Stability issues in topological computations on noisy data

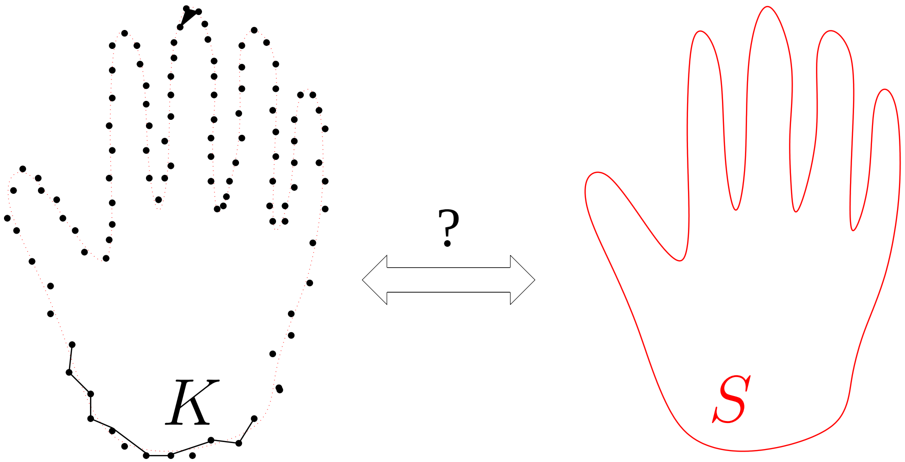
Frédéric Chazal (Institut de Mathématiques de Bourgogne)

joint works with

D. Cohen-Steiner (INRIA) and A. Lieutier (Dassault Systèmes)

An important issue in geometric computin

How topology and geometry of objects behave under approximation?



Given a data set K that approximates a geometric object S, what can we tell about the geometry/topology of S from K?
$\rightarrow$ robustness of geometry/topology

Motivations

- Robust geometric/topologic computing:

  $\longrightarrow$ necessary to design of robust algorithms.

  $\longrightarrow$ allows discretization of geometric objects.

- Certified geometric computing:

  $\longrightarrow$ critical issue in some applications.

- “geometric data analysis”:

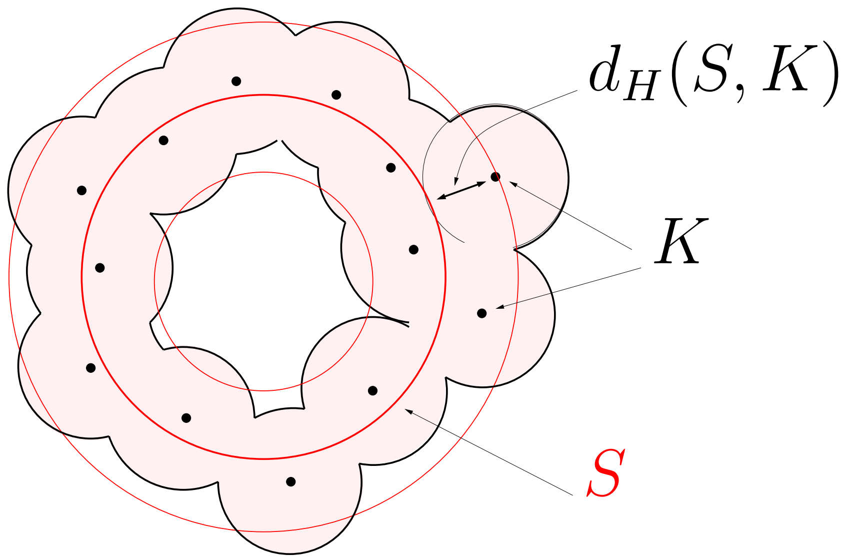
  $\longrightarrow$ Given a data set K, does there exist relevant geometric information associated to K?

- Geometric approximation theory.

Aim of the talk

- Illustrate the stability problems with two examples:
  - medial axis approximation,
  - distance functions and non smooth surface reconstruction.
- Show how one can put stability problems in a rigourous mathematical framework to solve them.

