The Hierarchy of Cosmic Voids

José Gaite

Instituto de Matemáticas y Física Fundamental,

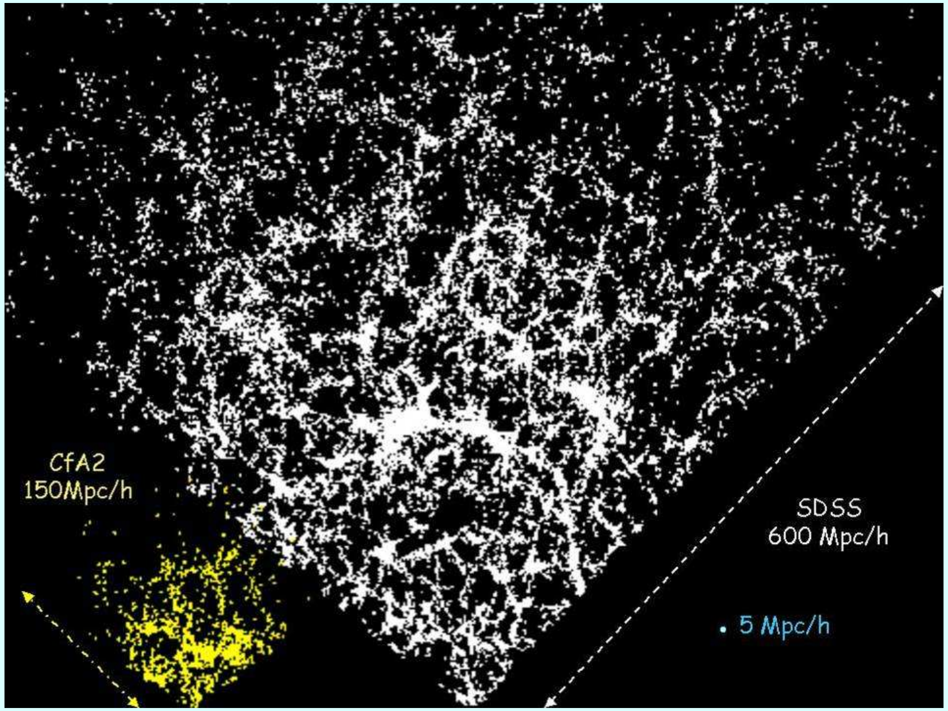
CSIC, Madrid (Spain)

PLAN OF THE TALK

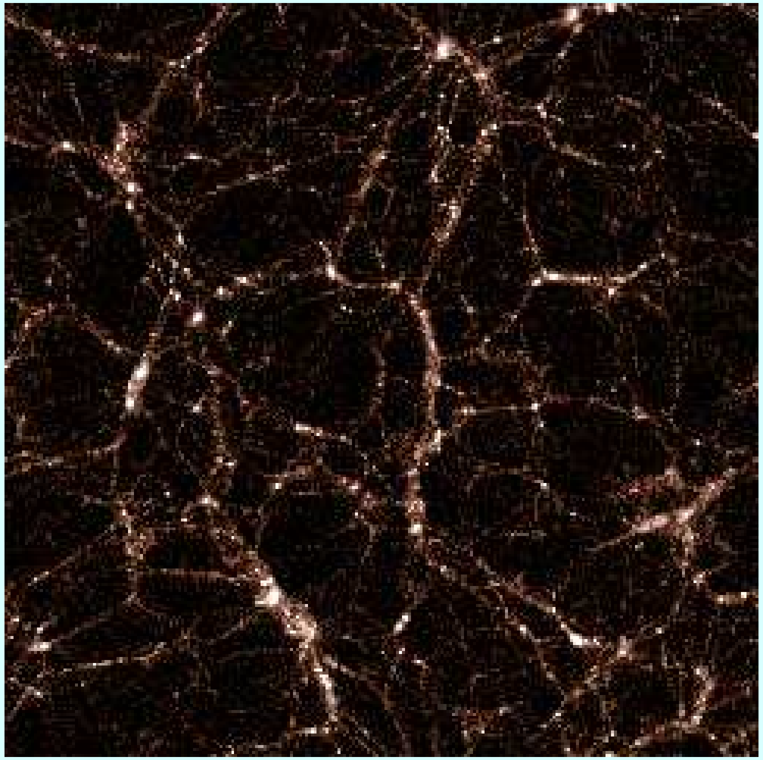
1. Voids in the Cosmic Structure
2. The hierarchy of fractal voids
3. Zipf's Law
4. "Cut-out sets"
5. Void finders
6. The Zipf law for Cosmic Voids
7. Voids in a multifractal
8. Conclusions

The Cosmic Structure

Matter clustered on large scales $\longrightarrow$ large voids (50 Mpc$/h$ in Boötes)



Structure displayed by old and new galaxy surveys



"Cosmic Web" produced by N-body cosmological simulation

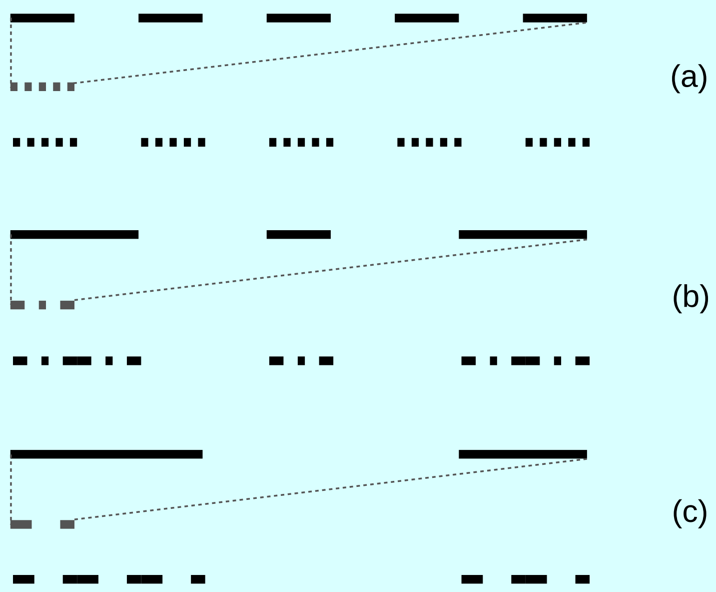
Fractal voids

Mandelbrot introduced *tremas* = fractal holes

- Morphology related to void structure $\longrightarrow$ *lacunarity*
- Same dimension fractals can have different void structure

Simple in one dimension:

Three Cantor type fractals and their gaps



Zipf's law

- Mandelbrot: number of length $U > u$ voids fulfill $N(U > u) \propto u^{-D} \Leftrightarrow$ Zipf's law
- General Zipf's law: size $\propto$ rank$^{-\alpha}$.
- $N(U > u)$ is the rank: $u \propto$ rank$^{-1/D}$.
- Zipf's law is equivalent to the probability of finding a given size be a *power law*.

