.

Part II: Void Structure.

The structure of a void is described by a few quantities. These are the density profiles, both the specific density profile $\delta)r,t)$ and mean density profile $\Delta(r,t)$. In addition, we need to know the velocity profile (both full velocit and peculiar velocity) $v(r,t)$.

The evolution of the mean density contrast $\Delta(r,t)$ of a shell around the void is described by:

$$\begin{aligned} 1+\Delta(r,t) &= \frac{1+\Delta_i(r_i)}{\mathcal{R}^3}\,\frac{a(t)^3}{a_i^3}\,, \\ &= \frac{9}{2} f(\Phi_r) \end{aligned} \tag{12}$$

with the function $f(\Phi_r)$ for an unbound shell given by

$$f(\Phi) = \frac{(\sinh\Phi_r - \Phi_r)^2}{(\cosh\Phi_r - 1)^3}\,. \tag{13}$$

The corresponding (specific) density profile $\delta(r,t)$ for an unbound shell can likewise be obtained from the expression

$$\delta(r,t) = \frac{1+\delta_i(r_i)}{R^3\left(1+x_i\dfrac{R'}{R}\right)}\,\frac{a(t)^3}{a_i^3} - 1\,, \tag{14}$$

where $x_i = r_i/a_i$ is the comoving initial radius of the shell. The derivative R'/R for an open shell can be computed from the following expression,

$$\begin{aligned} \frac{R'}{R} &= \frac{1}{R}\frac{dR}{dx_i} \qquad (15) \\ &= \frac{\Delta'_{ci}}{\alpha_i - \Delta_{ci}} - \frac{\alpha'_i - \Delta'_{ci}}{\alpha_i - \Delta_{ci}} + \left[\frac{3}{2}\frac{\alpha'_i - \Delta'_{ci}}{\alpha_i - \Delta_{ci}} - \frac{\Delta'_{ci}}{\alpha_i - \Delta_{ci}}\right]\frac{\sinh\Phi_r\,(\sinh\Phi_r - \Phi_r)}{(\cosh\Phi_r - 1)^2}\,. \end{aligned}$$

Note that the expressions for the spatial gradients Δ'_{ci} of the initial mean density profile and α'_i of the initial velocity parameter are given in Part I.

The velocity profile of the profile is given by

$$v(r,t) = H_i r_i\,(\alpha_i - \Delta_{ci})^{1/2}\,\frac{\sinh\Phi_r}{\cosh\Phi_r - 1} \tag{16}$$

III: the initial void density profiles.

In this computer experiment we will test the expansion of voids forming around two different initial underdensities, in an Einstein-de Sitter Universe.

1) One set of void models is the ones that for around an initial tophat underdensity profile $\delta(r, t_i)$,

$$\delta_a(r, t_i) = \begin{cases} \delta_i & r < R_i \\ 0 & r > R_i \end{cases} \tag{17}$$

where R_i is the typical size of the initial underdensity. You may test the subsequent evolution for a range of values of the initial density deficit δ_i, and values of its initial size R_i.

Take three cases: $\delta_i = -0.01$, $\delta = -0.02$ and $\delta = -0.03$ at initial expansion factor $a_i = 0.01$.

2) A second set of void models are the ones that form around a Gaussian shaped initial underdensity,

$$\delta_a(r, t_i) = \delta_i \exp\left(-r^2/2R_i^2\right) \tag{18}$$

where R_i is the typical size of the initial underdensity. You may test the subsequent evolution for a range of values of the initial density deficit δ_i, and values of its initial size R_i. See the suggested values above.

IV: the computer task.

For each of the different initial void profiles (at least one tophat one, and one Gaussian one), the task consists of the following:

1) Plot the initial density profile $\delta(r, t_i)$ for $0 < r < 2.5R_i$.
2) Compute and plot the initial mean density profile $\Delta(r, t_i)$ for the same range, $0 < r < 2.5R_i$.
3) For a set of 25 shells, with initial radii $0 < r_i < 2.5R_i$, compute and plot the expansion factor $\mathcal{R}(r_i, t)$.
4) For the same set of 25 shells, with initial radii $0 < r_i < 2.5R_i$, compute and plot the expanding shell radius $r(r_i, t)$.
5) For 20 epochs in between $0 < a < 1.0$, compute and plot the mean density profile $\Delta(r, t)$ of the emerging void.
6) For the same 20 epochs, compute and plot the peculiar velocity profile $v_{pec}(r, t)$ of the emerging void.
7) For the courageous: for the same 20 epochs, compute and plot the specific density profiles $\delta(r, t)$ of the emerging void.
8) What do you notice with the models for $\delta_i = -0.03$?

V: Recepy for void profile & evolution computation.

To follow the evolution of a void, given its initial density profile, you proceed with the following sequence of steps.

1) Choose an initial time t_i, or expansion factor a_i, or redshift z_i. Determine Ω_i: for an EdS universe, this is $\Omega_i = 1$. The Hubble parameter H_i at the initial time is:

$$\begin{aligned} H_i &= \frac{H_0}{a_i^{3/2}} . \\ H_i t_i &= \frac{2}{3} \end{aligned} \tag{19}$$

Note that in an Einstein-de Sitter Universe, always $H(t)t = \frac{2}{3}$.

