A Discrete Representation of Gravitation: The Fundamental Role of Dual Tessellations in Regge Calculus

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Tullio Regge presented with Dirac Medal

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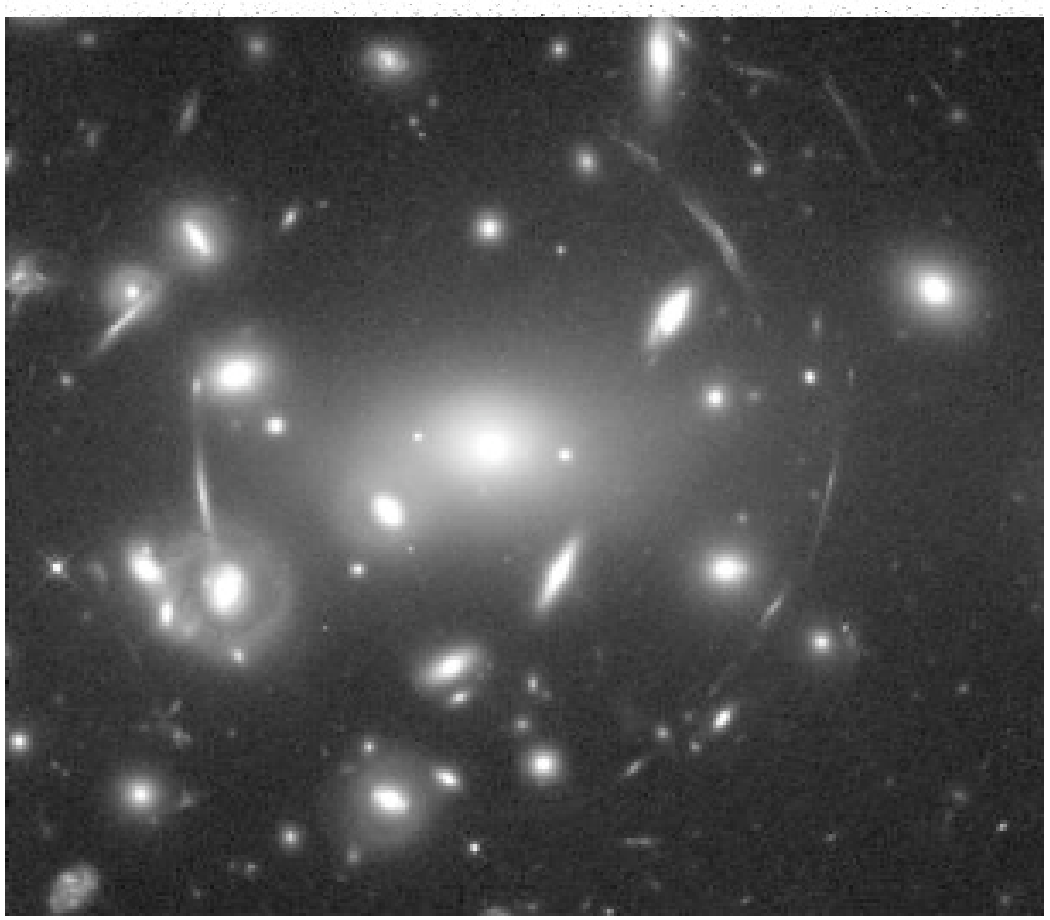
Why Regge Calculus?

If nature is indeed fundamentally discrete; built out of a finite number of elementary quantum phenomena. Then one may hope that by studying the discrete representations of the most beautiful geometric theory of nature we know, gravitation, one may be able to glean some of the fundamental features of the discretization that may yield way points to a true understanding of the basic building blocks of nature. We are not aware of a more pure, geometrically-based discrete model of gravity than Regge Calculus.

The laws of gravitation appear to be encoded locally on the lattice in with less complexity than in the continuum.

Gravitation



$$\underbrace{G_{\mu\nu}}_{\substack{Curved \\ Spacetime \\ Geometry}} = \underbrace{8\pi T_{\mu\nu}}_{\substack{Matter \\ and \\ Fields}}$$

In general relativity the gravitational interaction is modeled by a 4-dimensional curved spacetime geometry represented by a

4×4 metric $g_{\mu\nu}$

(or by a tessellated geometry as the case may be).

Curvature by Parallel Transport

$$Curvature = \frac{\left(\begin{array}{c} Angle\ Vector \\ Rotates \end{array}\right)}{\left(\begin{array}{c} Area \\ Circum- \\ navigated \end{array}\right)} = \frac{\frac{\pi}{2}}{\frac{1}{8}\left(4\pi R_{\oplus}^{2}\right)} = \frac{1}{R_{\oplus}^{2}}$$

Curvature in 4-D Spacetime

$$-\frac{1}{4}\overbrace{e_\alpha \wedge e_\beta}^{\substack{Rotation\\Bivector}}\quad R^{|\alpha\beta|}{}_{|\mu\nu|}\quad\underbrace{dx^\mu \wedge dx^\nu}_{\substack{Orentation\\of\,area\\Circumnavigated}}$$

Many more combinations in 4-dimensions

2-D: x–y

3-D: x–y, x–z and y–z

4-D: t–x, t–y, t–z, x–y, x–z and y–z

The Einstein-Hilbert Action

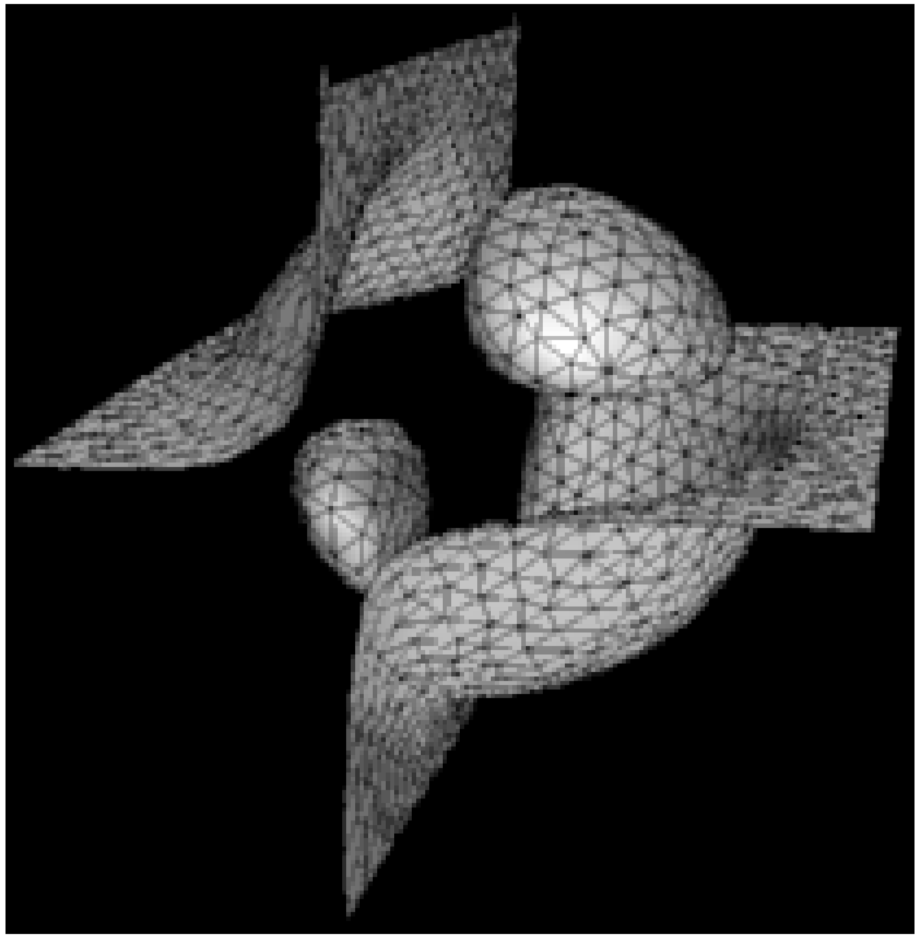
$$I = \frac{1}{16\pi} \int {}^{(4)}R \, d^{(4)}V_{proper}$$

