Generating Spatial Point Processes

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1 Introduction

Any stochastic process may be described by a probability distribution, and may be thought of as the mapping of a sequence of random variables to a new set of states. Examples of systems that may be modelled by a stochastic process, are stock markets, images, brownian motion, landscapes, galaxies and cosmological density fields. Although the term process first brings to mind a time series it can be generalized to any suitable parameter space. When the space is a spatial volume we refer to its as a spatial random field.

A point process is a form of stochastic or random process. It may be thought of as a set of random points in a space, with a certain probability defined over the same space (Formally it should be called a point field, but let us just use both names as stochastic variation on the theme). We restrict ourselves to point processes in $\Re^3$, and to the bounded region in which a point field is located; *the window*. Our window here will the unit cube. Many other common windows are being used, for example the observationally defined pie slices. In general we can broadly distinguish two kinds of point process, homogeneous and inhomogeneous process. If the intensity is not a function of location $\mathbf{x}$, then we speak of a homogeneous poisson process.

For astronomy point processes are very relevant, many observables may be modelled by them. For example the spatial distribution of stars and galaxies can be thought of as a point process. Also X-ray observations generate a point process: they consist of a discrete number of highly energetic photons. They define a limited and discrete set of spatially distributed points at the locations of incidence.

1.1 Poisson Field

One of the simplest and fundamental point processes is the spatial Poisson point process. The points are stochastically independent and the probability of the number of points $N(A)$ in a region A, is given by the Poisson distribution:

$$P(N(A) = k) = \frac{(\lambda V(A))^k}{k!} e^{-\lambda V(A)} \tag{1}$$

Here λ is the intensity of the point distribution, which is the mean of the distribution. Note: In general the mean of the distribution is definded as the average value at a certain point in space over **many realizations**. Only if one assumes Ergodicity, the spatial mean is equal to the average of the probability distribution. Ergodicity is of utmost importance to cosmology as our Universe is the only one sample we have.

Generating a Poisson field is straightforward. For a realization in a window A, determine with a random poisson deviate, the number of points lying in A. Distribute these points randomly over the window. This can be done by using a uniform random number generator. In the Cartesian system each coordinate is just one random deviate, though in other coordinate system this might require a volume preserving mapping. Remember that for distributions we have the following rule for coordinate transformations.

$$p(y1, y2, ...)dy1dy2... = p(x1, x2, ...) \left| \frac{\partial(dx1dx2..)}{\partial(dy1dy2...)} \right| dy1dy2... \tag{2}$$

Note that if we had fixed the total of number of points in A a priori, then strictly speaking we would be generating a binomial point field. In practice, for large N, this will be approximately the same.

1.2 General Poisson Field

The previous process may be generalized by letting λ be a function of $\mathbf{x}$. This is also called the inhomogeneous poisson point process. The probability of finding $N(A)$ points is again given by equation (1), with the substition of $\lambda V(A)$ by;

$$\Lambda(A) = \int_A \lambda(\mathbf{x}) d\mathbf{x} \tag{3}$$

1.3 Intermezzo: Galaxies as a Poisson Process

We may node make the connection between a cosmological density field and the discrete spatial distribution of galaxies. Imagine that the density field takes the role as intensity function for an inhomogeneous point process. Then the points are considered to be the galaxies.

$$\lambda(\mathbf{x}) = \bar{n}[1 + \delta((x))] \tag{4}$$

In cosmology this is called the Poisson model. It is a doubly stochastic process both the density field and the following point process are random processess.

1.4 Segment Cox Process

Another example of a double stochastic random field are the Cox processes. The segment process is a very simple version of a Cox process. Here needles are radomly thrown over space, with random positions and orientations. Subsequently we sample points on each segment, see Fig. 1a . So this process of two poisson process one for the needle and the other for the points on the needle. The generator consists of three steps.

1. First draw a random position,
2. Then choose a random orientation
3. and lastly populate the segment with random points.

Note that all of these steps can be done by using uniform number generators.