Cut-out sets

- Mandelbrot and Falconer's "cut-out sets": obtained from an open interval by removing an infinite sequence of disjoint open intervals.
- The *box-counting* dimension depends only on the void set.
- Every one-dimensional fractal is a cut-out set.
- Theorem: E, with a sequence of *ordered* void intervals ℓ_k $(k = 1, 2, \ldots)$, such that $\sum_{k=1}^{\infty} \ell_k = |E|$, has

$$\dim_B E = -1/ \lim_{k\to\infty} (\log \ell_k / \log k) \, .$$

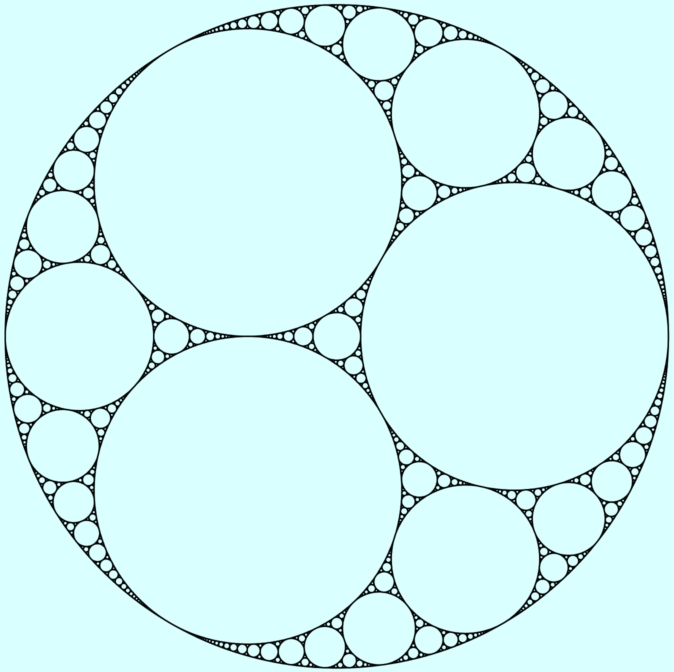
Equivalent to Zipf's law $\ell_k \sim k^{-1/D}, \; D = \dim_B E$.

Example: Cantor set

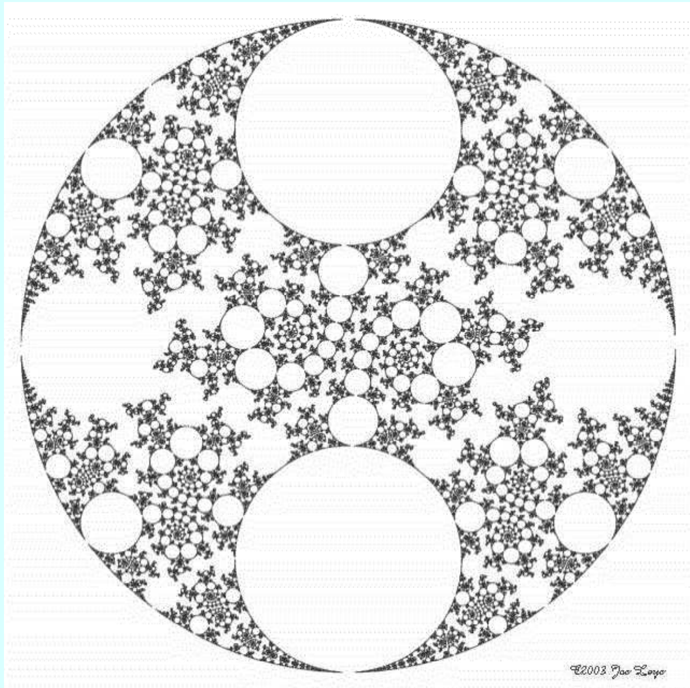
- Cantor set obtained by removing the sequence of (disjoint) open middle-third intervals.
- Length $\ell_k = (1/3)^n,\ 2^{n-1} \leq k < 2^n \ \Rightarrow\ \frac{\log \ell_k}{\log k} = \frac{n \log(1/3)}{(n-1)\log 2} \ \Rightarrow\ \dim_B E = \frac{\log 2}{\log 3}$.
- In this case, $\dim_B E = \dim_H E$, but this does not hold in general. Take the set $E = \{1/3, 1/3+1/9, 1/3+1/9+1/9, 1/3+1/9+1/9+1/27, \ldots\}$. It has $\dim_H E = 0$, since it is countable.
- In general, arrangement of voids matters.

Fractal voids in more than one dimension

- Voids in $d > 1$ can have different shapes; but $\dim_B E$ still depends only on the sequence of sizes.
- In $d > 1$, cut-out sets must have topological codimension one: curves in $d = 2$, surfaces in $d = 3$, etc.



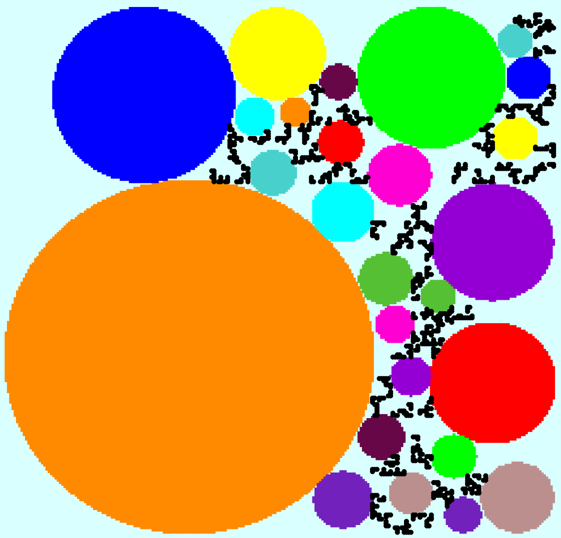
Apollonian packing of disks

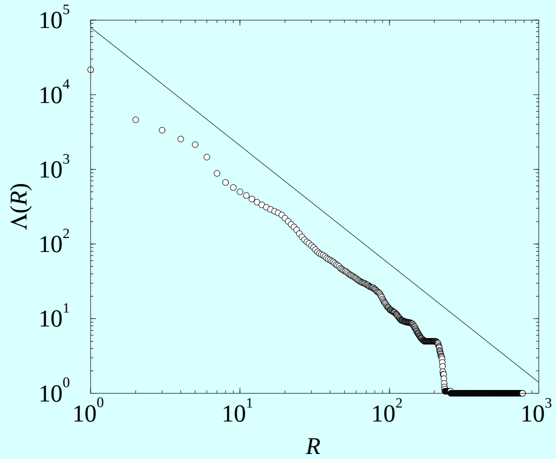


Limit set of a Kleinian group

Voids in arbitrary fractals

- Gaite & Manrubia, 2002, MNRAS: Voids of constant shape: discs, squares, etc.





Random Cantor fractal, its voids, and Zipf's law

- Drawback: Constructed cut-out set can have larger dimension. With discs, we need $\dim_B$fractal $> \dim_B$cut-out set (1.31).

Solutions:
- Voids must touch the fractal.
- Voids of arbitrary shape $\longrightarrow$ void finders.