$$\underbrace{\delta I = 0}_{g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu}}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Regge Calculus (RC)

In Regge Calculus the spacetime geometry is represented by a simplicial lattice. The discrete geometry is built of internally flat 4-dimensional triangles (simplices).



Hilbert Action

$$I = \int {}^{(4)}R \, d^{(4)}V_{proper}$$

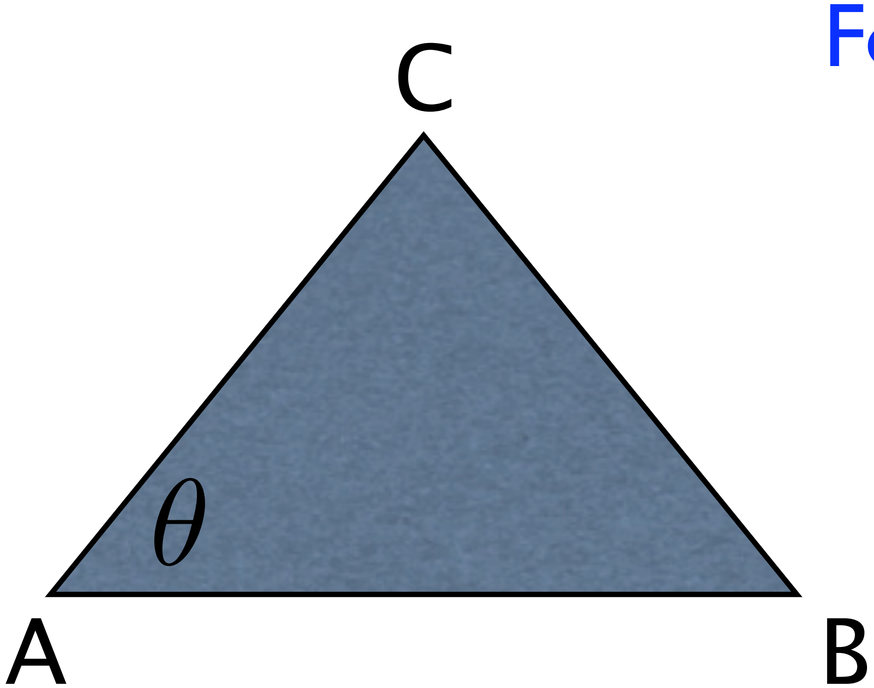
Regge-Hilbert Action

$$I_R = \frac{1}{8\pi} \sum_{\substack{triangle \\ hinges,\ h}} \epsilon_h \, A_h$$

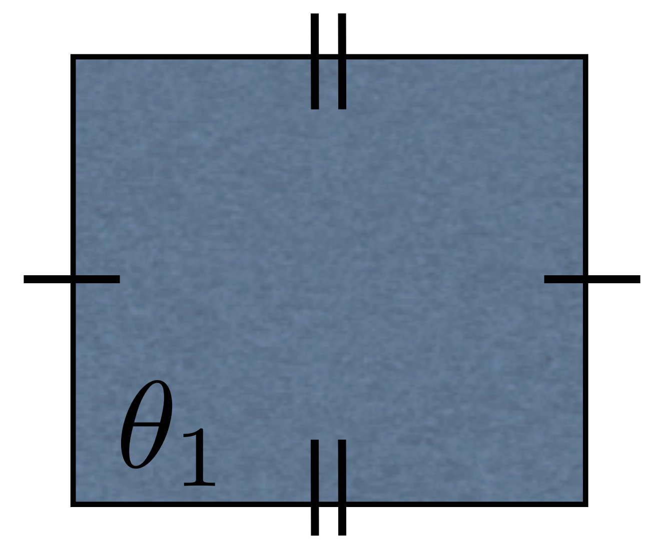
The RC Spacetime Building Blocks

Simplicies are often used because the geometry can be determined by the edgelengths and solely the egdelengths. The length of an edge of a simplex is the proper distance between its two vertices, and represent the RC analog of the metric.