1.5 Matern Process

The Matern process is much like segment Cox process. Instead of randomly distributed segments, spheres of fixed size R are generated. Each sphere is subsampled with a Poisson point distribution, with mean μ, see Figure 1b. The Matern processes can be easily modified be changing the spatial point distribution of each cluster. For example we could also have used a normal distribution at each position (Thomas process).

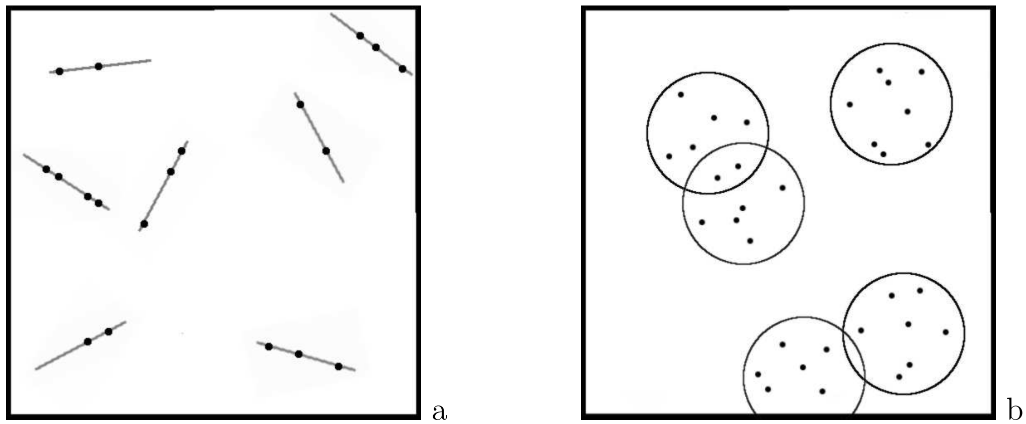


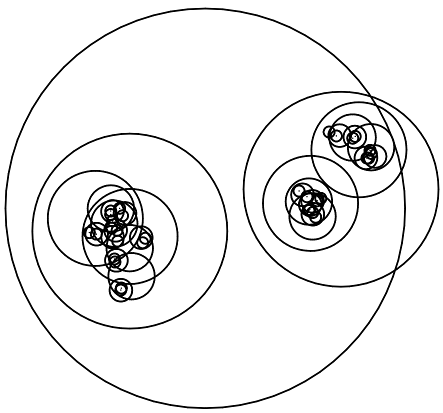
Figure 1: An illustration of the generation of a segment cox process (a) and a matern process (b).

1.6 Soneira-Peebles Fractal

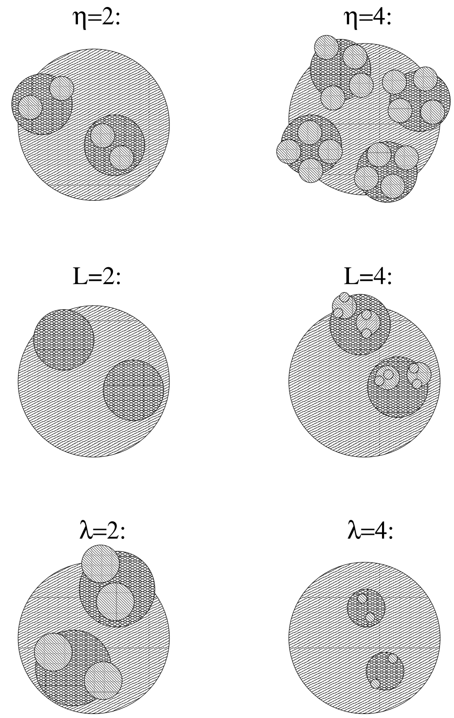
The Soneira-Peebles model is a fractal-like pont distribution involving hierarchically embedded levels of ever larger point density, see Figure (2)a. It was introduced by Soneira and Peebles to model the galaxy distribution obeying various clustering measures. A realization is generated in the following manner.

