Sterrewacht Leiden

An Analytic Expression for the Evolution of Voronoi Features

Vincent Icke

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Voronoi Tessellation Primer

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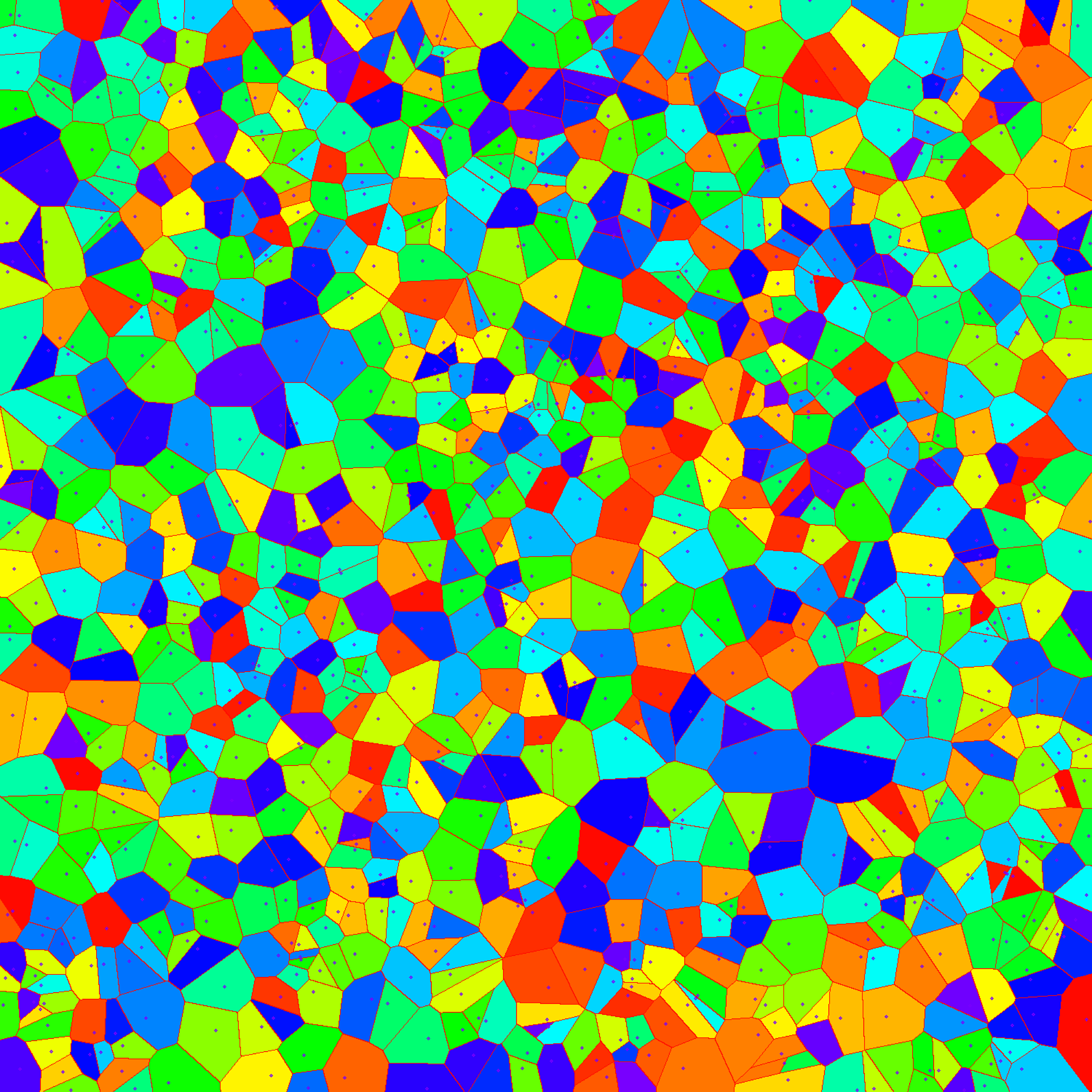

