Chapter 4:
Linear Perturbation Theory

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1. Gravitational Instability

The generally accepted theoretical framework for the formation of structure is that of gravitational instability. The gravitational instability scenario assumes the early universe to have been almost perfectly smooth, with the exception of tiny density deviations with respect to the global cosmic background density and the accompanying tiny velocity perturbations from the general Hubble expansion.

The minor density deviations vary from location to location. At one place the density will be slightly higher than the average global density, while a few Megaparsecs further the density may have a slightly smaller value than on average. The observed fluctuations in the temperature of the cosmic microwave background radiation are a reflection of these density perturbations, so that we know that the primordial density perturbations have been in the order of 10^{-5}. The origin of this density perturbation field has as yet not been fully understood. The most plausible theory is that the density perturbations are the product of processes in the very early Universe and correspond to *quantum fluctuations* which during the *inflationary phase* expanded to macroscopic proportions[1]

Originally minute local deviations from the average density of the Universe (see fig. 1), and the corresponding deviations from the global cosmic expansion velocity (the Hubble expansion), will start to grow under the influence of the involved gravity perturbations. The gravitational force acting on each patch of matter in the universe is the total sum of the gravitational attraction by all matter throughout the universe. Evidently, in a homogeneous Universe the gravitational force is the same everywhere. In a universe with minute density perturbations this will be no longer true, the density perturbations will induce local differences in gravity. In the vicinity of a region with a higher density the surplus of matter will exert an attractive gravitational force larger than the average value, near low density regions a deficit in matter will lead to a weaker force. Because of the differences in gravitational force the extent to which the expansion of the Universe will be decelerated will differ per location (or, from the moment onward that dark energy attains a dominant cosmic influence, it will not accelerate anymore to the same extent). During its early evolution an overdensity will experience a gradually stronger deceleration of its expansion velocity so that its its initial expansion will increasingly slow down with respect to the global Hubble expansion. Because matter gets attracted slightly more by a region of higher density it will also have the tendency to move towards that region. The mass of the overdensity will increase correspondingly, the slow-down of the initial cosmic expansion gets correspondingly stronger. When the region has become sufficiently overdense the mass of the fluctuation will have grown so much that its expansion may even come to a halt. The region decouples completely from the Hubble expansion, it turns around and starts to contract. If or as long as pressure forces are not sufficient to counteract the infall, the overdensity will grow without bound, and assemble more and more matter by accretion of

[1] As the result of a phase transition the very early Universe went through a phase of an astonishingly rapid exponential expansion. During this phase the universe expanded by a factor e^{100}. A good candidate for this phase transition is the GUT (Grand Unified Theory) transition, about 10^{-36} sec after the Big Bang. In this phase transition the *strong nuclear force* splitted itself off from the *electroweak force*. Not only would *inflation* offer an explanation for why the cosmos has a *flat geometry*, but also for the origin of the primordial density fluctuations and thus for the origin of all structure in the Universe.

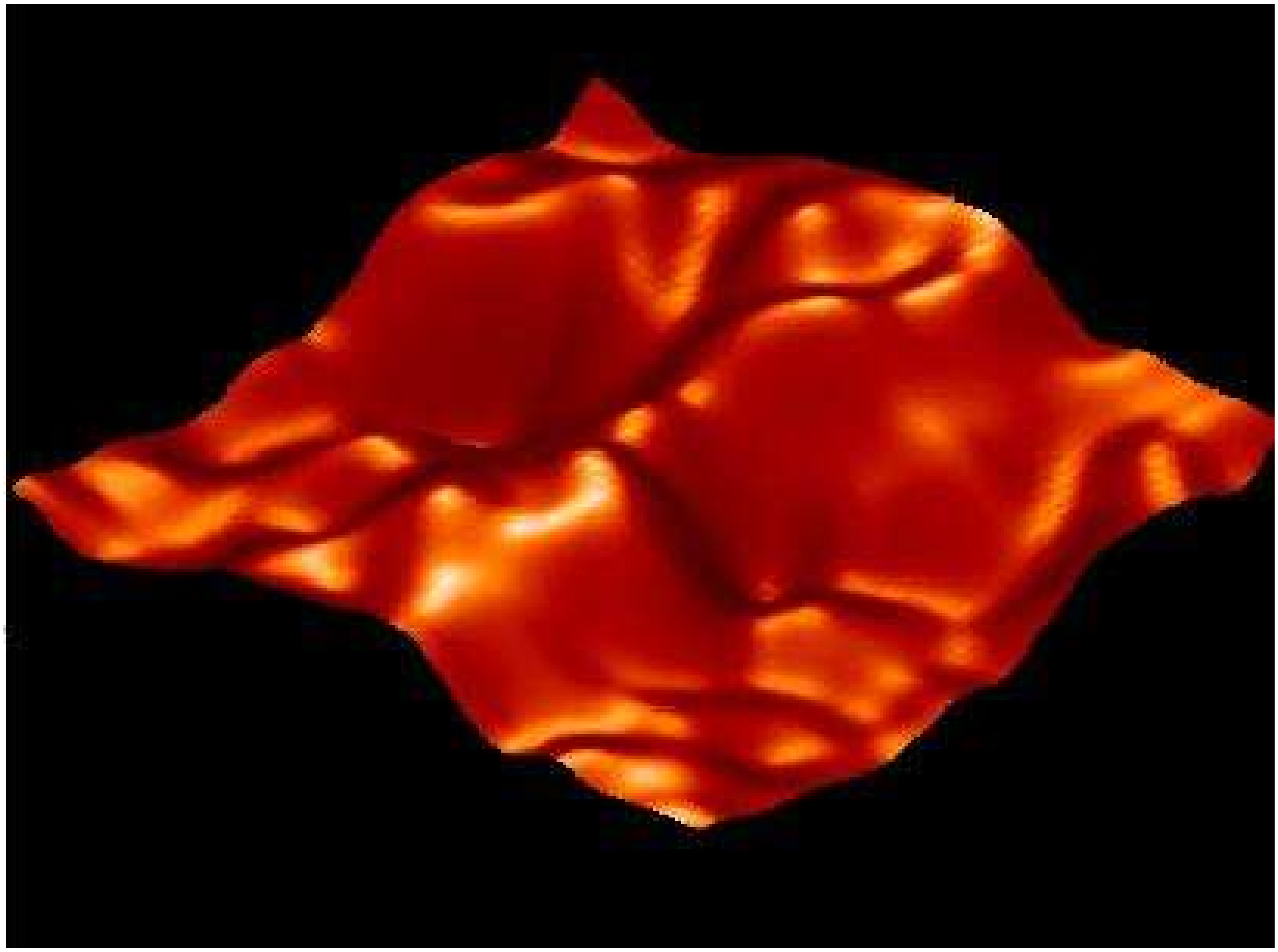


Figure 1. *Example of a random field of Gaussian density fluctuations. The figure illustrates the corresponding density field realization in a plane by means of a surface plot.*

matter from its surroundings (for an illustration of the process see fig. 3. Ultimately this will turn into a full *collapse* to form a gravitationally bound object. By means of the mutual exchange of energy the matter in the collapsed overdensity will seek to reach *virial equilibrium*. Once it has settled as such a genuine recognizable *cosmic object* has formed. Their precise nature (galaxy, cluster, etc.) and physical conditions are determined by the scale, mass and surroundings of the initial fluctuation.

The opposite tendency may be seen to occur in the case of primordial density depressions. Because they contain less matter than on average, the deceleration of the matter in and around such an underdense region is less than that of the global Hubble expansion. Matter will therefore tend to get displaced somewhat further, with the net result of matter streaming out of the interior of the underdensities and them expanding with respect to the global Universe. As the process continues and becomes more pronounced the gravitational instability process results in the gradual emergence of a void in the matter distribution.

The early linear stages of structure formation have been succesfully and completely worked out within the context of the linear theory of gravitationally evolving cosmological density and perturbation fields (Peebles 1980). At every cosmologically interesting scale, it aptly and succesfully describes the situation in the early eons after the decoupling of radiation and matter at recombination. However, we should also be aware of the fact that linear theoretical predictions fail soon after gravity surpasses its initial moderate imprint and nonlinear features start to emerge. Primordial density perturbations on a small scale appear to have a much higher amplitude than those on larger scales. This leads to a hierarchical process of structure formation, with small-scale perturbations being the first one to become nonlinear and develop into cosmic objects. This also implies that at any cosmic epoch we can identify spatial scales over which the spatially averaged perturbations still reside in a linear phase and the spatial density field resembles that of a panorama of gently sloping hills. At present this concerns scales larger than $\approx 10h^{-1}$Mpc. If we assume, or impose, the condition that one may discard the smaller scale structures which such large linear perturbations contain (the smaller-scale nonlinear and linear structures), the linear analysis of structure evolution is at the present cosmic epoch still valid for these large Megaparsec scales. By implication, the study of the cosmic mass distribution and cosmic flows on scales larger than $\approx 10\,h^{-1}$Mpc is therefore based on the framework of linear perturbation theory.

We will thus first set out to analyze the early linear phase of structure formation. It will help us to develop an intuition for the processes involved with gravitational instability. It will also provide us with an important set of tools for analyzing the observations of structure of the Universe on scales exceeding $\approx 10h^{-1}$Mpc, and as well that of the structure of the primordial Universe as observed through the angular distribution of the microwave background temperature.

2. Perturbation Quantities

The description of the formation and evolution of structure in the Universe, against the background of the global, expanding, and uniform FRW Universe it is preferrable to focus on the quantities that specify the development of the corresponding density and velocity deviations from the global cosmic background. In other words, we wish to relegate the background FRW Universe also literally to the background of our formalism. Note, however, that the reality of the Hubble expansion will always be present and will therefore also appear, in a different disguise, in the resulting equations.

2.1. Comoving Coordinates

The location of an object of parcel of matter/radiation in the Universe is specified by its physical coordinates $\mathbf{r}$. In an expanding Universe, its evolution is dictated by the Hubble expansion. For an ideal uniform FRW Universe, only the Hubble expansion changes the coordinate. Because the Hubble expansion is uniform throughout the Universe we have seen that it can be encrypted in a universal expansion factor $a(t)$, such that the location $\mathbf{r}$ of any object moves along,

$$\mathbf{r}(t) \,=\, a(t)\,\mathbf{x}\,. \tag{1}$$

By convention, we have chosen the dimensionless expansion factor $a(t)$ such that $a(t_0) = a_0 = 1$ for the present cosmic epoch. By definition of course $a(t = 0) = 0$ at the very time of the Big Bang itself. The comoving position $\mathbf{x}$ remains fixed in an FRW Universe, one may see it as the location at which an object is pinned to the expanding background Universe and subsequently moves along with the expansion of that background.

While in a pure FRW Universe $\mathbf{x}$ remains fixed in time, in the context of structure formation it will change due to the corresponding displacements in comoving space. Once there are gravity perturbations inducing motions of objects with respect to the background Universe the position $\mathbf{r}$ of an object will not only evolve through the development of $a(t)$: also the comoving coordinate $\mathbf{x}$ becomes a time-dependent quantity $\mathbf{x}(t)$.

It is therefore much more convenient, optimizing the visibility of the displacement of an object in comoving space, to focus on its **comoving position** $\mathbf{x}(t)$

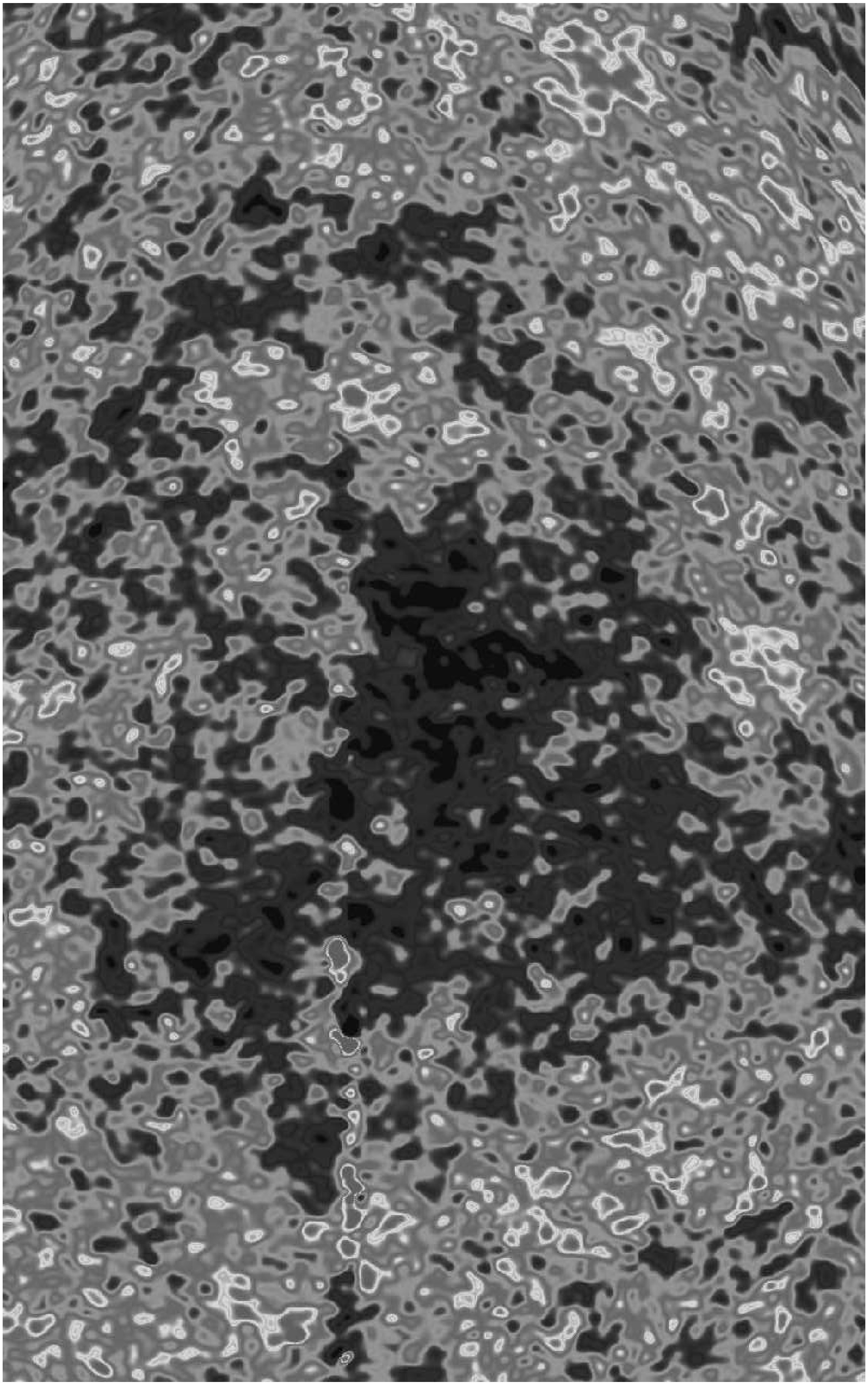


Figure 2. *A region of the CMB sky as observed by the WMAP microwave background satellite. The grayscale map represents temperature fluctuations in the CMB, with an amplitude of* $\Delta T \approx 10^{-5}$K, *reflecting underlying primordial density and velocity perturbations. These are the seeds of the structure observed in the present Universe.*

$$\boxed{\mathbf{x}(t) \,=\, \frac{\mathbf{r}(t)}{a(t)}} \qquad (2)$$

2.2. Density Perturbations

Perturbations from the global Universe may come in a variety of ways (see later chapter on perturbation character, treating issues such as *adiabatic fluctuations*). The primary perturbation mode involves the **(energy) density** $\rho(\mathbf{r},t)$ at a particular cosmic location $\mathbf{r}$.

By virtue of the *Cosmological Principle* the background FRW Universe has a global uniform density $\rho_u(t)$. In an unperturbed Universe the density has the same value throughout the Universe, $\rho(\mathbf{r},t) = \rho_u(t)$. For notational purposes, we distinguish the background density at a cosmic time t from the local density values $\rho(\mathbf{r},t)$ and denote it by $\rho_u(t)$.

The density perturbation at a comoving location $\mathbf{x}$ is most conveniently characterized by its fractional difference $\delta(\mathbf{x},t$ with respect to the the background Universe,

$$\boxed{\delta(\mathbf{x},t) \,\equiv\, \frac{\rho(\mathbf{x},t)-\rho_u(t)}{\rho_u(t)}} \qquad (3)$$

Arguably, the quantity δ may be considered to be the **key** quantity of this course. Structure formation is all about the growth of δ in an expanding and evolving Universe ! Evidently, in an unperturbed Universe with $\rho(\mathbf{r},t)=\rho_u(t)$ everywhere $\delta(\mathbf{x},t)=0$. In fact, this will be true at any time by lack of any gravitational source to generate fluctations (see eqn. 108).

Note that positive density fluctuations may in principle grow limitless: galaxies correspond to $\delta \approx 10^6$ fluctuations, clusters of galaxies to $\delta = 1000$ fluctuations on a scale of $R_c = 1.5h^{-1}$Mpc correspond and in principle there is nothing to prevent collapse to $\delta = \infty$. Negative density perturbations, on the other hand, have a strict lower limit, $\delta = -1$. Emptier than empty does not exist. Nonetheless, theoretically we will see that we *do* in fact sometimes make calculations with negative δ values, for example corresponding to a hypothetical *linear growth* of primordial perturbations. Thus, always take care to appreciate the context ! In a full and detailed treatment of (energy) density perturbations in the Universe we should differentiate between the contributions of the different components of the Universe. Radiation, Dark Matter, Baryonic Matter and Dark Energy have their own individual cosmological history. The total energy density $\rho(\mathbf{r},t)$ is the **sum** of the various components in the Universe,

$$\rho(\mathbf{r},t) \,=\, \rho_b(\mathbf{r},t)+\rho_{DM}(\mathbf{r},t)+\rho_{rad}(\mathbf{r},t)+\rho_v(\mathbf{r},t)$$

Cosmological epochs are identified by the components which are gravitationally dominant and dictate their dynamical evolution. In terms of their global gravitational influence dark matter and baryonic matter contribute and evolve equivalently. On cosmological scales we may therefore combine them into a total matter density $\rho_m = \rho_u + \rho_{DM}$. In this chapter we will make no further distinction between baryonic and dark matter and include both components in the total matter contribution.

The corresponding perturbations in the energy density of the Universe are therefore composed of perturbations in the various cosmic components,

$$\rho_u(t)\delta(\mathbf{r},t) \,=\, \rho_{b,u}\delta_b \,+\, \rho_{DM,u}\delta_{DM} \,+\, \rho_{rad,u}\delta_{rad} \,+\rho_{v,u}\delta_v\,. \qquad (4)$$

Each component may have its own (primordial) perturbation character. Dark energy does not have any density fluctuations, i.e. $\delta_v = 0$ always, a result of its negative pressure and repulsive gravity. Cosmological perturbations will evolve quite differently for different **perturbation modes**, each identified on the basis of the nature of radiation perturbations with respect to the matter perturbations. The mode of **isothermal perturbations** only involves matter perturbations, radiation would remain distributed uniformly throughout the Universe. On the other hand, the primordial perturbations δ_m and δ_{rad} of matter and radiation may be completely equivalent (except for a constant proportionality factor), corresponding to a zero fluctuation in the entropy. These are called **adiabatic perturbations**. A third mode of perturbations is that of **isocurvature perturbations** in which radiation and matter perturbations cancel each other such that the local curvature remains equal to the global cosmological curvature. Moreover, the evolution of the various components is complicated to a considerable extent by their mutual interactions. A simple illustration is that of baryonic matter initially without density perturbations: while dark matter creates ever deeper potential wells baryonic matter will fall in and experience increasing density perturbations. Also, in the *pre-recombination* epoch baryonic matter is closely coupled to radiation so that its evolution is seriously affected by the corresponding radiation pressure gradients.

In the analysis in this chapter of the linear evolution of perturbations we will mainly concentrate on the evolution of individual components of the Universe. As most of the structure formation happens during the *matter-dominated era*, this mainly involves an assessment of linear evolution of matter perturbations. While this single component analysis is superior in providing insight into the essentials of structure formation, it is clear that a formal treatment will be more complex. Towards the end of the chapter we will treat a simple situation of perturbation evolution in a Universe filled with radiation and matter.

2.3. Pressure Perturbations

Together with perturbations in the spatial energy density distribution we also find perturbations in the pressure of corresponding cosmic media. The resulting pressure gradients will have a considerable impact on the evolution of structure, the pressure forces do counteract gravitational contraction. Proper and complete treatment of gravitational instability should include this. A more appropriate name for "gravitational instability" therefore goes by the name of **"Jeans Instability"**.

We will see in chapter 6 that structure will only be able to form when gravity prevails over the pressure forces. This will be possible when the corresponding mass scale is larger than the characteristic Jeans Mass, the mass set by the sound speed in the medium: as long as pressure waves – soundwaves – can travel through the entire perturbation within in less than its collapse timescale gravitational collapse will be withhold by the pressure gradients in the medium.