Approximation and Hausdorff distance

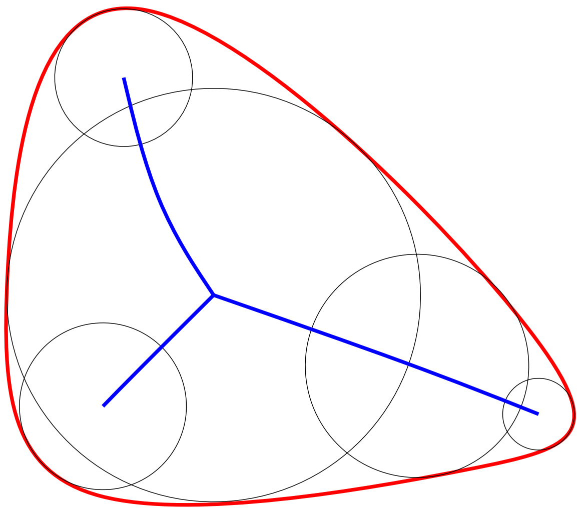


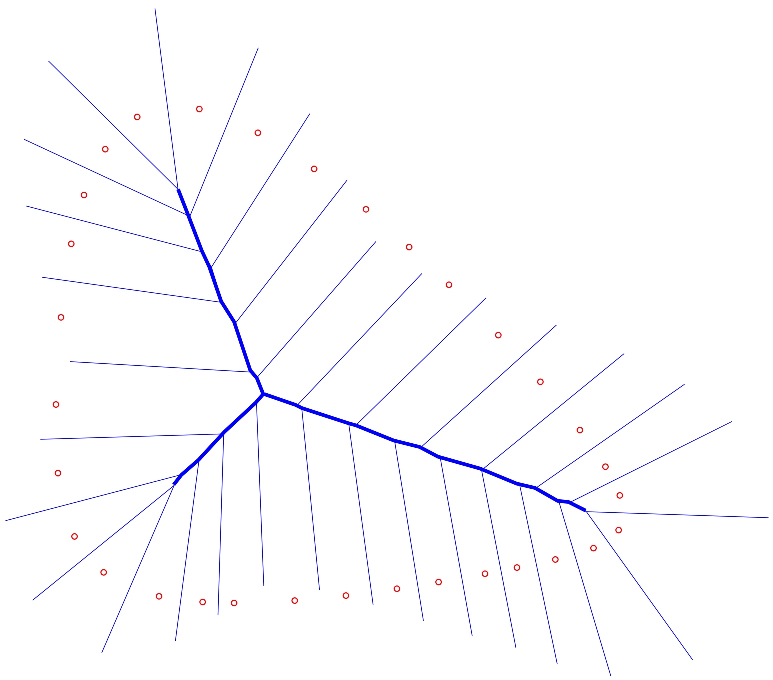
- r-thickening A^r of a set A = $A \oplus B(0, r)$ = union of balls of radius r and center in A.
- Hausdorff distance:

$$d_H(S, K) = \inf\{r \geq 0 : S \subset (K)^r \text{ and } K \subset S^r\}.$$

Example 1

Medial axis approximation





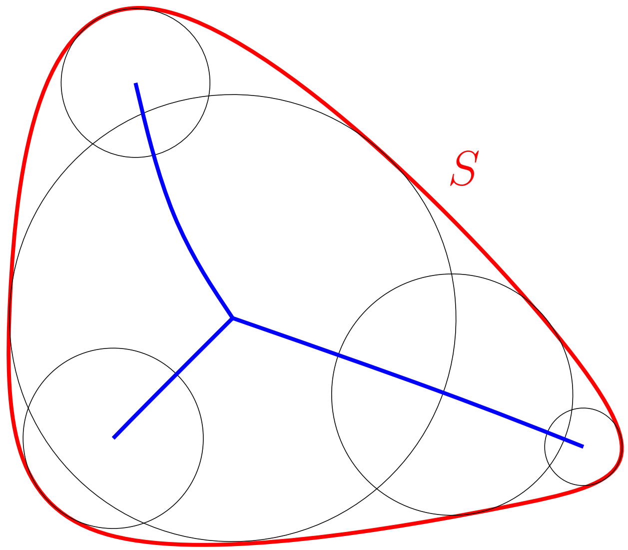
Medial axis

Let $S \subset \mathbb{R}^n$ be a compact set.