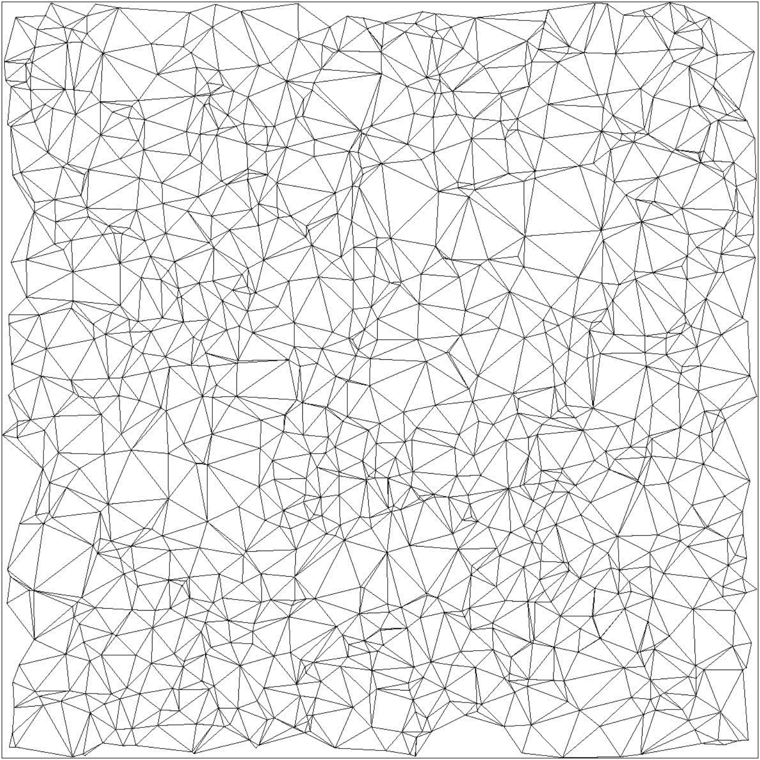

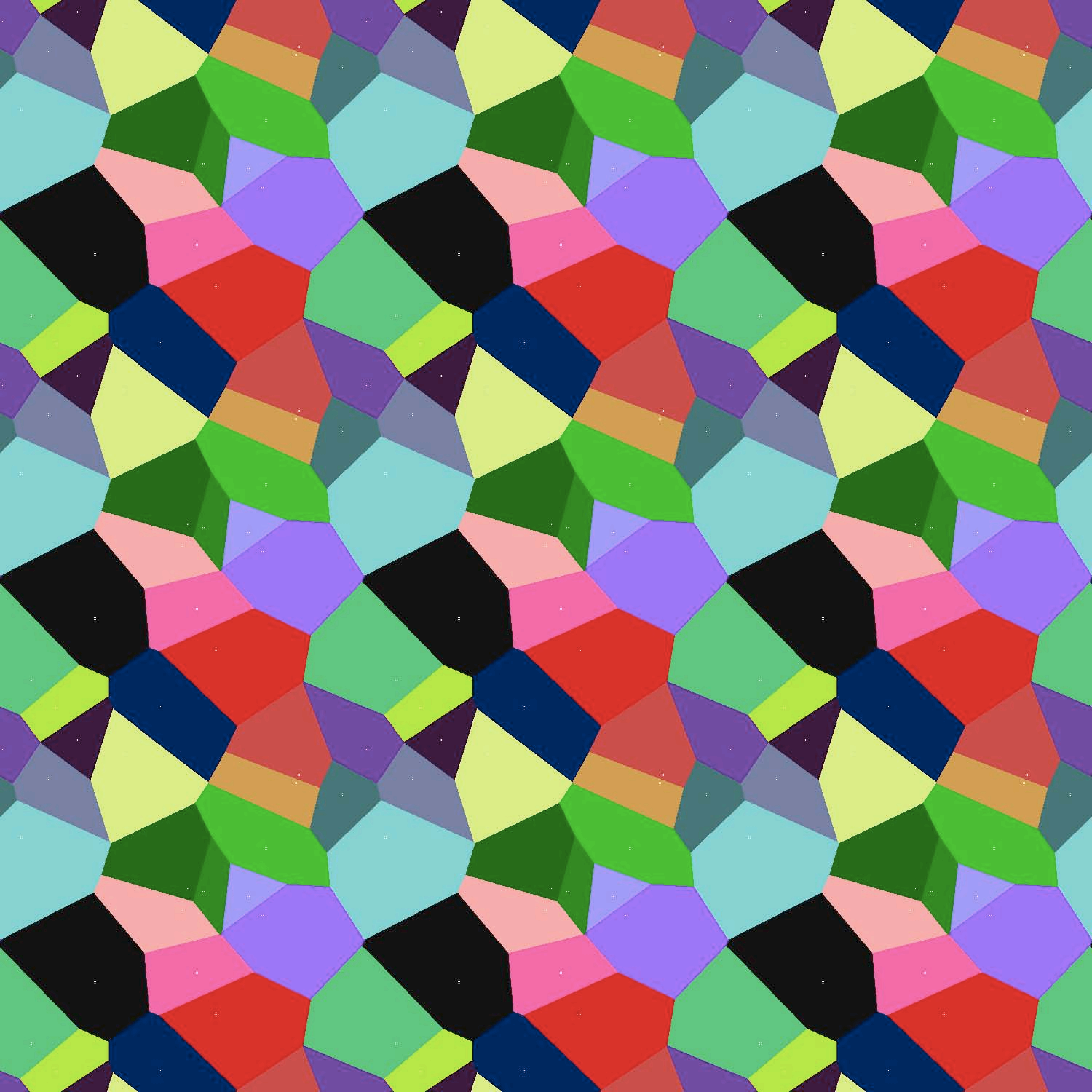

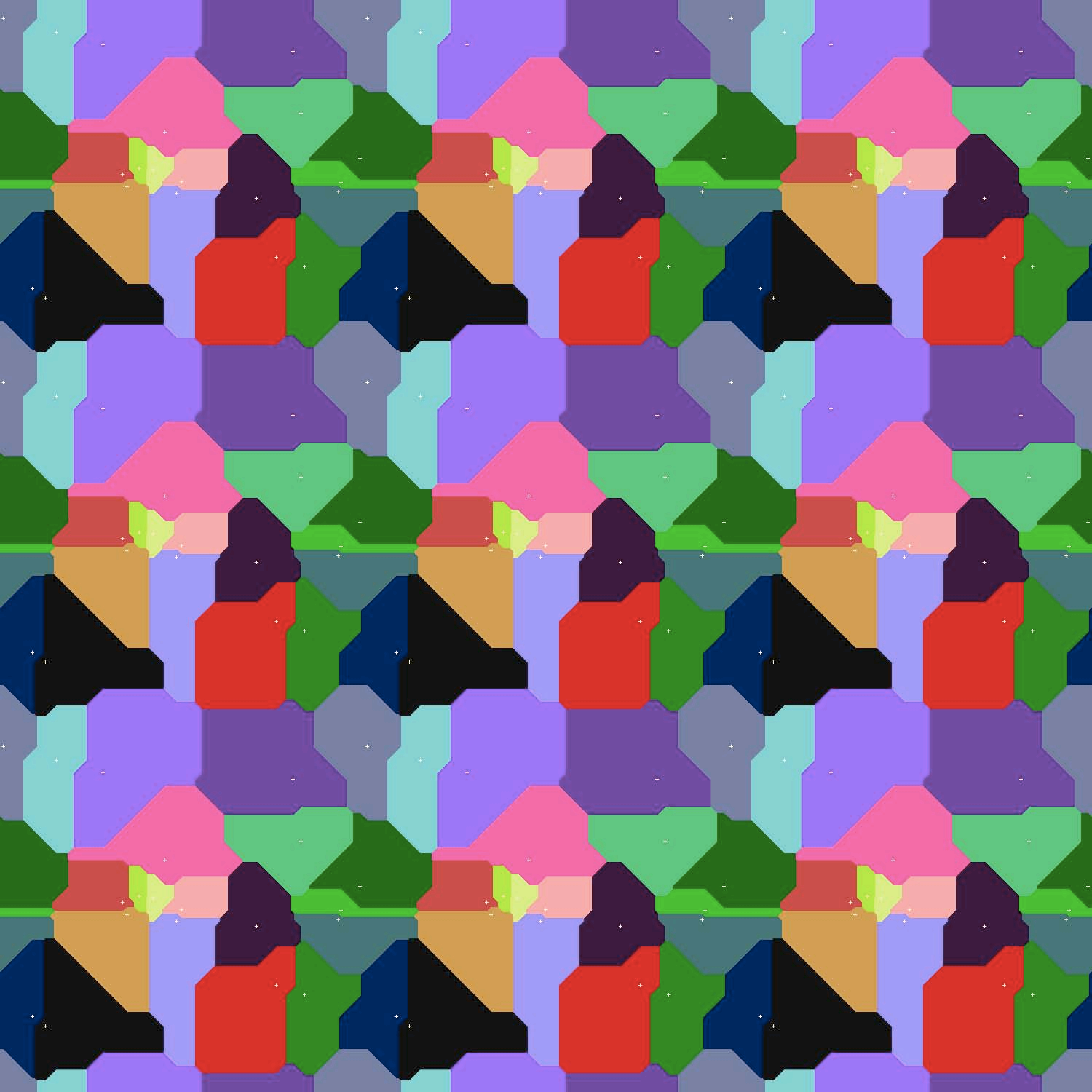

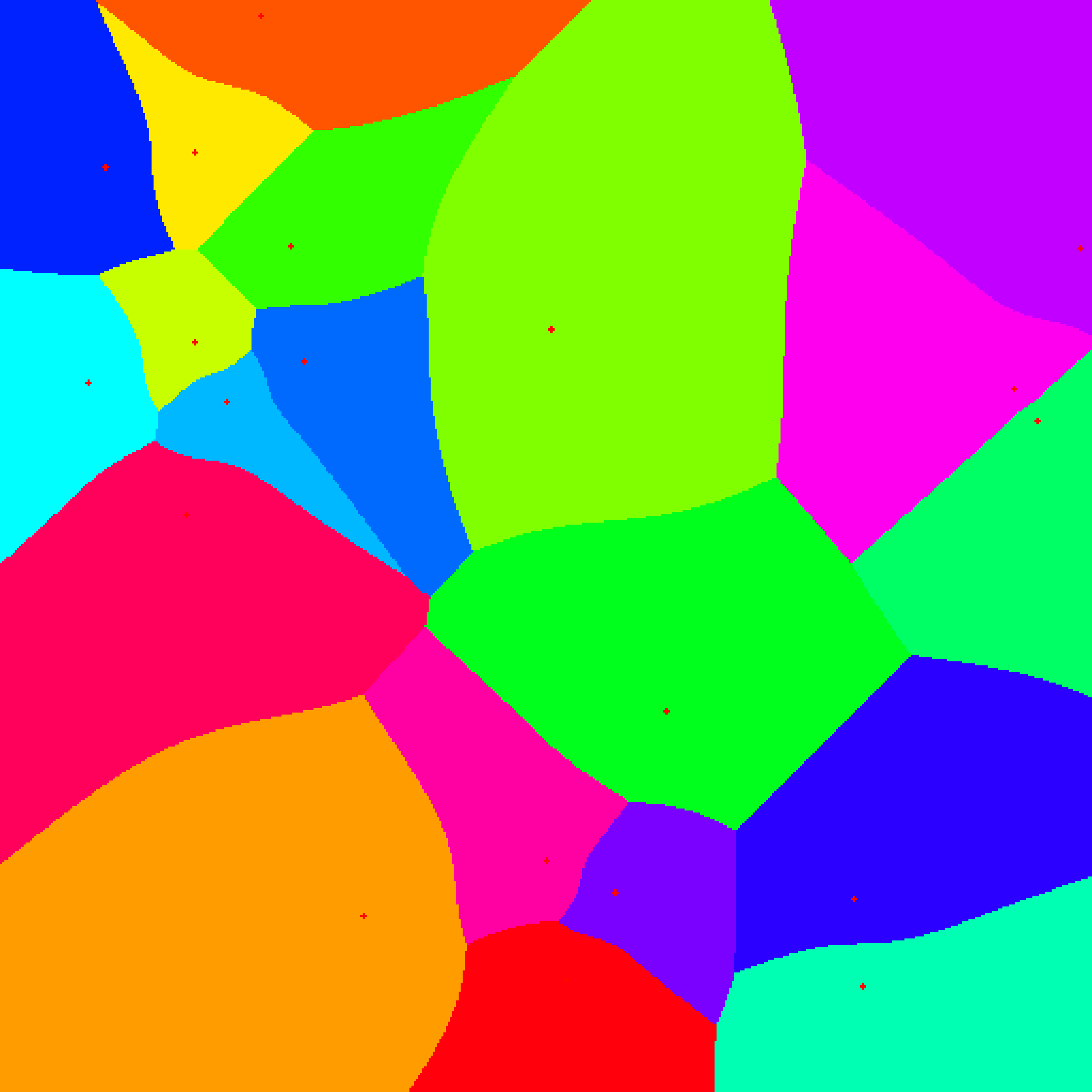