With respect to the total pressure $P(\mathbf{r},t)$ we may identify the pressure perturbation $p(\mathbf{r},t)$ with respect to the global pressure P_u,

$$p(\mathbf{r},t) \,=\, P(\mathbf{r},t) \,-\, P_u\,, \tag{5}$$

whereby we should remark that physically the difference between P and p is only in the relativistic source term for the gravitational field (Poisson equation). For the pressure force, the result of pressure gradients, there is not distinction between p and P.

Cosmic pressure includes three contributions. **Dark Matter** is assumed to consist of weakly interacting particles and to form a pressureless medium (note that strictly speaking this would not be true for a medium of relativistic cosmic neutrinos). Total pressure is therefore the sum of the pressure in the baryonic matter, in radiation and in the dark energy,

$$P(\mathbf{r},t) \,=\, P_b(\mathbf{r},t) + P_{rad}(\mathbf{r},t) + P_v(\mathbf{r},t)\,.$$

As for its contribution to the gravitational field the contribution by the baryonic pressure can be ignored. It is always much less than the corresponding energy density, $P_b \ll \rho_b/c^2$. Meanwhile, because dark energy remains a perfectly uniformly distributed medium (see sect. **??**), the contributions to pressure perturbations (and gradients) are confined to those in the baryonic matter and radiation,

$$p(\mathbf{r},t) \;=\; p_b(\mathbf{r},t) + p_{rad}(\mathbf{r},t)\,.$$

Pressure forces are very important during the *pre-recombination era*. During the *radiation-dominated* epoch this is self-evident. After matter takes over as gravitationally dominant component and until *recombination*, radiation pressure remains the main agent for pressure forces through the tight coupling between baryonic matter and radiation.

2.4. Velocity Perturbations

In addition to density perturbations we usually are dealing with **velocity perturbations**. They are usually intimately coupled to the density perturbations. The corresponding gravity perturbations do induce velocities. Theoretically, there might be extra velocity perturbation contributions. Mostly, we will consider the socalled *growing mode* situation in which they are entirely coupled.

In an unperturbed FRW Universe all matter is moving along with the Hubble expansion, characterized by the Hubble parameter $H(t)$,

$$H(t) \;=\; \frac{\dot{a}}{a}\,. \tag{6}$$

An object at a location $\mathbf{r}$ has a Hubble velocity $\mathbf{v}_H(\mathbf{r})$,

$$\mathbf{v}_H(\mathbf{r}) \;=\; H(t)\,\mathbf{r} \tag{7}$$

which in terms of its comoving coordinate is $v_H = \dot{a}\mathbf{x}$. In the generic perturbed case there is an additional *velocity perturbation* $\mathbf{v}$, often known by the name of *peculiar velocity*. The total velocity $\mathbf{u}$ of an object is then given by

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{v}_H(\mathbf{x},t) \;+\; \mathbf{v}(\mathbf{x},t)\,. \tag{8}$$

Given the fact that $\mathbf{r} = a(t)\mathbf{x}(t)$, we may easily see that

$$\begin{aligned}\mathbf{u} = \frac{d\mathbf{r}}{dt} \;&=\; \dot{a}\mathbf{x} \;+\; a\dot{\mathbf{x}}\\ &=\; \mathbf{v}_H(\mathbf{x},t) \;+\; a(t)\dot{\mathbf{x}}\,.\end{aligned} \tag{9}$$

In other words, the peculiar velocity $\mathbf{v}$ is the quantity describing the change in comoving position $\mathbf{x}$,

$$\boxed{\mathbf{v} \;=\; a(t)\dot{\mathbf{x}}} \tag{10}$$

2.5. Potential Perturbations

The perturbed gravitational field corresponding to the density perturbations will be specified by a location-dependent gravitational potential Φ. It is related to the density/energy fluctuations via the Poisson equation. In the generic cosmological situation we have to take into account of the contribution by matter, but also by that of the relativistic media of *radiation* and *dark energy*. Strictly speaking one needs to resort to a fully general relativistic treatment. However, in the situation we will be considering the radiation and dark energy fields are so weak that we can resort to a special relativistic treatment and use Newtonian gravity with a relativistic source term,

$$\nabla_r^2\Phi \;=\; 4\pi G\,\left(\rho(\mathbf{r},t) \;+\; \frac{3P}{c^2}\right)\,. \tag{11}$$

In the epoch of most interest for structure formation, the matter-dominated era, we may neglect the relativistic pressure contributions because its contribution is considerably smaller than the energy density of the Universe,

$$P \ll \rho c^2 . \tag{12}$$

This allows us to restrict ourselves to the conventional Newtonian continuity equation

$$\nabla_r^2 \Phi \;=\; 4\pi G\,\rho(\mathbf{r},t) . \tag{13}$$

In the equations above we already explicitly tagged the ∇^2 operator with an index r to indicate it is with respect to the physical coordinate system $\mathbf{r}$. Because we are only interested in the perturbation with respect to the background, we split the potential Φ in a background contribution Φ_u and a potential perturbation component ϕ. The background potential

$$\Phi_u \;=\; \frac{1}{2}\,a\ddot{a}x^2 , \tag{14}$$

includes the contribution by the background energy densities $\rho_{m,u}$, $\rho_{rad,u}$ and $\rho_{v,u}$ as well as that of the global relativistic pressure contributions $P_{rad,u}$ and $P_{v,u}$. The potential perturbation ϕ is therefore given by

$$\boxed{\phi(\mathbf{x},t) \;=\; \Phi(\mathbf{r},t) \;-\; \frac{1}{2}\,a\ddot{a}x^2} \tag{15}$$

Because the complete inventory of the uniformly distributed dark energy is included in the background potential Φ_u, it does not contribute anything to the potential perturbation ϕ. The perturbed Poisson equation (see section 3.3) specifies the relation between the potential perturbation ϕ and its sources, the perturbations in the matter density as well as those in the radiation energy density and pressure.

2.6. Peculiar Gravity

Having defined the potential perturbation $\phi(\mathbf{x},t)$, we may proceed to define the corresponding *peculiar acceleration*. The peculiar gravitational acceleration $\mathbf{g}$ is the extra acceleration with respect to the FRW background, and thus the gradient of the *potential perturbation*,

$$\boxed{\mathbf{g}(\mathbf{x},t) \;\equiv\; -\,\frac{\nabla\phi}{a}} \tag{16}$$

Note that the gradient in this equation is also with respect to the comoving coordinate system $\mathbf{x}$, leading to the extra factor $a(t)$ in the divisor. In addition, one may easily appreciate that with

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{a}\mathbf{x} + 2\dot{a}\dot{\mathbf{x}} + a\ddot{\mathbf{x}} \\ &= \ddot{a}\mathbf{x} + \mathbf{g}(\mathbf{x},t) \\ \\ \dot{\mathbf{v}} &= \dot{a}\dot{\mathbf{x}} + a\ddot{\mathbf{x}} \end{aligned} \tag{17}$$

we find that the *peculiar velocity* $\mathbf{v}$ and the *peculiar acceleration* $\mathbf{g}$ are related through the relation

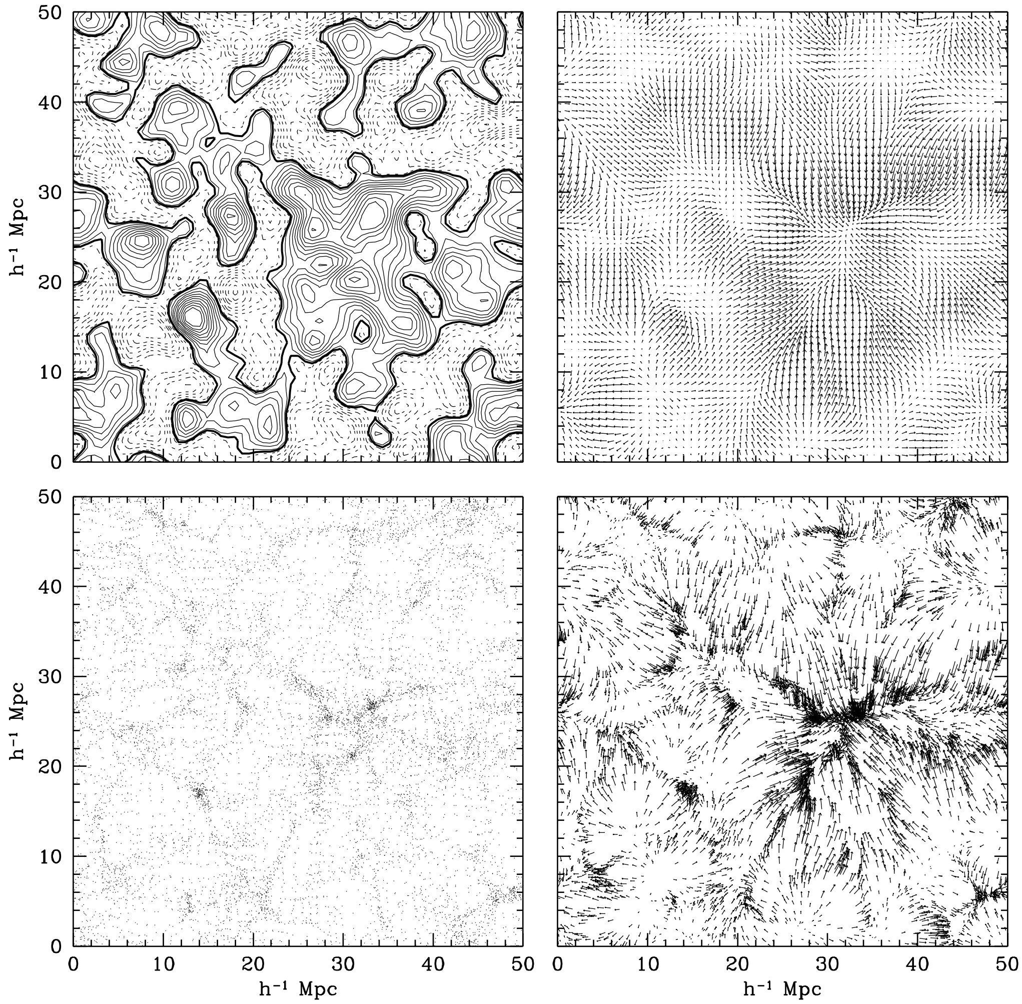


Figure 3. *Gravitational Instability: schematic presentation of process. Top lefthand: contour map of a (Gaussian) stochastic density field. Top righthand: the resulting gravitational force field. Lower lefthand: resulting (nonlinear) particle distribution. Lower righthand: vector map corresponding velocity field*

$$\boxed{\mathbf{g}(\mathbf{x},t) \;=\; \frac{1}{a}\,\frac{d(a\mathbf{v})}{dt}} \tag{18}$$

Thus while one easily recognizes the conventional relation of the gravitational acceleration being the time derivative of the velocity, through the definition of the perturbation quantities and the choice to work in comoving coordinates $\mathbf{x}$ we introduce extra factors $a(t)$.

3. Equations of Motion

Having established the physical *perturbation quantities*, we can set out to describe the full evolution of a system of coupled cosmic density-, velocity- and gravity fields.

On the large Megaparsec scales we are studying the formation of structure we may consider the matter and radiation content of the Universe as a *continuous fluidum*. In other words, we may gloss over the details of a discrete matter distribution. The dynamics of the Megaparsec Universe appears to be mainly governed by *dark matter*. Hypothetically the most plausible possibility is that it exists of as yet unidentified weakly interacting elementary particles. With discreteness restricted to microscopic levels, a continuous matter distribution is a perfectly good approximation on Megaparsec scales. Also the *baryonic matter* distribution is perfectly suited for a description in terms of a fluidum approximation. At later epochs, however, the approximation will break down on scales smaller than a hundred kiloparsec. At these small scales discrete objects, galaxies and stars, have condensed out of the cosmic matter distribution. As an illustration consider the evolution of a globular cluster: containing in the order of 10^5-10^6 stars its dynamical evolution may not be properly understood without taking into account the discrete nature of its matter distribution. Nonetheless, on cosmological Megaparsec scales the fluidum approximation remains a reasonable and very useful one. Even the evolution of perturbations in the *radiation* energy density, consisting of a sea of photons (and also cosmic background neutrinos), may be analyzed in terms of a fluidum.

The evolution of a fluid is dictated by three fluid equations. The **continuity equation** or **energy equation** describes the conservation of energy (mass). The **Euler equation** is the force-law describing the acceleration of the fluid elements as a result of the gravitational force and pressure (gradient) in the fluid. The sources for the gravitational field are specified by the **Poisson equation**. The prevailing pressure of a medium is obtained through the **equation of state**, specifying the nature of the cosmic fluid.In the following we will introduce these equations. We do this in the *physical coordinate* system $\mathbf{r}$ and on the basis of the full physical quantities. These are the density $\rho(\mathbf{r},t)$ at a location $\mathbf{r}$, the corresponding total velocity $\mathbf{u}(\mathbf{r},t)$, and total gravitational potential $\Phi(\mathbf{r},t)$ as well as the pressure $P(\mathbf{r},t)$ of the medium.

In the following we will first treat the fluid equations for a strictly matter-dominated Universe. As has been mentioned earlier the Universe does contain two important relativistic media, **radiation** and **dark energy**. Strictly speaking for these media we should resort to a fully general relativistic treatment, but because the cosmologically interesting situations involve weak fields for these media we may use specially relativistic fluid mechanics and Newtonian gravity with relativistic source terms. Ignoring radiation, a rather decent approximation at the present epoch given that $\Omega_{r,0} \approx 10^{-5}$, we can even assert that the perturbation fluid equations are valid for a Universe filled with **matter** and **dark energy**. Because of its *negative pressure* and *repulsive gravity* perturbations in the dark energy component will not be able to develop so that it remains a purely uniform medium. Following the transformation to the fluid equations in comoving space, subtracting the background components, we will be left with exactly the same equations as in the pure matter-dominated situation.

3.1. Continuity Equation

The first equation, the **Continuity Equation** or **Energy Equation**, assures the conservation of energy (mass). It guarantees that the growth (or decrease) of mass in a particular volume of space is equal to the netto amount of matter flowing into the volume (the flux). For the non-relativistic matter component in the Universe we have the conventional Newtonian continuity equation,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot \rho \vec{u} = 0} \tag{19}$$

Within the context of our assessment of gravitational instability, the continuity equation establishes the significant link between the density $\rho(\mathbf{r},t)$ of a medium and the corresponding velocity flow $\mathbf{u}(\mathbf{r},t)$.

Both **radiation** and **dark energy** are cosmic components whose nature makes it necessary to use a modified, special relativistic, energy equation. The corresponding inertia term needs to include the pressure inertia contribution,

$$\rho \quad \longmapsto \quad \rho + \frac{P}{c^2}\,. \tag{20}$$

and the **Energy Equation** becomes

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot \left(\rho + \frac{P}{c^2}\right) \vec{u} \,=\, 0} \tag{21}$$

3.2. Euler Equation

The **Euler equation**, of **force equation**, specifies the sources of the velocity flows in our cosmic fluidum. Evidently, one needs forces to set the fluidum into motion. The two we take into account are the gravitational force, the integrated gravitational attraction by all matter in the Universe, and the force due to the pressure in the medium. We will make the reasonable approximation to discard any influence of magnetic fields on these scales (be it that we may not forget about them !).

The Euler equation embodies *Newton's second law* for a fluidum, specifying the acceleration of a particular parcel of fluid on the lefthand side and the accelerating force terms on the righthand side,

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}_r\right) \vec{u} \,=\, -\frac{1}{\rho} \vec{\nabla}_r P \,-\, \vec{\nabla}_r \Phi} \tag{22}$$

Note that here the fluid equations are written in **Eulerian** language. That is, we act as if we are fixed to a particular location **r** and then look how a quantity changes at the spot. Often it is more insightful in terms of the physics involved to take a **Lagrangian** point of view. This implies us to travel along with a particular fluid element. The acceleration of such a fluid element is precisely the full lefthand term of the Euler equation:

$$\frac{d\vec{u}}{dt} \equiv \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}_r)\,\vec{u}\,. \tag{23}$$

For the **radiation component** the Euler equation concerns a slightly modified version as it involves the contribution by pressure to the inertia of the medium,

$$\rho \quad \longmapsto \quad \rho + \frac{P}{c^2}\,. \tag{24}$$

so that the special relativistic **Euler Equation** is specified by the relation

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}_r\right) \vec{u} \,=\, -\frac{1}{\left(\rho_{rad} + P_{rad}/c^2\right)} \vec{\nabla}_r P \,-\, \vec{\nabla}_r \Phi} \tag{25}$$

3.3. Poisson Equation

Having established that the gravitational potential Φ and the pressure p are the agents of the velocity flows in the cosmic fluidum, we need to establish what their ultimate source is.

The gravitational field Φ is induced by the total cosmic matter and energy distribution $\rho(\mathbf{r}, t)$. The most general case involves the contributions by radiation and dark energy components as sources for the gravitational field. This means that we also need to take into account their pressure contributions,

$$\boxed{\nabla_r^2 \Phi \;=\; 4\pi G \left(\rho(\mathbf{r}, t) \,+\, \frac{3P}{c^2}\right)\,.} \tag{26}$$

Notice that the source term is the **sum** of all components in the Universe, including the contributions by the energy density of baryonic matter, dark matter, radiation and dark energy, as well as the pressure contributions by baryonic matter, radiation and dark energy,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \rho_b(\mathbf{r}, t) + \rho_{DM}(\mathbf{r}, t) + \rho_{rad}(\mathbf{r}, t) + \rho_v(\mathbf{r}, t) \\ P(\mathbf{r}, t) &= P_b(\mathbf{r}, t) + P_{rad}(\mathbf{r}, t) + P_v(\mathbf{r}, t) \\ &\approx P_{rad}(\mathbf{r}, t) + P_v(\mathbf{r}, t) \end{aligned} \tag{27}$$

In the above we assume that **dark matter** consists of weakly interacting particles, forming a pressureless medium. Because the pressure of a normal baryonic gas is always much less than its density, $P_b \ll \rho_u/c^2$, we may discard its contribution to the gravitational potential. In a matter-dominated Universe we only need to take into account the matter density of matter as source of the gravitational potential in the Universe, leaving us the pure **Newtonian form** of the **Poisson equation**,

$$\boxed{\nabla_r^2 \Phi \;=\; 4\pi G\, \rho} \tag{28}$$

By subtracting the background contribution from the Poisson equation, we obtain the Poisson equation for for the perturbed potential ϕ,

$$\begin{aligned} \nabla_x^2 \phi &= 4\pi G\, a^2 \left[\rho_m(\mathbf{x}, t) - \rho_{m,u}(t)\right]\,, \\ &= 4\pi G\, a^2\, \rho_{m,u} \delta_m(\mathbf{x}, t)\,. \end{aligned} \tag{29}$$

Because dark energy does not contribute to the potential perturbation, this equation is also valid for a Universe consisting of **matter** and **dark energy**. This is indeed the situation pertaining at the current epoch (see sect 8).

When we also have to take into account the gravitational influence of radiation, essentialy before and around the *matter-radiation equivalence epoch* (and mostly up to the recombination era), the Poisson equation gets modified through the presence of the radiation pressure contribution (see sect 9.2):

$$\begin{aligned} \nabla_x^2 \phi &= 4\pi G\, a^2\, \left[\rho_m(\mathbf{x}, t) - \rho_{m,u}(t)\right] \,+ \\ &\quad\; 4\pi G\, a^2\, \left[\rho_{rad}(\mathbf{x}, t) - \rho_{rad,u}(t)\right] \,+\, 4\pi G\, a^2 \left[\frac{3p_{rad}(\mathbf{x}, t)}{c^2}\right] \\ &\Downarrow \\ \nabla_x^2 \phi &= 4\pi G\, a^2\, \left[\rho_{m,u} \delta_m(\mathbf{x}, t) \,+\, 2\rho_{rad,u} \delta_{rad}(\mathbf{x}, t)\right] \end{aligned} \tag{30}$$

In this we have used the equation of state for radiation (see next section),

$$P_{rad} = \frac{1}{3} \rho_{rad} c^2 . \tag{31}$$

3.4. Equation of State

The pressure P in a particular medium of density ρ depends on its nature. Its value depends on the density of the medium as well as its entropy, specified through the the **equation of state**,

$$\boxed{P = P(\rho, S)} \tag{32}$$

Note that in addition to the density ρ the equation of state is also dependent on the **entropy** S of the fluid.

Within a cosmological context, the equation of state for the different components is usually characterized by a constant w,

$$\boxed{P(\rho) = w\, \rho c^2} \tag{33}$$

In the case of matter, strictly speaking for the approximation of "cosmic dust", pressure is assumed to be negligible on cosmological scales: $w_m = 0$. On (comoving) scales smaller $\approx 1 Mpc$ this approximation is not really appropriate, pressure forces in the baryonic matter component will become a significant influence. Here we will neglect it. Radiation is known to have $w_{rad} = \frac{1}{3}$. Dark energy, seemingly dominant at the current cosmic epoch, has a value $-1 < lew_v < -\frac{1}{3}$.