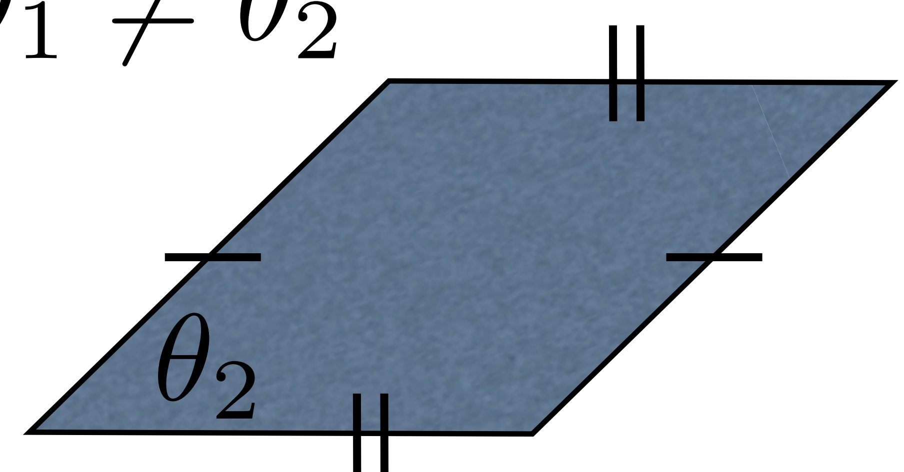
For Example in 2-D



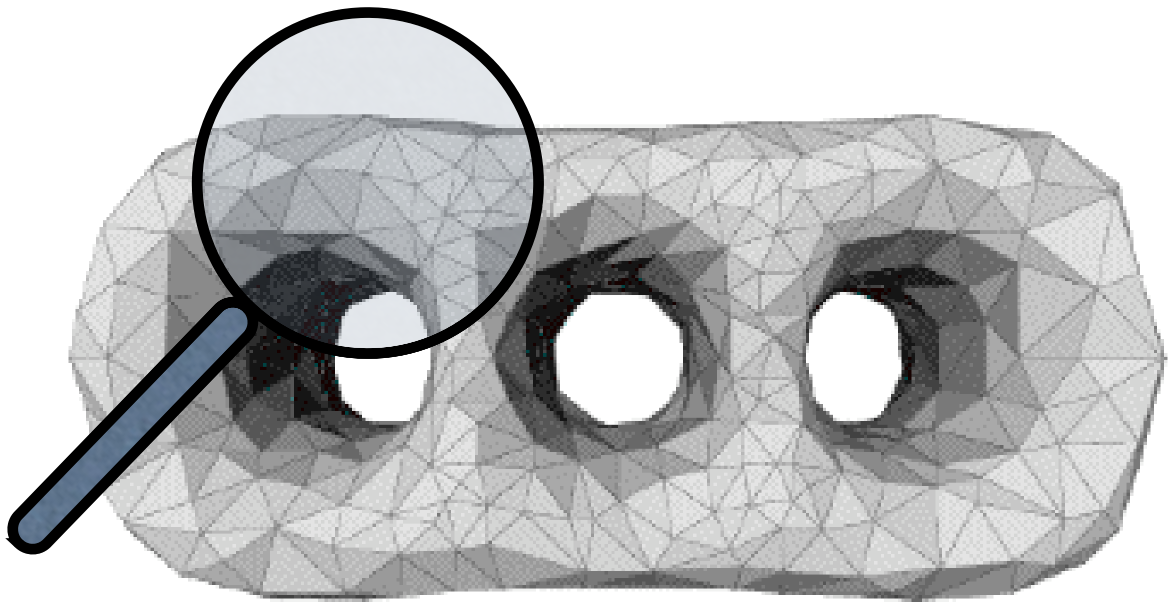
$$\sin(\theta) = \frac{2\triangle_{ABC}}{\overline{AB}\,\overline{AC}}$$



$$\theta_1 \neq \theta_2$$

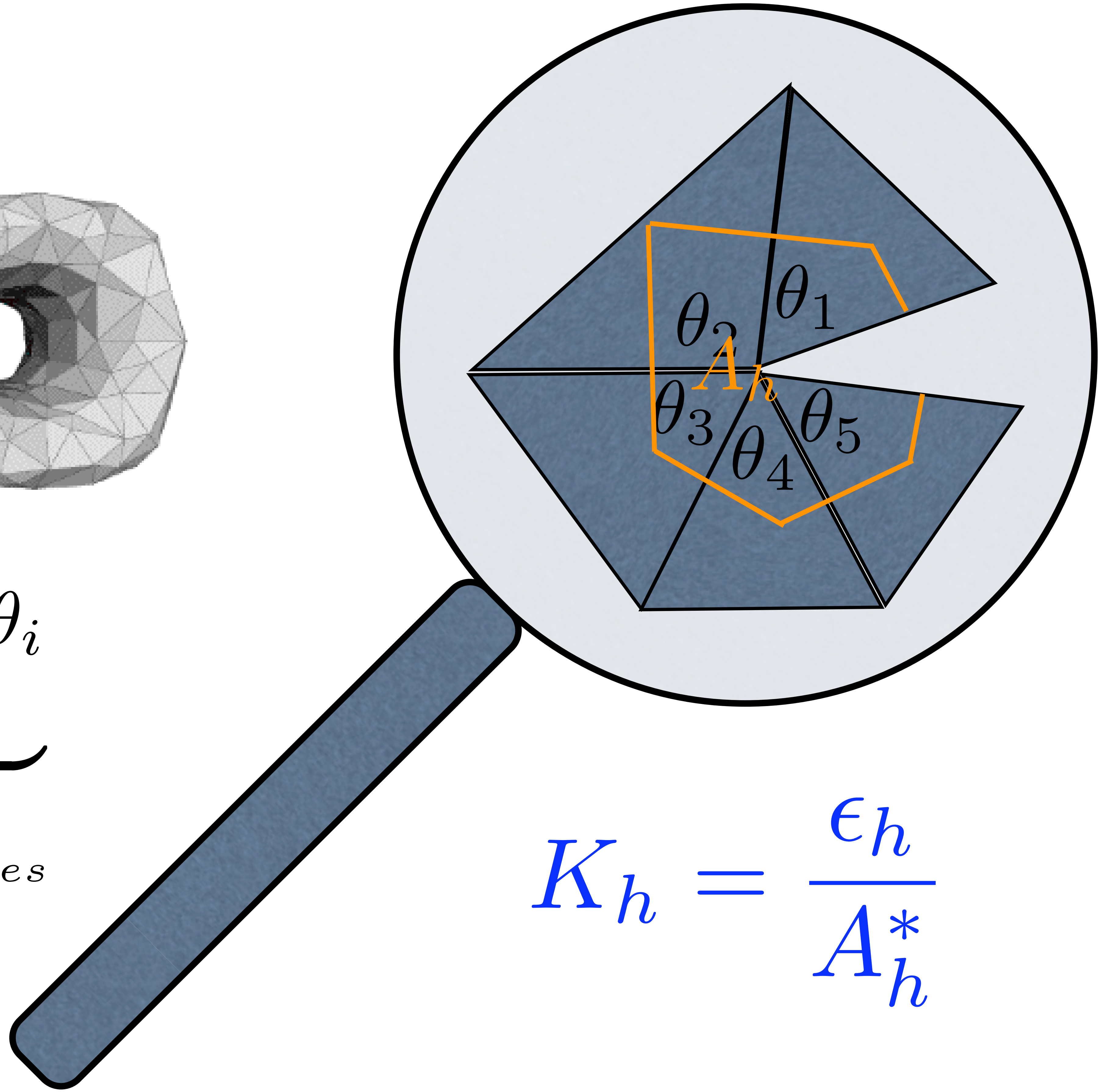


Curvature in RC (2–D)