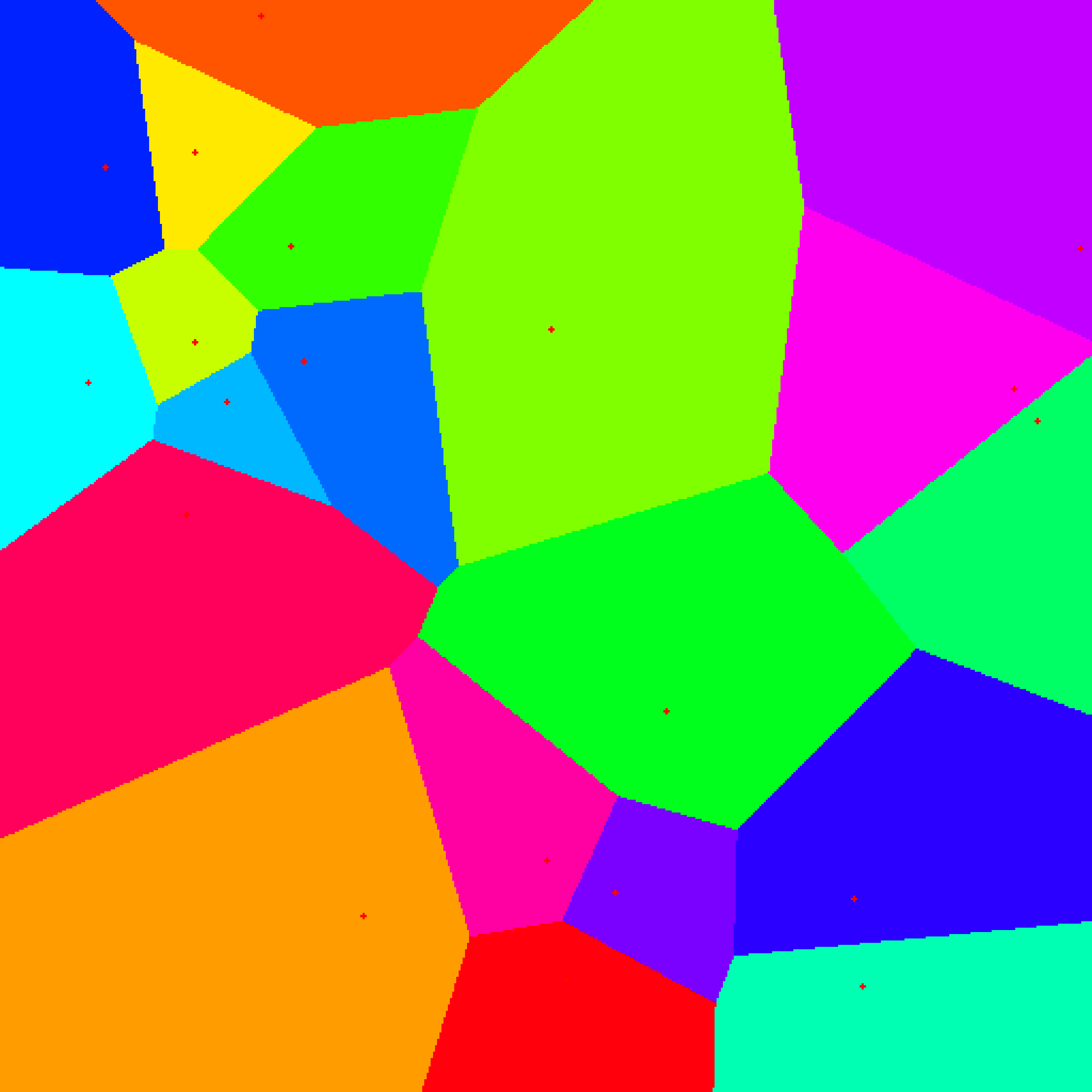

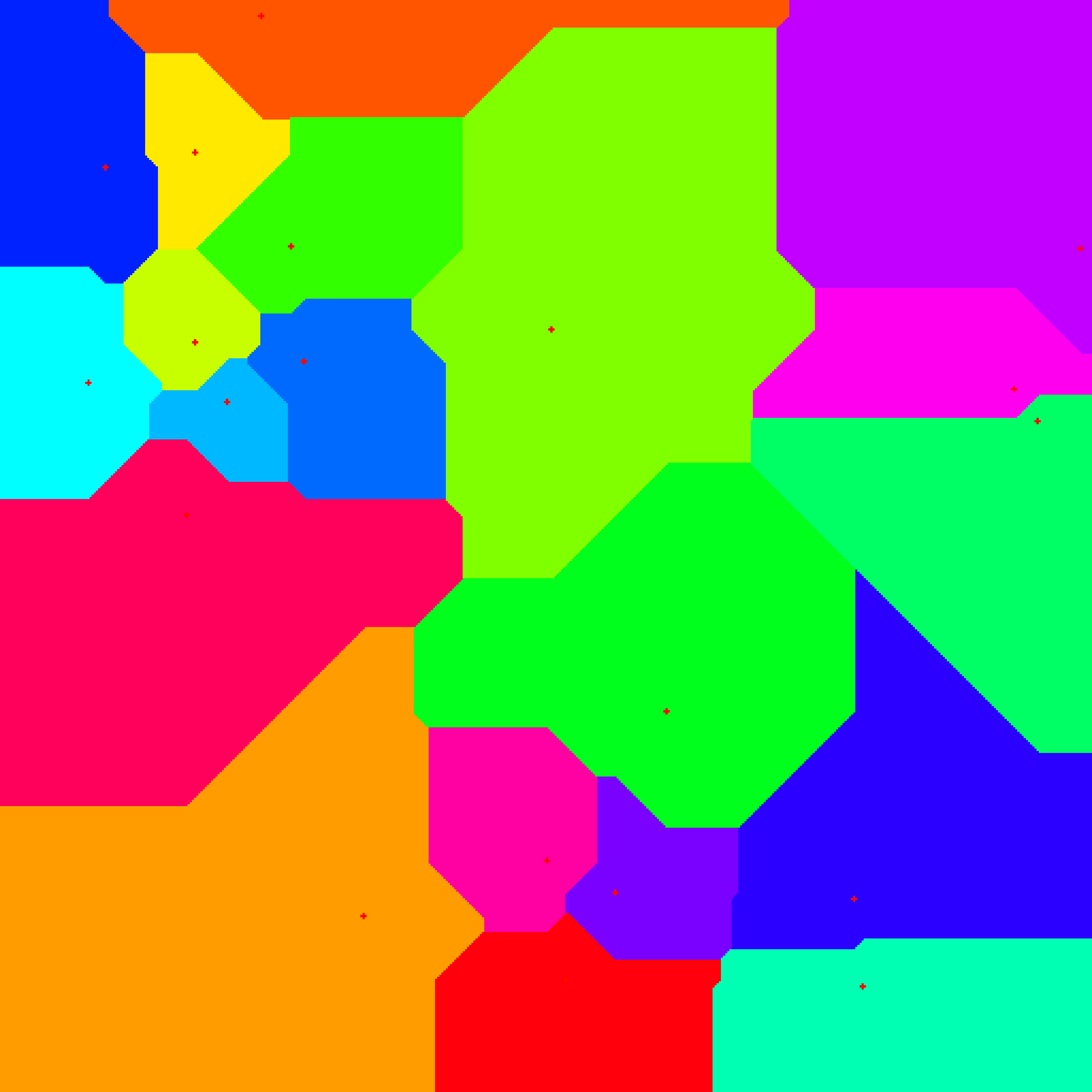

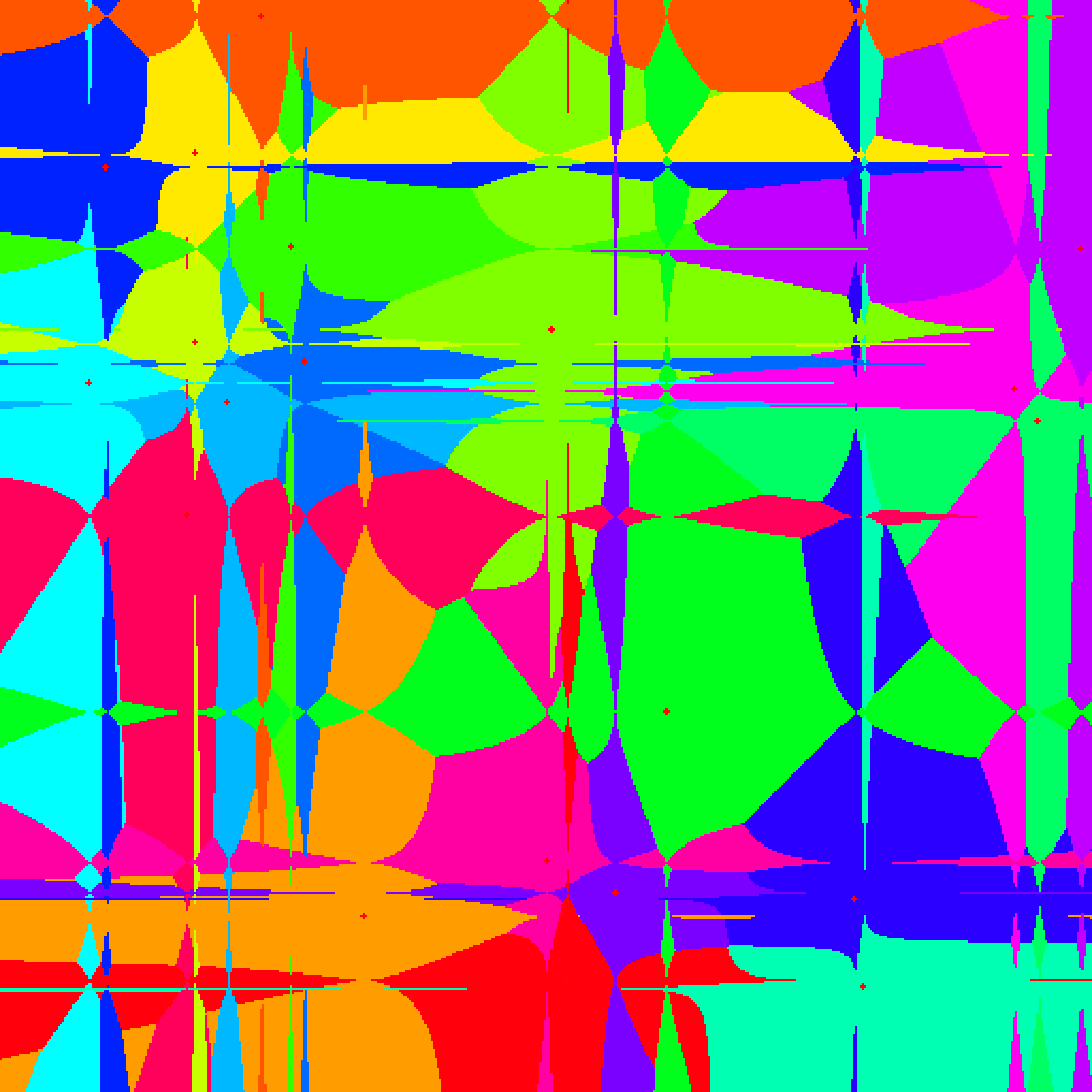

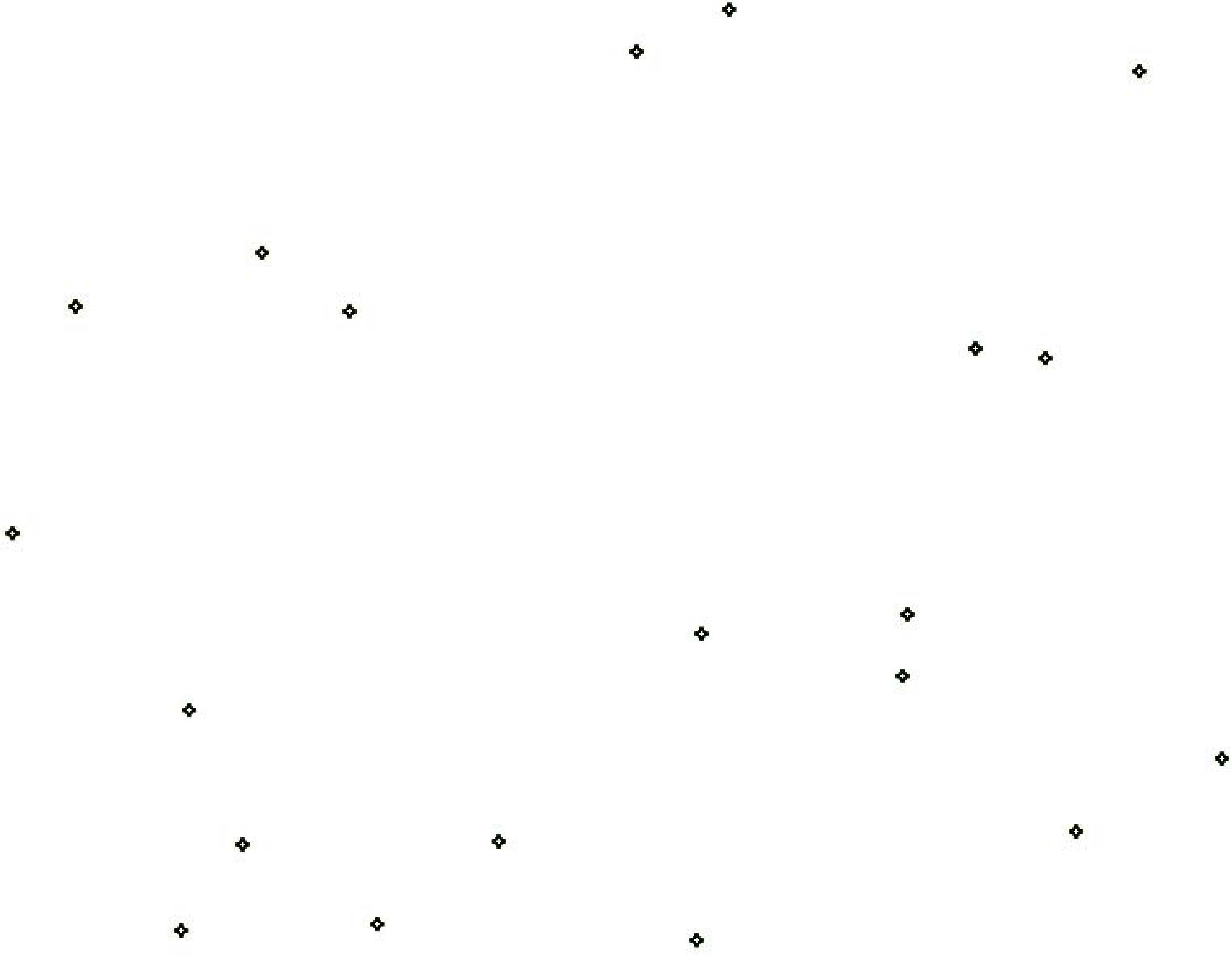

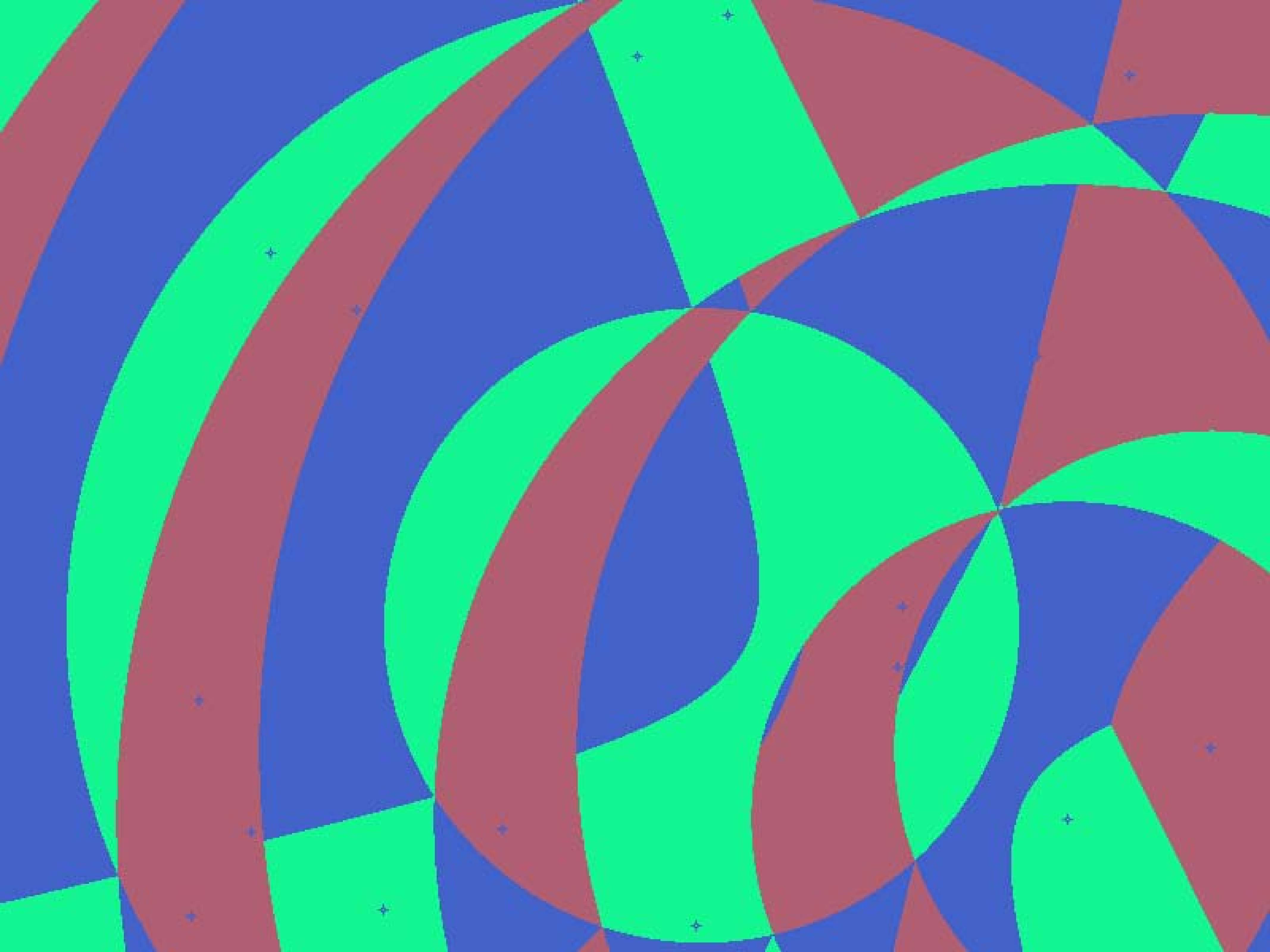