3.5. Fluid Equations: total

In summary, for a particular cosmological component j, with an energy density ρ_j and equation of constant parameter w_i, the resulting full set of three fluid equations is

$$\boxed{\begin{aligned} \frac{\partial \rho_j}{\partial t} + \vec{\nabla}_r \cdot (1+w_j)\rho_j \vec{u} &= 0 \\ \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}_r\right) \vec{u} &= -\frac{1}{(1+w_j)\rho_j} \vec{\nabla}_r P - \vec{\nabla}_r \Phi \\ \nabla_r^2 \Phi &= 4\pi G \left\{ \sum_l (1+3w_l)\, \rho_l \right\} \end{aligned}}$$

in which the Poisson equation includes the sum over all components (matter, radiation and dark energy) in the Universe. In the analysis in this chapter we will discard the role of pressure forces $-\vec{\nabla}_r P$, and thus continue without this term in the Euler equation.

4. Perturbations: from physical to comoving coordinates

The purpose of our analysis is to study the evolution of perturbations with respect to the background FRW Universe. Such an analysis is considerably facilitated by translating the above three fluid equations, formulated with respect to *physical coordinates* and in terms of full *physical* quantities, to a set of fluid equations with respect to *comoving coordinates* and in terms of *perturbation* quantities. In other words, instead of evaluating the

- density $\rho(\mathbf{r}, t)$,
- velocity $\mathbf{u}(\mathbf{r}, t)$,
- gravitational potential $\Phi(\mathbf{r}, t)$,

with respect to the physical coordinate system $\mathbf{r}$, we will evaluate the evolution of the

- density perturbation $\delta(\mathbf{x}, t)$,
- peculiar velocity $\mathbf{v}$,
- potential perturbation ϕ.

For a cosmic component j (matter, radiation or dark energy) the resulting set of fluid equations is:

$$\boxed{\begin{aligned}
\frac{\partial \delta_j}{\partial t} + \frac{(1+w_j)}{a}\vec{\nabla}_x \cdot (1+\delta_j)\vec{v} &= 0 \\
\frac{\partial \vec{v}}{\partial t} + \frac{1}{a}\left(\vec{v}\cdot\vec{\nabla}_x\right)\vec{v} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}_x \phi \\
\nabla_x^2 \phi &= 4\pi G\, a^2 \left\{\sum_j (1+3w_j)\,\rho_{j,u}\,\delta_j\right\}
\end{aligned}}$$

In the above set of equations we still explicitly tagged the gradient and nabla operators by an index $''x''$. Because these equations are going to be the basis of our further evaluations, this index will be dropped in the remainder of this chapter. Unless otherwise stated or indicated these operators are defined with respect to comoving coordinates $\mathbf{x}$.

Because energy density fluctuations in the dark energy component do not exist (see section 8), the sum in the Poisson equation only needs to include the component of matter and radiation. During the matter-dominated epoch of structure formation, the most important one, also the radiation contribution may be neglected.

Of the comoving fluid equations the Poisson equation is the one most resembling its original form in physical coordinates. The background cosmology figures in via the FRW background density $\rho_{j,u}$ and the cosmological expansion factor $a(t)$. Also the *continuity equation* in comoving space $\mathbf{x}$ retains a close resemblance to its form in physical space $\mathbf{r}$. The background cosmology enters via the expansion factor $a(t)$ in front of the divergence term. The Euler equation is the one mostly affected, it includes an extra factor

$$\frac{\dot{a}}{a}\mathbf{v}\,. \tag{34}$$

This term may be considered as a *"Hubble expansion drag term"*. Due to the expansion of the background FRW Universe velocity perturbations get gradually damped: the extra displacement involved with the peculiar velocity will bring the motion of a particle more and more in agreement with the background Hubble expansion and the motion of the particle gets more and more in line with the expected Hubble expansion. The only way to sustain and increase velocity perturbations is therefore the righthand source term, the influence of the gravitational field.

One may appreciate this most directly by rewriting the Euler equation into its Lagrangian formulation. Through the translation from a **comoving Eulerian** to a **comoving Lagrangian formulation**

$$\frac{d}{dt} \quad \Rightarrow \quad \frac{\partial}{\partial t} + \frac{1}{a}\,\mathbf{v}\cdot\nabla\,, \tag{35}$$

we can rewrite the Euler equation

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{\dot{a}}{a}\,\mathbf{v} \quad &= \quad -\frac{1}{a}\,\nabla\phi \\ &\Downarrow \\ \frac{d\mathbf{v}}{dt} \quad &= \quad -\frac{\dot{a}}{a}\,\mathbf{v} - \frac{1}{a}\,\nabla\phi \end{aligned} \tag{36}$$

This leads to an immediately recognizable equation of motion,

$$\frac{d\,a\mathbf{v}}{dt} = -\nabla\phi\,. \tag{37}$$

EXERCISE: Show how you can transform the fluid equations in **physical coordinate system** (eq: 34) to fluid equations in **comoving coordinate system** (eq: 34). It is most convenient to do this by restricting yourself to one medium, matter, in the matter-dominated epoch.

5. Linear Evolution

While the set of three fluid equations form a fully selfconsistent description of the cosmic fluidum, in the general situation an analytical evaluation is not feasible. We may understand this already from a superficial evaluation of eqn. (34). The presence of higher order terms like

- $\delta(\mathbf{x})\,\mathbf{v}$ in the continuity equation
- $\left(\vec{v}\cdot\vec{\nabla}_x\right)\,\vec{v}$ in the Euler equation

reflects the nonlinearity of a generally evolving system. Nonlinear couplings between the various perturbation quantities will render the system insolvable for generic density and velocity fields.

A very important and crucial exception to this is the situation in which the density and velocity perturbations are still very small. In that case the coupling terms are negligible and can be discarded, yielding a *linearized* set of fluid equations. This linear system can be fully solved. The linear regime is defined by density and velocity perturbations of a small amplitude. In explicit terms this means

$$\begin{aligned} \delta &\ll 1 \\ \left(\frac{vt_{exp}}{d}\right)^2 &\ll \delta \end{aligned} \tag{38}$$

where d is the coherence length for spatial variations of δ, v the characteristic fluid velocity and t_{exp} the expansion time $\sim (G\rho_u)^{-1/2}$ (see Peebles 1980, § 10). The **Linear Regime** of gravitational clustering is **extremely important** for a variety of reasons.

- First, we know that on all scales primordial fluctuations were extremely small, $\delta \ll 1$. On all scales the first instances of structure formation were linear in character.
- Also, the linear stage of structure development is a relatively long lasting one. Once a perturbation is entering the nonlinear regime, $\delta \approx 1$, we will notice ever shorter timespans in which particular higher order moments are dominant.
- One may always find large scales on which the density and velocity perturbations still reside in the linear regime. A the current cosmic epoch this concerns spatial scales larger than $\approx 10\, h^{-1}$Mpc.
- With experiments measuring the temperature fluctuations in the cosmic microwave background, we have established a probe to directly measure the prevailing linear density fluctuations at the recombination era. By working out the structure growth in the linear regime we will be enabled to translate these into the amplitude of fluctuations at the current epoch, against which we can compare the measured large scale structure in the galaxy distribution.

Under the conditions of linearity (eq. 39) we may discard the higher order terms $\delta\mathbf{v}$ and $\left(\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}$. Because both δ and $\mathbf{v}$ are small perturbed quantities, mutual quadratic and higher order products of these quantities are in turn negligible with respect to the first order perturbed quantities themselves. This leaves the following set of **Linearized Fluid Equations**:

$$\boxed{\begin{aligned}
\frac{\partial \delta_j}{\partial t} + \frac{(1+w_j)}{a}\vec{\nabla}_x \cdot \vec{v} &= 0\\
\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}\phi\\
\nabla^2\phi &= 4\pi G\, a^2 \left\{\sum_j (1+w_j)\rho_{j,u}\,\delta_j\right\}
\end{aligned}}$$

In the following sections we will treat a few of the most relevant situations of linear perturbation evolution. First, we will treat the evolution of matter perturbations in a pure matter-dominated Universe.

6. Matter-Dominated FRW Universes: Linear Perturbations

Most of structure formation has been proceeding during the matter-dominated era of the Universe. In the early stages after the *matter-radiation equivalence epoch*, until *recombination*, baryonic matter and radiation were still tightly coupled. In the meantime, density and velocity perturbations in the gravitationally dominant dark matter component could grow almost uninterruptedly. On scales larger than the *Jeans mass* this was equally true for the baryonic matter component.

If indeed we appear to live in a Universe closely characterized by the socalled *Concordance Model*, with $\Omega_{m,0} \approx 0.3$ and $\Omega_{\Lambda,0} \approx 0.7$, at a rather recent redshift of $z \approx 0.7$ dark energy has taken over as the component dominating the cosmic expansion. This is even later than the redshift $z_g \approx 1.3$ at which large scale structure growth ceases (see sect. 8). This justifies a concentration in detail on the matter-dominated epoch for our first assessment of the process of cosmic structure formation.

As most of matter in the Universe appears to consist of *collisionless* dark matter we are also justified in ignoring the effects of pressure and concentrate our analysis on a pure *"cosmic dust"* medium. Even for the baryonic matter component this is a good approximation in the aftermath of the *recombination era*, after which the Jeans mass dropped from a value of around $M_J \approx 10^{12}\, M_\odot$ to a mere $M_J \approx 10^5\, M_\odot$ (see chapter 6). We are therefore justified in concentrating on purely gravitating matter, neglecting any pressure effects, in identifying the principal characteristics of the early and large scale structure formation process. Perhaps the main virtue for our purposes is that it has the great benefit of providing the most illuminating insights into the evolution of a gravitationally unstable system.

6.1. Fluid Equations for Matter Perturbations

In our approximation we discard completely the influence of dark energy and radiation, so that the resulting set of fluid equations (cf. eqn. 34) becomes

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot \rho \vec{u} &= 0 \\ \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}_r\right) \vec{u} &= -\vec{\nabla}_r \Phi \\ \nabla_r^2 \Phi &= 4\pi G\, \rho \end{aligned}}$$

Transforming to comoving coordinates, these become

$$\boxed{\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}_x \cdot (1+\delta)\vec{v} &= 0 \\ \frac{\partial \vec{v}}{\partial t} + \frac{1}{a}\left(\vec{v} \cdot \vec{\nabla}_x\right) \vec{v} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}_x \phi \\ \nabla_x^2 \phi &= 4\pi G\, a^2\, \rho_u\, \delta \end{aligned}}$$

For the linear regime we then find the resulting linearized fluid equations,

$$\boxed{\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}_x \cdot \vec{v} &= 0 \\ \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}\phi \\ \nabla^2 \phi &= 4\pi G\, a^2\, \rho_u\, \delta \end{aligned}}$$

6.2. Perturbation Evolution Equation for Matter Perturbations

From the three fluid equations we can directly infer the time evolution of the density perturbation $\delta(\mathbf{x}, t)$. By taking the divergence of the Euler equation,

$$\frac{\partial}{\partial t}\,(\nabla \cdot \mathbf{v}) + \frac{\dot{a}}{a}\,(\nabla \cdot \mathbf{v}) = -\frac{1}{a}\,\nabla^2 \phi\,,$$

and combining this with the continuity equation for the relation between the velocity divergence and δ and the Poisson equation for relating the potential ϕ and the density perturbation δ,

$$\begin{aligned} \nabla \cdot \mathbf{v} &= -a\,\frac{\partial \delta}{\partial t}\,, \\ \nabla^2 \phi &= 4\pi G\, a^2\, \rho_u\, \delta\,, \end{aligned} \tag{39}$$

we obtain a second order partial differential equation for the density perturbation δ,

$$\boxed{\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = \frac{3}{2}\Omega_0 H_0^2\,\frac{1}{a^3}\,\delta} \tag{40}$$

The above second order partial differential equation is the **linearized equation for the growth of density perturbations** $\delta(\mathbf{x}, t)$. It forms one of the key equations within the linear theory of gravitational instability. The fact that it concerns a second order differential equation in time coordinate t implies two things. *First*, one may see that it has *two solutions*,

$$\delta(\mathbf{x}, t) \quad = \quad \delta_1(\mathbf{x}, t) + \delta_2(\mathbf{x}, t)$$

Secondly, the fact that the equation only includes differential terms in time t implies that the time evolution is equivalent throughout the cosmic volume. In other words, the time evolution is independent of cosmic location $\mathbf{x}$ and the corresponding solutions can be separated into a spatial part $\Delta(\mathbf{x})$ and a temporal part $D(t)$, $\delta(t) = D(t)\Delta(\mathbf{x})$,

$$\boxed{\begin{aligned} \delta_1(\mathbf{x}, t) &= D_1(t)\,\Delta_1(\mathbf{x}) \\ \delta_2(\mathbf{x}, t) &= D_2(t)\,\Delta_2(\mathbf{x}) \end{aligned}} \tag{41}$$

The time evolution of $D(t)$ is specified by the evolution equation

$$\boxed{\frac{d^2D}{dt^2} + 2\,\frac{\dot{a}}{a}\,\frac{dD}{dt} = \frac{3}{2}\Omega_0 H_0^2\,\frac{1}{a^3}\,D} \tag{42}$$

In all, the implied **general linear solution** of the evolution equation may be written as the sum of two terms of separated time and spatial functions:

$$\boxed{\delta(\mathbf{x},t) = D_1(t)\,\Delta_1(\mathbf{x}) + D_2(t)\,\Delta_2(\mathbf{x})} \tag{43}$$

where $D_1(t)$ and $D_2(t)$ are the universal **density growth factors** for the linear evolution of density perturbations, and $\Delta_1(\mathbf{x})$ and $\Delta_2(\mathbf{x})$ represent the corresponding spatial configuration of the cosmic primordial matter distribution. From this results we may immediately appreciate that the rate with which the primordial densities are to grow in the linear regime is the same everywhere, solely dependent on the global time factors $D_1(t)$ and $D_2(t)$. From this we can infer that

- The density fluctuations $\delta(\mathbf{x},t)$ will grow at the same rate at every location,
- The topology of the matter distribution will remain exactly the same (ie. the contours do not change in geometrical shape).
- The density values of the corresponding density contours **do evolve**, all developing at the same rate, the **growth factor** $D(t)$. In other words, only the corresponding density contrast in the density field will increase.

The density growth factors are dependent upon the cosmological background: in different FRW Universes the growth of structure will proceed differently. This may be immediately observed from the inspection of the density perturbation growth equation 42. The FRW cosmological background enters through two terms, explicitly involving the cosmic expansion factor $a(t)$. One is the '$2\dot{a}/a$ term on the lefthand side, the **"Hubble dragterm"** reflecting the cosmic expansion. The second term is the righthand term, that of the evolution of the cosmic background density ρ_u, which in a matter-dominated Universe evolves like

$$\rho_u(t) \propto \frac{1}{a^3}\,. \tag{44}$$

Before assessing the general expression for the density growth factor $D(t)$ we will first assess a few specific cases. This in order to get a better appreciation for their meaning.

7. Linear Density Growth factors in Matter-dominated FRW Universes

The linear density growth factors $D(t)$ in a matter-dominated FRW Universe can be computed by solving fully analytically the linear structure growth equation 42.

For appreciation of these solutions we will first consider two specific situations. One is an $\Omega_0 = 1$ Einstein-de Sitter Universe, the other is a totally empty and freely expanding $\Omega_0 = 0$ FRW Universe. The linear structure growth for these two situations provide a good illustration of generic structure growth behaviour.

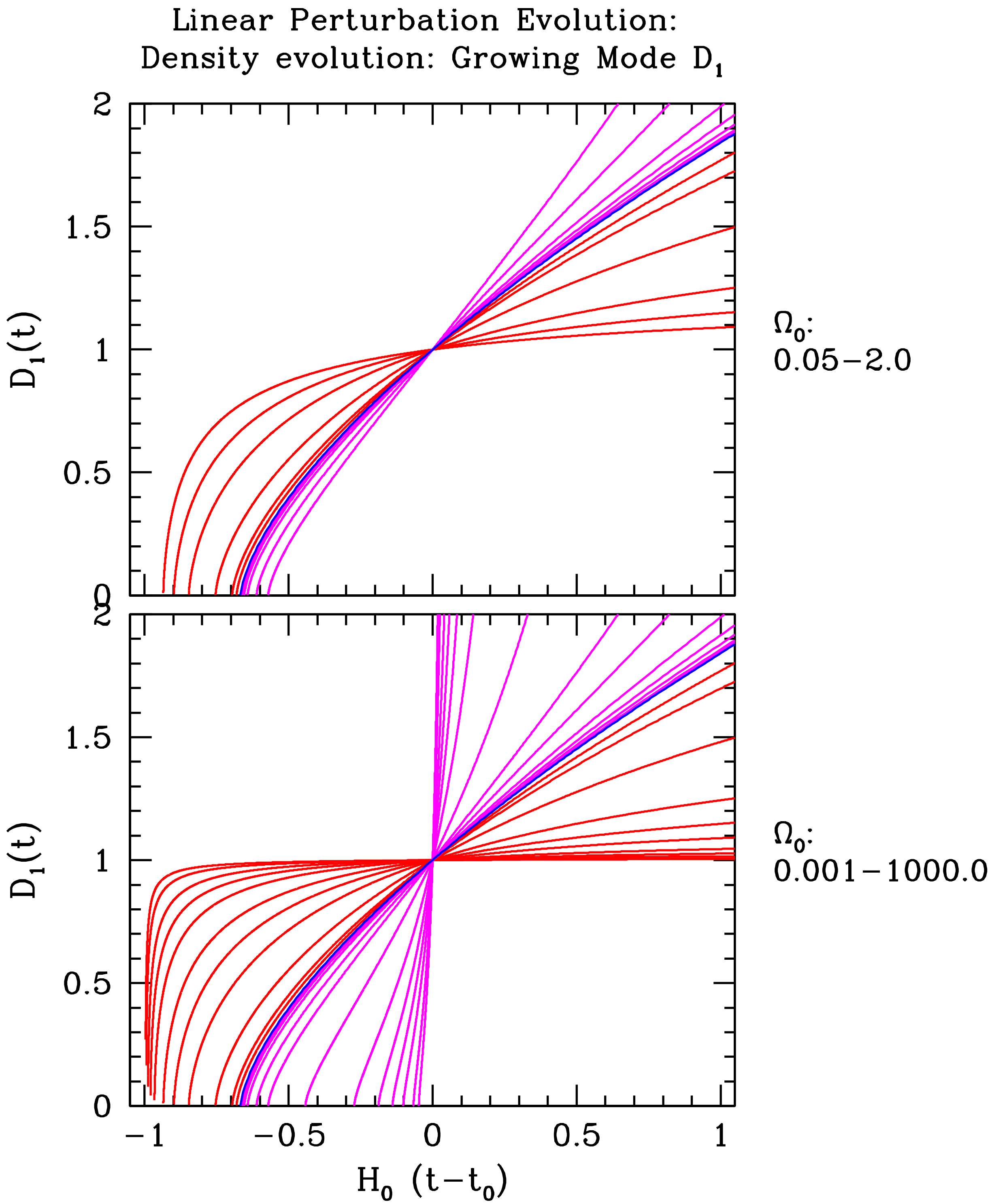


Figure 4. *Growing mode linear density growth factors $D_1(t)$ for a range of matter-dominated FRW Universes, plotted as a function of cosmic time. $D_1(t)$ has been normalized with respect to the current epoch, $D_1(t_0) \equiv 1$. Red: open Universes. Magenta: Closed Universes. Blue: Einstein-de Sitter Universes. Top range:* $0.05 < \Omega_0 < 2.0$*, Bottom range:* $0.001 < \Omega_0 < 1000.0$.

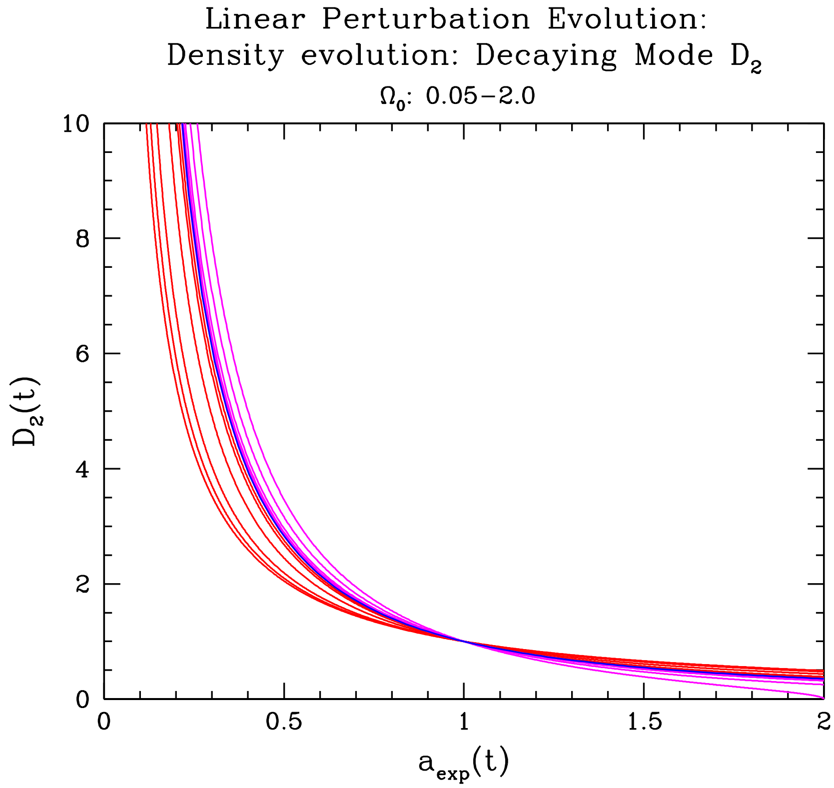


Figure 5. *Decaying mode linear density growth factors* $D_2(t)$ *for a range of matter-dominated FRW Universes, plotted as a function of expansion factor* $a(t)$*.* $D_2(t)$ *has been normalized with respect to the current epoch,* $D_2(t_0) \equiv 1$*. Red: open Universes. Magenta: Closed Universes. Blue: Einstein-de Sitter Universes. Range:* $0.05 < \Omega_0 < 2.0$.