General cut-out sets

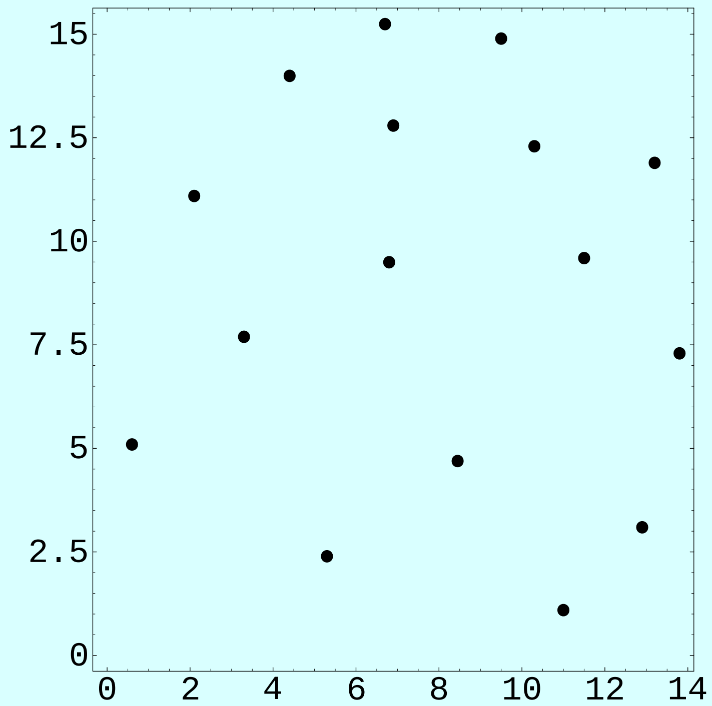
- Voids of arbitrary shape can have fractal boundary $\Rightarrow$ too general.
- Simple restriction: convex voids (as in, e.g., the “Voronoi foam” of Icke & van de Weygaert, 1987).
- Theorem (Gaite, 2006, Physica D): Cut-out set $E \in \mathbb{R}^d$ with $d - 1 \leq \dim_B E < d$ and convex non-degenerate voids fulfill Zipf’s law with exponent $d/\dim_B E$.
  Non-degenerate (in $d = 3$): shape-coefficient a_k^3/v_k^2 is bounded.
- Furthermore: for a statistically self-similar cut-out set, $\dim_H E = \dim_B E$ *almost surely*.

Void finder based on discrete geometry

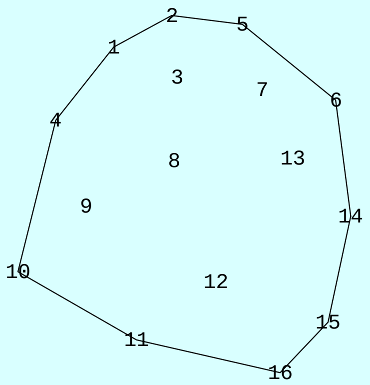
Computational problem: in a finite set of points, find filling voids of variable shape $\longrightarrow$ criterium for separation.

Methods of discrete stochastic geometry: Delaunay and Voronoi tessellations

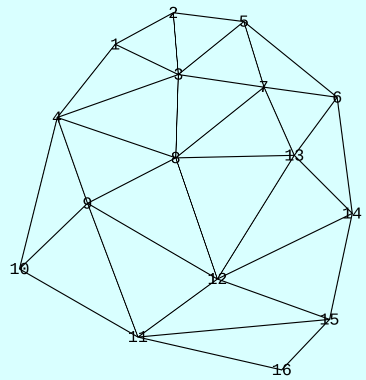
Consider a set of points (in the plane):



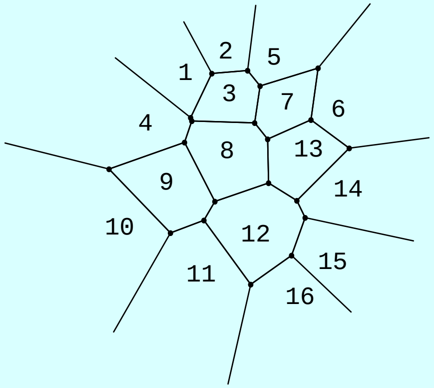
Void finder based on discrete geometry



Convex hull



Delaunay triangulation



Voronoi tesselation

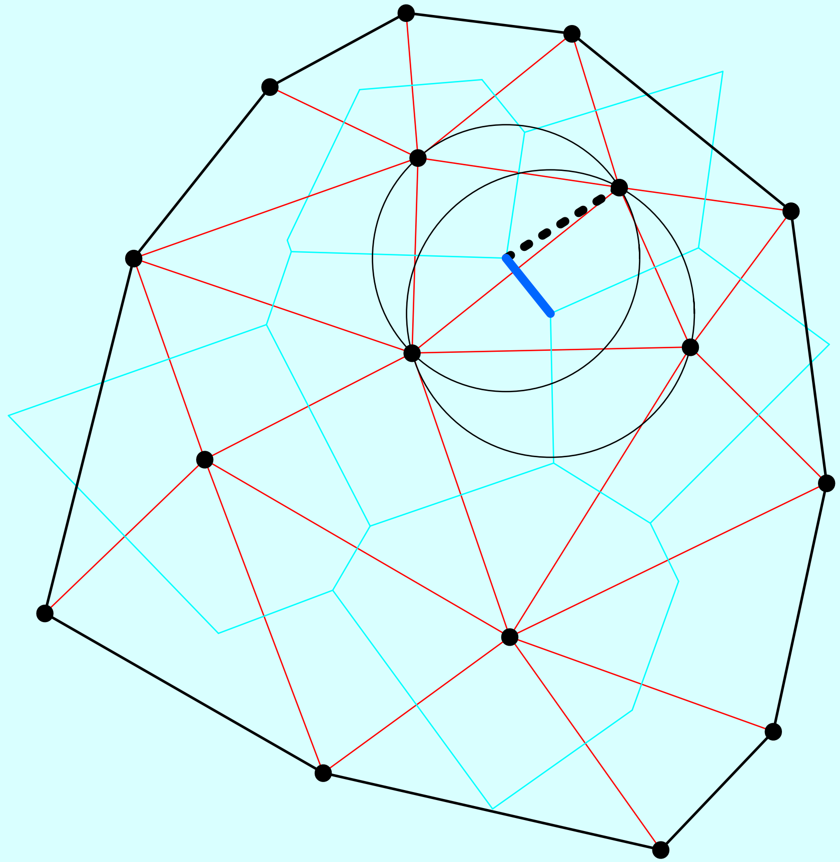
- The circle circumscribed to each Delaunay triangle does not contain more points of the set.
- The Delaunay triangulation maximizes the minimum angles.

Delaunay triangulation $\longleftrightarrow$ Voronoi tesselation.