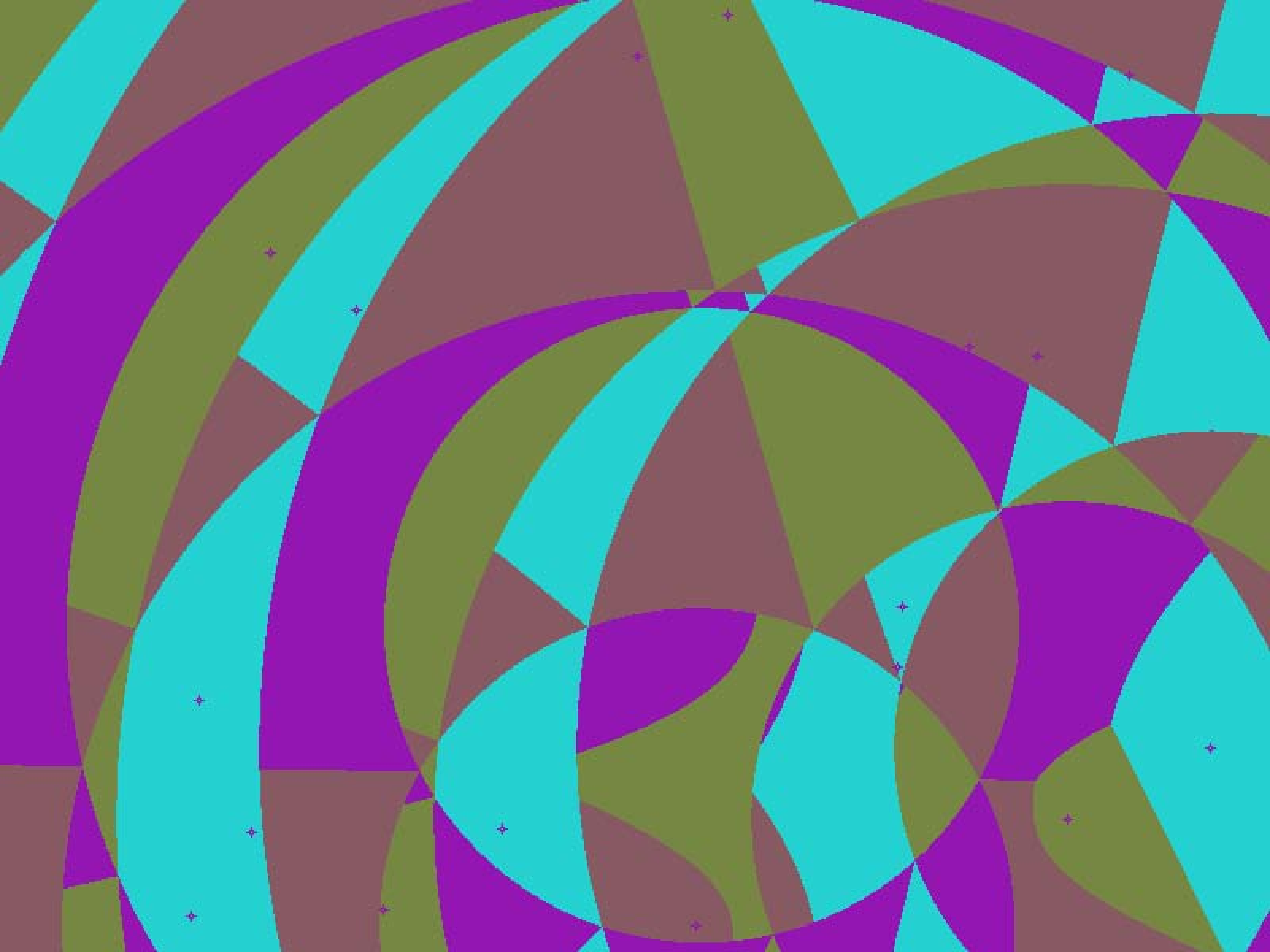

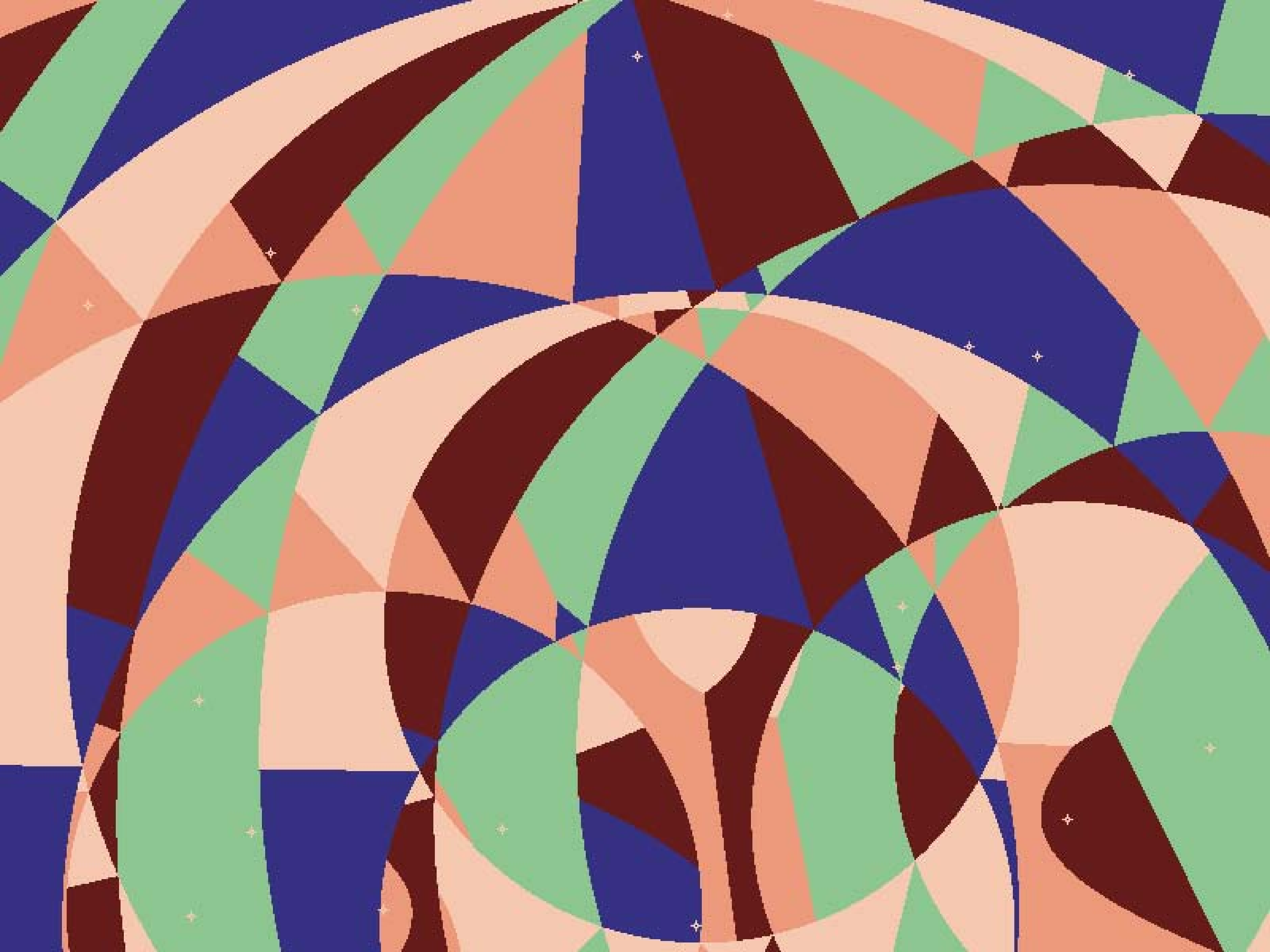

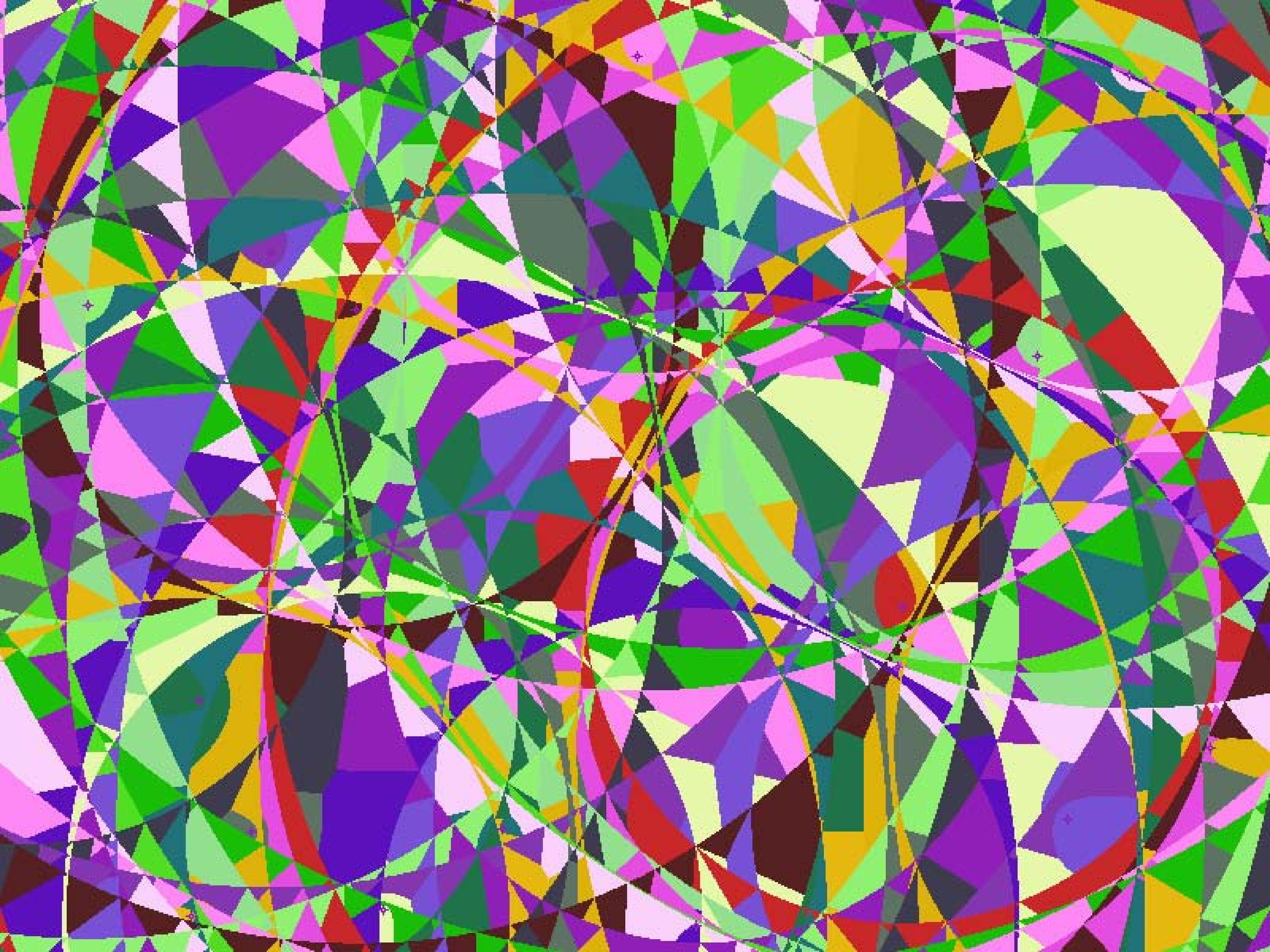

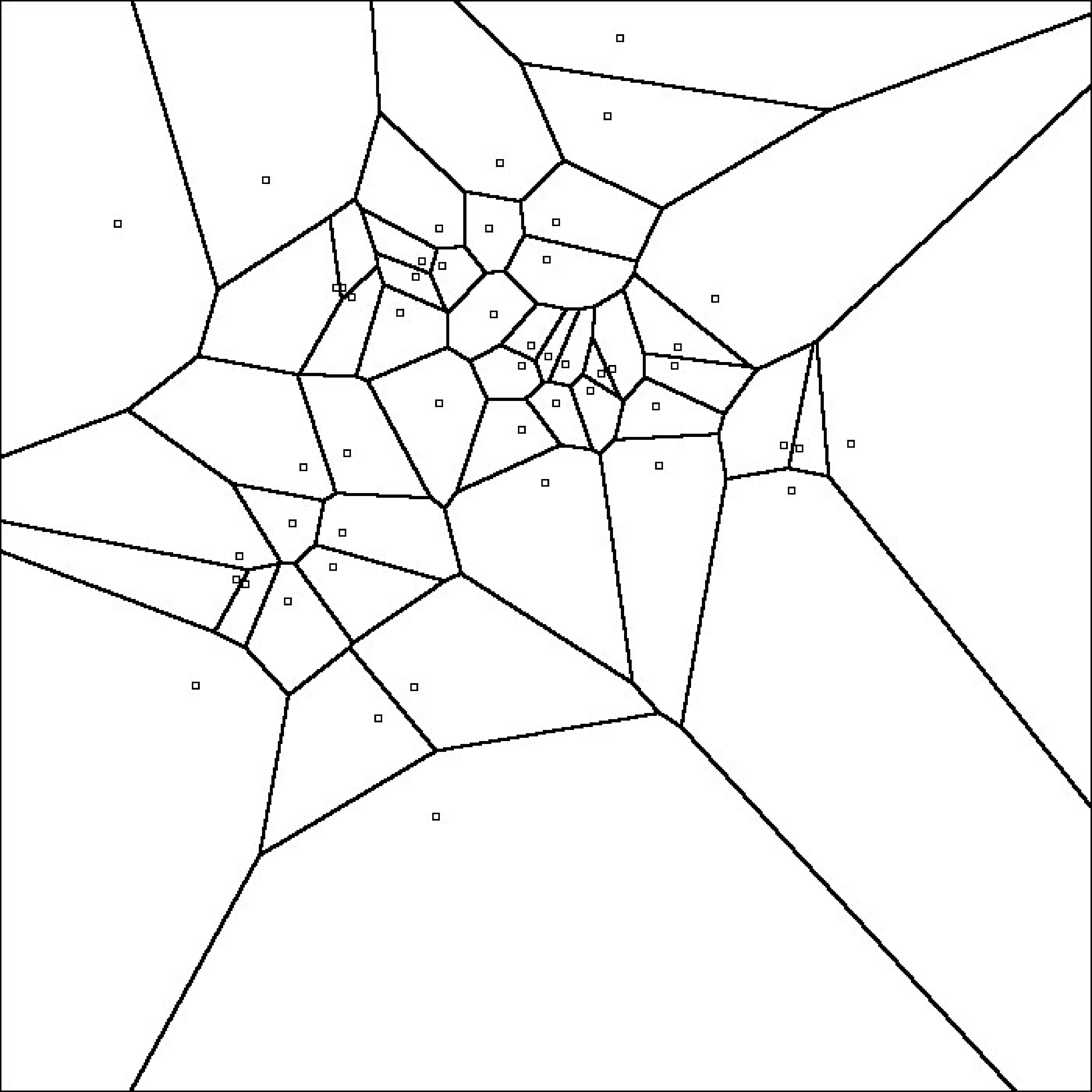