$$\Gamma(x) = \{y \in S : d(x, y) = d(x, S)\}$$

Medial axis of S:

$$\mathcal{M}(S) = \{x \in \mathbb{R}^n :| \Gamma(x) | \geq 2\}$$



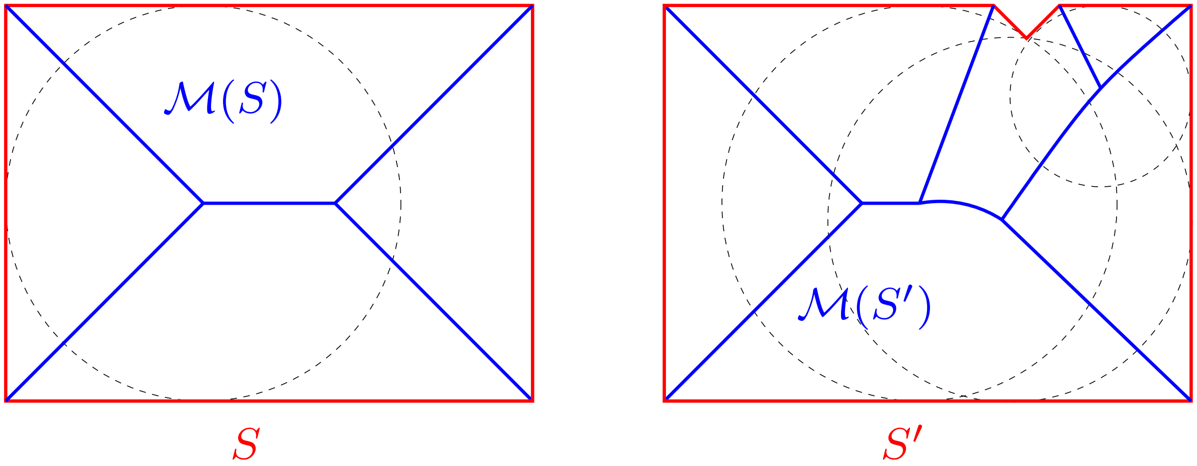
- "Medial axis = continuous version of Voronoï diagram"
- The medial axis "encodes" the topology of $\mathbb{R}^n \setminus S$ (they are homotopy equivalent).

Applications of medial axis

- Sampling conditions in reverse engineering ($\longrightarrow$ N. Amenta and T. Dey's talks),
- Motion planning,
- image analysis,
- shape recognition,
- ...

$\Rightarrow$ Big amount of literature on medial axis computation...