Also these solutions represent the asymptotic solutions in the two asymptotic regimes of an evolving open FRW Universe. An open Universe will start off as a Universe which is close to an Einstein-de Sitter Universe and will gradually unfold into a Universe with low Ω, ultimately evolving into a freely expanding empty Universe. Thus, we expect that at early cosmic times structure in the Universe will grow according to the density growth in an Einstein-de Sitter Universe, $D(t) \propto t^{2/3}$, gradually slow down and finally come to a halt like that in an empty Universe.

7.1. Einstein-de Sitter Universe

An Einstein-de Sitter Universe, with $\Omega(t) = \Omega_0 = 1$ and Hubble parameter $H(t) = H_0$, expands like

$$a(t) = \left(\frac{3}{2}\,H_0 t\right)^{2/3} \tag{45}$$

so that

$$\begin{cases} \dfrac{\dot{a}}{a} & = \dfrac{2}{3t} \\ \\ \frac{3}{2}\,\Omega_0 H_0^2\,\dfrac{1}{a^3} & = \dfrac{2}{3t^2} \end{cases}$$

so that the *linearized density growth equation* is given by

$$\boxed{\frac{\partial^2\delta}{\partial t^2} + \frac{4}{3t}\frac{\partial\delta}{\partial t} = \frac{2}{3\,t^2}\,\delta} \tag{46}$$

One may recognize that this partial differential equation has 2 solutions, $\delta_1(\mathbf{x},t)$ and $\delta_2(\mathbf{x},t)$. They are both power-law functions of cosmic time t,

$$\delta_m(t) \propto t^{\alpha}\,, \quad m = 1, 2\,, \tag{47}$$

whose temporal parts, the linear structure growth factors $D_m(t)$ are given by

$$\boxed{\begin{array}{l} D_1(t) \,\propto\, t^{2/3} \\[2em] D_2(t) \,\propto\, t^{-1} \end{array}} \tag{48}$$

The first solution $\delta_1(t)$ involves a growth of the densities in the Universe. Note that in this specific case of an Einstein-de Sitter Universe, the growth is proportional to the expansion factor of the Universe,

$$D_1(t) \propto t^{2/3} \propto a(t)\,. \tag{49}$$

This solution is known as the **Growing Mode Solution**. The second solution, on the other hand, leads to a continuously declining density: the primordial density contrast is diminishing in time. It is named **Decaying Mode Solution**. Often, once its share (between growing and decaying mode) in the primordial density fluctuations is fixed it is discarded from further considerations in the evolution of structure: its share gradually fades away and at the current epoch it will be no longer noticeable. It is convenient practice to normalize the *density growth factors* such that their current value is unity:

$$\begin{aligned} D_1(t) \quad &\equiv \quad \frac{D_1(t)}{D_1(t_0)} \;=\; \left(\frac{t}{t_0}\right)^{2/3} \\[1em] D_2(t) \quad &\equiv \quad \frac{D_2(t)}{D_2(t_0)} \;=\; \left(\frac{t}{t_0}\right)^{-1} \end{aligned} \tag{50}$$

By choosing this convention, the spatial functions $\Delta_1(\mathbf{x},t)$ and $\Delta_2(\mathbf{x},t)$ get to correspond to **linearly extrapolated** density fluctuations. They are the density values which a fluctuation would have if it would have continued growing **linearly** up to the present epoch (which usually they have not). Strictly, in fact, it would only be a valid assumption if the *growing* mode contribution $\Delta_1 \ll 1$. As for the *decaying* mode contribution, it would indeed have to be very small today in order to prevent it to be larger than unity in the primordial Universe.

In summary, we find that the general solution for the linear evolution of a density fluctuation in an Einstein-de Sitter Universe is given by the sum of a growing solution, specified by the normalized density growth factor $D_1(t) = a(t)$, and a decaying solution specified by $D_2(t) \propto t^{-1}$.

7.2. Empty Universe

Although the consideration of an empty matter-dominated FRW Universe, with $\Omega(t) = \Omega_0 = 0$, appears at first sight a mere academic exercise, it is indeed of genuine interest. It concerns the asymptotic limit for an $\Omega_0 < 1$ Universe. We know that while such a Universe will have an $\Omega \approx 1$ at early cosmic times, $a(t) \downarrow 0$, it will become increasingly empty and asymptotically evolve to $\Omega(t) \to 0$ as $a(t) \to \infty$. Once $\Omega(t) \ll 1$, the cosmic expansion turns into a *free expansion* once $\Omega(t) \ll 1$,

$$a(t) = H_0 t \tag{51}$$

so that

$$\begin{cases} \dfrac{\dot{a}}{a} & = \dfrac{1}{t} \\ \\ 4\pi G\,\rho_u(t) & = 0 \end{cases}$$

so that we end up with the following density growth equation,

$$\boxed{\frac{\partial^2 \delta}{\partial t^2} + \frac{2}{t}\frac{\partial \delta}{\partial t} = 0} \tag{52}$$

Also in this case we can identify two solutions δ_1 and δ_2, and each of them can be separated in a spatial and a temporal part.

$$\begin{aligned} \delta(\mathbf{x}, t) &= \delta_1(\mathbf{x}, t) + \delta_2(\mathbf{x}, t) \\ &= D_1(t)\Delta_1(\mathbf{x}) + D_2(t)\Delta_2(\mathbf{x}, t) \end{aligned} \tag{53}$$

In terms of *normalized density growth factors* $D_m(t)$ the solutions are

$$\boxed{\begin{aligned} D_1(t) &= cst. \\ \\ D_2(t) &\propto t^{-1} \end{aligned}} \tag{54}$$

Interestingly, instead of *growth*, the solution $\delta_1(\mathbf{x}, t)$ comes to a halt: density growth **freezes out** and structure retains the present configuration. On the other hand, the **decaying mode** solution behaves the same as in an Einstein Universe.

When extrapolating back in time, we see that the repercussion of such a scenario in terms of the formation of present-day structure is that it must have been in place in the primordial Universe. Evidently, this would be hard to reconcile with our knowledge of the early Universe (in particular, the isotropy of the cosmic microwave background). However, following the remark that an *empty* Universe may be regarded as the asymptotic state of an evolving *open* FRW Universe, we may conclude that the present-day structure has been in place since a much earlier cosmic epoch. This epoch is the time at which a Universe not yet empty evolved into a *freely expanding* empty Universe. In summary, in the linear regime:

$$\boxed{\text{Empty Universe:} \quad \text{NO Structure Evolution}}$$

7.3. General Matter-Dominated FRW Universes

From the linear structure growth equation 42 we observed that the two cosmological factors determining linear structure growth are the background density $\rho_u(t)$ and the Hubble parameter $\dot{a}/a$. As a function of the expansion factor $a(t)$ the first one evolves the same as in the case of the Einstein-de Sitter and the empty Universe, $\rho_u \propto a^{-3}$. The $\dot{a}/a$ term, however, has a more complex behaviour. Its evolution is set by a combination of two factors, the background density and curvature. In the case of the pure Einstein-de Sitter Universe the only influence was the density, in the case of the empty Universe it was the curvature.

7.3.1. The Friedmann Equation

To evaluate their contributions to the linear density growth solutions, we turn to the Friedmann equation,

$$\boxed{\frac{\dot{a}^2}{a^2} \;=\; \frac{8\pi G\,\rho_u}{3} \;-\; \frac{kc^2/R_0^2}{a^2}}$$

To rephrase the Friedmann equation into a form from which we can readily appreciate the balance between the two contributions, we express the curvature term kc^2/R_o^2 and the background density ρ_u in terms of the Hubble parameter H_0 and the cosmic density parameter Ω_0,

$$\begin{cases} \dfrac{8\pi G\,\rho_u}{3} \;=\; \Omega_0 H_0^2\, a^{-3} \\[2ex] \dfrac{kc^2}{R_0^2} \;=\; H_0^2\,(\Omega_0 - 1) \end{cases}$$

From this, we can proceed to rewrite the curvature term as

$$\begin{aligned} \frac{kc^2/R_0^2}{a^2} &= \frac{8\pi G\,\rho_u}{3}\,\frac{H_0^2\,(\Omega_0-1)}{\Omega_0 H_0^2 a^{-1}} \\ &= \frac{8\pi G\,\rho_u}{3}\,a(t)\left(1-\frac{1}{\Omega_0}\right)\,. \end{aligned} \tag{55}$$

so that the Friedmann equation reads

$$\frac{\dot{a}^2}{a^2} \;=\; \frac{8\pi G\,\rho_u}{3}\left\{1 + a(t)\left(\frac{1}{\Omega_0}-1\right)\right\}\,. \tag{56}$$

By parameterizing the evolution of the cosmic density factor $\Omega(t)$ in terms of the factor $x(t)$ (see Peebles 1980),

$$x(t) \equiv \begin{cases} \left(\dfrac{1}{\Omega(t)}-1\right) = a(t)\left(\dfrac{1}{\Omega_0}-1\right) & \Omega_0 < 1 \\[2ex] 0 & \Omega_0 = 1 \\[2ex] \left(1-\dfrac{1}{\Omega(t)}\right) = a(t)\left(1-\dfrac{1}{\Omega_0}\right) & \Omega_0 > 1 \end{cases} \tag{57}$$

we find the following useful expression for the Friedmann equation

$$\boxed{\frac{\dot{a}^2}{a^2} = \begin{cases} \dfrac{8\pi G\,\rho_u}{3}\,(1\,+\,x(t)) & \Omega_0 < 1 \\[2ex] \dfrac{8\pi G\,\rho_u}{3} & \Omega_0 = 1 \\[2ex] \dfrac{8\pi G\,\rho_u}{3}\,(1\,-\,x(t)) & \Omega_0 > 1 \end{cases}}$$

For the three distinct cases the value of the factor $x(t)$ varies within the range

$$x(t) \;=\; \begin{cases} [0 \rightarrow \infty) & \Omega_0 < 1 \\[2ex] 0 & \Omega_0 = 1 \\[2ex] [0 \rightarrow 1 \rightarrow 0) & \Omega_0 > 1 \end{cases} \tag{58}$$

It is the value of $x(t)$ which determines the balance between density and curvature. If $|x| < 1$ the density is clearly dominant, while the curvature dominates as $|x| \gg 1$. The latter only happens in the case of an open Universe, once it gets into free expansion.

7.3.2. The Einstein-de Sitter Universe

The Einstein-de Sitter Universe represents the asymptotic situation in which $x(t) = x_0 = 0$,

$$\frac{\dot{a}^2}{a^2} \;=\; \frac{8\pi G\,\rho_u}{3}\,. \tag{59}$$

7.3.3. Closed Universe

In a **closed Universe** $x < 0$. This results in a lower value of the Hubble dragterm $\dot{a}/a$. From eqn. (42) we can readily appreciate that this results in a speeding up of the density evolution. Perhaps not surprisingly structure in a closed Universe grows more rapidly, mainly due to the higher background density of the Universe and corresponding higher mass content of density fluctuations.

7.3.4. Open Universe

In an open Universe we see the reverse, structure growth will proceed less rapidly than in a Einstein-de Sitter Universe. For $a(t) \gg 1$ the $x(t)$ term becomes all-dominant so that the open Universe has proceeded towards *free expansion*,

$$\dot{a}^2 \;\approx\; \frac{8\pi G\,\rho_0}{3}\left(\frac{1}{\Omega_0} - 1\right) \;=\; \text{const.}. \tag{60}$$

Linear density growth in an open Universe will therefore proceed from a primordial situation in which it resembles that in a Einstein-de Sitter Universe, $D(t) \propto a(t)$, towards a halting of structure growth akin to that in a freely expanding empty Universe, $D(t) = \text{const.}$. We may therefore identify a **characteristic epoch** a_g at which the initial evolving structure in such a Universe comes to a halt. It is set by the equality of the two contributing terms, i.e. $x(t) \approx 1$,

$$a_g \;=\; \left(\frac{1}{\Omega_0} - 1\right)^{-1}\,. \tag{61}$$

corresponding to a characteristic redshift

$$z_g \equiv \frac{1}{\Omega_0} - 2\,. \tag{62}$$

In other words, the redshift z_g marks the transition from the phase in which structure grows (for redshifts $z > z_g$) to one in which this process comes to a halt ($z < z_g$),

$$\begin{cases} 1+z_g & \gg \left(\frac{1}{\Omega_0} - 1\right) \qquad\qquad \textit{structure growth} \\ \\ 1+z_g & \ll \left(\frac{1}{\Omega_0} - 1\right) \qquad\qquad \textit{structure growth halts} \end{cases}$$

If we were to live in a open Universe at the time of the transition, it would imply $\Omega_0 \approx 0.5$. If $\Omega_0 \approx 0.3$, as observations seem to indicate, structure did stop growing at around $z_g \approx 1.3$. Note that this refers to linear structure growth. At the current cosmic epoch this involves spatial scales larger than $\approx 10h^{-1}$Mpc. On smaller spatial scales nonlinear collapse of structure may still proceed.

A comparison of this crude analytical result to the outcome of large computer simulations of structure formation in open Universes does confirm the effect of linear growth coming to a halt. Moreover, as we will see later on scales much smaller than the horizon the effect of dark energy on the growth of structure is considerably smaller than that of dark matter. This is due to the fact that dark energy remains a uniformly distributed medium which itself cannot clump and thus accelerate the corresponding gravitational collapse. In simulations of a Universe with a finite Λ and a $\Omega_m < 1$ we therefore find practically similar behaviour.

7.3.5. Structure Growth Equation

Having introduced the parameter $x(t)$ in phrasing the evolution of $\dot{a}/a$ in a matter-dominated FRW Universe, we may also observe that for $\Omega_0 \neq 1$ it is linearly proportional to the expansion factor $a(t)$. For $\Omega_0 \neq 1$, we therefore change the time variable from expansion factor $a(t)$ to that of $x(t)$,

$$x(t) = \begin{cases} a(t)\left(\frac{1}{\Omega_0} - 1\right) & \Omega_0 < 1 \\ \\ a(t)\left(1 - \frac{1}{\Omega_0}\right) & \Omega_0 > 1 \end{cases} \tag{63}$$

Following this change of variables the equation for the linear structure growth factor $D(x)$ (cf. eqn. 42) becomes

$$\boxed{\begin{array}{ll} \dfrac{d^2D}{dx^2} + \dfrac{3+4x}{2x(1+x)}\dfrac{dD}{dx} = \dfrac{3D}{2x^2(1+x)} & \qquad \Omega_0 < 1 \qquad (64) \\ \\ \dfrac{d^2D}{dx^2} + \dfrac{3-4x}{2x(1-x)}\dfrac{dD}{dx} = \dfrac{3D}{2x^2(1-x)} & \qquad \Omega_0 > 1 \qquad (65) \end{array}}$$

This second order differential equation can be solved analytically (see Peebles 1980).

7.3.6. Open Universe: linear structure growth factors

The **growing mode** solution $D_1(t)$ for $\Omega_0 > 1$ of equation (66) is (Peebles 1980)

$$\boxed{D_1(t) \;=\; 1 + \frac{3}{x} + \frac{3\sqrt{1+x}}{x^{3/2}} \log\sqrt{1+x} - \sqrt{x}} \tag{66}$$

An alternative expression for the growing mode solution $D_1(t)$ is in terms of the development angle Φ_u of the Universe, related to $x(t)$ via

$$x(t) \;=\; \frac{\cosh\Phi_u - 1}{2}\,, \tag{67}$$

yielding

$$D_1(t) \;=\; \frac{3\sinh\Phi_u\;(\sinh\Phi_u - \Phi_u)}{(\cosh\Phi_u - 1)^2} \;-\; 2 \tag{68}$$

The corresponding **decaying mode** solution $D_2(t)$ is

$$\boxed{D_2(t) \;=\; \frac{\sqrt{1+x}}{x^{3/2}}} \tag{69}$$

7.3.7. Closed Universe: linear structure growth factors

In the case of a closed Universe we need to distinguish separate solutions for the expanding regime and the recollapse regime of cosmic evolution. In terms of the cosmic development angle

$$\begin{cases} 0 \;<\; \Phi_u \;<\; \pi & \textit{cosmic expansion} \\ \\ \pi \;<\; \Phi_u \;<\; 2\pi & \textit{cosmic recollapse} \end{cases} \tag{70}$$

For the **growing mode** the solutions $D_1(t)$ are

$$\boxed{D_1(t) \;=\; \begin{cases} -1 \;+\; \frac{3}{x} \;-\; \frac{3\sqrt{1-x}}{x^{3/2}}\;\arctan\left(\frac{x}{1+x}\right)^{1/2} & 0 < \Phi_u < \pi \\ \\ -1 \;+\; \frac{3}{x} \;-\; \frac{3\sqrt{1-x}}{x^{3/2}}\;\left[\arctan\left(\frac{x}{1+x}\right)^{1/2} \;-\; \pi\right] & \pi < \Phi_u < 2\pi \end{cases}} \tag{71}$$

while the corresponding **decaying mode solution** $D_2(t)$ is given by

$$
\boxed{D_2(t) \;=\; \begin{cases} \dfrac{\sqrt{1-x}}{x^{3/2}} & 0 < \Phi_u < \pi \\ & \\ -\,\dfrac{\sqrt{1-x}}{x^{3/2}} & 0 < \Phi_u < \pi \end{cases}} \tag{72}
$$

8. Linear Perturbations: FRW Universe with Dark Energy

Earlier we have observed that at the present epoch the main contribution to the energy density of the Universe is that by **dark energy**. Usually this is identified with the energy content of the **vacuum** of the Universe, be it that any simple interpretation in these terms leads to the conclusion its energy density should be a factor 10^{120} higher than has been inferred from cosmological measurements. Even though its nature remains a mystery, its impact on the dynamics of the Universe is unmistakable and bizarre: cosmic expansion finds itself in a state of **acceleration** !!!

The one requirement for all options of **dark energy** is that they involve a cosmic acceleration, that is

$$
\rho_v \,+\, \frac{3P_v}{c^2} < 0\,. \tag{73}
$$

With a **dark energy** equation of state

$$
\begin{aligned}
P_v \;=\; w\,\rho_v c^2 \qquad &\Rightarrow \quad 1) \qquad w < -\frac{1}{3} \\
&\Rightarrow \quad 2) \qquad \rho_v \;=\rho_{v,0}\,a^{-3(1+w)}
\end{aligned} \tag{74}
$$

As yet cosmological observations seem to indicate that the most plausible interpretation is that in terms of a “conventional” **cosmological constant** Λ, i.e. $w = -1$,

$$
P_\Lambda \;=\; -\,\rho_\Lambda c^2\,. \tag{75}
$$

This implies a rather rigid *dark energy* medium, whose energy density would remain constant while the Universe expands: the expansion of the Universe creates dark energy as a result of its **negative pressure**.