Shear Flow for Pedestrians

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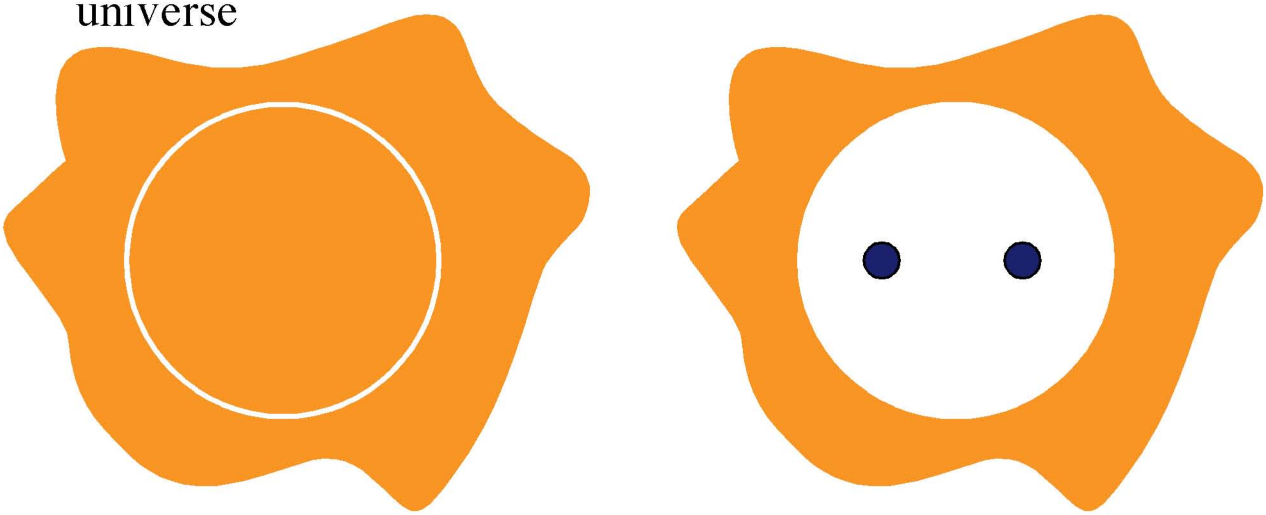
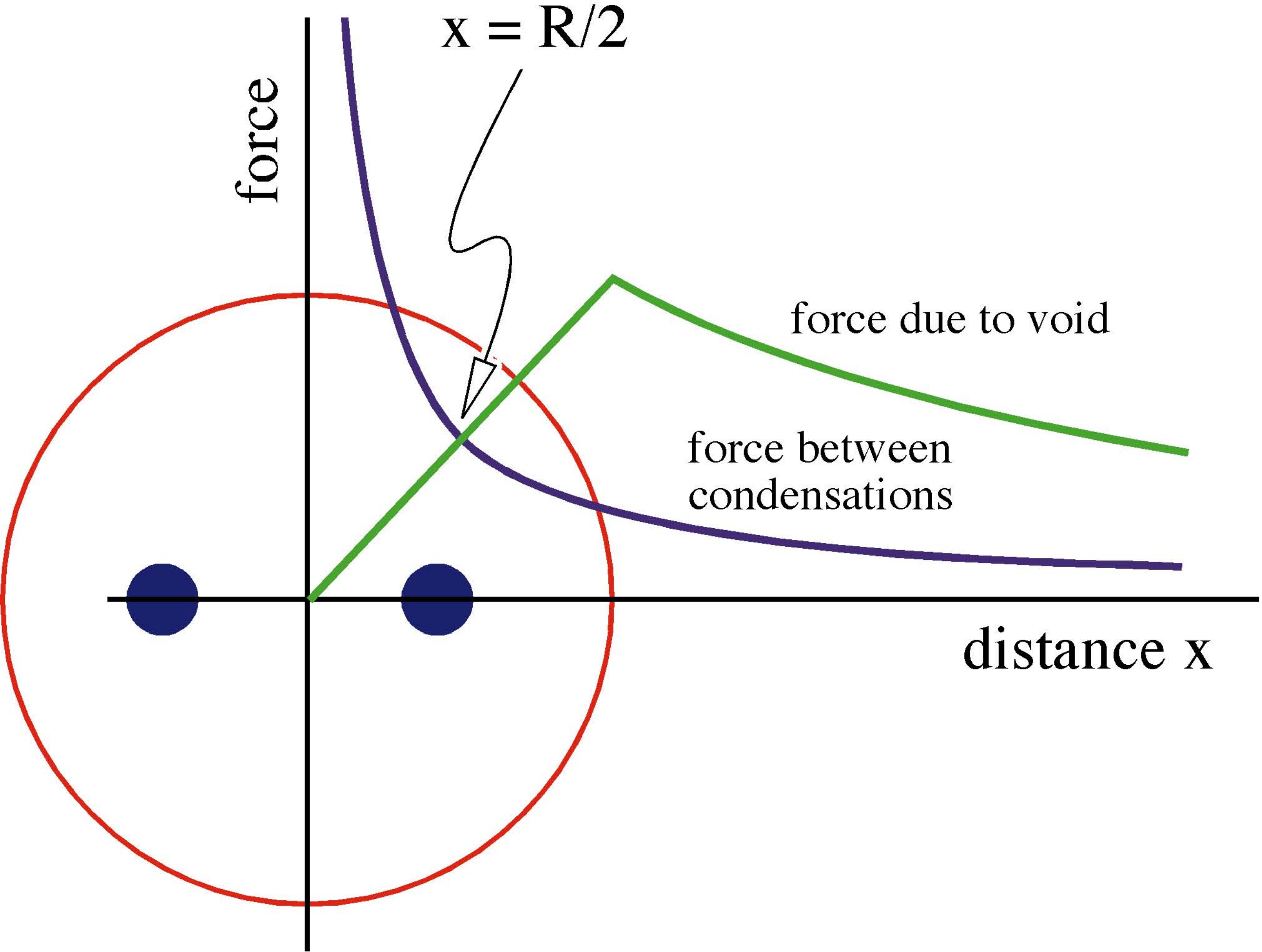
Bubble Theorem