Unstability of medial axis

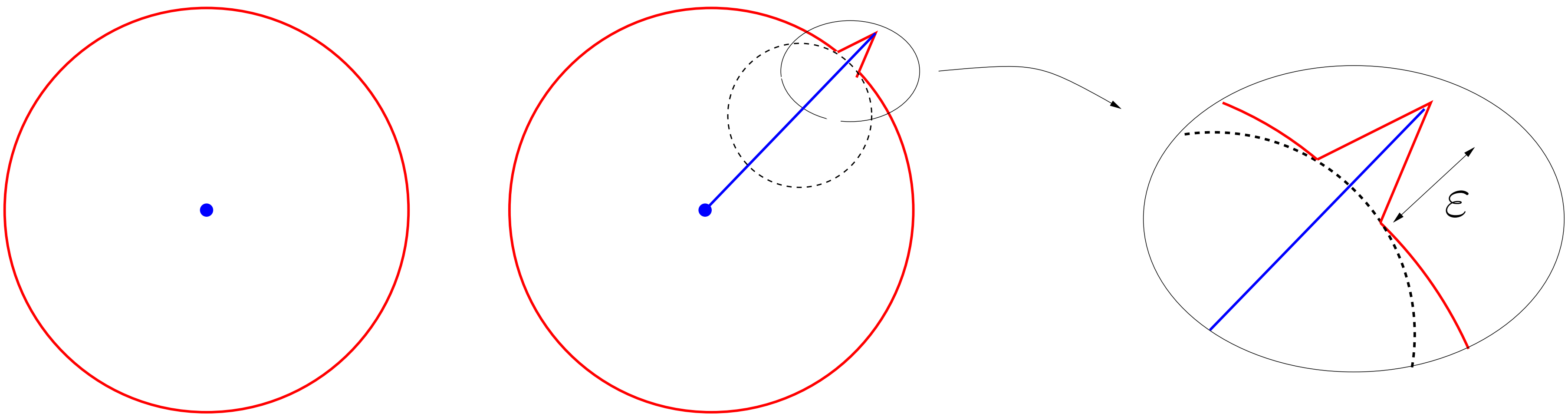


Main drawback of medial axis: it is unstable under Hausdorff perturbations:

$$d_H(\mathcal{M}(S), \mathcal{M}(S')) \not\to 0 \text{ when } d_H(S, S') \to 0$$

$\Rightarrow$ pb to compute/approximate the medial axis.

How to remove Unstability

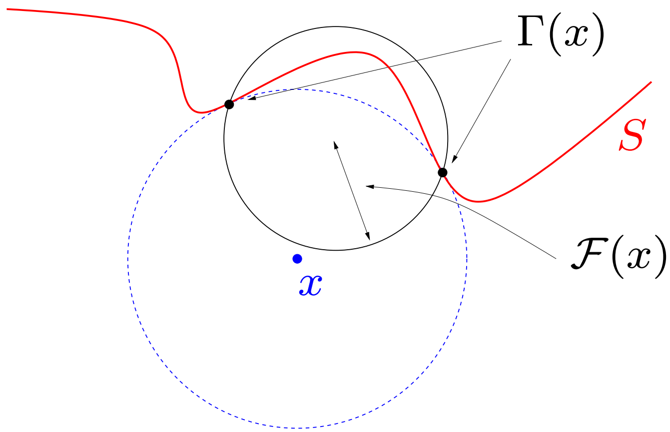


Unstable parts of the medial axis correspond to spheres that meet S in points that are very near from each other.

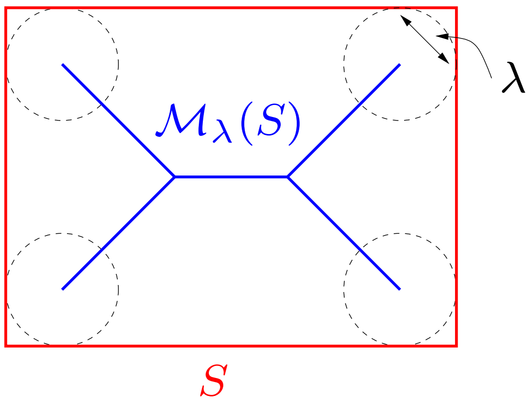
$\rightarrow$ Filter medial axis by removing the points x such that $\Gamma(x)$ is contained in a ball of small radius.

The λ-medial axis

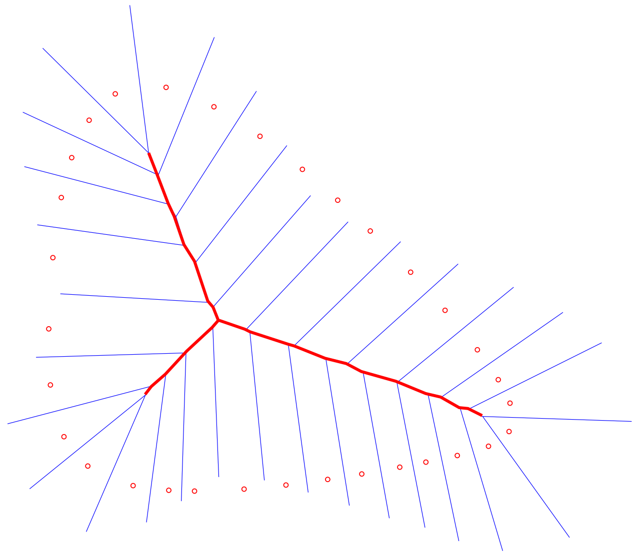
For any $x \in \mathbb{R}^n$, $\mathcal{F}(x) =$ radius of the smallest ball that contains $\Gamma(x)$.