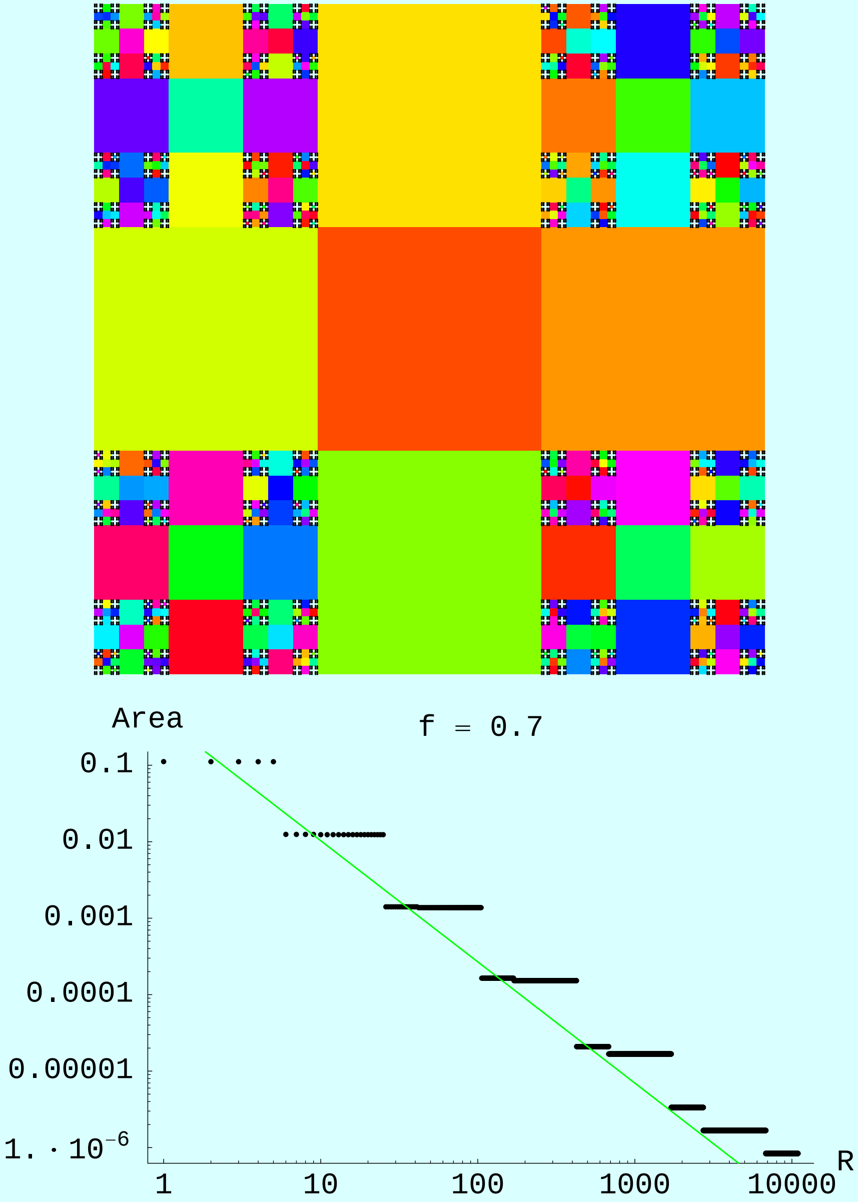
Void finder method

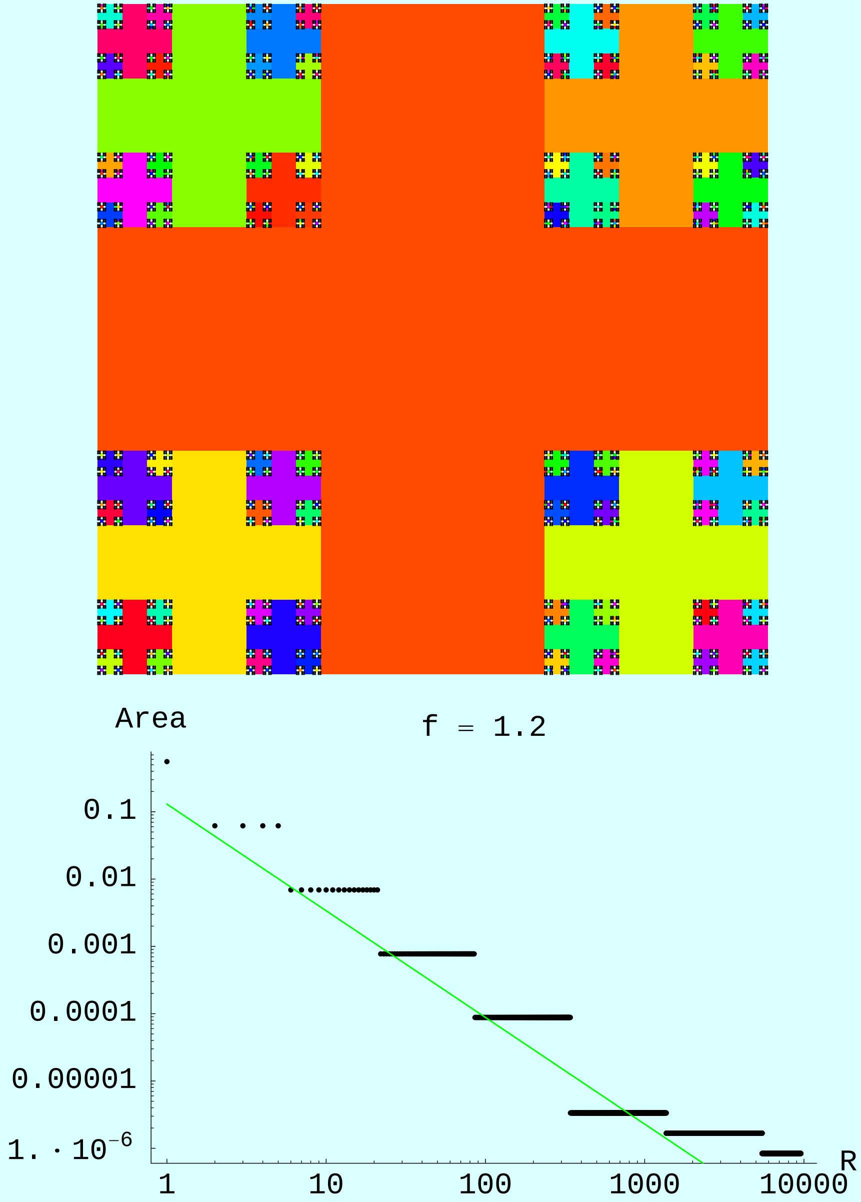
One criterium to join *basic voids* (Gaite, 2005, EPJB):

Circles circumscribed to two adjacent triangles, with the segment linking their centers and the radius of the smaller circle $\longrightarrow$ overlap parameter f is a bound to the ratio of both lengths.