forces in a spherical void





The potential Φ near any point (x, y, z) of a self-gravitating medium can be written as

$$\Phi = \sum_{ijk} a_{ijk} x^i y^j z^k$$

Near a density maximum one has

$$\Phi = Ax^2 + By^2 + Cz^2 + \cdots$$

$$\Phi = Ax^2 + By^2 + Cz^2 + \cdots$$

Neglecting terms of higher than second order, this is the potential of a homogeneous ellipsoid with axes (a, b, c).

They evolve according to $(aX(t), bY(t), cZ(t))$, and the density ρ obeys

$$\rho(t) = \rho_0 / XYZ$$

If $a > b > c$, it follows from Poisson's Equation $\Delta\Phi = 4\pi G\rho$ and the equations of motion that

$$-\frac{1}{X}\frac{d^2X}{dt^2} < -\frac{1}{Y}\frac{d^2Y}{dt^2} < -\frac{1}{Z}\frac{d^2Z}{dt^2}$$

Consequently, the axial ratios $a : b : c$ always increase with time. *Slight initial asphericities are amplified during the collapse.*

Now consider the evolution of the *low*-density regions. These are the progenitors of the voids.

A void is a region of *negative density* in a uniform background, so the voids expand as the overdense regions collapse. The sense of the deformation is reversed:

Slight asphericities decrease as the voids become larger ("Bubble Theorem", Icke 1984).

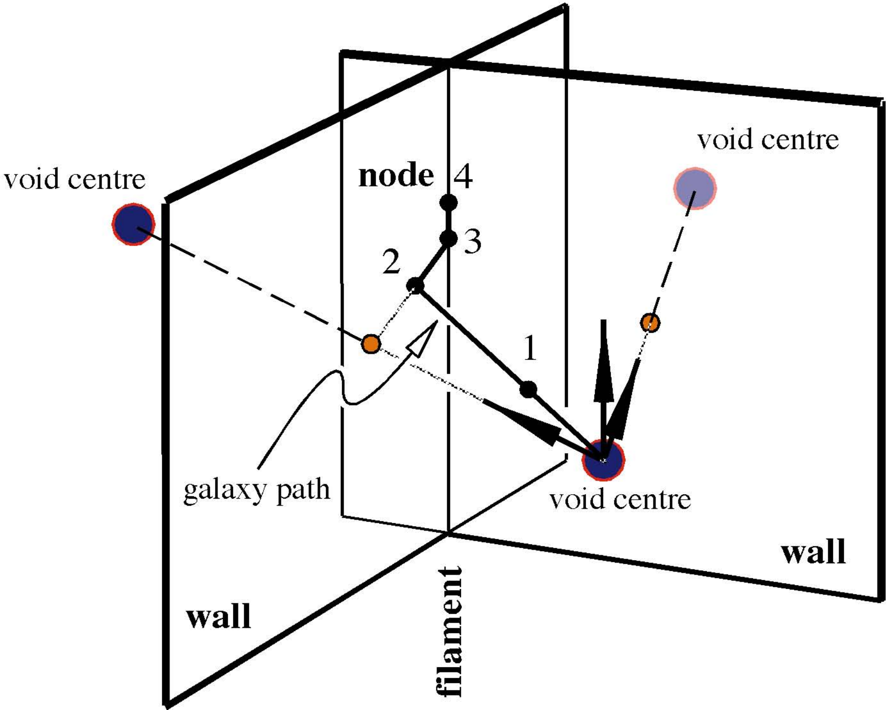
Low-density fluctuations form a packing of 'super-Hubble bubbles':

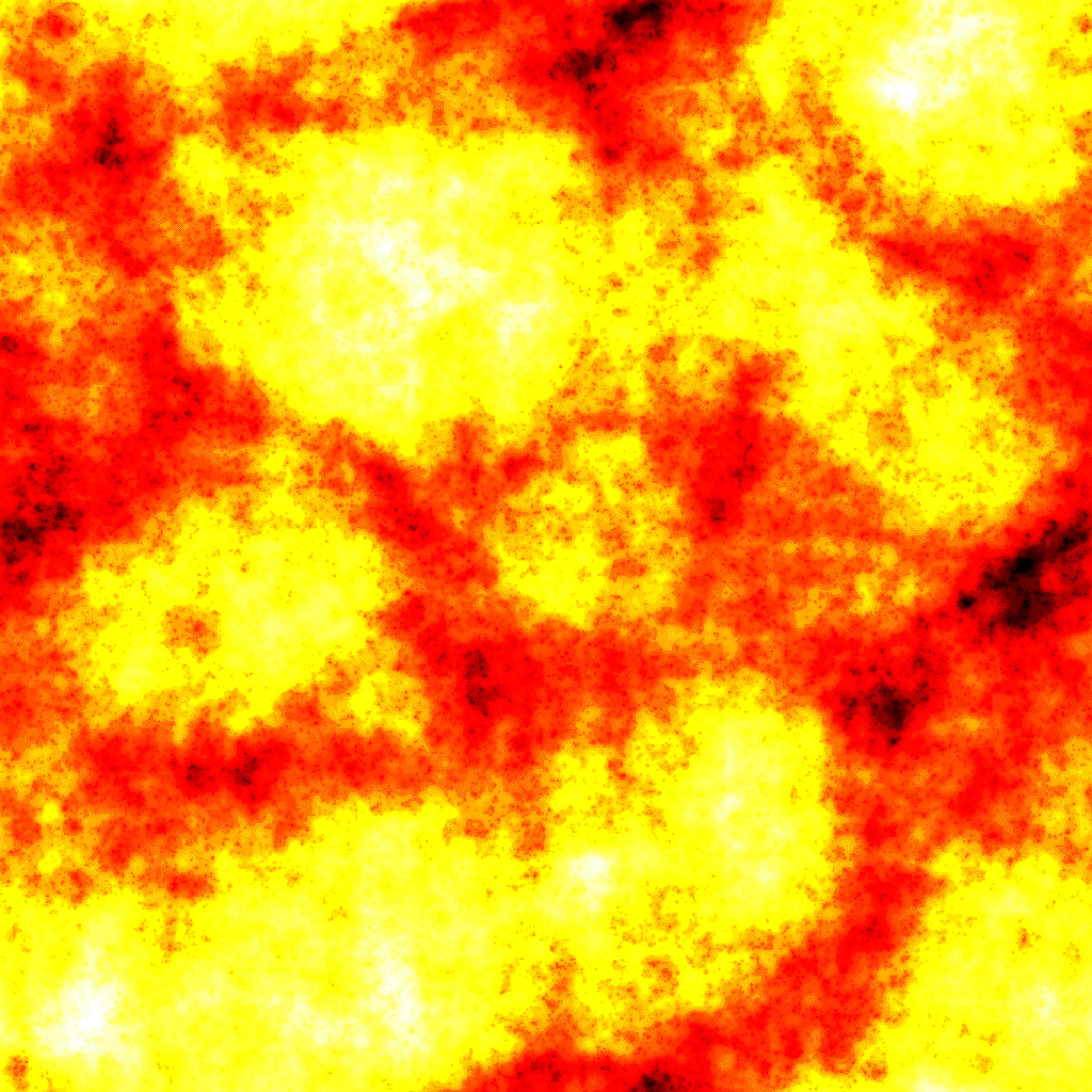
Voronoi Foam

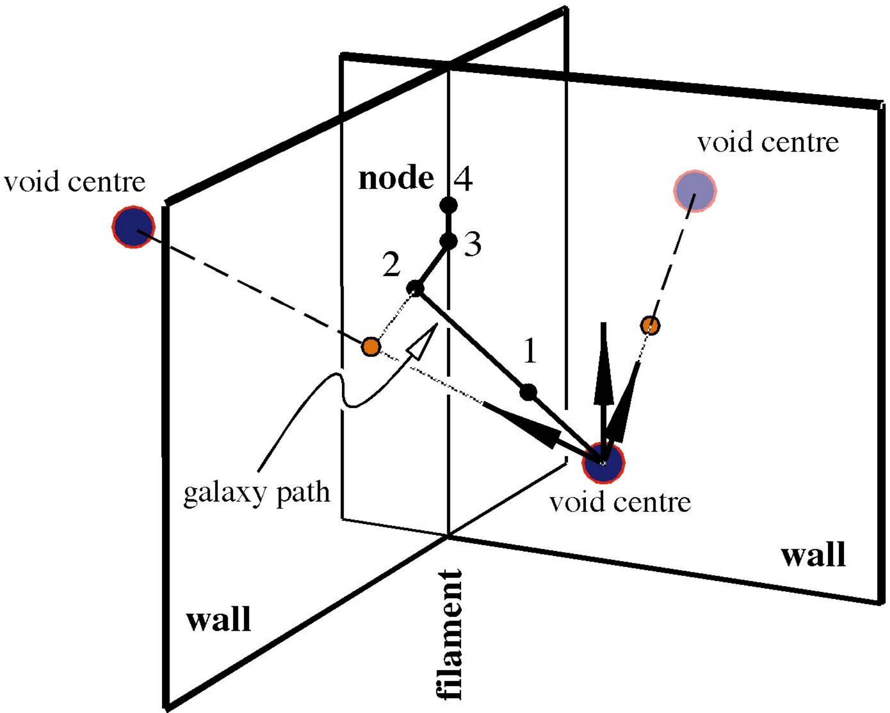
Evolution of Voronoi Features

Growth Laws for Voronoi Foam

Underdensities become more and more spherical when they expand, which they do a little faster than the Hubble expansion. Consequently, they form convex voids, from which the mass flows towards the zones separating the low-density regions. The asymptotic shape of such a mass distribution is a Voronoi foam.


void centre
void centre
node
4
3
2
1
galaxy path
void centre
wall
wall
filament




void centre
node
4
void centre
3
2
1
galaxy path
void centre
wall
wall
filament

Let m_v, m_w, m_f, m_n be the mass in voids, walls, filaments, and nodes, respectively. The first three features lose mass in a Voronoi "cascade":

$$\text{void} \implies \text{wall} \implies \text{filament} \implies \text{node}$$

The mass loss term in N dimensions has the form

$$\text{mass change} = -NmH^*\,dt \tag{1.1}$$

where H^* is the excess Hubble expansion and dt is the time increment.