$$\epsilon_h = 2\pi - \sum_{i=1}^{5} \theta_i$$

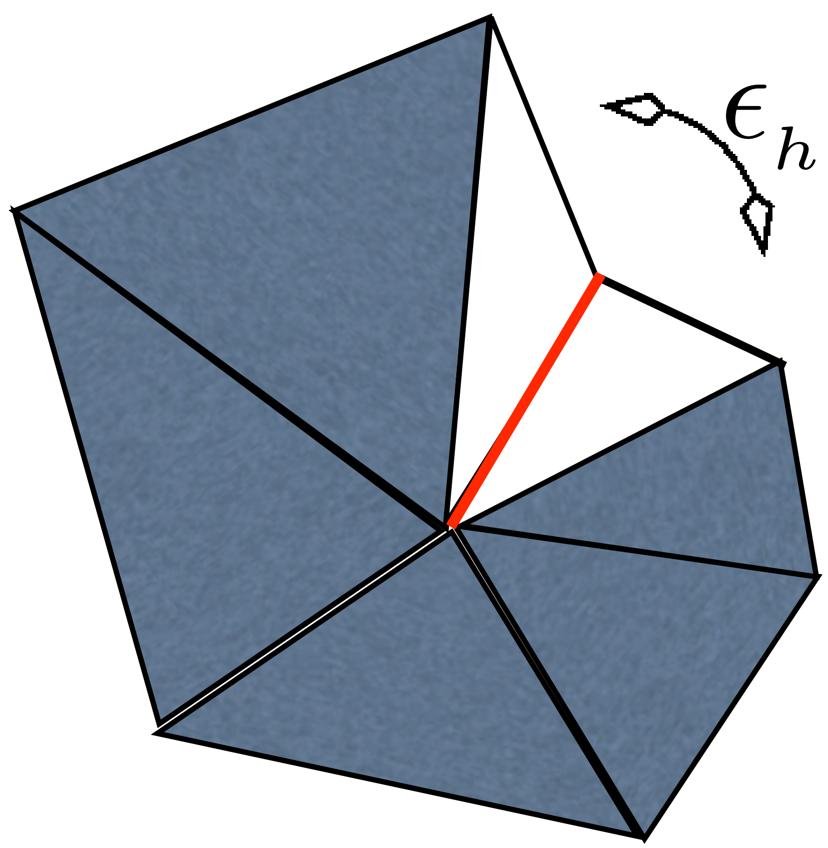
Deficit Angle
Angle Vector Rotates

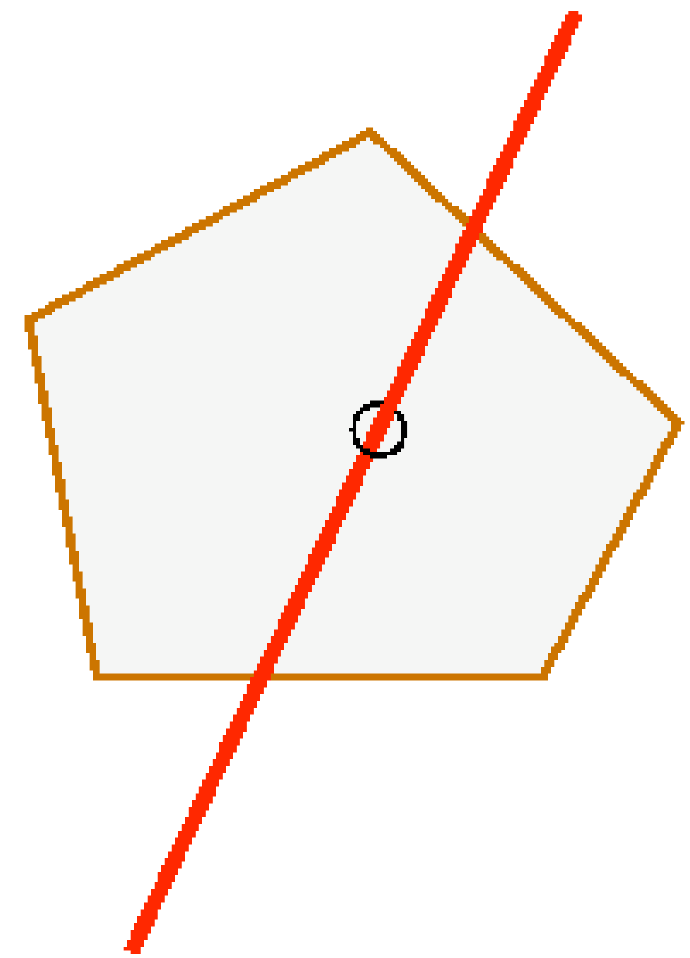


$$K_h = \frac{\epsilon_h}{A_h^*}$$

Curvature in RC (3-D)

The Building block is a tetrahedron.
The hinge is the edge common to the five tetrahedrons.