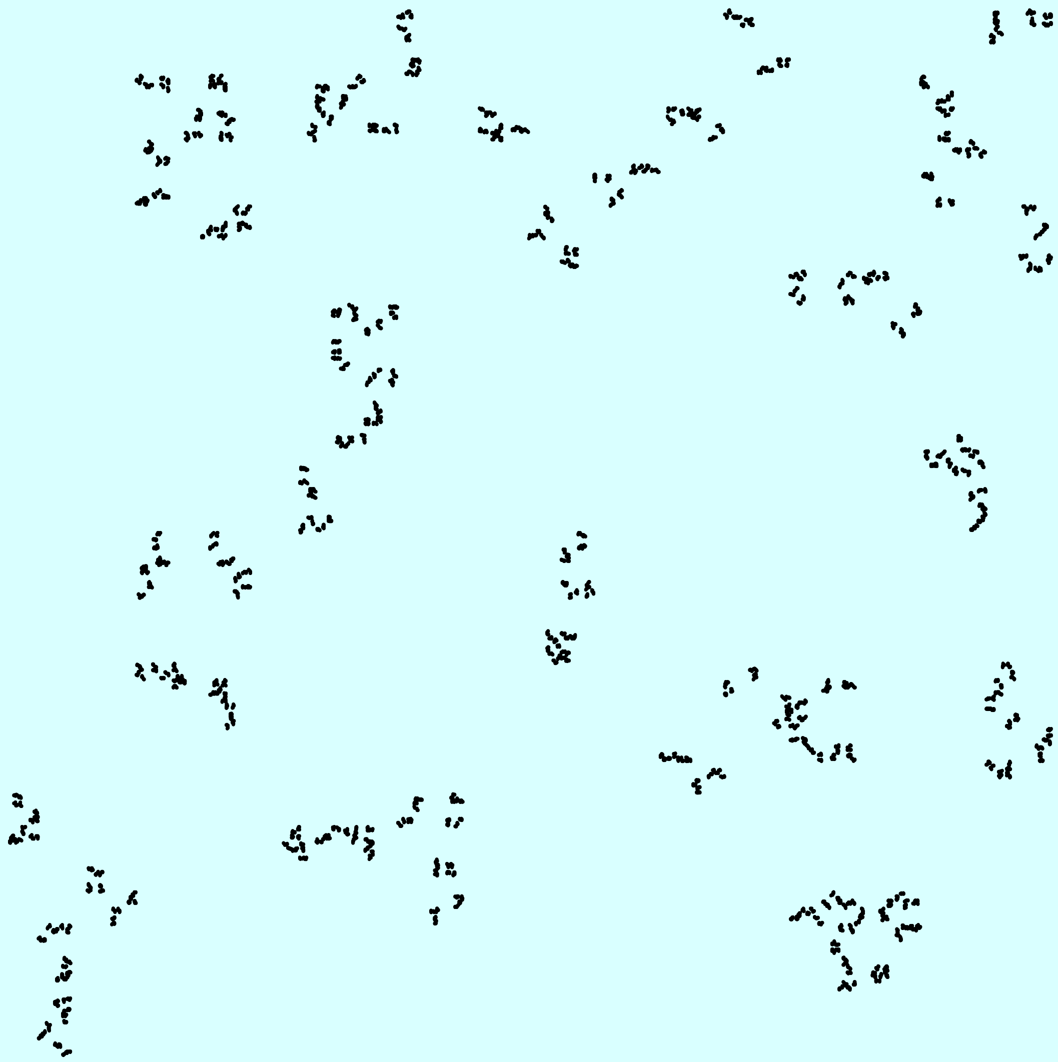
Application to self-similar fractal





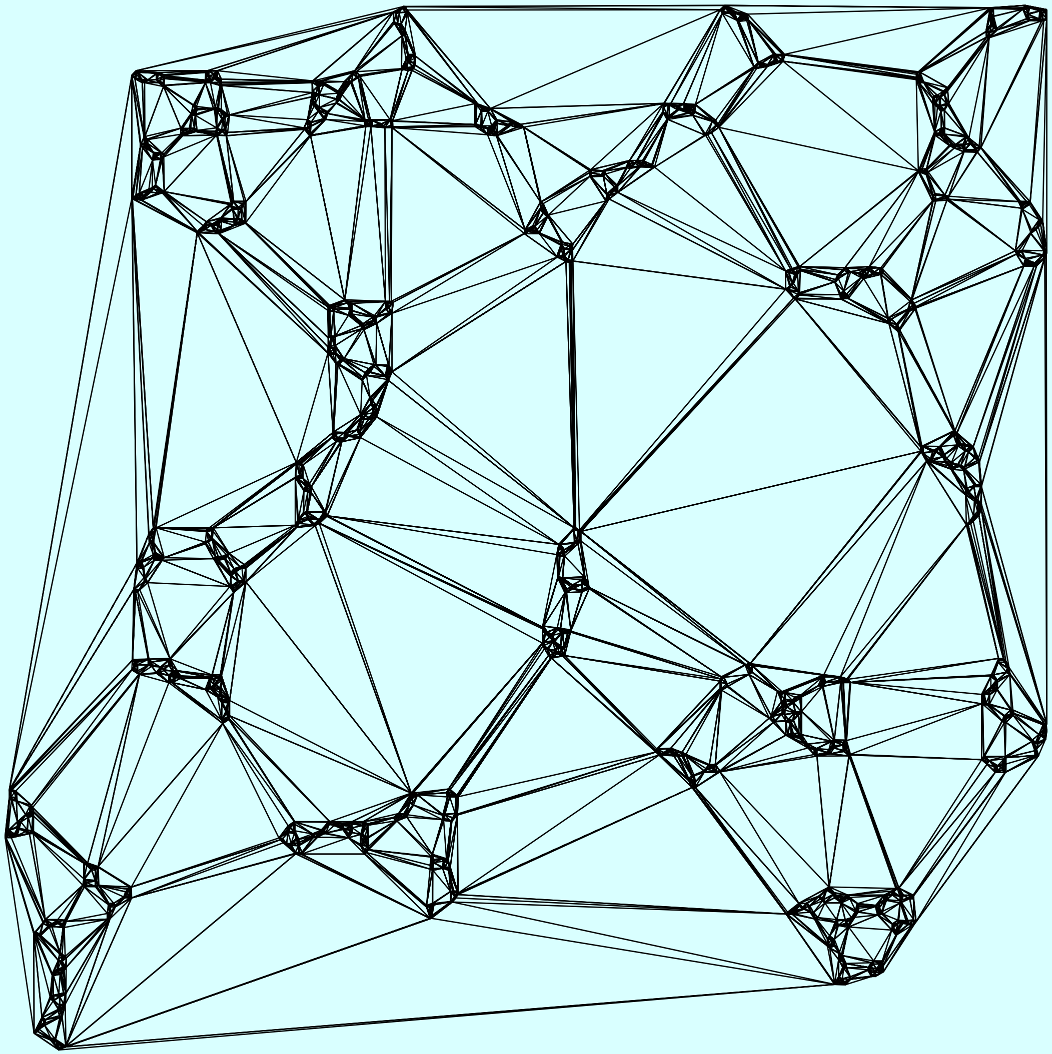
Random fractal: Example 1

Random Cantor fractal ($12\,288$ points) with $D = 1$ and transition to homogeneity.