One may absorb H^* into the time, so that

$$\theta \propto t^{2/3} \tag{1.2}$$

The mass loss in N dimensions is then

$$dm = -Nm\,d\theta \tag{1.3}$$

The components parallel and perpendicular to the wall are

$$cH^* \cos\theta = aH^* \tag{1.4}$$

$$cH^* \sin\theta = bH^* \tag{1.5}$$

Thus, *the excess velocity in any Voronoi feature is simply found by multiplying* H^* *with the length along the feature.* This allows us to use the above formula for $N = 3$, 2, 1, and 0.

The mass gain is found simply by reversing the sign of the loss term of the feature higher in the hierarchy. This gives the following equations:

$$\frac{dm_v}{d\theta} = \quad -3m_v \tag{1.6}$$

$$\frac{dm_w}{d\theta} = 3m_v - 2m_w \tag{1.7}$$

$$\frac{dm_f}{d\theta} = 2m_w - m_f \tag{1.8}$$

$$\frac{dm_n}{d\theta} = m_f \tag{1.9}$$

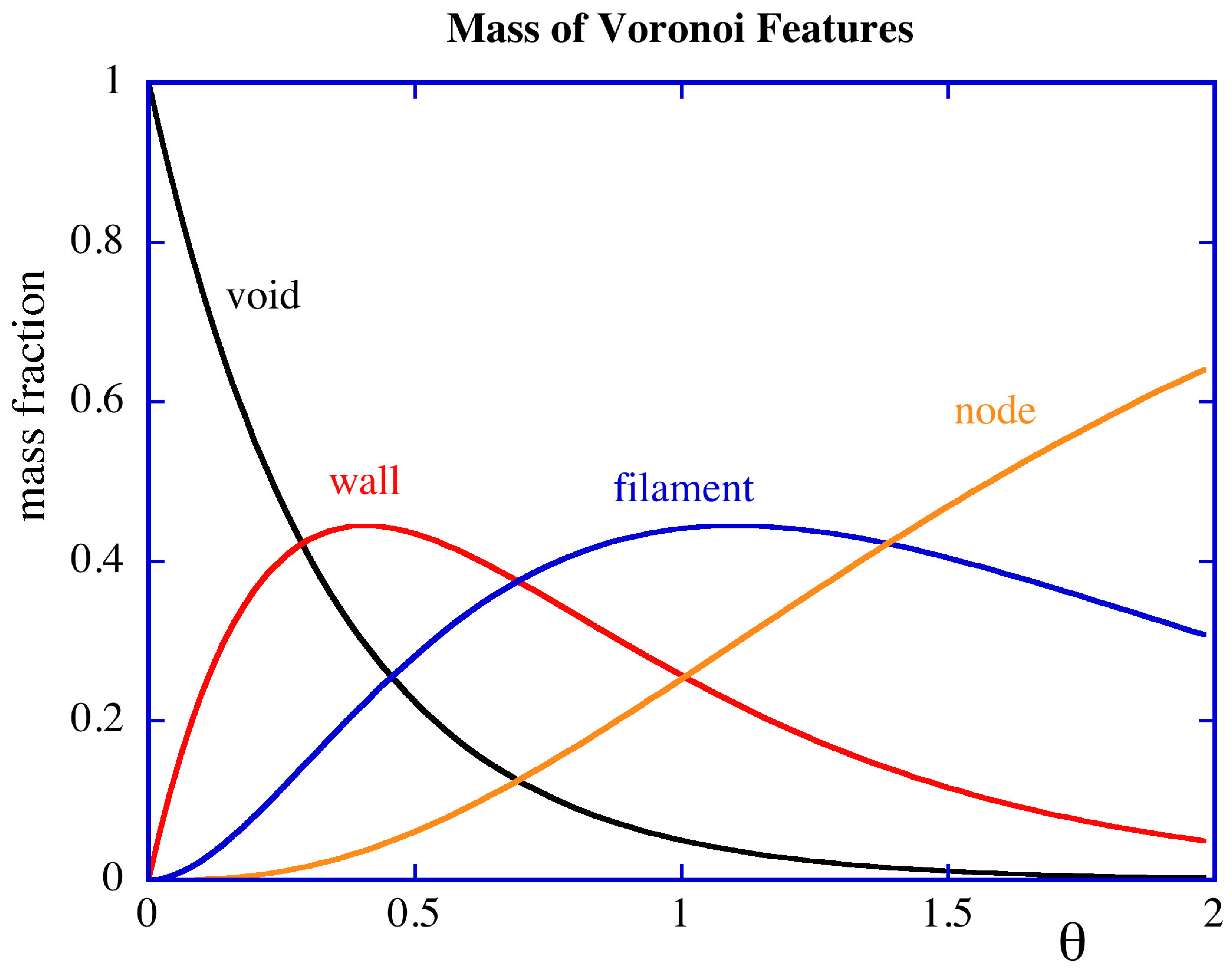
These equations are simply solved by noting that the N-dimensional mass loss equation has a solution of the type $\psi \exp(-N\theta)$:

$$m_v = \mathrm{e}^{-3\theta} \tag{1.10}$$

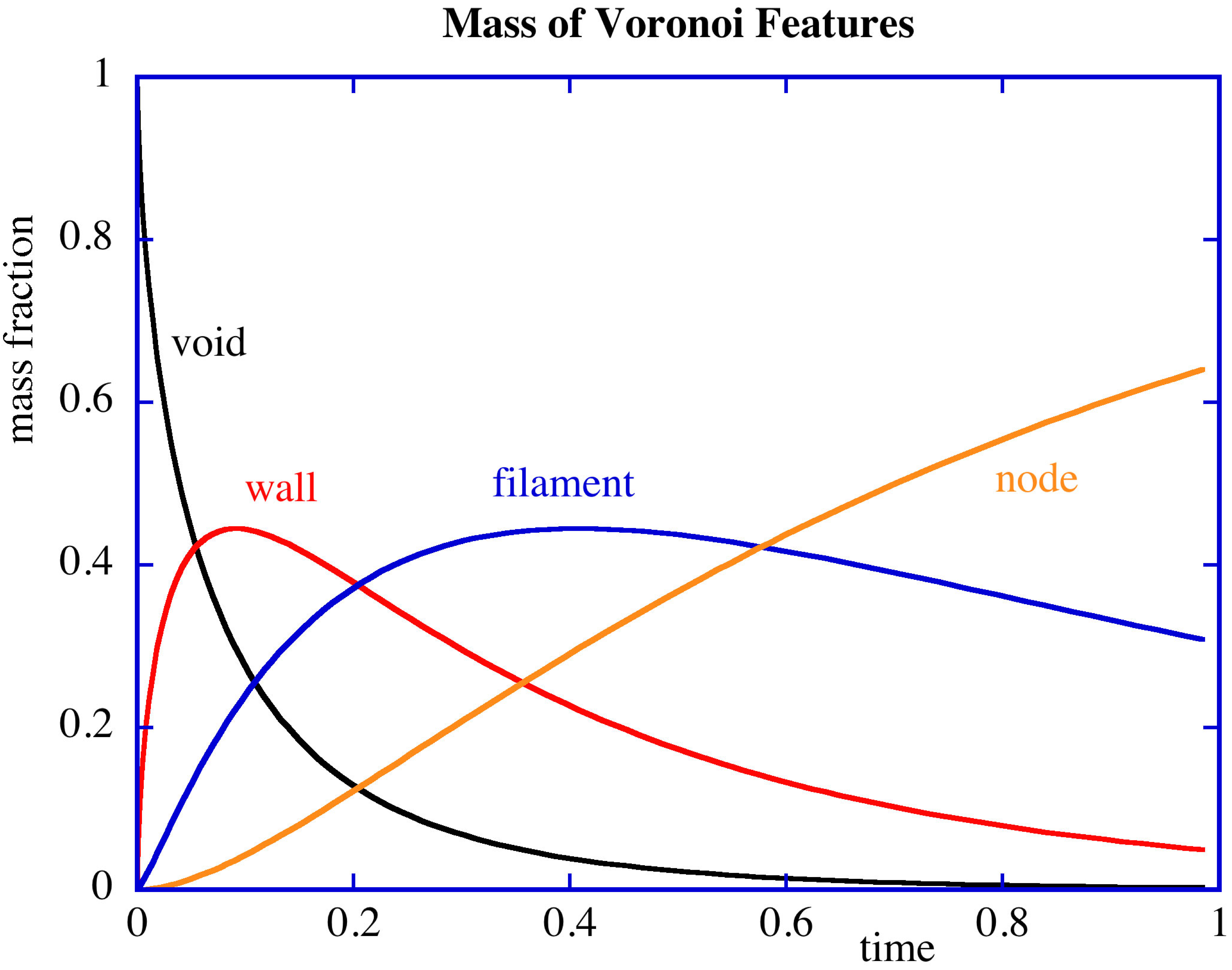
$$m_w = 3\mathrm{e}^{-2\theta}(1 - \mathrm{e}^{-\theta}) \tag{1.11}$$