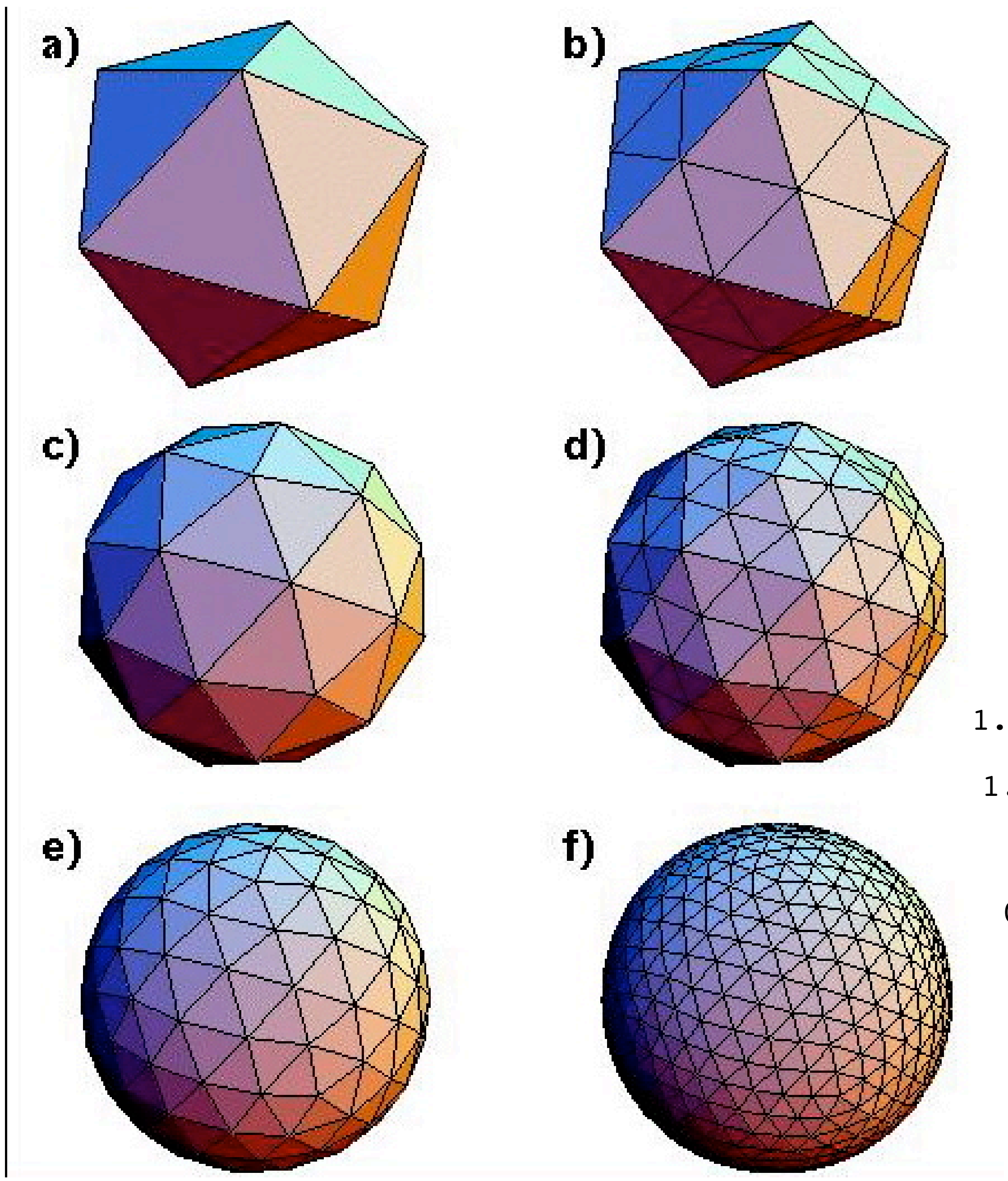


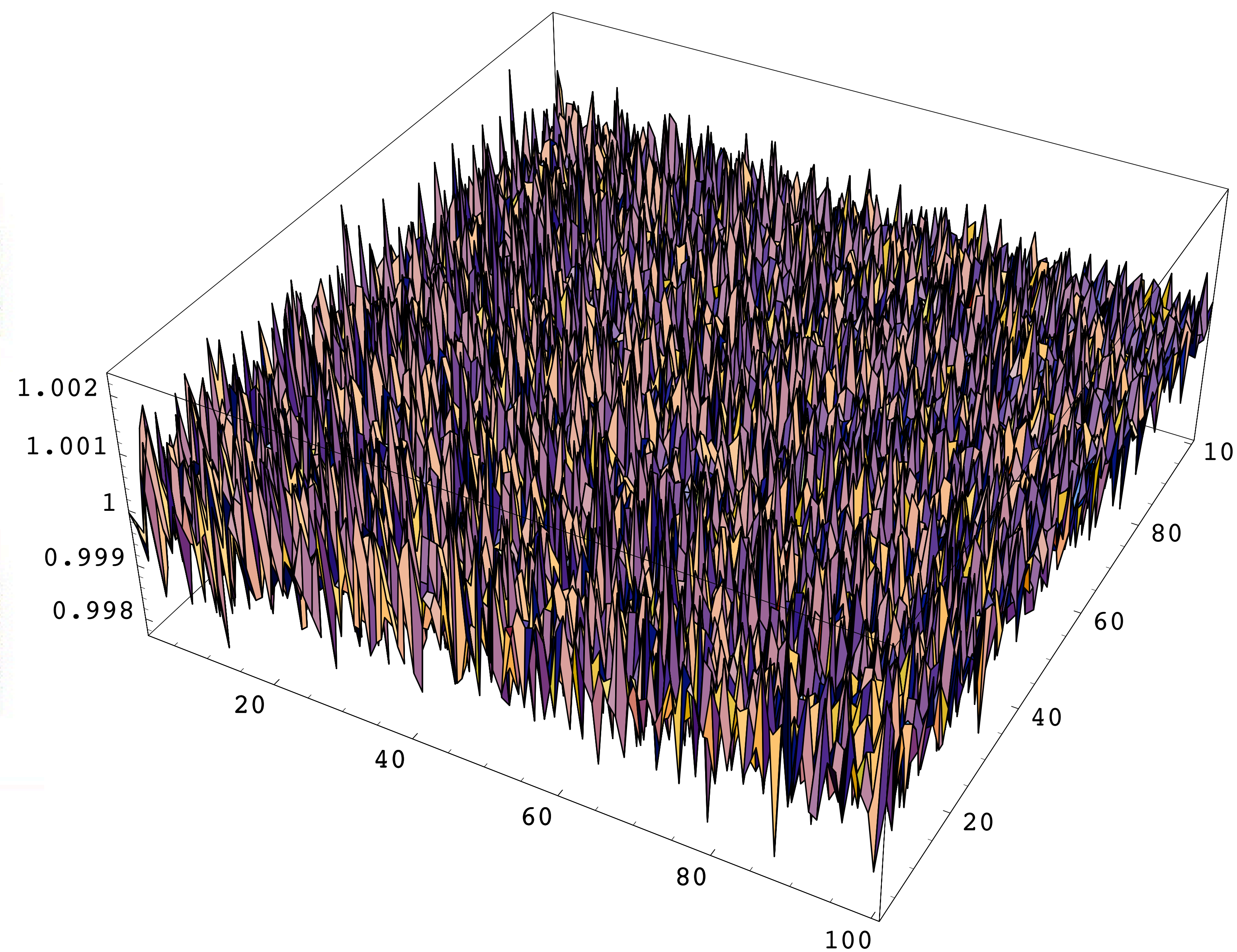


$$^{(3)}K_h = \frac{\epsilon_h}{A^*_h}$$

Scalar Curvature Example

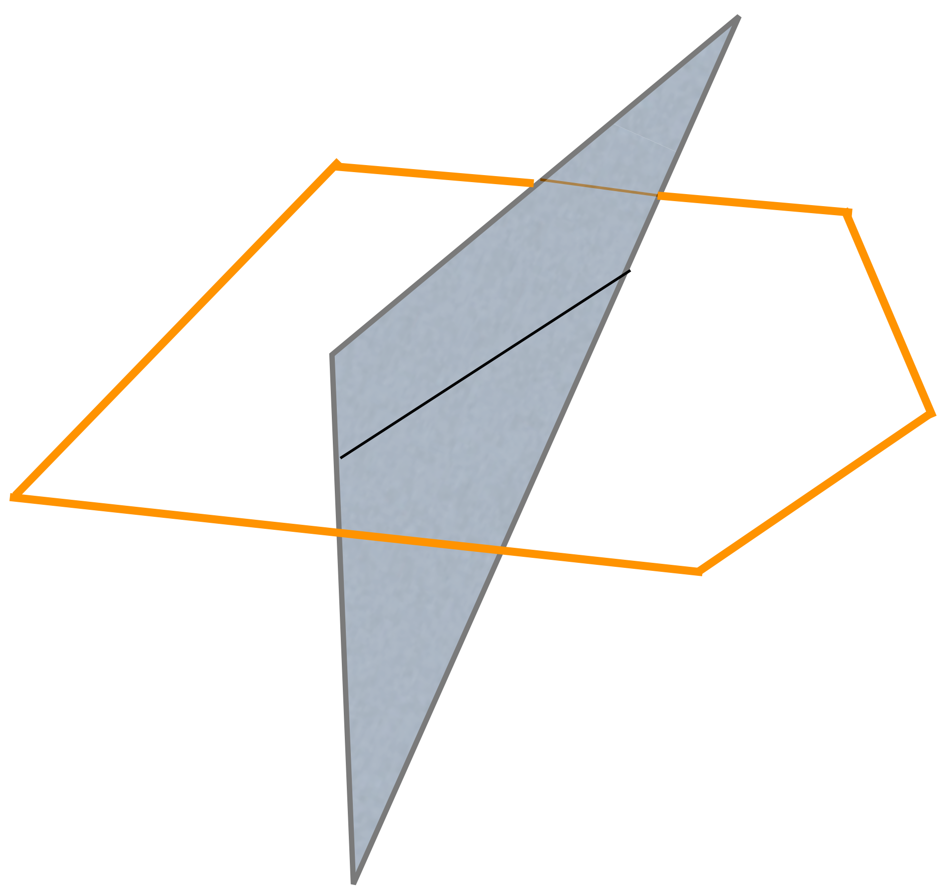
$$K_h = \frac{\epsilon_h}{A_h^*}$$





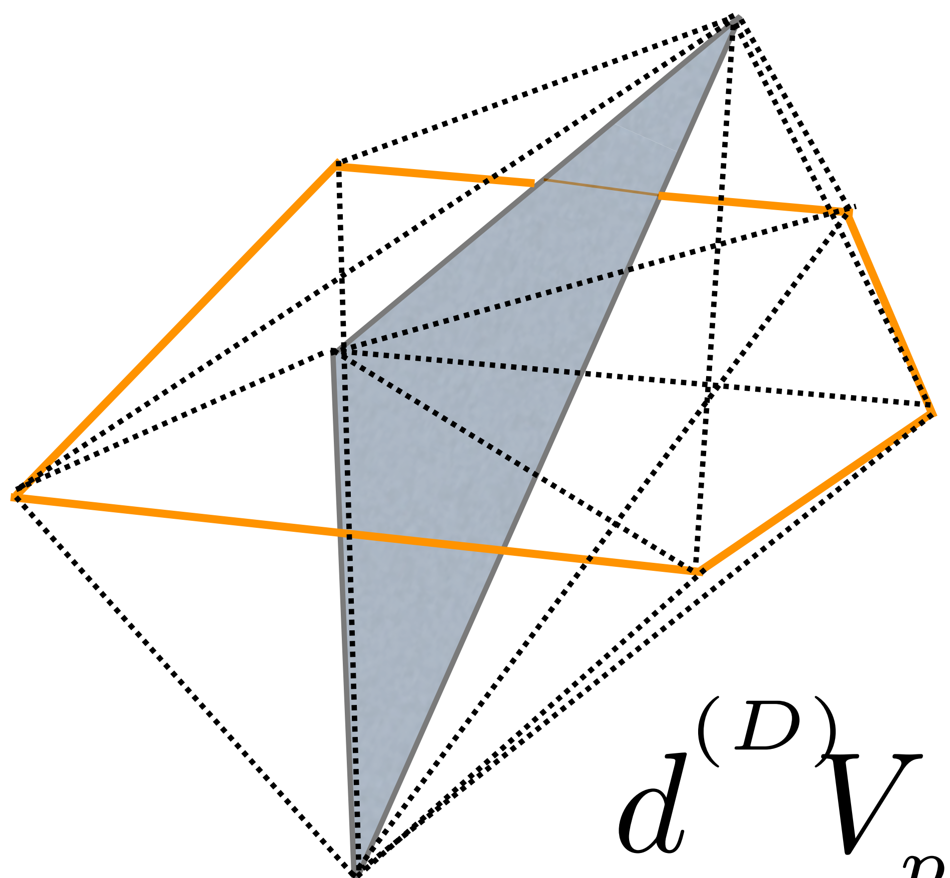
Curvature in RC (4-D)

The Building block is a simplex and the hinge (h) is the triangle common to all the simplicies



$$^{(4)}K_h = \frac{\epsilon_h}{A^*_h}$$

A Fundamental Block Coupling the Voronoi and Delaunay Lattices Together



$$d^{(D)}V_{proper} = \frac{2}{D(D-1)} A_h A_h^*$$

A co-dimension 2 version of $\frac{1}{2}\, base \times altitude$