2) If the density profile $\delta(r, t)$ is a linearly extrapolated density profile at $t = t_0$, transform it to the initial time t_i.

$$\delta_i(r_i) = \frac{D(a_i)}{D(a_0)} \delta_0(r_i) \tag{20}$$

Note: in this tutorial task, you got the values of δ_i at the initial epoch specified directly. You can therefore skip this step. One example is

3) Choose N shells r_i over a range $0 < r_i < L_i$. Of these shells, also determine the comoving value x_i of the shell radius,

$$x_i = \frac{r_i}{a_i} \tag{21}$$

4) Compute the mean density profile Δ_i at the initial time t_i,

$$\Delta_i(x_i) = \frac{3}{x_i^3} \int_0^{x_i} \delta_i(y) y^2 dy . \tag{22}$$

4b) In an Einstein-de Sitter Universe,

$$\Delta_{ci}(x_i) = \Delta_i(x_i) . \tag{23}$$

Thus, Δ_{ci} for each shell is easily computed.

5) Compute the gradient of the mean density profile, $\Delta_i'(x_i)$,

$$\Delta_i'(x_i) = \frac{3}{x_i} \left(\delta_i(x_i) - \Delta_i(x_i) \right) . \tag{24}$$

6) Determine for each shell, at initial time t_i, the value of the velocity/kinetic energy parameter α_i, assuming linear initial conditions,

$$\alpha_i = -\frac{2}{3}\Delta(r_i, t_i) \,. \tag{25}$$

6b) In addition, determine the gradient of the velocity/kinetic energy parameter at the initial time t_i,

$$\alpha_i' = -\frac{2}{3}\Delta_i', . \tag{26}$$

7) For each radius, determine time constant t_{r0}. First, from condition that shell expansion factor at time t_i is $\mathcal{R}(t_i) = 1$, we obtain the initial shell development angle $\Phi_{r,i}$,

$$\Phi_{r,i} = \operatorname{arccosh}\left(1 + \frac{2(\alpha_i - \Delta_i)}{1+\Delta_i}\right) \,. \tag{27}$$

Then,

$$H_i t_{r0} = H_i t_i - \frac{1}{2}\frac{1+\Delta_i}{(\alpha_i - \Delta_i)^{3/2}}\left(\sinh\Phi_{ri} - \Phi_{ri}\right) \tag{28}$$

$$= \frac{2}{3} - \frac{1}{2}\frac{1+\Delta_i}{(\alpha_i - \Delta_i)^{3/2}}\left(\sinh\Phi_{ri} - \Phi_{ri}\right) \tag{29}$$

8) Determine cosmic expansion factor(s) $a = a(t)$ which you wish to determine the void profile(s). Determine the Hubble parameter at time t,

$$\begin{aligned} H_i(t - t_{r0}) &= \frac{H_i}{H(t)} H(t)t - H_i t_{r0} \\ &= \left(\frac{a}{a_i}\right)^{3/2}\frac{2}{3} - H_i t_{r0} \end{aligned} \tag{30}$$

9) Determine the development angle $\Phi_{r,t}$ for each individual shell. To that end solve the following equation numerically for $\Phi_{r,t}$

$$Q(\Phi_{r,t}) = \sinh\Phi_{r,t} - \Phi_{r,t} = \frac{2(\alpha_i - \Delta_{ci})^{3/2}}{1+\Delta_{ci}} H_i\,(t - t_{r0}) \,. \tag{31}$$

You may achieve this by determining the zero-point $\Phi_{r,t}$ of the equation $Q'(\Phi_{r,t}) = 0$ by means of the Newton-Raphson method, with

$$Q'(\Phi_r) = Q(r) - \frac{2(\alpha_i - \Delta_{ci})^{3/2}}{1+\Delta_{ci}} H_i\,(t - t_{r0}) \,. \tag{32}$$

Subsequently, with these parameters you may determine the following void structural functions,

10) Determine the expansion factor $\mathcal{R}(t, r_i)$ of each of the shells at time t,

$$\mathcal{R}(t, r_i) \;=\; \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})}\;\;(\cosh\Phi_r - 1) \tag{33}$$

11) Determine the radius $r(t, r_i)$ of each of the shells at time t,

$$r(t, r_i) \;=\; \mathcal{R}(t, r_i)\, r_i \;=\; \frac{1}{2}\, r_i\, \frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})}\;\;(\cosh\Phi_r - 1) \tag{34}$$

12) Determine the mean density perturbation $\Delta(t, r_i)$ within the shell at time

$$1+\Delta(r,t) \quad=\quad \frac{1+\Delta_i(r_i)}{\mathcal{R}^3}\,\frac{a(t)^3}{a_i^3}\,, \tag{35}$$

13) Determine the specific density perturbation $\delta(t, r_i)$ at shell, with radius $r(t, r_i)$,

$$\delta(r,t) \;=\; \frac{1+\delta_i(r_i)}{R^3\left(1 + x_i \dfrac{R'}{R}\right)}\,\frac{a(t)^3}{a_i^3}\;-1\,, \tag{36}$$

14) Determine the velocity $v(t, r_i)$ if shell,

$$v(r,t) \;=\; H_i r_i\;(\alpha_i - \Delta_{ci})^{1/2}\;\frac{\sinh\Phi_r}{\cosh\Phi_r - 1} \tag{37}$$