λ-medial axis: given $\lambda > 0$,
$\mathcal{M}_\lambda(S) = \{x \in \mathcal{M}(S) : \mathcal{F}(x) \geq \lambda\}$

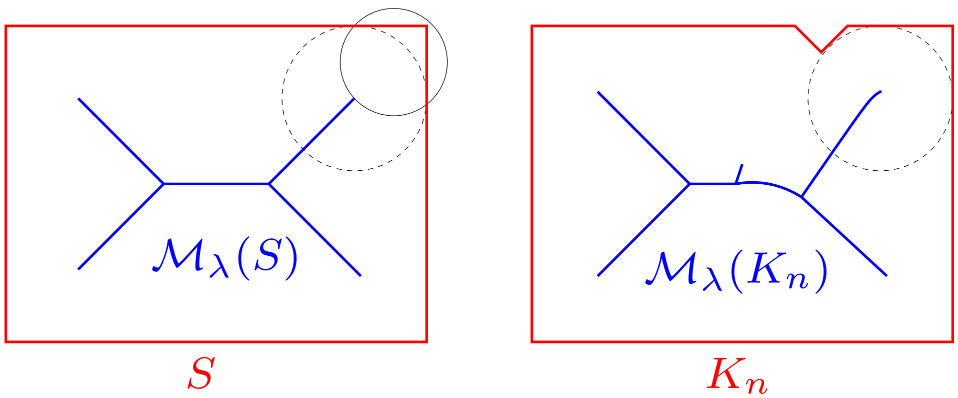


λ-medial axis and Voronoï diagrams



If K is a finite set of points, $\mathcal{M}_\lambda(K)$ is an easy to compute subcomplex of $Vor(K)$: the function $\mathcal{F}$ is constant on each cell of $Vor(K)$.

Stability of λ-medial axis

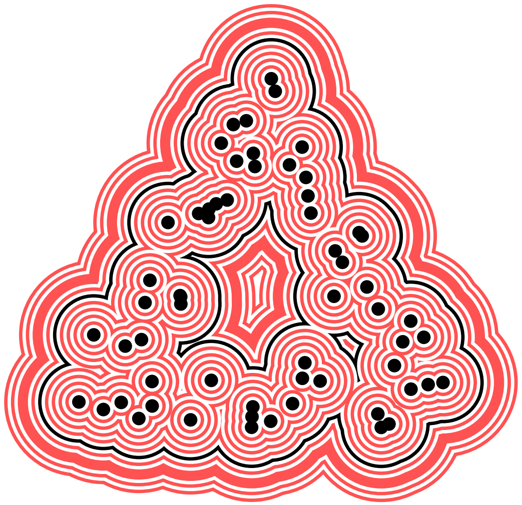


Thm: Let $S \subset \mathbb{R}^n$ and $\lambda_0 > 0$ be s.t. $\lambda \to \mathcal{M}_\lambda(S)$ is continuous at λ_0. If K_p is a sequence of compact sets s.t. $d_H(S, K_p) \to 0$ then $d_H(\mathcal{M}_{\lambda_0}(S), \mathcal{M}_{\lambda_0}(K_p)) \to 0$.

Rmk: Moreover, if S is smooth then $\mathcal{M}_{\lambda_0}(K_p)$ is homotopy equivalent to $\mathcal{M}_{\lambda_0}(S)$ as soon as $d_H(S, K_p)$ is “small enough”.

Example 2

Stability of “wave fronts”