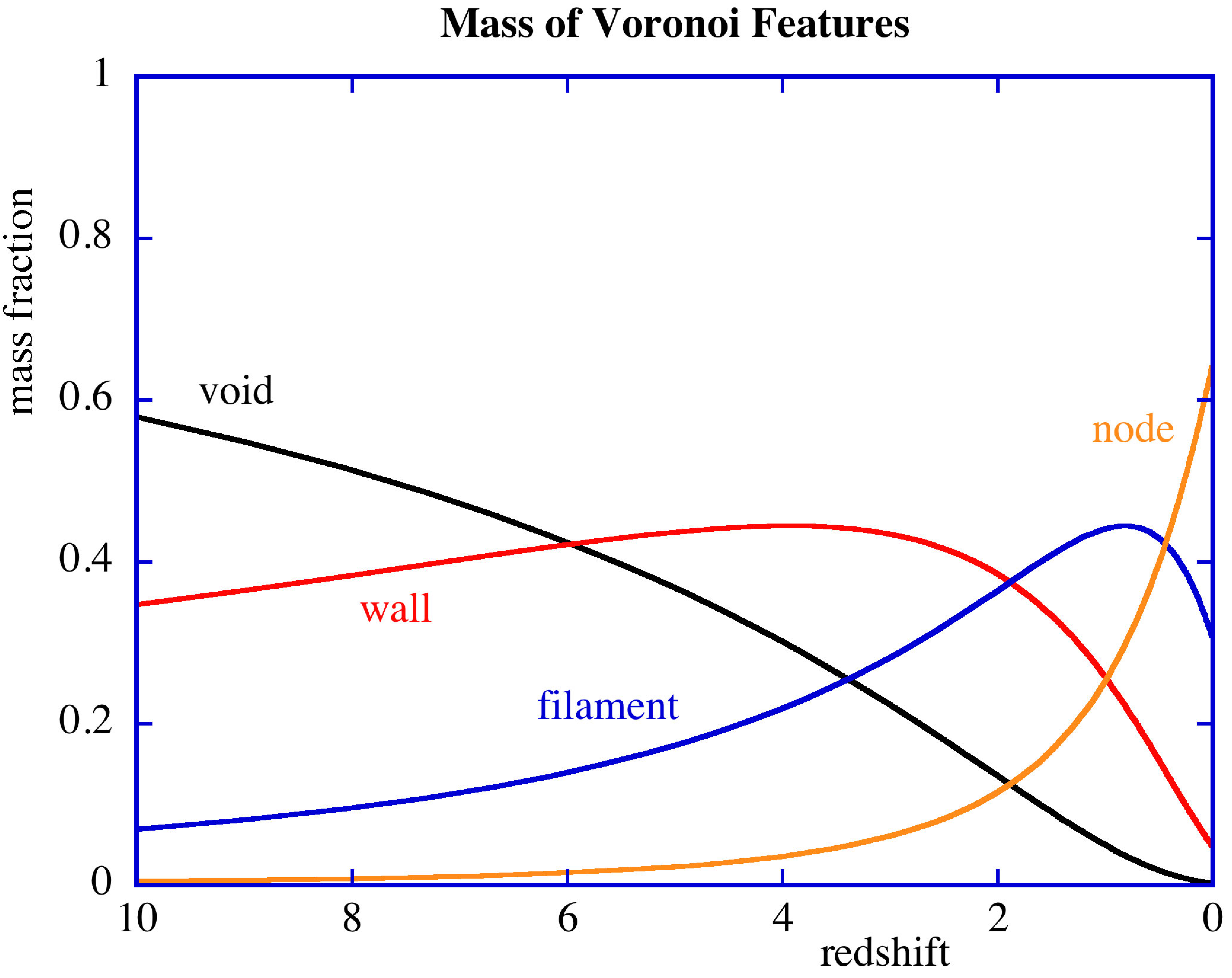
$$m_f = 3\mathrm{e}^{-\theta}(1 - \mathrm{e}^{-\theta})^2 \tag{1.12}$$

$$m_n = (1 - \mathrm{e}^{-\theta})^3 \tag{1.13}$$

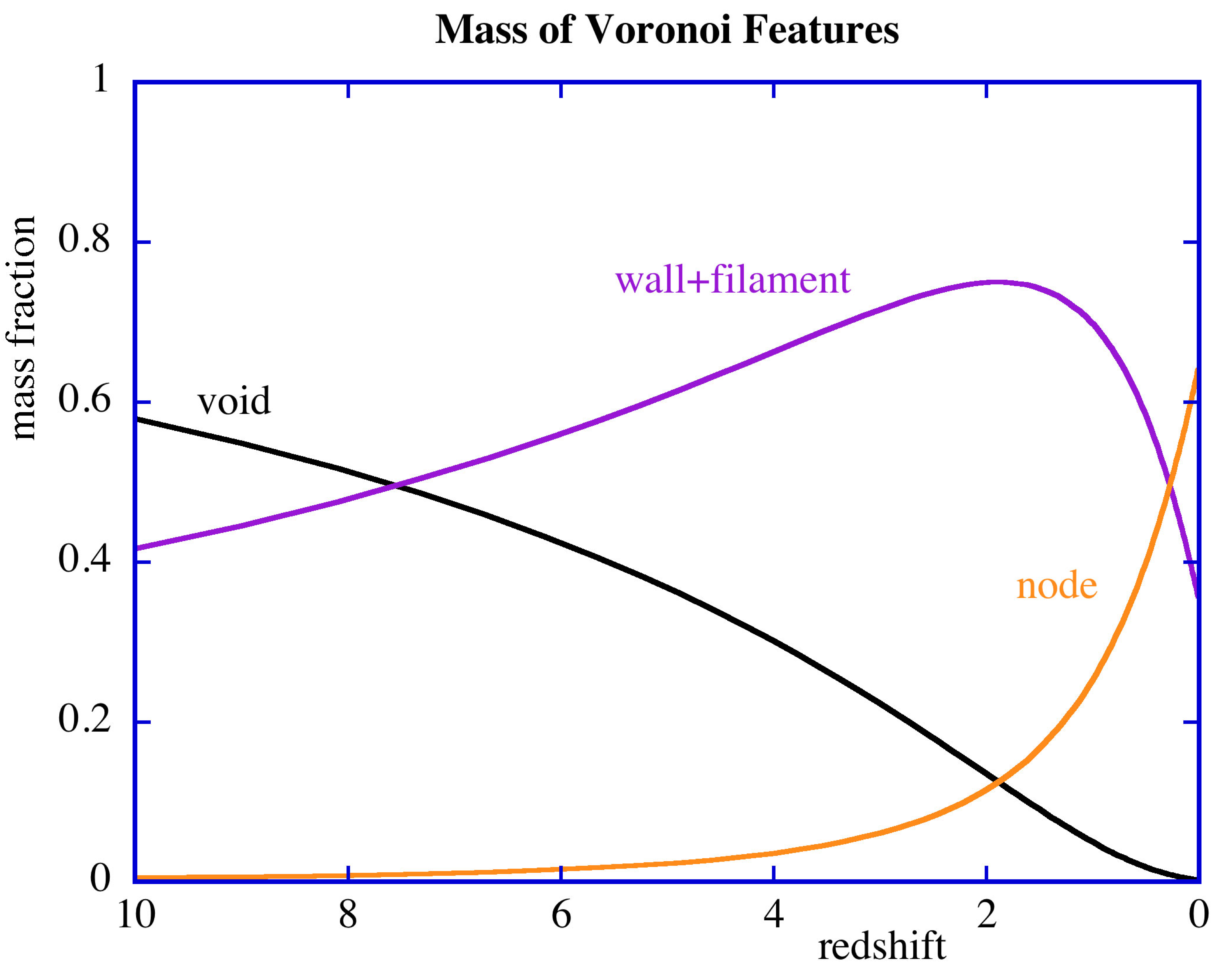


Mass of Voronoi Features
mass fraction
θ
void
wall
filament
node
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2

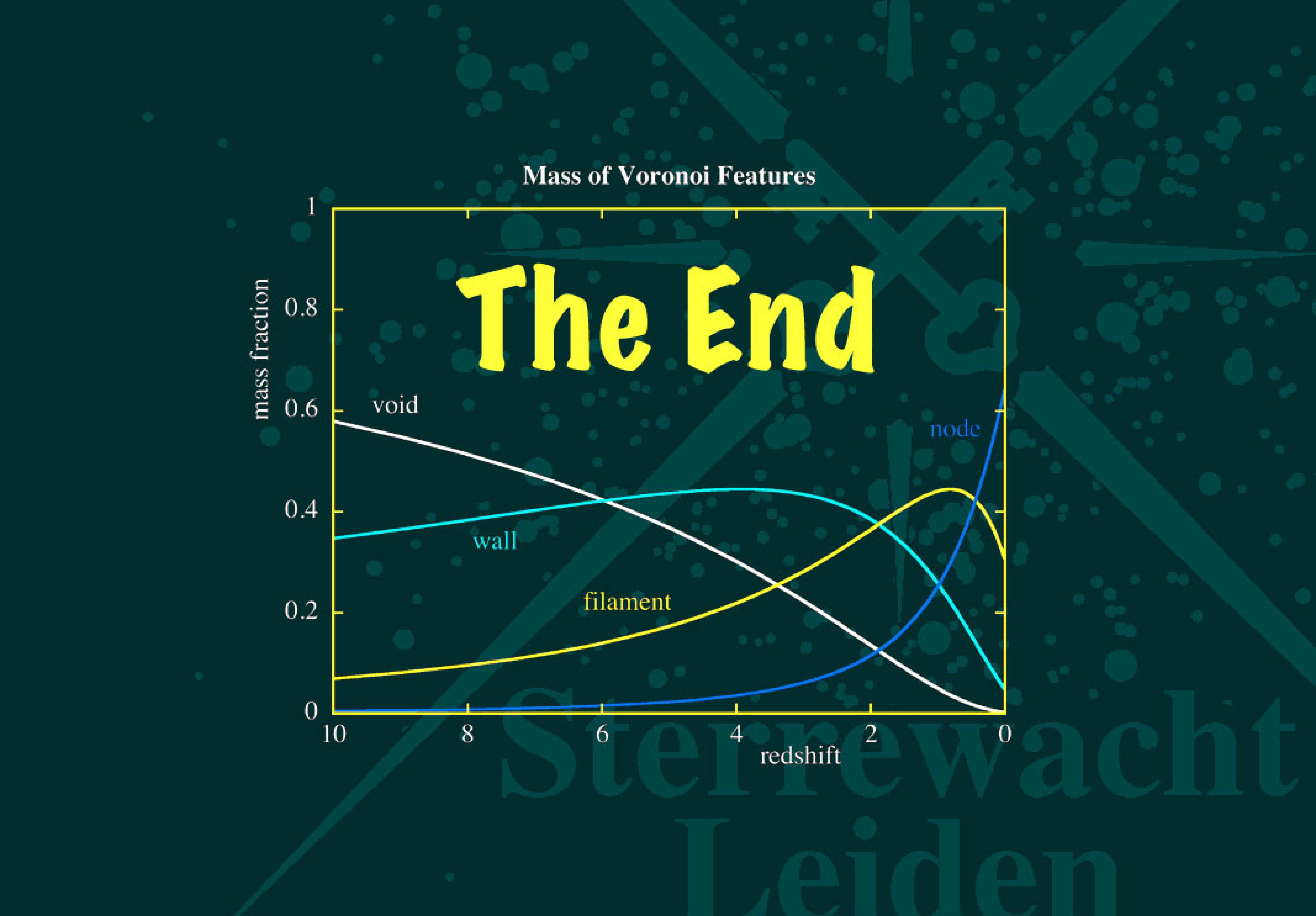

Mass of Voronoi Features
mass fraction
time
void
wall
filament
node
0
0.2
0.4
0.6
0.8
1



Mass of Voronoi Features
mass fraction
redshift
void
wall
filament
node
1
0.8
0.6
0.4
0.2
0
10
8
6
4
2
0



Mass of Voronoi Features
mass fraction
1
0.8
0.6
0.4
0.2
0
10
8
6
4
2
0
redshift
void
wall+filament
node


The End
Mass of Voronoi Features
mass fraction
redshift
void
wall
filament
node
0
0.2
0.4
0.6
0.8
1
10
8
6
4
2
0
Sterrewacht
Leiden