8.1. Dark Energy Perturbations

Energy perturbations in a Universe with a cosmological constant Λ are not an issue for our study. They simply do not exist, nor would they grow. We may immediately appreciate this from the continuity equation, for which of course we need to use the relativistic form,

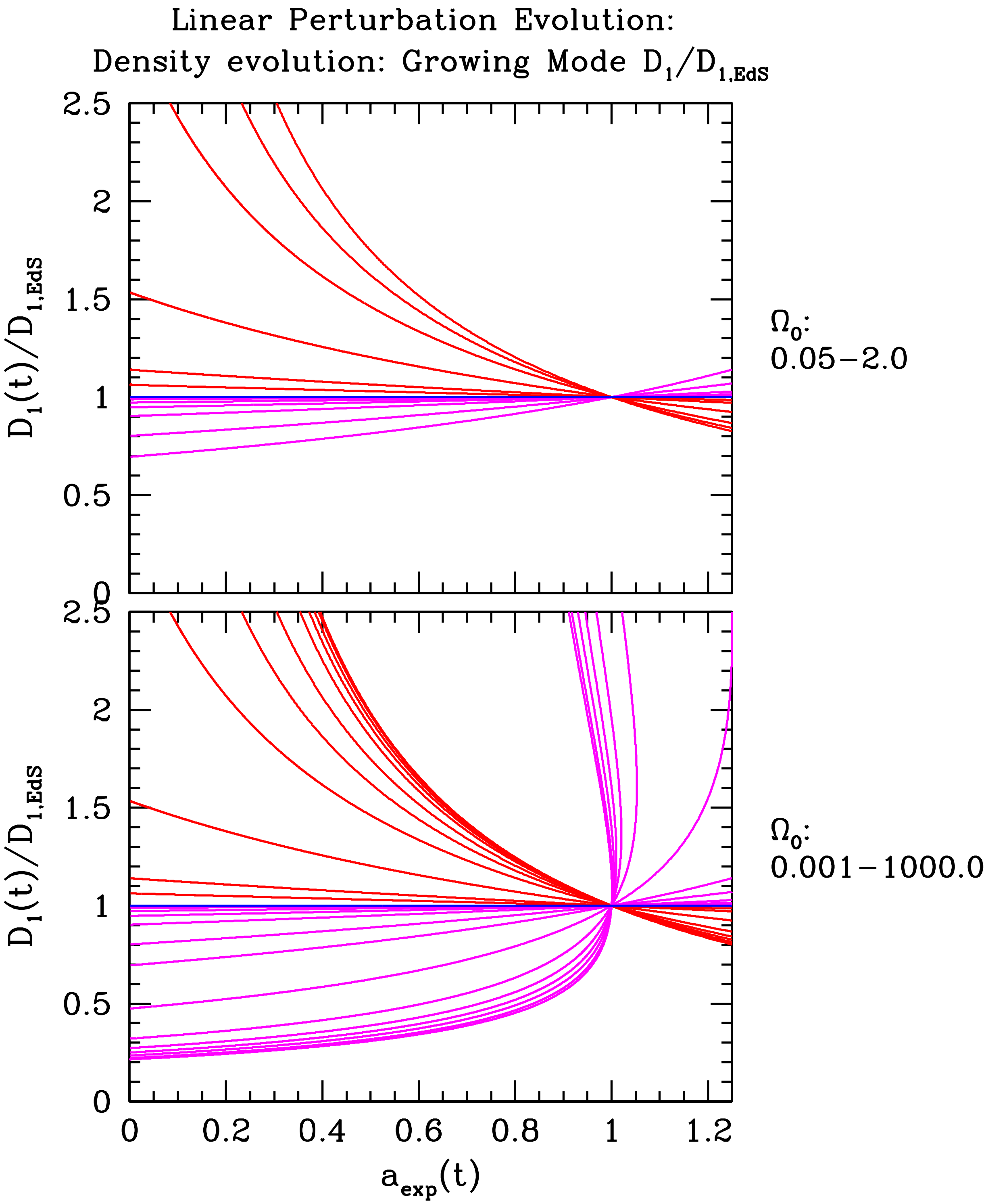


Figure 6. *Growing mode linear density growth factors $D_1(t)$ for a range of matter-dominated FRW Universes, plotted as a function of cosmic time. $D_1(t)$ has been normalized with respect to the current epoch, $D_0 \equiv 1$. Red: open Universes. Magenta: Closed Universes. Blue: Einstein-de Sitter Universes. Top range:* $0.05 < \Omega_0 < 2.0$, *Bottom range:* $0.001 < \Omega_0 < 1000.0$.

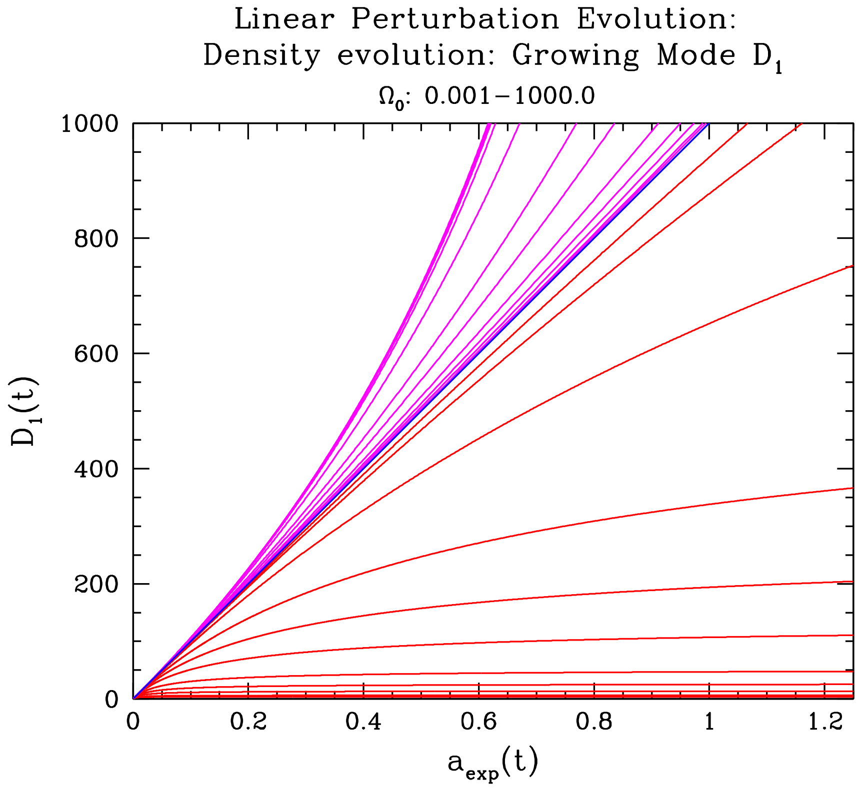
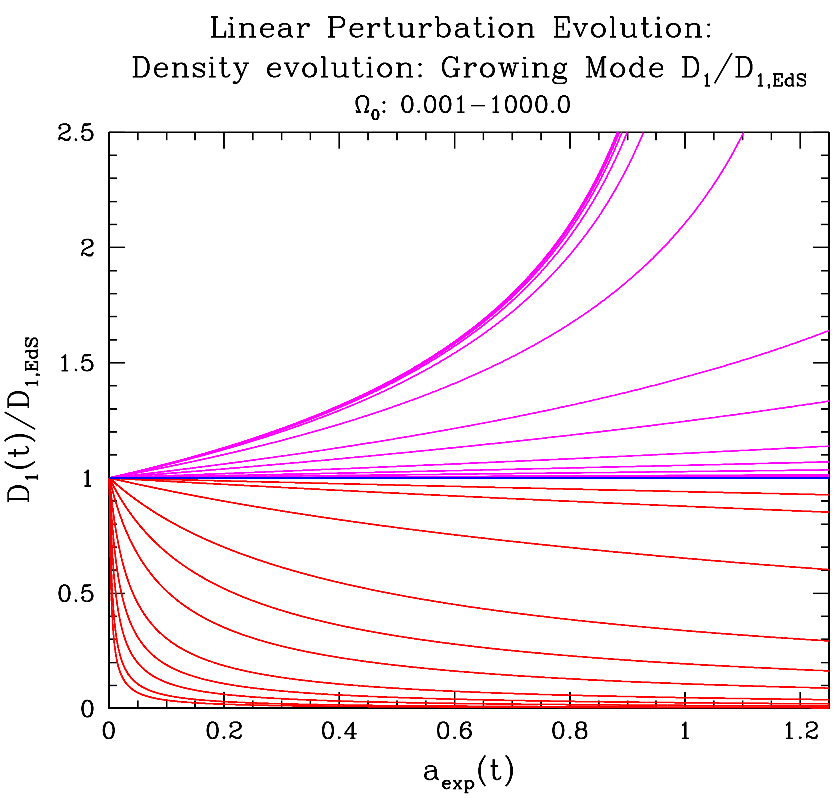


Figure 7. *Decaying mode linear density growth factors $D_2(t)$ for a range of matter-dominated FRW Universes, plotted as a function of expansion factor $a(t)$. $D_2(t)$ has been normalized with respect to the current epoch, $D_2(t_0) \equiv 1$. Red: open Universes. Magenta: Closed Universes. Blue: Einstein-de Sitter Universes. Range: $0.05 < \Omega_0 < 2.0$.*



$$\boxed{\begin{aligned} \frac{\partial \rho_\Lambda}{\partial t} + \vec{\nabla}_r \cdot \left(\rho_\Lambda + \frac{P_\Lambda}{c^2} \right) \vec{u} &= 0 \\ \Downarrow \\ \frac{\partial \rho_\Lambda}{\partial t} &= 0 \end{aligned}} \tag{76}$$

For the cosmological constant, the implication is therefore that $\rho_{Lambda} = cst$. While the matter density within a certain region will increase or decrease due to the contraction ($\nabla \cdot \mathbf{v} < 0$) or expansion ($\nabla \cdot \mathbf{v} > 0$) of that region, nothing like this will happen for Λ dark energy. Its nature would guarantee its energy density to remain constant, so that local deviations would never be able to grow (or decay). We can therefore see the presence of a cosmological constant as the presence of a background medium, itself completely inactive with respect to the formation of structure. Also in the more general situation of a dark energy medium whose equation of state has $-1 < w < -1/3$ there would be no growth of structure. The reason for this is to be found in its dynamical influence: its repulsive gravitational effect.

$$\nabla_r^2 \Phi \;=\; 4\pi G \left(\rho_v(\mathbf{r}, t) \;+\; \frac{3P_v}{c^2} \right) . \tag{77}$$

If locally the dark energy density would be slightly higher than average, it would result in an extra repulsive force and therefore in an expansion (instead of contraction) of the region. This would quickly even out the energy density perturbation. The net result is an immediate halt to any structure growth in such a dark energy medium.

However, we still find ourselves living in a present-day Universe filled with a dominant dark energy medium. One therefore needs to understand the growth of structure in **matter** against the background of an "inert" dark energy medium.

8.2. FRW Universes containing Matter and Cosmological Constant: Linear Matter Perturbations

In the presence of a cosmological constant, within a generally curved FRW Universe, the linear growth of density perturbations in the matter distribution can be evaluated directly throuhg an integral expression. This expression allows us to calculate the linear density growth factor $D(t)$ for matter by means of a simple numerical integration.

In principal we could also include the effect of radiation to obtain a fully general theory of linear structure growth in a FRW Universe. However, in the presence of a significant contribution by radiation to the cosmic energy density the evolution of matter and radiation perturbations are coupled (see sect. 9), complicating the analysis somewhat. In the corresponding *pre-recombination era* dark energy was totally insignificant. In section 9 we will concentrate on the linear evolution of radiation perturbations (larger than the Jeans mass). Here we will focus on the evolution of matter perturbations in a general FRW Universe, filled with matter and a cosmological constant, with a general curvature (also see Hamilton 2000). At a particular stage of the analysis we will neglect the influence of radiation.

We assume a FRW Universe with a Hubble parameter H_0, for which the contributions to the cosmic density parameter are

$$\begin{cases} \Omega_{rad,0} & \text{radiation} \\ \Omega_{m,0} & \text{matter} \\ \Omega_{\Lambda,0} & \text{cosmological constant} \\ & \\ \Omega_0 \,=\, \Omega_{rad,0} \,+\, \Omega_{m,0} \,+\, \Omega_{\Lambda,0} & \text{curvature} \end{cases} \tag{78}$$

The Hubble parameter $H(t)$ at any cosmic time t in such a FRW Universe is given by the relation

$$H^2(t) \,=\, H_0^2 \,\left\{\Omega_{rad,0}a^{-4} \,+\, \Omega_{m,0}a^{-3} \,+\, (1-\Omega_0)a^{-2} \,+\, \Omega_{\Lambda,0}\right\} \tag{79}$$

Differentiating the above expression both once and twice yields

$$\begin{aligned} 2H\dot{H} \quad &= \quad H_0^2\,\frac{\dot{a}}{a}\,\left\{-4\frac{\Omega_{rad,0}}{a^4} - 3\,\frac{\Omega_{m,0}}{a^3} - 2\,\frac{(1-\Omega_0)}{a^2}\right\} \\ &= \quad H_0^2 H\,\left\{-4\frac{\Omega_{rad,0}}{a^4} - 3\,\frac{\Omega_{m,0}}{a^3} - 2\,\frac{(1-\Omega_0)}{a^2}\right\} \end{aligned} \tag{80}$$

$$\ddot{H} \quad = \quad H_0^2 H\,\left\{8\frac{\Omega_{rad,0}}{a^4} + \frac{9}{2}\,\frac{\Omega_{m,0}}{a^3} + 2\,\frac{(1-\Omega_0)}{a^2}\right\} \tag{81}$$

Adding both expressions we find that

$$\begin{aligned} \ddot{H} \,+\, 2H\dot{H} \quad &= \quad H_0^2 H\,\left\{3\,\frac{\Omega_{m,0}}{2a^3} \,+\, 4\frac{\Omega_{rad,0}}{a^4}\right\} \\ &\approx \quad H_0^2 H\,\frac{3}{2}\,\frac{\Omega_{m,0}}{a^3} \qquad \text{for} \quad \Omega_{rad,0} \ll 1 \end{aligned} \tag{82}$$

so that we find that

$$\ddot{H} \,+\, 2H\dot{H} \,=\, 4\pi G\, H\, \rho_{m,u},. \tag{83}$$

Thus, $H(t)$ evolves according to exactly the same equation as the density growth factor $D(t)$ in a Universe with matter (see eqn. 42):

$$\ddot{D} \,+\, 2H\dot{D} \,=\, 4\pi G\, D\, \rho_{m,u}\,. \tag{84}$$

Multiplying the equation for the evolution of H by $D(t)$ and subtracting $H(t)$ times the equation for $D(t)$ we have

$$\begin{aligned} D\ddot{H} \,-\, H\ddot{D} \quad &+ \quad 2H\left(D\dot{H} - H\dot{D}\right) \,=\, 0 \\ &\downarrow \end{aligned} \tag{85}$$

$$a^2\,\frac{d}{dt}\left(\dot{D}H - D\dot{H}\right) \quad + \quad \frac{da^2}{dt}\left(\dot{D}H - D\dot{H}\right) = 0 \tag{86}$$

which leads to the second-order differential equation

$$\boxed{\frac{d}{dt}\,\left\{a^2 H^2\,\frac{d}{dt}\,\left(\frac{D}{H}\right)\right\} \,=\, 0} \tag{87}$$

whose solution is given by the integral equation

$$\boxed{D(t) \approx H(t) \int \frac{dt}{a^2 H^2(t)}} \tag{88}$$

The linear perturbation growth factor $D(t)$ can be calculated for any FRW Universe with matter and a cosmological constant – specified by the parameters $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ –by inserting the expression for the evolution of the Hubble parameter 79 into the above integral expression and computing the integral. For most cases this involves a numerical evaluation of the integral. It is customary to evaluate the growing mode growth factor $D(z)$ in terms of the redshift z. This is accomplished through the relation between redshift z and cosmic time t,

$$dt = -\frac{1}{(1+z)H(z)}\, dz \tag{89}$$

which one may infer on the basis of the fact that $a(t) = 1/(1+z)$. It leads to the integral expression

$$\boxed{D(z) = \frac{5\,\Omega_{m,0} H_0^2}{2}\, H(z) \int_z^{\infty} \frac{1+z'}{H^3(z')}\, dz'} \tag{90}$$

whereby the proportionality factor is chosen such that

$$D(z) \propto \frac{1}{1+z} \qquad \text{for} \quad z \to \infty\,. \tag{91}$$

because for early times the Universe will tend asymptotically towards an Einstein-de Sitter Universe. The integral (90) can be solved analytically for a pure matter-dominated Universe, with $\Omega = \Omega_m$ and $\Omega_\Lambda = 0$. The solutions are given by equations (103), (66) and (71) in the preceding sections. In the general situation of $\Omega_\Lambda \neq 0$, for most purposes a fitting formula provides a sufficiently accurate approximation. For the relative growth factor $g(t) \equiv D(t)/a(t)$ – relative with respect to the equivalent Einstein-de Sitter Universe – we find (see e.g. Lahav & Suto 2003),

$$g(z) \equiv (1+z)\, D(z) \approx \frac{5\Omega(z)}{2} \frac{1}{\Omega^{4/7}(z) - \Omega_\Lambda(z) + [1+\Omega_m(z)/2][1+\Omega_\Lambda(z)/70]}\,, \tag{92}$$

with $\Omega_m(z)$ and $\Omega_\Lambda(z)$ the values of the cosmological density parameter for matter and the cosmological constant at a redshift z,

$$\begin{aligned} \Omega_m(z) &= \Omega_{m,0}\,(1+z)^3 \left[\frac{H_0}{H(z)}\right]^2 \\ \Omega_\Lambda(z) &= \Omega_{\Lambda,0} \left[\frac{H_0}{H(z)}\right]^2 \end{aligned} \tag{93}$$

9. Linear Radiation Perturbations

Extending our analysis of perturbation growth to a broader class of FRW Universes, composed of other and/or more components than just **matter** makes it necessary to extend the analysis to the set of appropriate (special) relativistic fluid equations. Even this is a mere idealization and approximation, a truely generally applicable analysis should not be based upon the idealization of special relativistic fluid equations. Instead, it should be based upon a fully general relativistic treatment of the Boltzmann equation. CMBFAST, the code for computing temperature perturbations in the cosmic microwave background, is precisely doing that.

Even though at present the dynamical influence of radiation is negligible, it representing around $1/100,000th$ of the Universe's energy content, it has dominated the Universe's dynamics before the **epoch of radiation-matter equivalence**. In the first subsection 9.1 we will study the evolution of radiation perturbations in this regime in which radiation is the only cosmic component of importance.

Near the epoch of *radiation-matter equivalence* and up to the epoch of **recombination** and resulting **decoupling** between matter and radiation, radiation remains a significant factor in the evolution of perturbations. Around the equivalence epoch its energy density is still comparable to that of the matter density. A proper treatment of such perturbations should involve a coupled system of matter and radiation perturbations. In subsection 9.2 we will shortly indicate the simple situation in which we disregard the pressure force and limit the coupling to the mutual gravitational influence.

Finally, we know that even while the gravitational significance of radiation is quickly diminishing after *radiation-matter equivalence*, radiation perturbations remain responsible for a sizeable pressure force acting on the baryonic matter component. The repercussions of this will be discussed in extensio in our treatment of **Jeans Instabilities** in chapter 6.

9.1. Linear perturbations in a pure radiation-dominated Universe

Here we focus on perturbation growth in a pure radiation-dominated Universe, discarding the influence of matter (and dark energy).

The resulting special relativistic fluid equations include three extra terms. In the continuity equation and the Euler equation we need to take into account a pressure inertia term,

$$\rho \quad \longmapsto \quad \rho + \frac{P}{c^2} \,. \tag{94}$$

The Euler equation also includes a pressure force term, $-\nabla_r P$. Finally, the Poisson equation needs to take into account the gravitating role of pressure. The results in the following set of three fluid equations:

$$\boxed{\begin{aligned}
\frac{\partial \rho_{rad}}{\partial t} + \vec{\nabla}_r \cdot \left(\rho_{rad} + \frac{P_{rad}}{c^2}\right) \vec{u} &= 0 \\
\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}_r\right) \vec{u} &= -\frac{1}{(\rho_{rad} + P_{rad}/c^2)} \vec{\nabla}_r P_{rad} - \vec{\nabla}_r \Phi \\
\nabla_r^2 \Phi &= 4\pi G \left[\rho_{rad} + \frac{3P_{rad}}{c^2}\right]
\end{aligned}}$$

In the following we will discard the pressure force term in the Euler equation. This is a far more restrictive assumption than in the case of the matter-dominated Universe. The sound velocity in light, $c/\sqrt{3}$, is so large that during the radiation-dominated era the **Jeans Mass** is only slightly smaller than the **Horizon Mass** (see chapter 6). The analysis presented in this section is therefore hardly a realistic one, there are nearly no radiation perturbations whose evolution is not affected by the pressure in the radiation fluid. Nonetheless, the results of our analysis do provide an interesting contrast to the ones obtained for the matter fluctuations.