Random fractal: Example 1

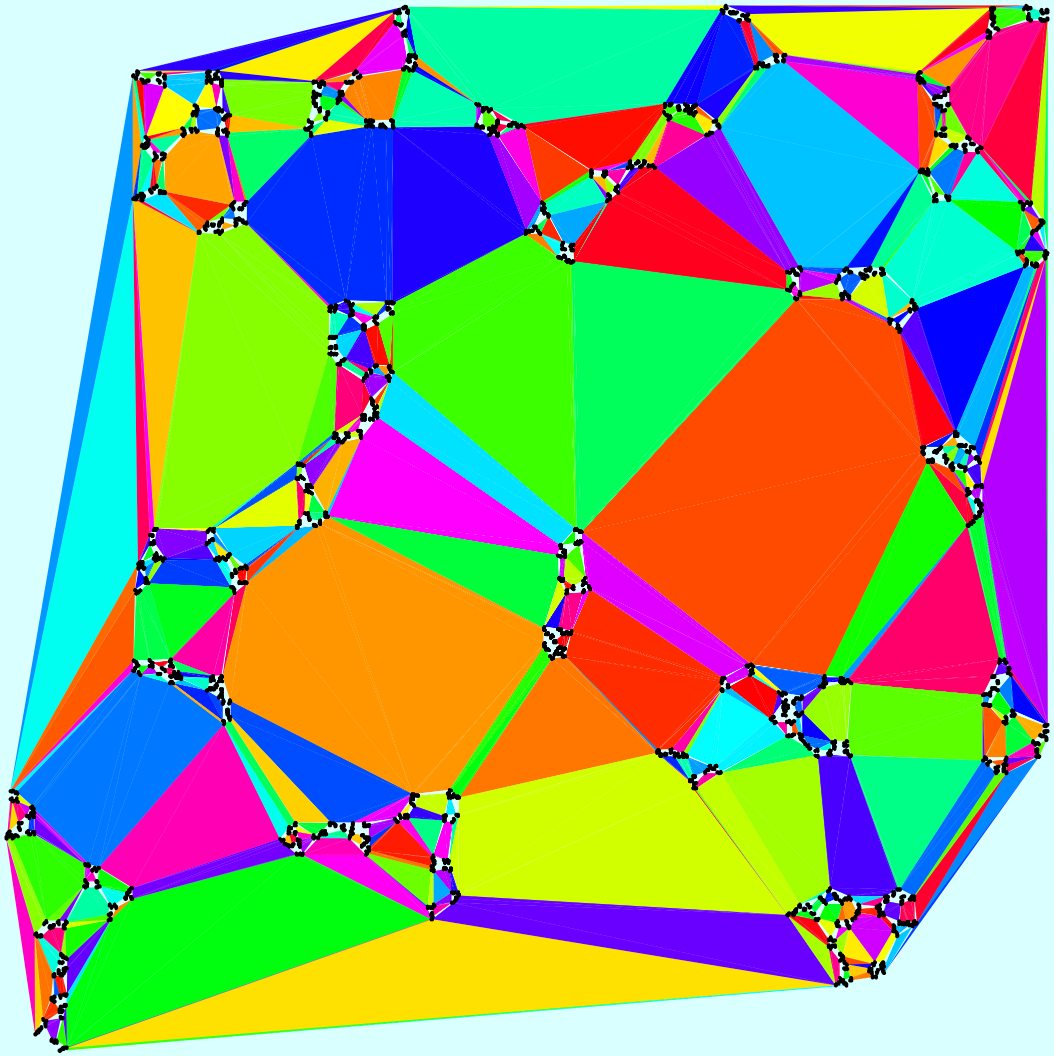
Delaunay triangulation of the fractal set:



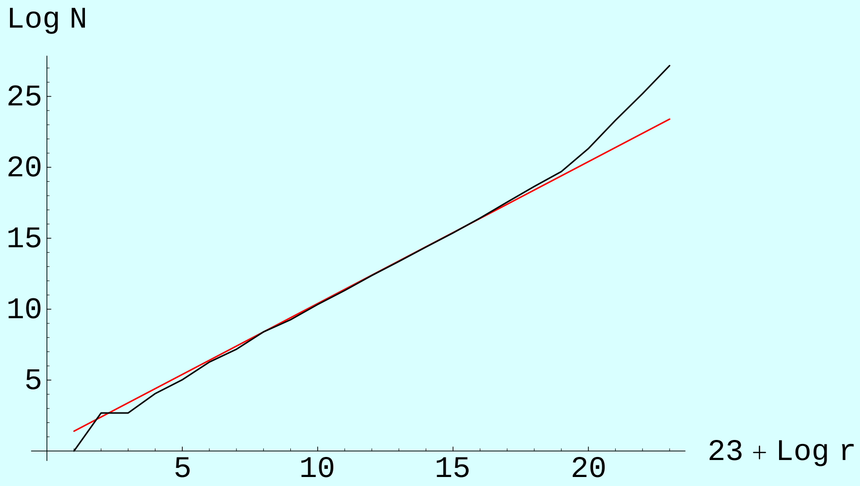
Random fractal: Example 1

Largest 500 voids with

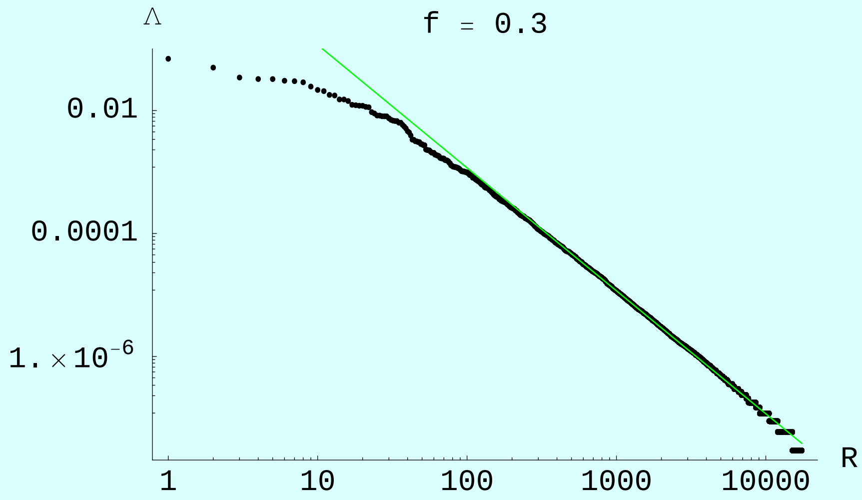
$f = 0.3$ overlap:



Random fractal: Example 1



$N(r)$ and fit $D = 1$



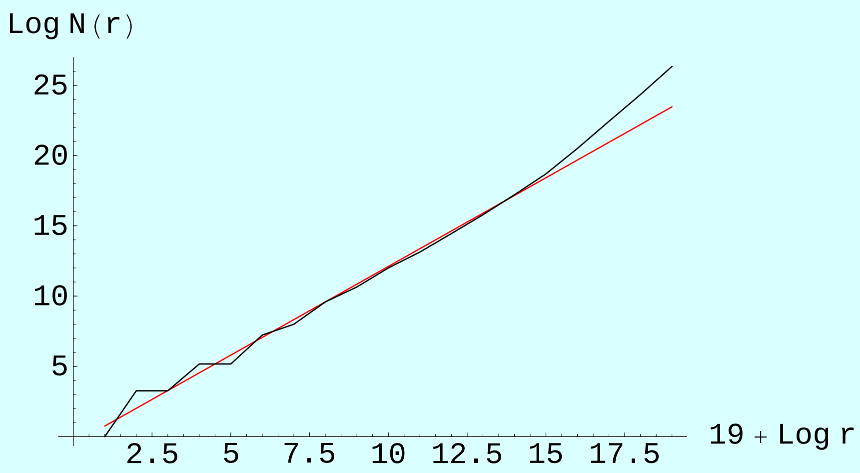
Zipf's law with slope $-2/D$

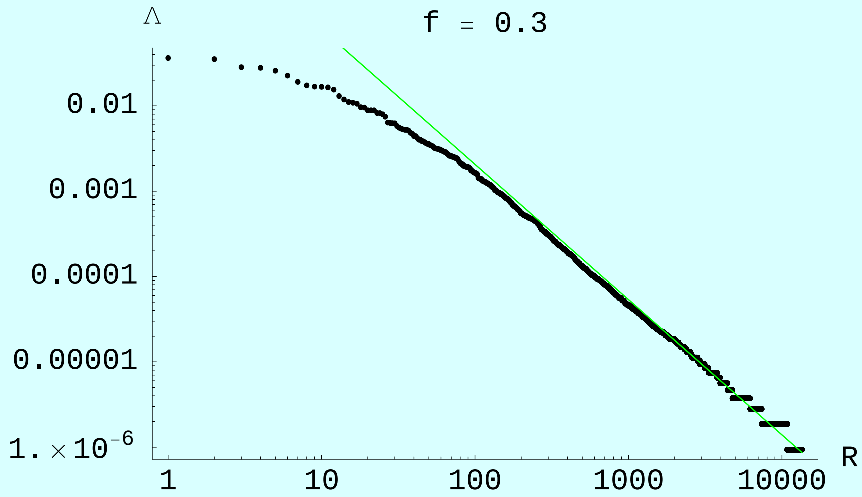
Compare:

- Scaling ranges of $N(r)$ and $\Lambda(R)$.
- Transition to homogeneity.

Random fractal: Example 2

Random fractal ($9\,216$ points) with $D = 1.26$





$N(r)$ and fit $D = 1.26$

Zipf's law with slope $-2/1.26$

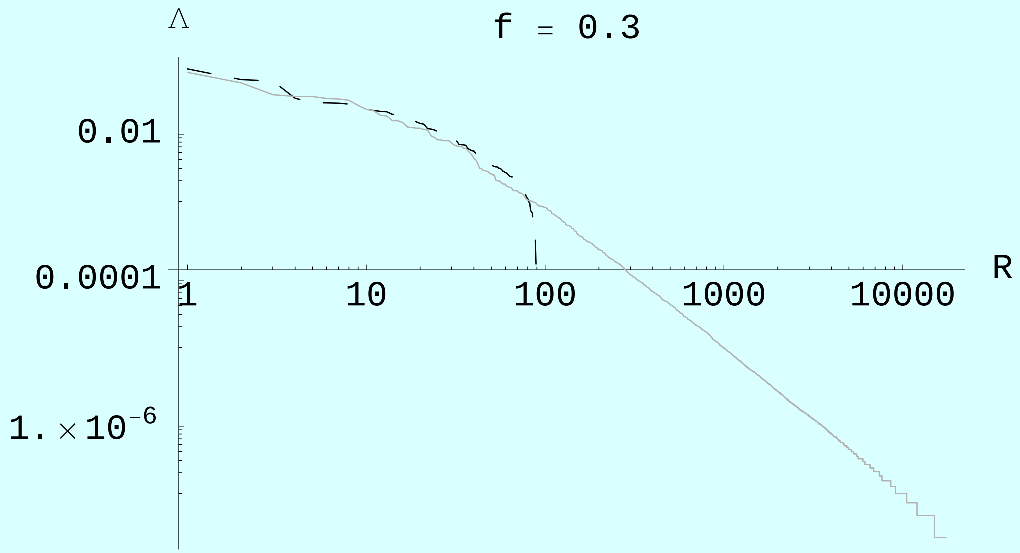
- Topological dimension 0 $\Rightarrow$ voids ill-defined.
- Cut-out set = Fractal + boundary of voids.
- Minimum value of D is 1 (boundary of voids).

Transition to homogeneity

If the fractal becomes homogeneous ($D = 3$) on large scales $\Rightarrow$ flattening of Zipf's law in the rank-ordering of voids.

Transition to homogeneity in Example 1:

Flattening of the Zipf law and voids of a random distribution of points (dashed line) having small-rank voids of the same size ($f = 0.3$ for both).



Fractal measures

Basic measure is the number-radius relation $N(r) = B\, r^D$, $r < r_0$ (scale of transition to homogeneity $D \to 3$).

- Gamma function (two-point correlation):

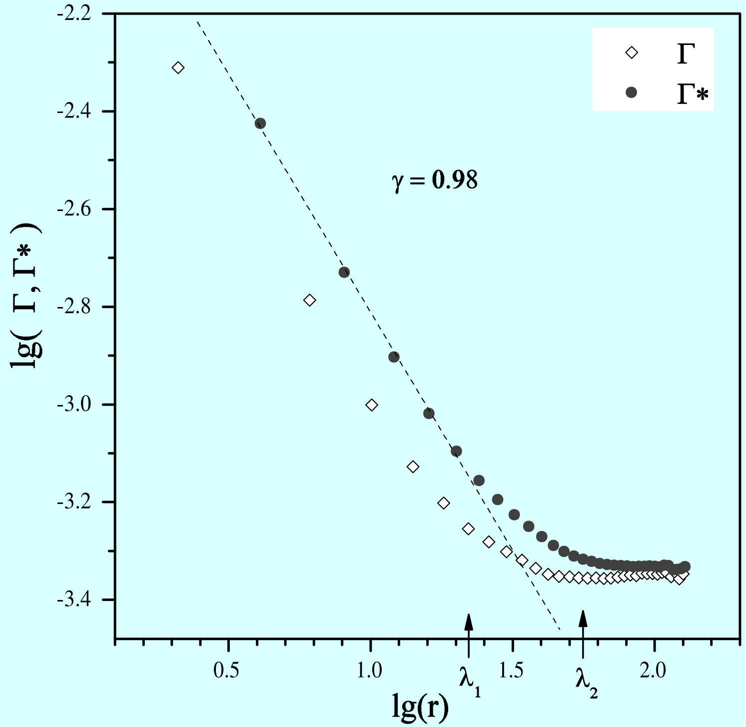
$$\Gamma(r) = \frac{1}{4\pi r^2}\frac{dN(r)}{dr} = \frac{BD}{4\pi} r^{D-3}.$$

One defines $\gamma = 3 - D > 0$.