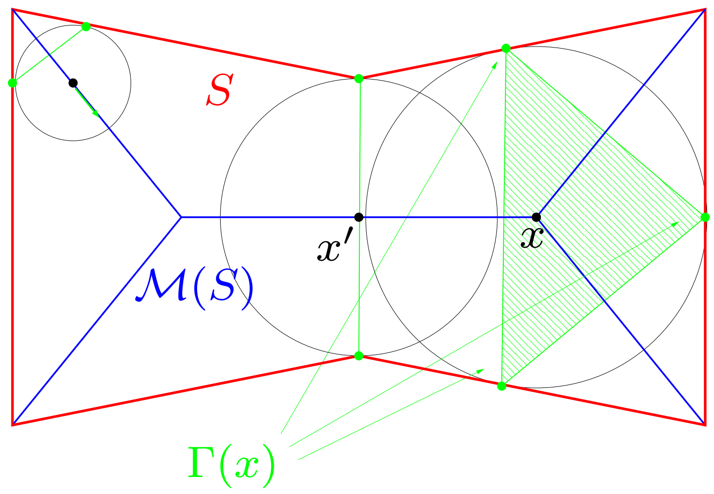
Let $S, K \subset \mathbb{R}^n$. Usually, even if $d_H(S, K)$ is very small, they have very different topologies.

What about the offsets $S^r = \{x : d(x, S) = r\}$
and $K^r = \{x : d(x, K) = r\}$?

Critical points of distance function

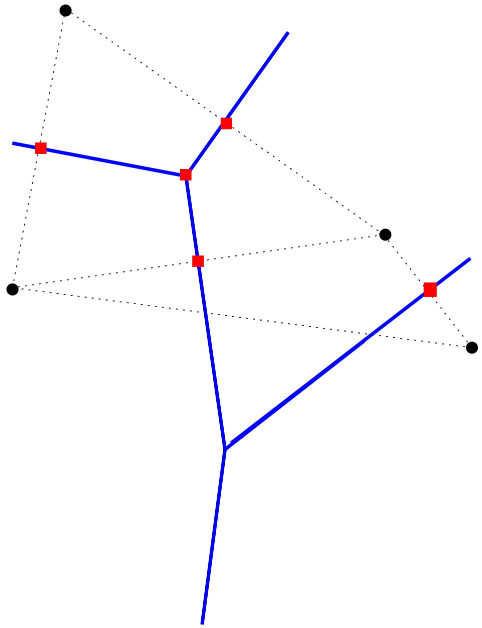
Def : $R_S(x) = d(x, S)$

$\Gamma(x) = \{y \in S : d(x,y) = R_S(x)\}$



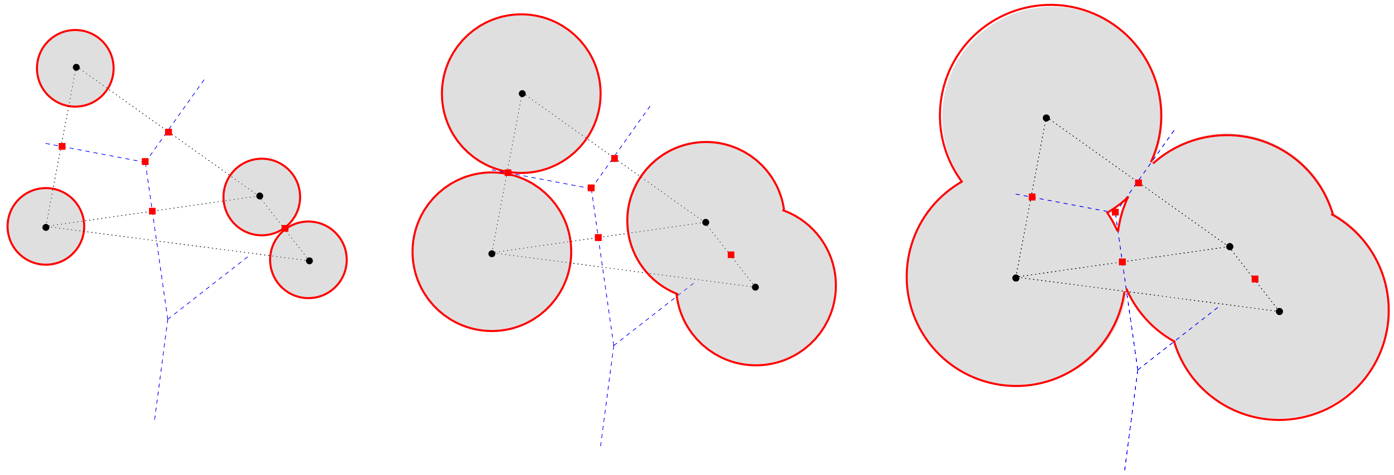
- R_S is not differentiable on $\overline{\mathcal{M}(S)}$
- "Critical point = equilibrium position": $x \in \mathbb{R}^n$ is a critical point for R_S iff it is contained in the convex hull of $\Gamma(x)$.
- $r \geq 0$ is a critical value iff there exists a critical point x s.t. $R_S(x) = r$.