Scalar Curvature of the Fundamental Block

- Curvature is concentrated on co-D-2 hinge, h
- Rotation in plane perpendicular to the hinge, h.
- Voronoi polygon A_h^* is perpendicular to hinge, h
- Locally the Regge spacetime is an Einstein space

$$ {}^{(D)}R_h = D(D-1)\,{}^{(D)}K_h = \frac{D(D-1)\epsilon_h}{A_h^*} $$

Hilbert Action

$$I = \frac{1}{16\pi}\int {}^{(4)}R\, d^{(4)}V_{proper}$$

$$\frac{1}{16\pi}\sum_{hinges,\ h}\left(\frac{D(D-1)\epsilon_h}{A_h^*}\right)\left(\frac{2}{D(D-1)}A_h A_h^*\right)$$

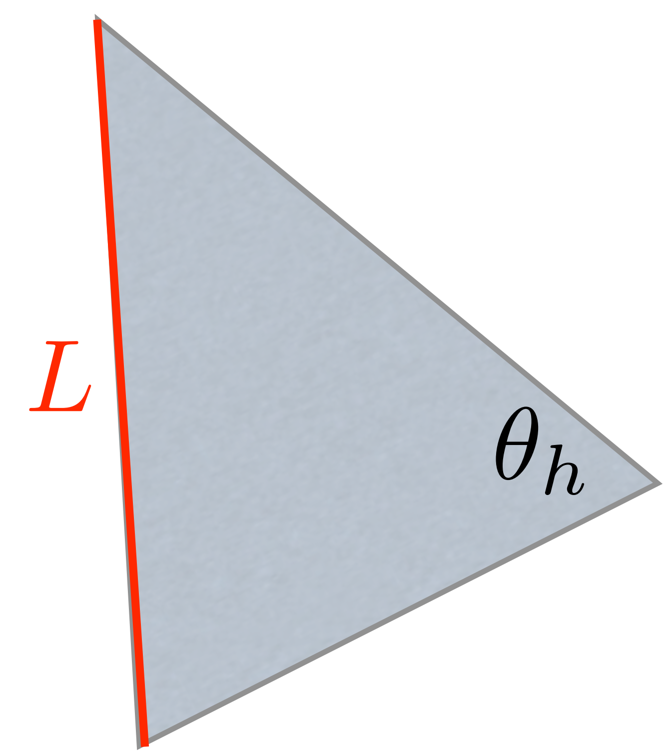
$$I_R = \frac{1}{8\pi}\sum_{\substack{triangle\\ hinges,\ h}} \epsilon_h\, A_h$$