Transforming the fluid equations from *spatial coordinates* and *physical quantities* into *comoving coordinates* and *perturbation quantities*, ignoring the pressure force and using the fact that for radiation,

$$P_{rad} = \frac{1}{3}\rho_{rad}c^2\,, \tag{95}$$

we find the following set of comoving radiation fluid equations:

$$\boxed{\begin{aligned} \frac{\partial \delta_{rad}}{\partial t} + \frac{4}{3}\frac{1}{a}\vec{\nabla}_x \cdot (1+\delta_{rad})\vec{v} &= 0 \\ \frac{\partial \vec{v}}{\partial t} + \frac{1}{a}\left(\vec{v}\cdot\vec{\nabla}_x\right)\vec{v} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}_x\phi \\ \nabla_x^2\phi &= 8\pi G\, a^2\, \rho_{rad,u}\,\delta_{rad} \end{aligned}}$$

For small linear perturbations we then obtain the following set of
Linearized Radiation Fluid Equations

$$\boxed{\begin{aligned} \frac{\partial \delta_{rad}}{\partial t} + \frac{4}{3}\frac{1}{a}\vec{\nabla}_x \cdot \vec{v} &= 0 \\ \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}\phi \\ \nabla^2\phi &= 8\pi G\, a^2\, \rho_{rad,u}\,\delta_{rad} \end{aligned}}$$

The evolution of the linear radiation perturbations can then be obtained by solving the second order differential equation

$$\boxed{\frac{\partial^2 \delta_{rad}}{\partial t^2} + 2\,\frac{\dot{a}}{a}\frac{\partial \delta_{rad}}{\partial t} = \frac{32\pi}{3}G\rho_{rad,u}\,\delta_{rad}} \tag{96}$$

The perturbation evolution equation for radiation is therefore almost equivalent to that for linear perturbations in the matter distribution. The difference concerns the factor $\frac{32\pi}{3}$ in front of the gravity term. For linear matter perturbations this factor was equal to 2π. The implied **general linear solution** for radiation energy density perturbations (larger than the Jeans mass) will therefore also be the sum of two terms of separated time and spatial functions:

$$\boxed{\delta(\mathbf{x},t) = D_{rad,1}(t)\,\Delta_1(\mathbf{x}) + D_{rad,2}(t)\,\Delta_2(\mathbf{x})} \tag{97}$$

where the spatial functions $\Delta_1(\mathbf{x})$ and $\Delta_2(\mathbf{x})$ represent the corresponding primordial radiation distribution and $D_1(t)$ and $D_2(t)$ are the universal linear **radiation energy density growth factors**. At the radiation-dominated epoch the Universe is very nearly flat, having a

$$\Omega_{rad} \approx 1\,, \tag{98}$$

with the Universe expanding according to

$$a(t) = (2H_0\,t)^{1/2} \quad \Rightarrow \quad \frac{\dot{a}}{a} = \frac{1}{2t}\,, \tag{99}$$

and the global radiation density $\rho_{rad,u}$ evolving according to

$$\rho_{rad,u}(t) \propto \frac{1}{a^4} \quad \Rightarrow \quad \frac{32\pi}{3} G\,\rho_{rad,u} = \frac{1}{t^2}\,. \tag{100}$$

This results into the radiation perturbation evolution equation

$$\boxed{\frac{\partial^2 \delta_{rad}}{\partial t^2} + \frac{1}{t}\frac{\partial \delta_{rad}}{\partial t} = \frac{1}{t^2}\,\delta} \tag{101}$$

Also the linear evolution of radiation perturbations is characterized by a **growing mode** solution and a **decaying mode** solution. The linear **growing mode** factor $D_{rad,1}(t)$ differs somewhat from that for matter perturbations in an equivalent Einstein-de Sitter Universe, while the **decaying mode** involves a substantially more rapid decay $D_{rad,2}(t)$,

$$\boxed{\begin{aligned} D_{rad,1}(t) &\propto t^{-\frac{1}{2}+\frac{1}{2}\sqrt{5}} && \approx t^{0.618} \\ D_{rad,2}(t) &\propto t^{-\frac{1}{2}-\frac{1}{2}\sqrt{5}} && \approx t^{-1.618} \end{aligned}} \tag{102}$$

9.2. Coupled Linear Matter-Radiation Perturbations

For a complete assessment of perturbation evolution we should write out the full system of couple fluid equations, involving separate **continuity/energy** equations and **Euler equations** for each component. The coupling between the components is established through the induced pressure forces and of course the combined gravitational field.

For the sake of illustration, here we focus on a simplified situation of perturbation growth in a matter-radiation fluidum. It involves the (unjustifiable) oversimplification of negligible pressure forces, so that it involves a situation which can hardly be identified with any realistic epoch in the Universe's history. Nonetheless, it proves to be illuminating with respect to the relationship between matter and radiation perturbations. Also, we assume that matter is cosmic "dust" and does have no pressure at all. In other words, matter is assumed to be collisionless dark matter. The **continuity/energy** equations, both physical and linearized, are

$$\begin{aligned} \frac{\partial \rho_m}{\partial t} + \vec{\nabla}_r \cdot \rho_m \vec{u} = 0 \quad &\Rightarrow \quad \frac{\partial \delta_m}{\partial t} + \frac{1}{a}\vec{\nabla}_x \cdot \vec{v} = 0 \\ \frac{\partial \rho_{rad}}{\partial t} + \vec{\nabla}_r \cdot \left(\rho_{rad} + \frac{P_{rad}}{c^2}\right)\vec{u} = 0 \quad &\Rightarrow \quad \frac{\partial \delta_{rad}}{\partial t} + \frac{4}{3}\frac{1}{a}\vec{\nabla}_x \cdot \vec{v} = 0 \end{aligned} \tag{103}$$

while the linearized **Euler equation** for both radiation and matter remains the same,

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\,\vec{v} = -\frac{1}{a}\,\vec{\nabla}\phi$$

The coupling between both components is established through the gravitational field, specified by the Poisson equation,

$$\begin{aligned} \nabla_r^2 \Phi &= 4\pi G \left[\rho_m + \rho_{rad} + \frac{3P_{rad}}{c^2}\right] \\ &\Downarrow \\ \nabla_x^2 \phi &= 4\pi G\, a^2 \left[\rho_{m,u}\,\delta_m + 2\,\rho_{rad,u}\,\delta_{rad}\right] \end{aligned}$$

From these coupled fluid equations it is straightforward to infer the coupled system of linearized perturbation evolution equations,

$$\boxed{\begin{aligned} \frac{\partial^2 \delta_m}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_m}{\partial t} &= 4\pi G\, a^2 \left[\rho_{m,u}\,\delta_m + 2\,\rho_{rad,u}\,\delta_{rad}\right] \\ \frac{\partial^2 \delta_{rad}}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_{rad}}{\partial t} &= 4\pi G\, a^2 \left[\frac{4}{3}\,\rho_{m,u}\,\delta_m + \frac{8}{3}\,\rho_{rad,u}\,\delta_{rad}\right] \end{aligned}}$$

It is particularly insightful to note that this equation may be written as a linear matrix equation, defined throught a linear operator L,

$$\boxed{L \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix} = 4\pi G \begin{pmatrix} \rho_{m,u} & 2\rho_{rad,u} \\ \frac{4}{3}\rho_{m,u} & \frac{8}{3}\rho_{rad,u} \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix}} \qquad (104)$$

in which the linear evolution operator L is defined by

$$L \equiv \frac{\partial^2}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial}{\partial t} \qquad (105)$$

Upon closer inspection of the coupled evolution equation **??** notice the special situation of the matter and radiation perturbations being related through a constant ratio $\frac{4}{3}$,

$$\boxed{\delta_{rad} = \frac{4}{3}\,\delta_m} \qquad (106)$$

If this is the case we find that the evolution of the matter perturbation δ_m and the radiation perturbation δ_{rad} are fully coupled: matter and radiation will retain the same constant ratio of energy density perturbations. This is an **extremely important** perturbation mode, the socalled **Adiabatic Perturbation Mode**. In chapter 6 we will discuss this in somewhat more detail. In this mode the **entropy per mass** $\mathcal{S}$, $\mathcal{S} = S/M$, remains the same: the implied perturbation in the entropy per mass is zero,

$$\frac{\delta \mathcal{S}}{\mathcal{S}} = \frac{3}{4}\delta_{rad} - \delta_m \quad \Rightarrow \quad \frac{\delta \mathcal{S}}{\mathcal{S}} = 0\,. \tag{107}$$

On the basis of the analysis of the acoustic angular fluctuations in the microwave background, measured by balloon experiments like Boomerang and to great precision by WMAP, it is almost sure that the primordial perturbations in our Universe are indeed adiabatic. This may be regarded as a confirmation of **inflation theory**, which does predict such perturbations (ie., most models of inflation).

10. Gravity Perturbations

Following the extensive analysis of the linear evolution of density perturbations, we may proceed to find the corresponding perturbations in the gravitational field. This involves both the gravitational potential perturbations $\phi(\mathbf{x}, t)$ and the peculiar gravitational acceleration $\mathbf{g}(\mathbf{x}, t)$. Given a field of density perturbations it is rather straightforward to determine these on the basis of the Poisson equation. Moreover, in the linear regime we may then easily infer the corresponding (universal) time evolution of the potential field and the peculiar gravitational acceleration.

10.1. Gravitational Potential Perturbations

The one-to-one relation between the (energy) density distribution $\rho(t)$ and the distribution of the gravitational potential $\Phi(t)$ in the Universe is established via the Poisson equation. In its general (special relativistic) form,

$$\boxed{\nabla_r^2 \Phi = 4\pi G \left(\rho(\mathbf{r}, t) + \frac{3P}{c^2}\right)\,.} \tag{108}$$

in which the energy density ρ is composed of matter, radiation and dark energy, and only radiation and dark energy provide a significant contribution to the pressure,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \rho_m(\mathbf{r}, t) + \rho_{rad}(\mathbf{r}, t) + \rho_v(\mathbf{r}, t) \\ P(\mathbf{r}, t) &\approx P_{rad}(\mathbf{r}, t) + P_v(\mathbf{r}, t) \end{aligned} \tag{109}$$

We will concentrate on the situation for a Universe with matter and a cosmological constant, and neglect the minor influence of radiation. Because dark energy remains a perfectly uniformly distributed medium, the resulting Poisson equation for the perturbed potential ϕ, in comoving coordinates, only involves a direct contribution by matter density perturbations,

$$\boxed{\nabla^2 \phi = \frac{3}{2}\Omega_m H^2\, a^2\, \delta_m(\mathbf{x}, t)} \tag{110}$$

The solution ϕ can then be simply found through the integral over the Green's function,

$$\boxed{\phi(\mathbf{x},t) \;=\; -\frac{3\Omega_m H^2}{8\pi}\, a^2 \int d\mathbf{x}'\; \delta_m(\mathbf{x}',t)\; \frac{1}{|\mathbf{x}'-\mathbf{x}|} \qquad (111)}$$

where the integral is over comoving space $\mathbf{x}'$. This expression for the potential ϕ is generally valid, and is independent of whether the perturbations reside in the **linear** or **nonlinear** regime.

Notice that density perturbations $\delta(\mathbf{x})$ throughout the whole of the observable Universe (i.e. within the horizon) contribute to the potential. Evidently, the nearby perturbations contribute more strongly but in principal we need to have mapped density perturbations throughout the whole cosmic volume. In reality, the spatial distribution and nature (sizes) of the density perturbations will determine the extent of the cosmic volume which having a significant dynamical influence. If there are no significant perturbations exceeding a particular coherence scale and if the perturbations constitute a random process (which they do, see chapter 5) the contribution from distant fluctuations will start cancelling each other. Although we have not yet firmly established this scale, it seems safe to suppose it does not exceed $100-200h^{-1}$Mpc. The integral expression also allows to immediately identify the time evolution of the potential perturbation. The three time-dependent factors concern the cosmic density $\rho_u \propto \Omega H^2$, the expansion factor a^2 and the evolving density perturbations $\delta(\mathbf{x},t)$. While the first two contributions yield

$$\Omega H^2\, a^2 \;\propto\; \frac{\Omega_0 H_0^2}{a^3}\, a^2 \;\propto\; \frac{1}{a}\,, \qquad (112)$$

the evolution of $\delta(\mathbf{x},t)$ is only universal in the **linear regime**. We have seen that in that case all density perturbations evolve according to the universal growth factor $D(t)$,

$$\delta(\mathbf{x},t) \;\propto\; D(t) \qquad (113)$$

If we restrict ourselves to the **growing mode** solution, we find therefore that in the linear regime potential perturbations ϕ also evolve according to a universal **potential perturbation growth factor** $D_\phi(t)$,

$$\phi(\mathbf{x},t) \;=\; D_\phi(t)\,\phi_0(\mathbf{x},t) \;=\; \frac{D(t)}{a(t)}\,\phi_0(\mathbf{x},t)\,, \qquad (114)$$

in which ϕ_0 is the linearly extrapolated potential perturbation. (extrapolated linearly towards the current epoch t_0). The linear potential perturbation growth factor D_ϕ is therefore given by

$$\boxed{D_\phi(t) \;=\; \frac{D(t)}{a(t)}} \qquad (115)$$

This finding has a very interesting repercussion. In the case of an Einstein-de Sitter Universe, we have found that the growth factor $D(t)$ is equal to the expansion factor $a(t)$. This means that for an Einstein-de Sitter Universe, the potential perturbations ϕ remain constant in time,

$$D_{\phi,EdS} \;=\; const. \qquad (116)$$

Because the cosmic microwave background temperature fluctuations are tightly coupled to the fluctuations in the gravitational potential we these results are of substantial practical significance.

10.1.1. Potential Fluctuations and the CMB

If a (microwave background) photon travels through the Universe from the surface of last scattering, at around the time of recombination, towards a telescope/detector on planet Earth, some 12.7 Gyrs later, its frequency will change while making its path through the potential landscape. As photons climb out of a potential well they get redshifted, resulting in a cooling of the CMB temperature. The resulting change in CMB temperature T is directly proportional to the shift in gravitational potential,

$$\frac{\delta T}{T} = \frac{1}{3}\frac{\delta\phi}{c^2}\,. \tag{117}$$

The shift in temperature is the combined effect of the corresponding gravitational redshift and time dilation (see chapter 1) and is known as the **Sachs-Wolfe** effect. Because in an Einstein-de Sitter Universe ϕ remains constant in time, the integral over the temperature shift along the path of the photon leads to a measured temperature shift with respect to the global CMB temperature of

$$\frac{\Delta T}{T} = \frac{1}{3}\frac{(\phi_o - \phi_e)}{c^2} \tag{118}$$

with ϕ_o the local potential perturbation and ϕ_e the potential perturbation at the time of emission at the surface of last scattering. We may therefore translate directly the map of measured CMB temperature fluctuations into a map of the gravitational potential perturbations at the epoch of recombination (and thus into a map of the corresponding density fluctuations).

For the same token, by evaluating the density fluctuations along the path of a CMB photon one may try to figure out the development of the potential growth factor $D_\phi(t)$. In other words, by measuring the integrated temperature shift along the photon's path, the socalled **integrated Sachs-Wolfe effect**, one may find the corresponding potential evolution. As we appear to live in a Universe with a cosmological constant and $\Omega_{m,0} \neq 1$, this function should not be constant in time. Instead, by inverting the relation we may potentially infer the underlying cosmological parameters implying the exciting prospect of measuring the value of the cosmological constant in a direct fashion.

10.2. Peculiar Gravity

The peculiar gravity $\mathbf{g}(\mathbf{x},t)$ is the gradient of the potential perturbation,

$$\boxed{\mathbf{g}(\mathbf{x},t) \equiv -\frac{\nabla\phi}{a}} \tag{119}$$

so that we can directly derive the integral expression for the peculiar gravitational acceleration,

$$\boxed{\mathbf{g}(\mathbf{x},t) = \frac{3\Omega_m H^2}{8\pi}\, a \int d\mathbf{x}'\, \delta_m(\mathbf{x}',t)\, \frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}} \tag{120}$$

This equation is the expression of the fact that the **peculiar gravitational acceleration** is the integrated effect of the excess gravity induced by **density fluctuations** throughout the observable Universe.

Both the equation for gravitational potential perturbations (111) and the one for the peculiar gravitational acceleration (120) are to be considered as the core of the theory of **Gravitational Instability**. For any density perturbation field $\delta(\mathbf{x}, t)$ they are universally valid.

It is straightforward to see that in the **linear regime**, the peculiar gravity therefore evolves according to a universal **gravity growth term**, D_g,

$$\boxed{D_g(t) \,=\, \frac{D(t)}{a^2(t)}} \tag{121}$$

It may be good to note that in nearly all situations of linear structure formation the corresponding peculiar gravitational accelerations appears to decrease in time. That even while the corresponding density perturbations are growing incessantly. However, it is mainly a result of the continuing expansion of the Universe and the corresponding increase in cosmic distances ! If there were no stucture growth, the situation in an empty Universe, we find that the peculiar acceleration decreases by the expected $1/a^2$. If it decreases less rapidly it implies the growing density fluctuations.

10.3. Peculiar Velocity

With the density distribution determined, and the corresponding gravitational potential and acceleration having been computed, we are set to study the repercussions of the induced forces. In the presence of peculiar gravitational forces matter will get displaced and move from its (comoving) location. We therefore expect a close link between the peculiar velocities and the gravitational force which has induced them. In particularly in the linear regime this relation is very strong.

The primary equation for the development of cosmic flows is the Euler equation (see eqn. 37),

$$\frac{da\mathbf{v}}{dt} \,=\, -\nabla\phi\,. \tag{122}$$

where the time derivative d/dt is the Lagrangian time derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\,. \tag{123}$$

Written in this form, the Euler equation immediately shows the direct relation between the presence of peculiar gravitational forces and the generation of cosmic flows. If there are no forces, peculiar velocities will not be able to develop. Before proceeding, we will look into this in somewhat more detail:

10.3.1. Velocity Decay

It is instructive to look at the imaginary situation of the presence of peculiar flows without corresponding gravitational perturbations. Of course this is a rather unrealistic situation, as soon matter starts to move density fluctuations will form which in turn will be a source for gravitationally induced peculiar flows.

Imagine a particle having a peculiar velocity $\mathbf{v}_p$ in a perfectly smooth Universe. The peculiar velocity of the particle will then evolve according to

$$\frac{da\mathbf{v}}{dt} \,=\, 0. \tag{124}$$

so that

$$\mathbf{v}_p(t) \,\propto\, \frac{1}{a(t)}\,. \tag{125}$$

It forms a clear illustration of the "Hubble drag" resulting from the expansion of the background Universe: as the Universe expands the particle's velocity will decrease proportionally. This can be understood from seeing the excess peculiar velocity of the particle against the background of the expanding Universe. As the particle moves away from its original location its velocity gets more and more in line with the local Hubble flow, so that its peculiar velocity will decrease accordingly.

10.3.2. Linear Peculiar Velocity: Lagrangian & Eulerian

An interesting observation is that in the linear regime we find that the **Lagrangian time derivative** of the peculiar velocity $\mathbf{v}$ is equivalent to the **Eulerian time derivative** of $\mathbf{v}$, because of discarding the second order term $\left(\vec{v} \cdot \vec{\nabla}_x\right) \vec{v}$. Therefore,

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}_x\right) \vec{v} \approx \frac{\partial \mathbf{v}}{\partial t} \\ &\Downarrow \end{aligned} \tag{126}$$

$$\frac{da\mathbf{v}}{dt} = \frac{\partial a\mathbf{v}}{\partial t} \tag{127}$$

The displacement of mass elements in the linear regime are so small that for the corresponding change of peculiar velocity $\mathbf{v}$ we do not need to take into account the displacement term $\left(\vec{v} \cdot \vec{\nabla}_x\right)$.