- Gamma-star function:

$$\Gamma^*(r) = \frac{N(r)}{4\pi r^3/3} = \frac{3B}{4\pi} r^{D-3}.$$

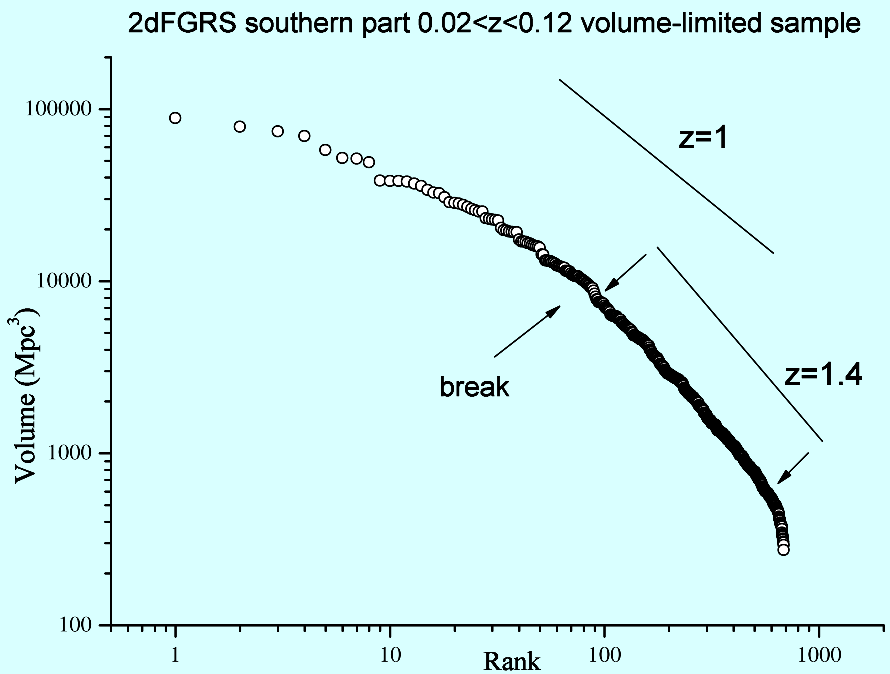
Scaling of galaxy clusters and voids



$\Gamma(r)$ of SDSS VL sample.

$r_0 \simeq 15$ Mpc (Tikhonov).

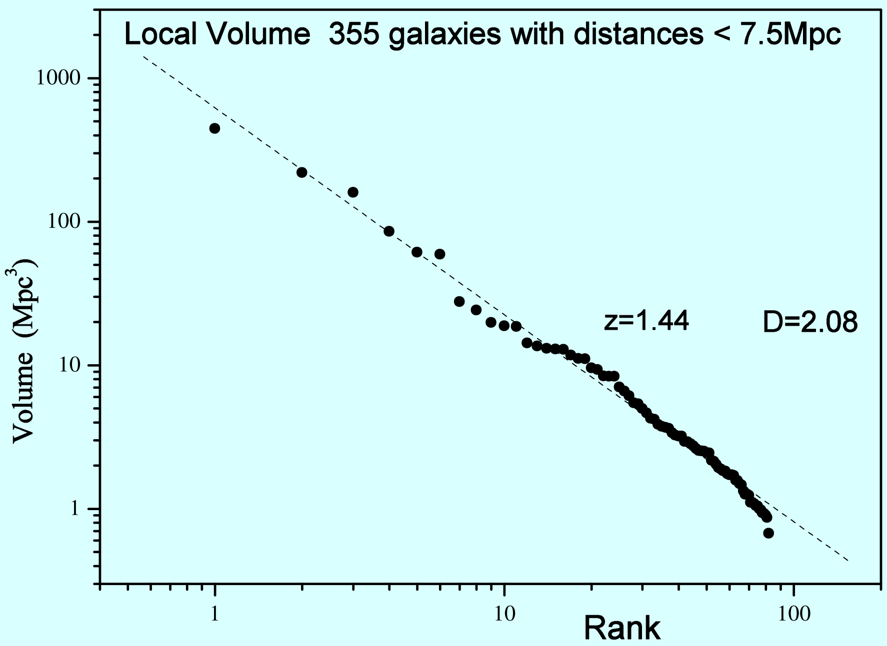
Slope $\gamma \to D \simeq 2$



The Zipf law for a 2dF VL sample (Tikhonov). Slope $z \Rightarrow D \simeq 2$

Scaling of galaxy clusters and voids

Rank ordering of local “minivoids” (Tikhonov & Karechentsev, 2006, ApJ):



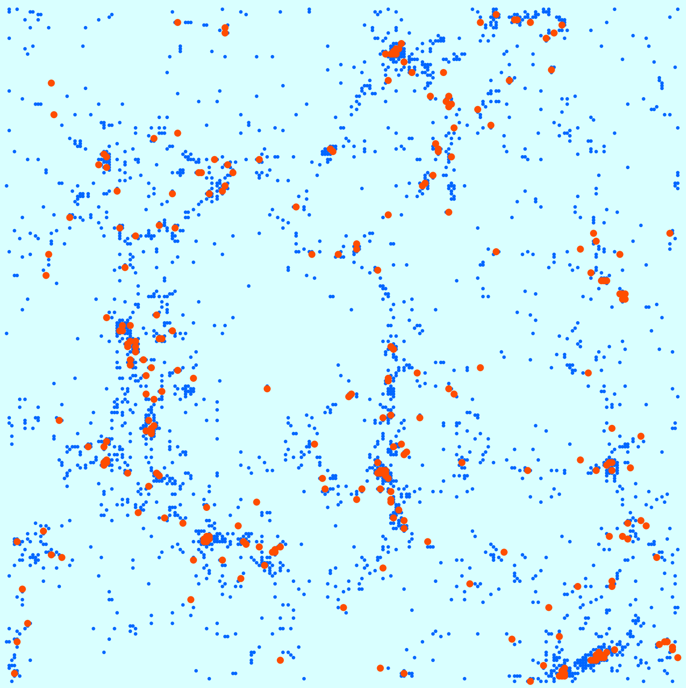
Slope $\gamma \Rightarrow D \simeq 2$. No transition to homogeneity

Six largest minivoids within the sphere of radius 7.5 Mpc

Voids in a multifractal

Full dark-matter distribution is multifractal (Gaite, 2007, ApJ) $\Rightarrow$ various (halo) populations with different clustering and fractal dimension.

Two halos populations in the GIF2 simulation (massive in red and light in blue). More clustered massive halos leave larger voids.



- Faint galaxies in voids $\longleftrightarrow$ galaxy formation (Peebles, 2001).
- Distribution of dark matter inside voids (Gottlöber et al, 2003).

SUMMARY and CONCLUSIONS

- Clustering $\Rightarrow$ hierarchy of voids. Scaling voids $\leftrightarrow$ Zipf's law.
- Rigorous definition of fractal voids $\longrightarrow$ cut-out sets, with topological codimension 1.
- Zipf's law flattening for largest voids $\longrightarrow$ transition to homogeneity.
- Void finders $\longrightarrow$ voids in fractals with topological codimension ≥ 1 (addition of boundaries).
- Recent analysis: large voids which fulfill Zipf's law but yield minimum $D = 2$.