Regge Action

Regge Equations

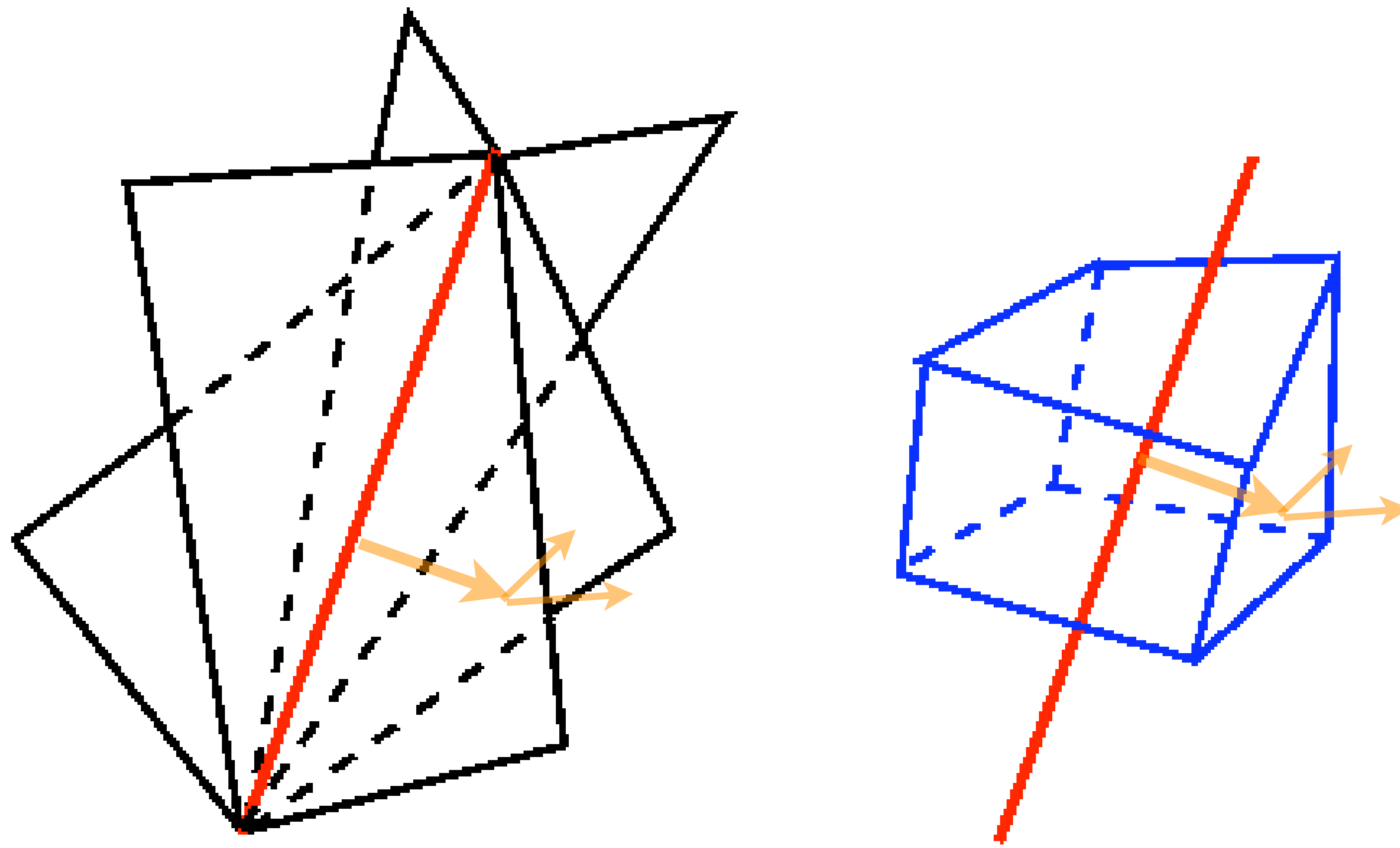
$$\delta(I_R) = \delta\left(\sum_{hinges,\,h} A_h\ \epsilon_h\right) = \sum_{hinges,\,h} \delta(A_h)\epsilon_h + A_h\delta(\epsilon_h) = 0$$

$$\delta(A_h) = \frac{\partial A_h}{\partial L}\,\delta(L) = \frac{1}{2} L \cot\left(\theta_h\right)$$



$$\underbrace{\sum_{\substack{hinges,\,h \\ \text{sharing edge } L}} \frac{1}{2} L \cot\left(\theta_h\right)\ \epsilon_h = 0}_{Regge\ Equation}$$