Because here we are dealing with the linear perturbation regime we will resort to the partial time derivative of the Eulerian formulation, and use the linear expression for the Euler equation,

$$\frac{\partial a\mathbf{v}}{\partial t} = -\nabla \phi \,. \tag{128}$$

10.3.3. Potential and Vorticity Flow

A velocity flow can always be decomposed into a **potential** flow component $\mathbf{v}_\parallel$ and a **rotational** flow component $\mathbf{v}_\perp$,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \,, \tag{129}$$

The potential flow $\mathbf{v}_\parallel$ can be written as the gradient of a scalar velocity potential ψ, while the "rotational" vorticity flow is the curl of some vector potential $\mathbf{A_v}$,

$$\begin{aligned} \mathbf{v}_\parallel &= \vec{\nabla} \psi \\ \mathbf{v}_\perp &= \nabla \times \mathbf{A_v} \,. \end{aligned} \tag{130}$$

Because

$$\nabla \cdot \mathbf{v}_\perp = \nabla \cdot (\nabla \times \mathbf{A_v}) = 0 \tag{131}$$

we can infer from the linearized continuity equation that in the linear regime it is only the potential flow $\mathbf{v}_\parallel$ which couples to the growing density perturbations,

$$\begin{aligned} \nabla \cdot \mathbf{v} &= -a \frac{\partial \delta}{\partial t} \\ &\Downarrow \\ \nabla \cdot \mathbf{v}_\parallel &= -a \frac{\partial \delta}{\partial t} \end{aligned} \tag{132}$$

Using this observation, we find that the Poisson equation implies that only $\mathbf{v}_\parallel$ couples to the gravitational potential ϕ.

$$\begin{aligned} \nabla \cdot \mathbf{v}_\parallel &= -a\,\frac{\partial \delta}{\partial t} \\ &= -a\;\nabla \cdot \frac{\partial}{\partial t}\left(\frac{\nabla\phi}{4\pi G\rho_u a^2}\right) \end{aligned} \tag{133}$$

The evolution of both the potential component and the vorticity component of the velocity field can thus be formulated separately, through the Euler equation,

$$\begin{aligned} \frac{\partial a\mathbf{v}_\parallel}{\partial t} &= -\nabla\phi \\ \frac{\partial a\mathbf{v}_\perp}{\partial t} &= 0 \end{aligned} \tag{134}$$

The second of these equations shows that the curl part of the velocity decays as

$$\mathbf{v}_\perp \propto \frac{1}{a} \tag{135}$$

as the Universe expands. If there were vorticity flows in the primordial Universe they would have decayed to zero, unless there were a mechanism to generate vorticity. It is only in advanced nonlinear stages of clustering that this may happen: in the **linear regime** there is **no vorticity** !!!! Thus, if only gravity operates (i.e. no pressure forces and dissipative effects,

in Linear Regime:
Peculiar Velocity = pure gradient Potential Flow

In other words, in the linear regime we can continue with the finding that

$$\mathbf{v} = \mathbf{v}_\parallel\,. \tag{136}$$

10.3.4. Peculiar Velocity & Peculiar Gravity

The relation between the peculiar gravitational acceleration,

$$\mathbf{g} = -\frac{\nabla\phi}{a}\,, \tag{137}$$

and the induced velocity flow $\mathbf{v}$ in the linear regime is expressed by the Poisson equation (134):

$$\begin{aligned} \nabla \cdot \mathbf{v} &= -a\,\nabla \cdot \frac{\partial}{\partial t}\left\{\frac{\nabla\phi}{4\pi G\rho_u a^2}\right\} \\ &= a\,\nabla \cdot \frac{\partial}{\partial t}\left\{\frac{\mathbf{g}}{4\pi G\rho_u a}\right\} \end{aligned}$$

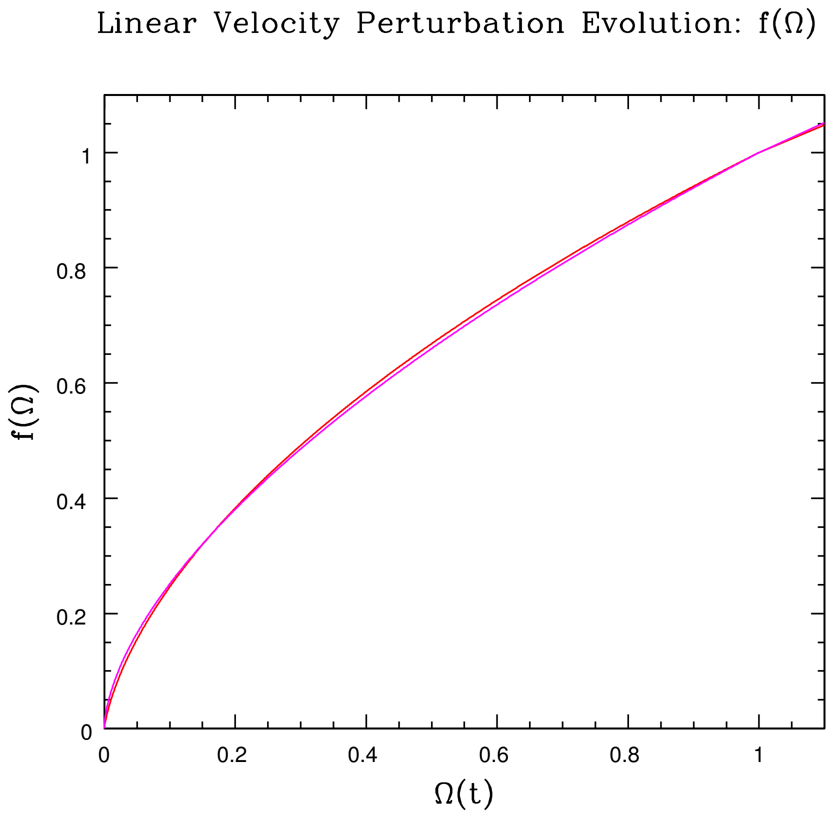


Figure 8. *Dimensionless Linear Velocity Growth Factor: comparison with the approximation* $f(\Omega) \approx \Omega^{0.6}$

Because both $\mathbf{g}$ and $\mathbf{v}$ are gradients of a potential (the gravitational potential ϕ and the velocity potential ψ respectively), this leads to the finding that:

$$\mathbf{v} = a \frac{\partial}{\partial t} \left\{ \frac{\mathbf{g}}{4\pi G \rho_u a} \right\} . \tag{138}$$

We have found earlier that in the linear regime $\mathbf{g}$ grows with a universal gravity growth factor D_g,

$$\mathbf{g}(t) \propto D_g(t) \propto \frac{D}{a^2} \qquad \Rightarrow \qquad \frac{\mathbf{g}}{4\pi G \rho_u a} \propto D(t) \tag{139}$$

in which $D(t)$ is the linear density growth factor. Thus,

$$\begin{aligned} \mathbf{v} &= a \frac{\partial}{\partial t} \left\{ \frac{\mathbf{g}}{4\pi G \rho_u a} \right\} \\ &= \frac{1}{D} \frac{dD}{dt} \left\{ \frac{\mathbf{g}}{4\pi G \rho_u} \right\} \end{aligned} \tag{140}$$

We have now arrived at one of the most important results in the theory of linear structure formation, the fact that the **induced peculiar velocity** is directly and linearly proportional to the **generating peculiar gravitational acceleration** !!!! The factor of proportionality between $\mathbf{v}$ and $\mathbf{g}$,

$$\frac{1}{D} \frac{dD}{dt} = \frac{1}{a} \frac{a}{D} \frac{dD}{da} \frac{da}{dt} = H(t) \frac{a}{D} \frac{dD}{da} \equiv H\, f\,, \tag{141}$$

which includes the **dimensionless linear velocity growth factor f**, undoubtedly one of the most fundamental concept within the theory of linear structure formation (see next subsection). The linear relation between peculiar velocity and gravitational acceleration is thus found to be:

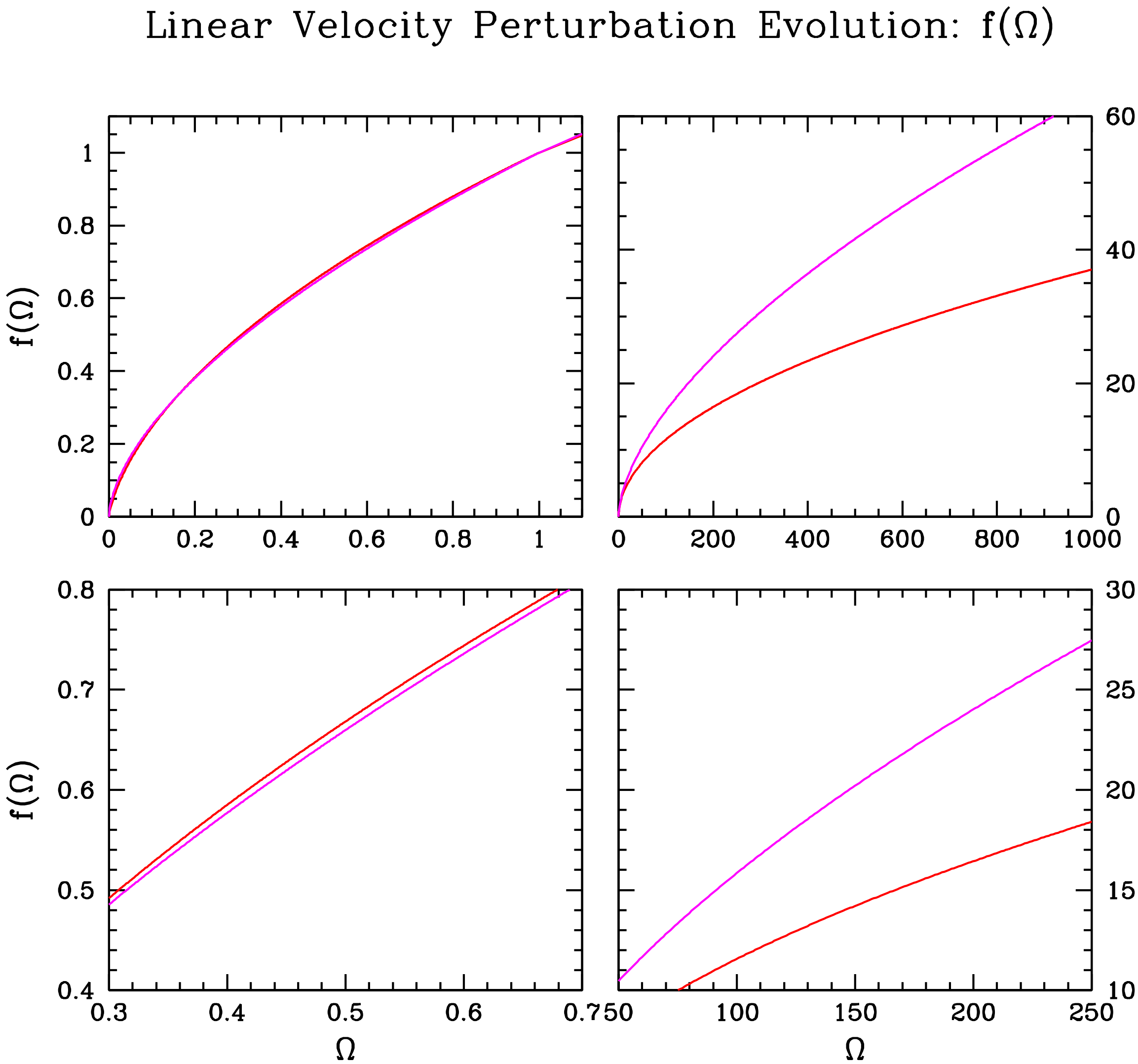


Figure 9. *Dimensionless Linear Velocity Growth Factor: Testing the approximation* $f(\Omega) \approx \Omega^{0.6}$ *in various regimes.*

$$\boxed{\mathbf{v} \;=\; \frac{H\,f}{4\pi G\,\rho_u}\,\mathbf{g} \;=\; \frac{2\,f}{3H\Omega}\,\mathbf{g}} \tag{142}$$

10.3.5. Dimensionless Linear Velocity Growth Factor $f(\Omega)$

In the above we found that the dimensionless linear velocity growth factor is a very important concept in the linear theory of structure formation,

$$\boxed{f \;\equiv\; \frac{a}{D}\frac{dD}{da} \;=\; \frac{d\log D}{d\log a}} \tag{143}$$

In the standard pressureless matter-dominated cosmology the dimensionless growth rate f is an analytical function of Ω_m (see fig. **??**). For $\Omega_m \lesssim 1$ Peebles (see Peebles 1980) found a celebrated approximation as a power law,

$$\boxed{f(\Omega_m) \approx \Omega_m^{0.6}} \tag{144}$$

The validity of this approximation may be appreciated from figure (**??**). It is an extremely good approximation for $\Omega_m \lesssim 1$ and therefore in nearly all conceivably plausible cosmologies. For academic purposes figure (9) shows that the approximation breaks down as Ω_m becomes very large.

We will notice that the factor $f(\Omega_m)$ and in particular its approximation by $f(\Omega_m) \approx \Omega_m^{0.6}$ will return in nearly every context involving the analysis of peculiar motions.

An approximation for $f(\Omega_m, \Omega_\Lambda)$ in the case of a Universe with matter and a cosmological constant Λ, Lahav et al. (1991) found an approximation

$$f(\Omega_m, \Omega_\Lambda) \approx \Omega_m^{0.6} + \frac{\Omega_\Lambda}{70}\left(1 + \frac{\Omega_m}{2}\right) \tag{145}$$

This clearly shows that the growth rate f mainly depends on the matter density Ω_m. It is only (very) weakly dependent on the cosmological constant. This confirms our expectation, given our earlier finding that the uniformly distributed dark energy component does not participate itself in the growth of perturbations and influences structure growth mainly through its impact on the expansion of the background Universe.

To first order we may therefore keep using the approximation $f(\Omega_m) \approx \Omega_m^{0.6}$ for linear velocities in a Universe with a cosmological constant.

10.3.6. Linear Velocity Growth

From the proportionality between peculiar velocity and peculiar gravity we can infer the linear growth rate for peculiar velocities,

$$\mathbf{v} = \frac{2f}{3H\Omega}\mathbf{g} \tag{146}$$

$$\propto \frac{Hf(\Omega_m)}{\Omega H^2}\frac{D}{a^2} \propto aDHf(\Omega_m)\,, \tag{147}$$

so that we may conclude that the linear velocity growth factor D_v in a matter-dominated Universe is equal to

$$\boxed{D_v(t) = aD\,Hf(\Omega_m)} \tag{148}$$

In an Einstein-de Sitter Universe, for which $H(t) \propto a^{-3/2}$ and $f(\Omega_m) = 1$, we therefore find that the peculiar velocity is continuously and vigorously growing with time,

$$\Omega_0 = 1 \quad \Rightarrow \quad v(t) \propto a^{1/2}\,. \tag{149}$$

It is also interesting to see that this implies that the "velocity" in comoving space is even slowing down in an Einstein-de Sitter Universe,

$$\dot{\mathbf{x}} = \frac{\mathbf{v}}{a} \propto \frac{1}{a^{1/2}}\,. \tag{150}$$

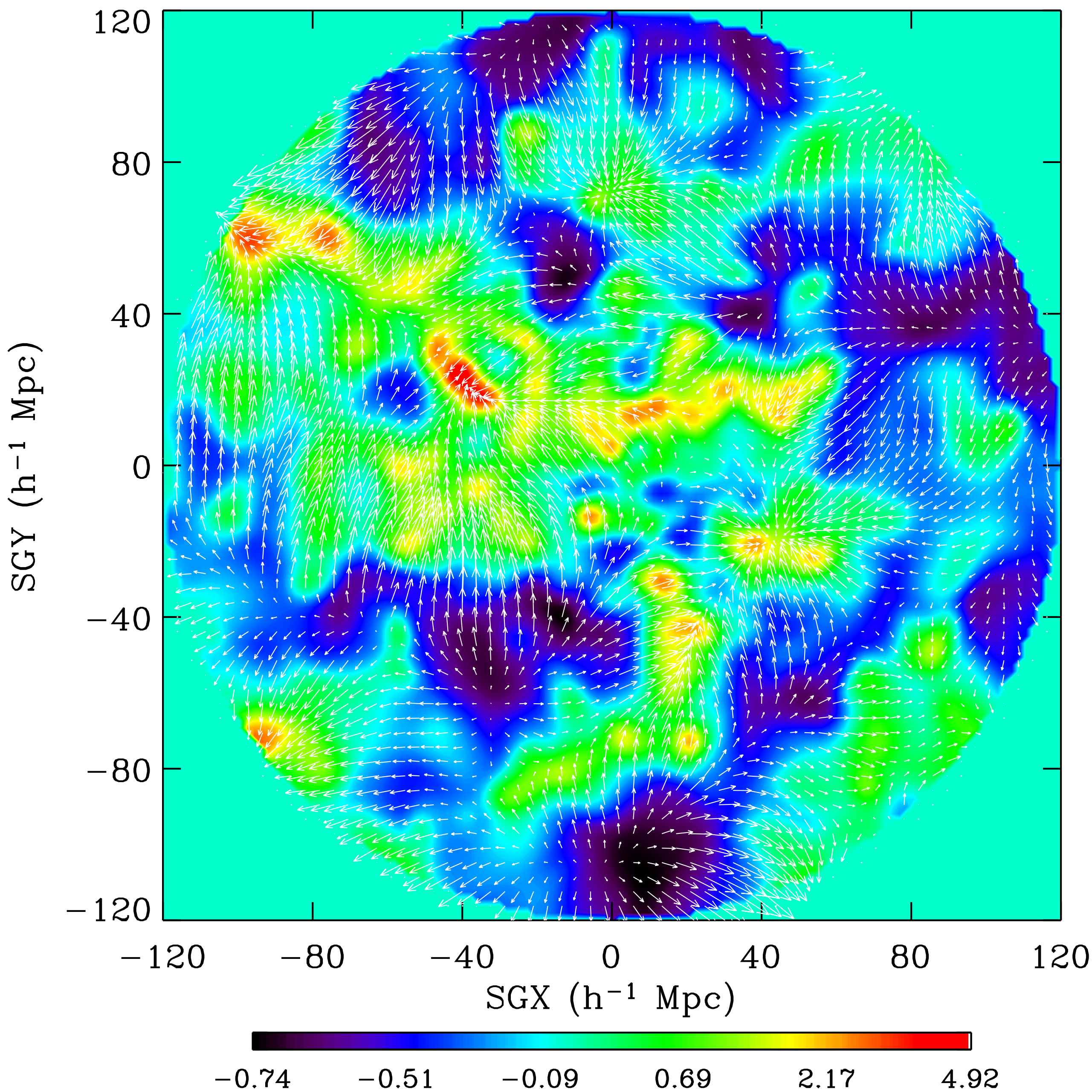


Figure 10. DTFE density and velocity fields projected along the $z-$supergalactic plane in a thin slice. The color bar indicates the plotted density scale. Velocities have been normalized to the maximum plotted velocity.

Indeed, this is what one observes when studying computer simulations of structure formation. Usually the particles are plotted in comoving coordinates (so that the simulation box retains the same size): the cosmic migrations on large linear scales are then gradually slowing down within these comoving volumes. On the other hand, in an open Universe with $\Omega_0 < 1$, the fact that structure growth will cease once the Universe gets into a free expansion mode also involves the decay and gradual diminishment of peculiar velocities. For the asymptotic limit $a \leftarrow \infty$, for which $\Omega(t) \propto \Omega_0/a$ and $H \propto 1/a$, we find that the peculiar velocity is decreasing according to

$$\Omega_0 < 1 \quad \Rightarrow \quad v(t) \propto a^{-0.6} \,. \tag{151}$$

10.3.7. Cosmic Flows and the Cosmic Density Fields

The direct measurement of gravitational accelerations $\mathbf{g}(\mathbf{x})$ throughout the (nearby) Universe is not really feasible. Evidently, one may seek to measure the spatial matter distribution in a suitably large volume of space and from this calculate the implied acceleration $\mathbf{g}$ within the region for which the measured mass distribution is sufficiently extended to be dynamically representative (this, by the way, is not a trivial issue and still leads to vigorous discussion). However, now we have found that there is a direct linear relationship between peculiar velocities $\mathbf{v}$ and the peculiar gravitational acceleration $\mathbf{g}$ (equation 142),

$$\mathbf{v} = \frac{2f}{3H\Omega}\mathbf{g} \tag{152}$$

we have found a wonderful means of mapping the gravitational acceleration throughout the local Universe. In fact, via this relation we can establish the direct link between the mass distribution $\delta(\mathbf{x}, t)$ in the Universe and $\mathbf{v}$ by using the integral expression (120) for the gravitational acceleration $\mathbf{g}$,

$$\boxed{\mathbf{v}(\mathbf{x},t) = \frac{H f(\Omega_m)}{4\pi} a \int d\mathbf{x}' \, \delta_m(\mathbf{x}',t) \, \frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3} \qquad (153)}$$

The repercussions of this finding are far-reaching and of great importance. It provides us with a means to *"weigh"* the Universe, i.e. a way to determine the cosmological density parameter Ω_m. It is a telltale illustration of the observation that one may infer the global properties of a system by studying its perturbations. In other words, an advanced affiliate of professor Zonnebloem's favorite instrument, the pendulum, for measuring the gravitational field of the Earth ! Also, it shows why the study of cosmic large scale structure plays such a central role within cosmology.

10.3.8. Peculiar Velocities and the Local Galaxy Distribution

First let us investigate how we may infer the cosmic density parameter Ω_m. Imagine that the spatial distribution of galaxies would be a fair discrete representation of the underlying mass distribution, and that we have access to the database of a large galaxy survey (such as the PSCz, 2dFGRS and SDSS redshift survey catalogues). If it would indeed be an unbiased reflection of the cosmic density field, it would mean the galaxy distribution to be a Poisson sample of the underlying continuous density $\delta(\mathbf{x})$ and its local number density $n(\mathbf{x}$ to be a direct reflection of $\delta(\mathbf{x})$,

$$\frac{\delta n(\mathbf{x})}{n_u} = \delta(\mathbf{x}) \,, \tag{154}$$

in which $n_u(t)$ is the average number density of the galaxy distribution.

By counting the galaxies at a large number of locations within the survey volume $\mathcal{V}_{galsur}$ we may thus infer the mass density field $\delta(\mathbf{x})$. Assume that $\mathcal{V}_{galsur}$ is indeed a sufficiently large volume. This means that it includes all structures whose influence is of importance for the dynamics of the local volume $\mathcal{V}_{galvel}$ in which we have measured peculiar velocities. This local volume $\mathcal{V}_{galvel}$ may even be only our own Galaxy or Local Group. Indeed, as we know via the dipole moment in the angular distribution of the cosmic microwave background the peculiar velocity of our own Galaxy to great accuracy, and we have a clear and well-defined mapping of the surrounding matter distribution there has been a lot of effort in determining Ω_m from these measurements (see subsection **??**). It is important to notice that the galaxy survey has to be cleanly defined, ideally with a uniform coverage throughout the survey volume, at least with a well-defined and well understood selection function. This allows us to transform the galaxy counts into a density field.

