The Hierarchy of Cosmic Voids

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The Cosmic Structure

Matter clustered on large scales $\rightarrow$ large voids (50 Mpc/$h$ in Boötes)

Structure displayed by old and new galaxy surveys

“Cosmic Web” produced by N-body cosmological simulation
Mandelbrot introduced *tremas* = fractal holes

- Morphology related to void structure $\rightarrow$ *lacunarity*
- Same dimension fractals can have different void structure

Simple in one dimension:

Three Cantor type fractals and their gaps
Mandelbrot: number of length $U > u$ voids fulfill

$$N(U > u) \propto u^{-D} \Leftrightarrow \text{Zipf’s law}$$

General Zipf’s law: size $\propto \text{rank}^{-\alpha}$.

$N(U > u)$ is the rank: $u \propto \text{rank}^{-1/D}$.

Zipf’s law is equivalent to the probability of finding a given size be a power law.
Mandelbrot and Falconer’s “cut-out sets”: obtained from an open interval by removing an infinite sequence of disjoint open intervals.

The box-counting dimension depends only on the void set.

Every one-dimensional fractal is a cut-out set.

Theorem: $E$, with a sequence of ordered void intervals $\ell_k (k = 1, 2, \ldots)$, such that $\sum_{k=1}^{\infty} \ell_k = |E|$, has

$$\dim_B E = -1/ \lim_{k \to \infty} \left( \log \ell_k / \log k \right).$$

Equivalent to Zipf’s law $\ell_k \sim k^{-1/D}$, $D = \dim_B E$. 

Example: Cantor set

- Cantor set obtained by removing the sequence of (disjoint) open middle-third intervals.

- Length $\ell_k = (1/3)^n$, $2^{n-1} \leq k < 2^n \Rightarrow \frac{\log \ell_k}{\log k} = \frac{n \log(1/3)}{(n-1) \log 2} \Rightarrow \text{dim}_B E = \frac{\log 2}{\log 3}$.

- In this case, $\text{dim}_B E = \text{dim}_H E$, but this does not hold in general. Take the set $E = \{1/3, 1/3+1/9, 1/3+1/9+1/9, 1/3+1/9+1/9+1/9+1/27, \ldots\}$. It has $\text{dim}_H E = 0$, since it is countable.

- In general, arrangement of voids matters.
Voids in $d > 1$ can have different shapes; but $\dim_B E$ still depends only on the sequence of sizes.

In $d > 1$, cut-out sets must have topological codimension one: curves in $d = 2$, surfaces in $d = 3$, etc.

Apollonian packing of disks  
Limit set of a Kleinian group
Voids in arbitrary fractals


Random Cantor fractal, its voids, and Zipf’s law

- **Drawback:** Constructed cut-out set can have larger dimension.

  With discs, we need $\dim_B^{fractal} > \dim_B^{cut-out\ set}$ (1.31).

- **Solutions:**
  - ♦ Voids must touch the fractal.
  - ♦ Voids of arbitrary shape $\rightarrow$ void finders.
General cut-out sets

- Voids of arbitrary shape can have fractal boundary \(\Rightarrow\) too general.

- Simple restriction: convex voids (as in, e.g., the “Voronoi foam” of Icke & van de Weygaert, 1987).

- Theorem (Gaite, 2006, Physica D): Cut-out set \(E \in \mathbb{R}^d\) with \(d - 1 \leq \dim_B E < d\) and convex non-degenerate voids fulfill Zipf’s law with exponent \(d / \dim_B E\).

  Non-degenerate (in \(d = 3\)): shape-coefficient \(a_k^3/v_k^2\) is bounded.

- Furthermore: for a statistically self-similar cut-out set,

  \[\dim_H E = \dim_B E\] almost surely.
Void finder based on discrete geometry

Computational problem: in a finite set of points, find filling voids of variable shape → criterium for separation.

Methods of discrete stochastic geometry: Delaunay and Voronoi tessellations

Consider a set of points (in the plane):
The circle circumscribed to each Delaunay triangle does not contain more points of the set.

The Delaunay triangulation maximizes the minimum angles.

Delaunay triangulation ↔ Voronoi tessellation.
Void finder method

One criterium to join *basic voids* (Gaite, 2005, EPJB):

Circles circumscribed to two adjacent triangles, with the segment linking their centers and the radius of the smaller circle $\rightarrow$ overlap parameter $f$ is a bound to the ratio of both lengths.
Application to self-similar fractal

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Random Cantor fractal (12,288 points) with $D = 1$ and transition to homogeneity.
Delaunay triangulation of the fractal set:
Random fractal: Example 1

Largest 500 voids with $f = 0.3$ overlap:
Random fractal: Example 1

\[ N(r) \] and fit \( D = 1 \)

Zipf’s law with slope \(-2/D\)

Compare:

- Scaling ranges of \( N(r) \) and \( \Lambda(R) \).
- Transition to homogeneity.
Random fractal: Example 2

Random fractal (9216 points) with $D = 1.26$

- $N(r)$ and fit $D = 1.26$
- Zipf’s law with slope $-2/1.26$

- Topological dimension 0 $\Rightarrow$ voids ill-defined.
- Cut-out set = Fractal + boundary of voids.
- Minimum value of $D$ is 1 (boundary of voids).
Transition to homogeneity

If the fractal becomes homogeneous \((D = 3)\) on large scales \(\Rightarrow\) flattening of Zipf’s law in the rank-ordering of voids.

Transition to homogeneity in Example 1:

Flattening of the Zipf law and voids of a random distribution of points (dashed line) having small-rank voids of the same size \((f = 0.3\) for both).
Basic measure is the number-radius relation $N(r) = B r^D$, $r < r_0$ (scale of transition to homogeneity $D \to 3$).

- **Gamma function (two-point correlation):**

$$\Gamma(r) = \frac{1}{4\pi r^2} \frac{dN(r)}{dr} = \frac{BD}{4\pi} r^{D-3}.$$ 

One defines $\gamma = 3 - D > 0$.

- **Gamma-star function:**

$$\Gamma^*(r) = \frac{N(r)}{4\pi r^3/3} = \frac{3B}{4\pi} r^{D-3}.$$
Scaling of galaxy clusters and voids

$\Gamma(r)$ of SDSS VL sample.

$r_0 \approx 15$ Mpc (Tikhonov).

Slope $\gamma \rightarrow D \approx 2$

The Zipf law for a 2dF VL sample (Tikhonov). Slope $z \Rightarrow D \approx 2$
Scaling of galaxy clusters and voids


Slope $\gamma \Rightarrow D \simeq 2$. No transition to homogeneity

Six largest minivoids within the sphere of radius 7.5 Mpc
Voids in a multifractal


Two halos populations in the GIF2 simulation (massive in red and light in blue). More clustered massive halos leave larger voids.

- Faint galaxies in voids ⇔ galaxy formation (Peebles, 2001).
- Distribution of dark matter inside voids (Gottlöber et al, 2003).
Clustering \(\Rightarrow\) hierarchy of voids. Scaling voids \(\leftrightarrow\) Zipf’s law.

Rigorous definition of fractal voids \(\rightarrow\) cut-out sets, with topological codimension 1.

Zipf’s law flattening for largest voids \(\rightarrow\) transition to homogeneity.

Void finders \(\rightarrow\) voids in fractals with topological codimension \(\geq 1\) (addition of boundaries).

Recent analysis: large voids which fulfill Zipf’s law but yield minimum \(D = 2\).