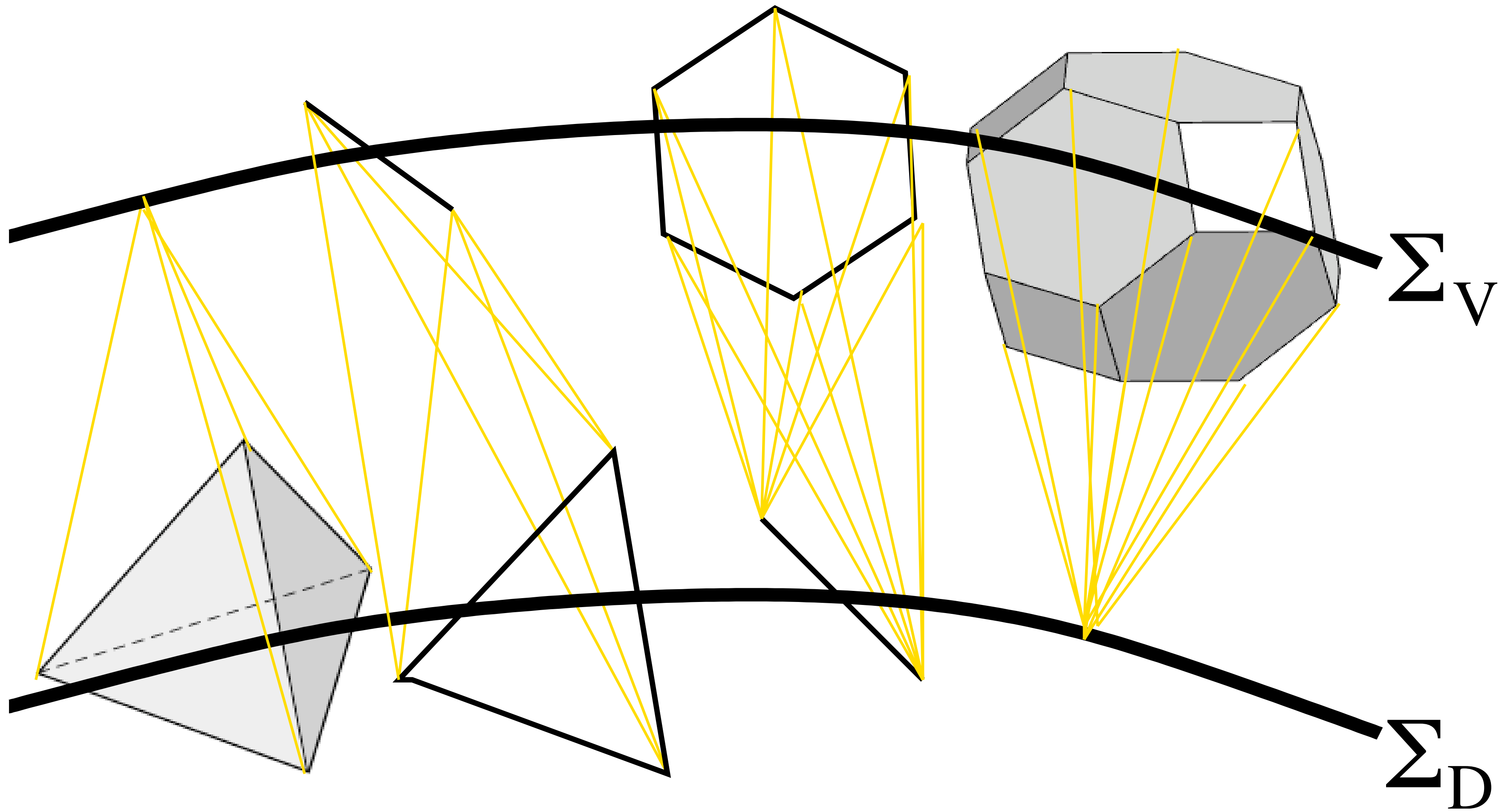
Cartan Moment-of-Rotation in RC

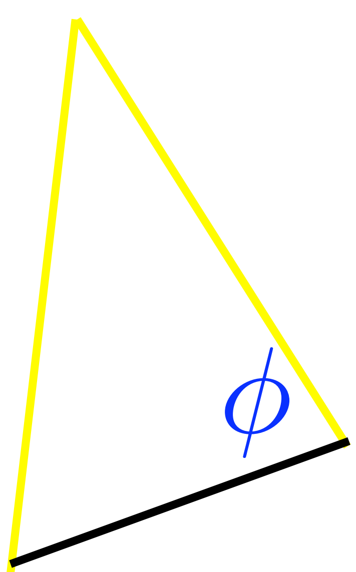


$$G_{LL}\,V^*_L = \sum_{\substack{hinges,\,h \\ \text{sharing edge } L}} \underbrace{\frac{1}{2} L \cot\left(\theta_h\right)}_{Moment\ \ Arm} \underbrace{\epsilon_h}_{Rot'n}$$

Einstein Field Equation is Diagonal, and Directed along L!
Identical to the variational equation!

An Application: Null-Strut RC





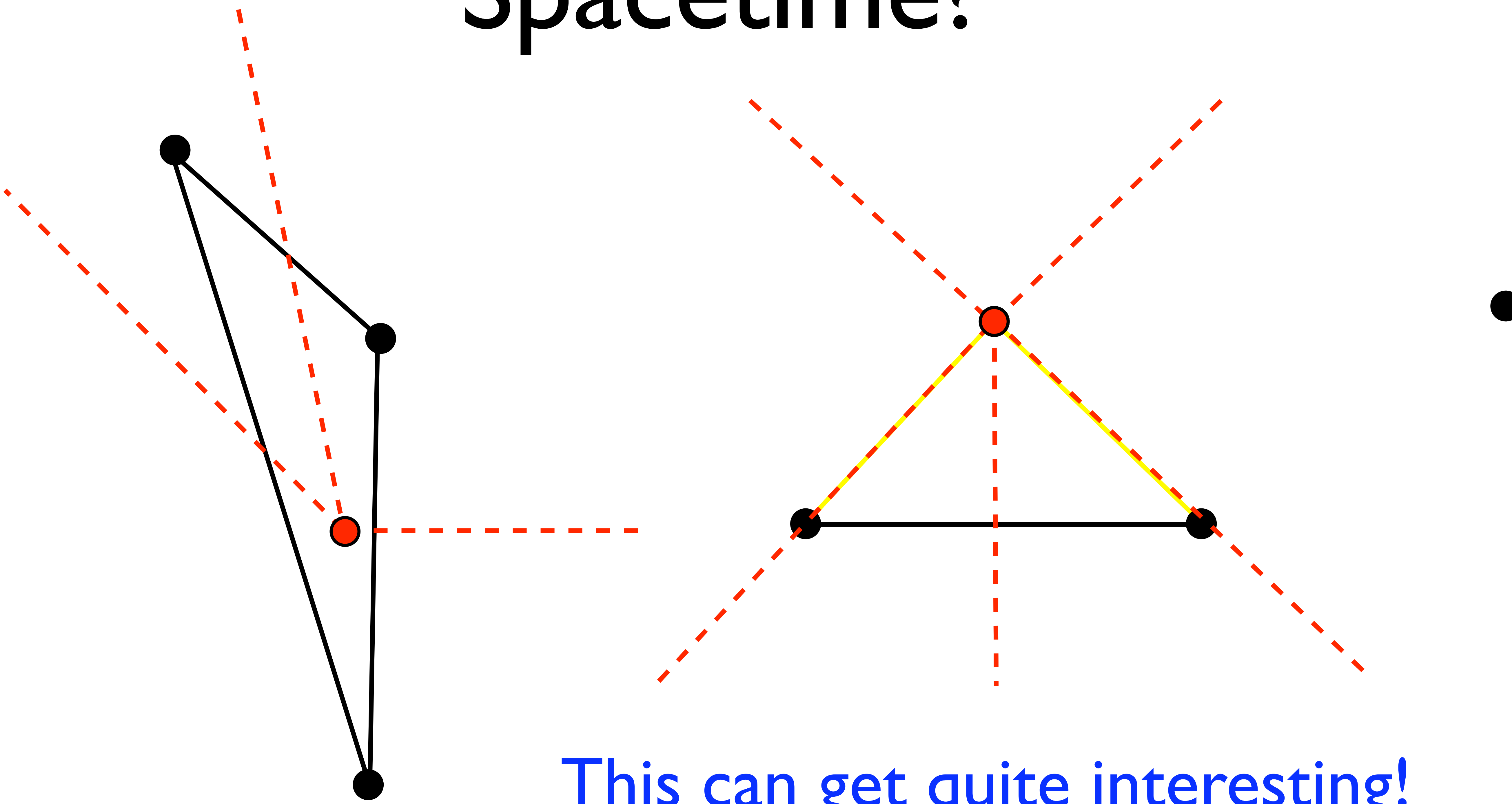
$$\underbrace{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_1^* + \epsilon_2^* + \epsilon_3^* = 0}_{Null-Strut\ \ Regge\ \ Equation}$$

Moment arms are all constant and equal for all the six triangles hinging on each null-strut edge!

IMPORTANT POINTS

- Principles of general relativity applied directly to the lattice geometry
- Provides a true finite representation of the theory based on the underlying physical principles
- Voronoi-Delaunay duality appears to be a salient feature of Regge Calculus, providing a new fundamental building block.
- The underlying discrete theory appears more austere via the underlying orthogonality; however, the full theory is recovered by convergence in mean.
- Can the Voronoi-Delaunay structure provide a platform for complementarity in quantum gravity?

Voronoi-Delaunay In Spacetime?



This can get quite interesting!