The velocity velocity field $\mathbf{v}(\mathbf{x})$ within the sampled cosmic volume may then be computed – through smart filtering and interpolation techniques – from the measured peculiar velocities $\mathbf{v}_i$ within the region $\mathcal{V}_{galvel}$. For this one does not necessarily need a uniformly defined sample of galaxy peculiar velocities, and in the case of the dipole motion of our own Galaxy we may even restrict the analysis just to one galaxies' velocity.

Having determined the velocity field $\mathbf{v}(\mathbf{x})$ within $\mathcal{V}_{galvel}$, and comparing this with the computed acceleration field for the density field $\delta(\mathbf{x}')$, by computing the integral (153) for the locations $\mathbf{x}$ in $\mathcal{V}_{galvel}$, we are left with one unknown factor. The ratio between the two fields involves the ratio

$$\frac{Hf(\Omega_m)}{4\pi} \tag{155}$$

and from this we can immediately derive the value of $f(\Omega_m)$ and thus of Ω_m !!!!

10.3.9. Galaxy Bias

In the above we made one crucial assumption which may not at all be warranted. We assumed that indeed the galaxy distribution is directly and linearly proportional to the matter density field. However, this is an a priori unjustified assumption. Because we do not have any compelling theory of galaxy formation we cannot be sure of the connection between the matter distribution and the galaxy distribution. It seems quite plausible to assume that on average a region with a higher matter density will be marked by the presence of more galaxies, as there was simply more matter available for these objects to condense out, but we cannot really say much more. Certainly, we may not say that we can simply infer the matter perturbation field $\delta(\mathbf{x})$ can be determined from the galaxy number densities,

$$\delta(\mathbf{x}) \,=\, \frac{\delta n(\mathbf{x})}{n_u}\,. \tag{156}$$

To circumvent this unwarranted assumption cosmologists have introduced a rather simplistic way of parameterizing their ignorance over the galaxy-matter relationship. This assumption, **Linear Bias**, says that the density fluctuations δ_{gal} in the galaxy distribution do form a biased reflection of underlying matter density fluctuations,

$$\delta_{gal}(\mathbf{x}) \,\equiv\, \frac{\delta n(\mathbf{x})}{n_u} \,\equiv\, b\,\delta(\mathbf{x})\,. \tag{157}$$

In other words, for positive $b > 1$ the galaxy distribution would be enhanced if located in a high-density region. In underdense void regions, on other hand, it would get extra suppressed. It is not inconceivable that something like this may have been a characteristic of the galaxy formation process. In special circumstances one may also think of **antibias**, $b < 0$). In the vicinity of a very bright quasar the formation of galaxies may have been suppressed because its intense radiation may have prevented collapse of gas clouds.

Even while the concept of the **linear bias factor b** is of course a strong oversimplification, there are both some theoretical reasons as well as observational indications for it not to be a useful approximation for galaxy clustering on Megaparsec scales. In chapter 5 we will see that cosmic structure has arisen out of a random field of primordial Gaussian perturbations. Cosmic objects will probably have formed from the peaks in these primordial fields, the nature of the object first and foremost determined by the scale of the fluctuations. In 1984 Kaiser proved that peaks in such random fields are more strongly clustered than the average mass distribution. Moreover, mathematically he demonstrated that a reasonable approximation for the clustering of moderately high peaks, corresponding to galaxies (or halos of galaxies), is equal to that expected for a *linearly biased population* of objects. Tying in with this idea is the expectation that different objects will be differently biased: they will have formed from a different set of peaks. This is indeed what one appears to observe in reality. Clustering is seen to be dependent on galaxy type (and may be dependent on the luminosity of galaxies, though this has proved harder to confirm). Early-type galaxies have a considerably more pronounced spatial distribution than the more smoothly distributed late-type galaxies. The factor b for the early-type galaxies may therefore be expected to be higher. We should think about a factor $b \approx 2$ for early-type galaxies, while late-type galaxies seem to tend towards $b \approx 1$. However, as we start to get a better insight into the nature of the galaxy distribution, and thus touch upon the more profound aspects of galaxy formation, there appear more and more complications for the linear bias description.

If we subsequently work from the galaxy density field δ_{gal} and seek to find the corresponding velocity field $\mathbf{v}$, we arrive at a modified relation of eqn. (153)

$$\boxed{\mathbf{v}(\mathbf{x},t) \;=\; \frac{H}{4\pi}\,\frac{f(\Omega_m)}{b}\,a\int d\mathbf{x}'\;\delta_{gal}(\mathbf{x}',t)\;\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3} \qquad (158)}$$

If we then combine measured peculiar velocities of galaxies with the galaxy density distribution δ_{gal} we cannot infer $f(\Omega)$ directly. We are left with the socalled β parameter which combines the cosmic mass density parameter Ω_m and the linear bias factor b,

$$\boxed{\beta \;\equiv\; \frac{f(\Omega_m)}{b} \;\sim\; \frac{\Omega_m^{0.6}}{b}} \qquad (159)$$

There is no way to break the degeneracy between Ω_m and b in the β parameter. Thus, while we do not have an a priori idea of the value of b we are set with a corresponding uncertainty in our estimate of Ω_m. Only via other techniques we may hope to solve this issue. One possibility is to look into the deviations from Gaussianity in the velocity field, whose first orders involve quantities solely dependent on Ω_m (in particular the third order moment called S_3, the skewness). An even more promising way is the exciting possibility to map the matter distribution directly through the study of its influence on the the paths of photons through the Universe. The study of gravitational lensing has in recent years already lead to impressive breakthroughs and is due to yield much more in the years ahead.

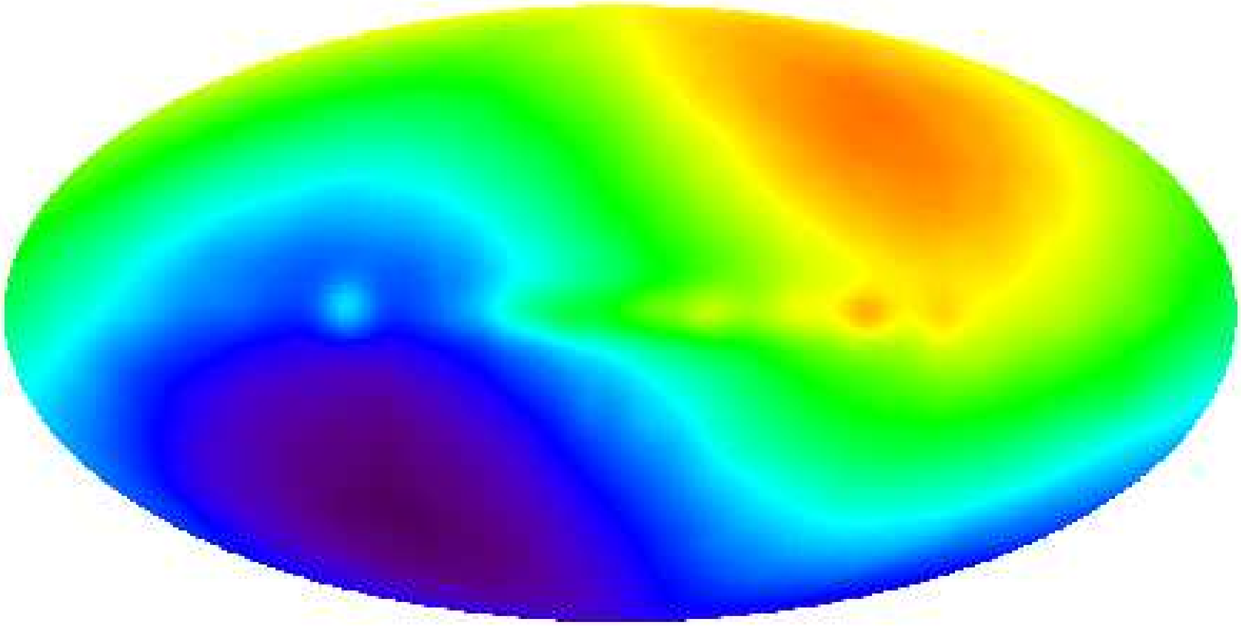


Figure 11. The Cosmic Microwave Background dipole as measured by the DMR instrument of the COBE microwave background satellite (see also Kogut et al. 1993)

10.3.10. Peculiar Velocities and the Galaxy Distribution

There are a few ways in which one may exploit the measured peculiar velocities of galaxies. One may seek to determine Ω_m by determining the peculiar velocities of galaxies and combining these with the mapping of the spatial matter distribution in the dynamically relevant region (i.e. the region containing all features in the matter distribution involved in generating the measured galaxy regions. One may also seek to invert the relationship and determine the mass distribution from the measured velocity field, or at least identify the

10.3.11. Mapping the Dynamics of the Local Universe

10.3.12. The Local Cosmic Acceleration: the Cosmic Dipole

While measuring the peculiar velocities of galaxies is extremely cumbersome, there is one peculiar velocity known to great precision. This is the velocity of our own Galaxy, or our Local Group, with respect to the Universe.

Because of our motion through the Universe the isotropically distributed CMB radiation gets Doppler shifted, resulting in a dipolar pattern over the sky.

and as well computed involving the densities over the volume $\mathcal{V}_{galsur}$ for the locations in the volume.

in the nearby Universe. We could then combine the local velocity field with the measured mass distribution and infer from equation (153) the as yet unknown factor $f(\Omega_m)$ and from this the cosmic density Ω_m.

Once it became possible to determine relatively accurately the peculiar velocities of galaxies, from the mid eighties of the last century onward, astronomers had a way to determine the cosmic matter density ! It may be no surprise that at the time this triggered tremendous activity in the community. Lately the enthusiasm has lured somewhat as it is hardly possible to measure peculiar velocities of galaxies to better than 20% accuracy.

the same volume

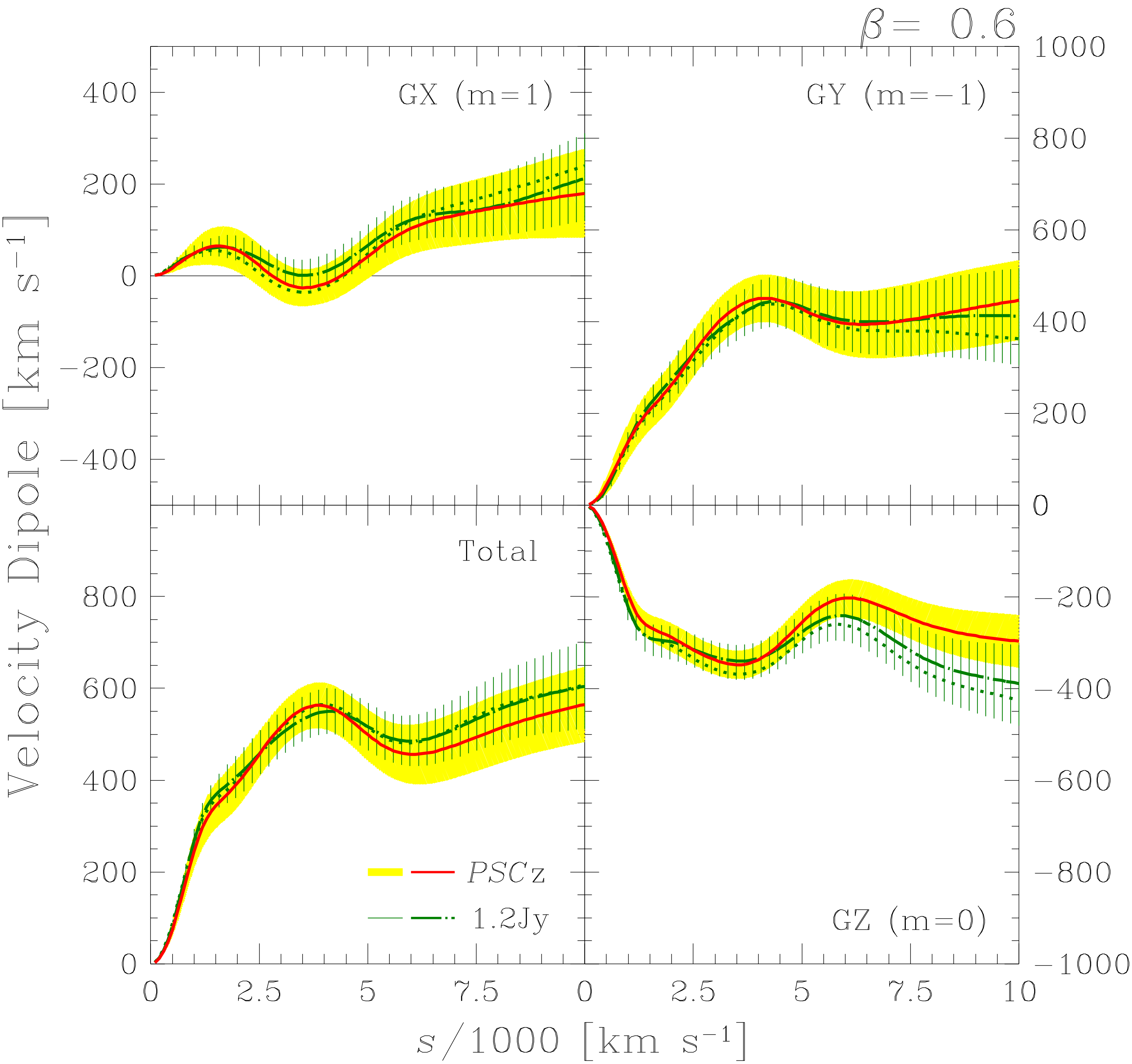


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Figure 12. The velocity dipole coefficients inferred from the PSCz (continuous) and 1.2 Jy QDOT survey. The inferred dipole velocities along the three Galactic Cartesian components GX, GY and GZ are shown in the top-left, top-right and bottom-right panel respectively. The total amplitude is shown

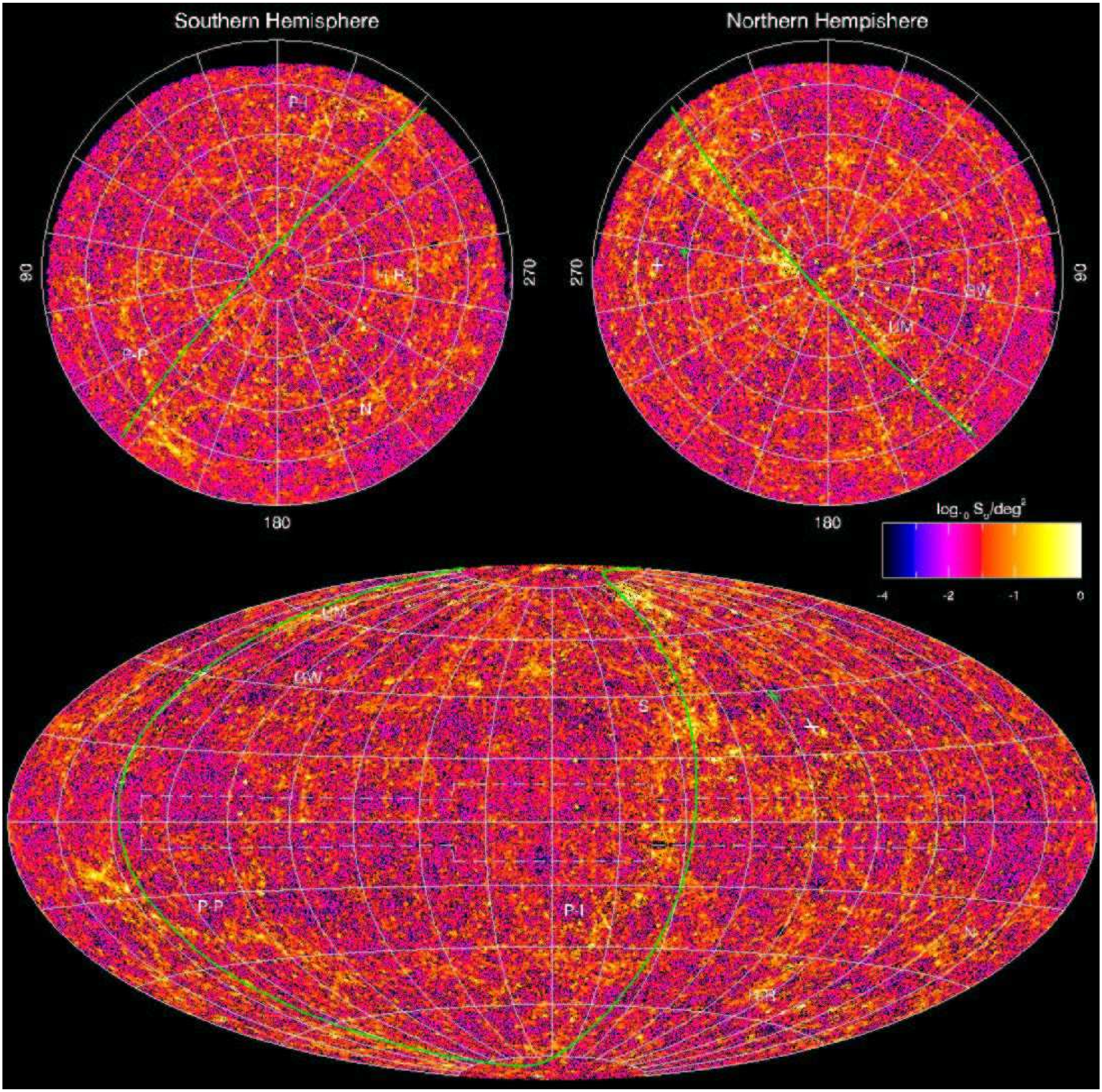

Figure 13. The 2mass near infrared flux distribution

We should stress the "in principle", the practical complications involved with measuring peculiar velocities are enormous.

Measuring the peculiar velocities throughout a representative volume of the nearby Universe is feasible.

11. Linear Theory: Fourier Mode Evolution

Once density perturbations become of the order unity, $\delta(\mathbf{x}, t) \approx 1$, it is no longer possible to use the basic linear theory of clustering. Different modes of the density fields will start to interact, resulting in mutual "power transfer".

Restricting ourselves to a pure matter-dominated Universe, the linear fluid evolution equations,

$$\boxed{\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}_x \cdot \vec{v} &= 0 \\ \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} &= -\frac{1}{a}\vec{\nabla}_x \phi \\ \nabla_x^2 \phi &= 4\pi G\, a^2\, \rho_u\, \delta \end{aligned}} \tag{160}$$

Rewriting these equations in their corresponding Fourier expressions provides a direct insight into the complications that go along with the growing nonlinearity of a gravitationally evolving system. The Fourier expressions for the density field $\delta(\mathbf{x}, t)$, velocity field $\mathbf{v}(\mathbf{x}, t)$, and potential field $\phi(\mathbf{x}, t$ are

$$\boxed{\begin{aligned} \delta(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\,\hat{\delta}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}} & \Longleftrightarrow \quad \hat{\delta}(\mathbf{k}) &= \int d\mathbf{x}\ \delta(\mathbf{x})\,e^{i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{v}(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\,\hat{\mathbf{v}}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}} & \Longleftrightarrow \quad \hat{\mathbf{v}}(\mathbf{k}) &= \int d\mathbf{x}\ \mathbf{v}(\mathbf{x})\,e^{i\mathbf{k}\cdot\mathbf{x}} \\ \phi(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\,\hat{\phi}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}} & \Longleftrightarrow \quad \hat{\phi}(\mathbf{k}) &= \int d\mathbf{x}\ \phi(\mathbf{x})\,e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}} \tag{161}$$

By inserting the Fourier definitions (162) in the linear fluid equations (161), and using the virtuous circumstance that spatial derivatives correspond to mere multiplications in Fourier space,

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\ \hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \nabla f(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\ -i\mathbf{k}\,\hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}} && (162) \\ \nabla^2 f(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}\ k^2\,\hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}} && (163) \end{aligned}$$

it is straightforward to infer the set of fluid equations formulated in Fourier space. The Fourier versions of the linearized continuity/energy equation, the Euler equation and the Poisson equation are

$$\boxed{\begin{aligned} \frac{d\hat{\delta}(\mathbf{k})}{dt} - \frac{1}{a} i\mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{k}) &= 0 \\ \frac{d\hat{\mathbf{v}}(\mathbf{k})}{dt} + \frac{\dot{a}}{a}\mathbf{v}(\mathbf{k}) &= \frac{1}{a} i\mathbf{k} \cdot \hat{\phi}(\mathbf{k}) \\ \frac{\hat{\phi}(\mathbf{k})}{a^2} &= -4\pi G\rho_u \frac{\hat{\delta}(\mathbf{k})}{k^2} \end{aligned}} \tag{164}$$

This brings us to the **crucial** observation that in the linear regime each Fourier mode $\mathbf{k}$ is evolving independently of the other modes !!!

$$\boxed{\begin{array}{c} \text{Linear Regime:} \\ \\ \text{All Fourier Modes evolve independently} \end{array}}$$