1. The starting point is a level-0 sphere of radius R.
2. In this sphere η level-1 spheres are placed with radius R/λ and $\lambda > 1$. The new spheres are placed at a random position inside the level-0 circle, such that their centers fall inside the original level-0 sphere.
3. Within each of these η level-1 spheres, one places η level-2 spheres of radius R/λ^2.
4. This process is repeated until one ends up with in total η^L level-L spheres of radius R/λ^L. At the center of each of these level-L spheres a point is placed.

One therefore ends up with in total η^L points, which in the Soneira-Peebles model represent galaxies. This procedure is illustrated in the top panel of Figure 2.



(a) The Soneira-Peebles model. Inside a level-0 sphere η level-1 circles are placed with a radius which is smaller by a fixed factor. This process is repeated until one ends up with η^L level-L circles. At the center of these level-L circles η^L points are placed, which form the resulting Soneira-Peebles point distribution.



(b) The physical meaning of the three defining parameters η, L and λ of the Soneira-Peebles model. The upper row shows the effect of varying η, the number of circles which is placed in each circle. The central row shows the effect of varying L, the total number of levels. The bottom row shows the effect of varying λ, the ratio of the radius of each circle with the radius of subsequent circles of one level higher.

Figure 2: Definition and Parameters of the Soneira-Peebles model

The Soneira-Peebles model is controlled through three parameters, η, L and λ. The effect of varying these parameters on the resulting point distribution is illustrated in the 2nd to 4th row of Figure 2. For a given number of points, η determines the dynamic range of the resulting point distribution. For a small value of η, many levels are needed to reach a fixed number of points, while a large value of η results in a smaller number of levels. A small value of η also results in a smaller filling fraction of space with spheres than a high value of η (2nd row in Figure 2). L denotes the total number of levels and therefore determines the range of densities and scales in the resulting point distribution. For a fixed value of η, L also determines the total number of points (third row in Figure 2). Finally, for given values of η and L, λ determines the range of spatial scales. A value of λ close to 1 means that subsequent spheres of higher levels are of comparable size. Values of λ much larger than one mean that each subsequent level consists of spheres which are significantly smaller than the spheres in the preceding level (bottom row in Figure 2).

An important property of the Soneira-Peebles model is that it is one of the few analytic self-similar models of the galaxy distribution for which the two-point correlation function can be analytically evaluated.

2 Tasks

The intention of this computer task is that you learn to generate a couple of random point processes on the computer. In upcoming computer tutorials we will ask you to analyze these data sets. And compare your 'measurements' with the theoretical predicted results.

- Make a program that generates a homogeneous Poisson Point Process in the unit cube with intensity λ (you may assume that N is large). Produce a xy scatter plot for $\lambda = 100000$.
- Write a program that generates a random deviate drawn from a poisson process, with mean μ. (Hint: you may want to consult chapter 7 of numerical recipes). For $\mu = 6$ draw 1000 deviates and make a plot of the histogram and overplot the theoretical distribution.
- Write a program that generates random points on the surface of a sphere, and in a sphere. For a sphere of unit diameter and 10000 point create an xy scatter plot and a yz scatter plot. Show with the radial distribution that you have produced a uniform distribution.
- Make a program that produces; the segment Cox process, the Matern point distribution and the Soneira-Peebles fractal.
- Plot for the following parameters xy scatter plots that illustrates the above point sets.
  - Segment: $\lambda_s = 1000$, $l = 0.1$ $\mu = 12$
  - Matern : $\lambda_c = 1000$, $r = 0.05$ $N = 12$
    Matern : $\lambda_s = 1000$, $r = 0.05$ $N = 100$
  - Soneira-Peebles: $\lambda = 6$, $\eta = 3$ $L = 6$
    Soneira-Peebles: $\lambda = 3$, $\eta = 1.7$ $L = 10$
    Soneira-Peebles: $\lambda = 4$, $\eta = 1.9$ $L = 8$
    Soneira-Peebles: $\lambda = 2$, $\eta = 1.5$ $L = 15$