Critical points of distance function



When K is a finite set of points, critical points of R_K = intersection points of Delaunay simplices of $Del(K)$ whith their dual Voronoï cells.

Properties of distance functions



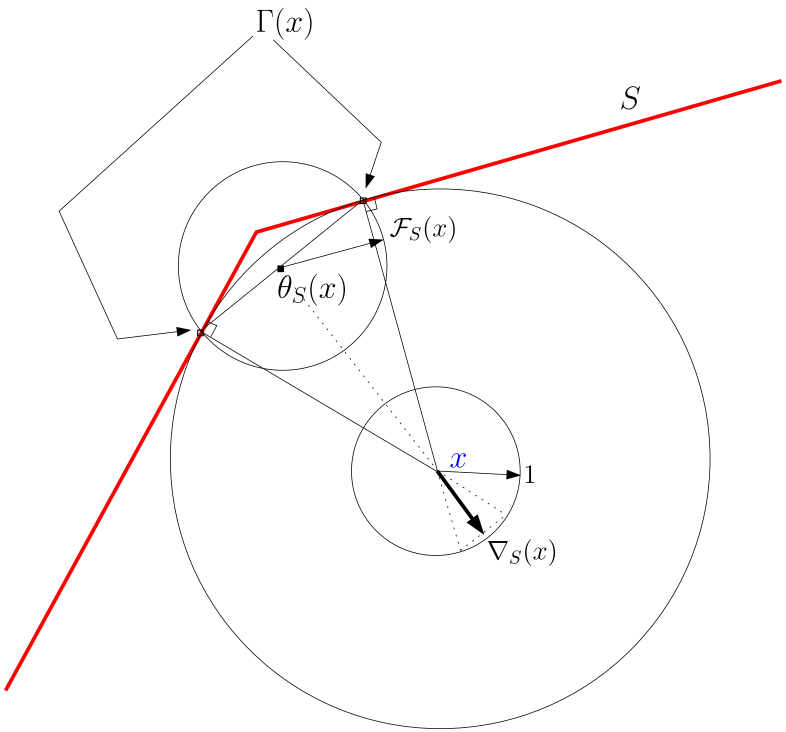
- if r is not a critical value of R_S, the level sets $R_S^{-1}(r) = \{x \in \mathbb{R}^n : R_S(x) = r\}$ are manifolds.
- the topology of the level sets of R_S change only at critical points.

The gradient of R_S

Let $x \in \mathbb{R}^n$ and let $\Theta_S(x)$ be the center of the smallest ball enclosing $\Gamma(x)$.

Gradient vector field of R_S at x:

$$\nabla_S(x) = \frac{x - \Theta_S(x)}{R_S(x)}$$

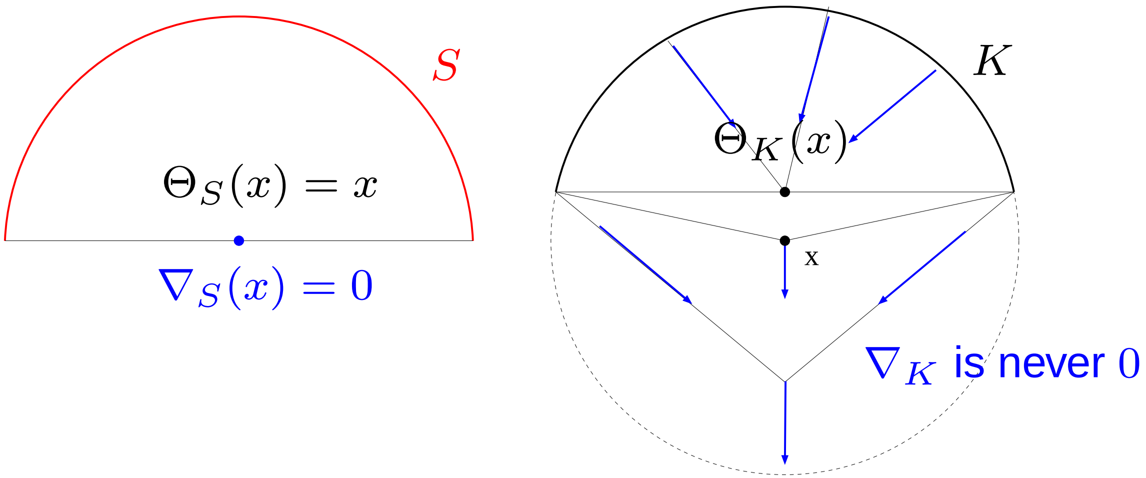


Rmk: $\nabla_S(x) = 0$ iff x is a critical point of R_S.

$$\|\nabla_S(x)\|^2 = 1 - \frac{\mathcal{F}_S(x)^2}{R_S(x)^2}$$

μ-critical points

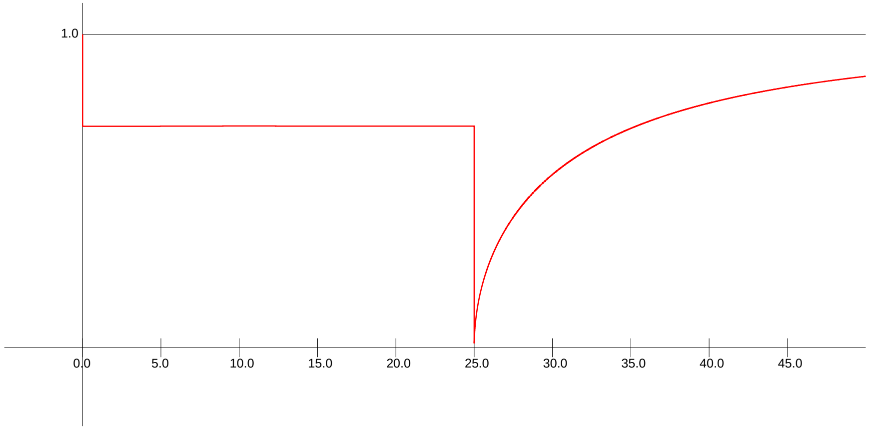
Critical points of R_S are not stable under perturbation but...

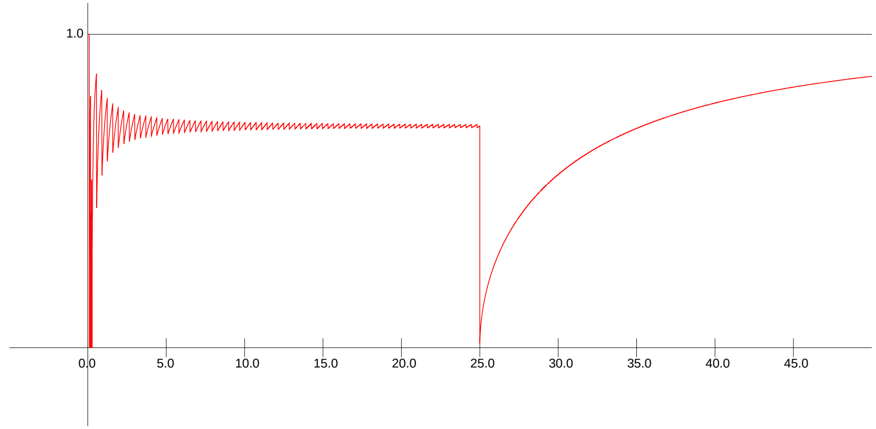


Def: x is a μ-critical point of R_S if $\|\nabla_S(x)\| \leq \mu$.

Thm (stability of μ-critical points): if $d_H(S, K) < \varepsilon$, for any μ-critical point x of S, there is a $(2\sqrt{\varepsilon / R_S(x)} + \mu)$-critical point of K at distance at most $2\sqrt{\varepsilon R_S(x)}$ from x.

The critical function





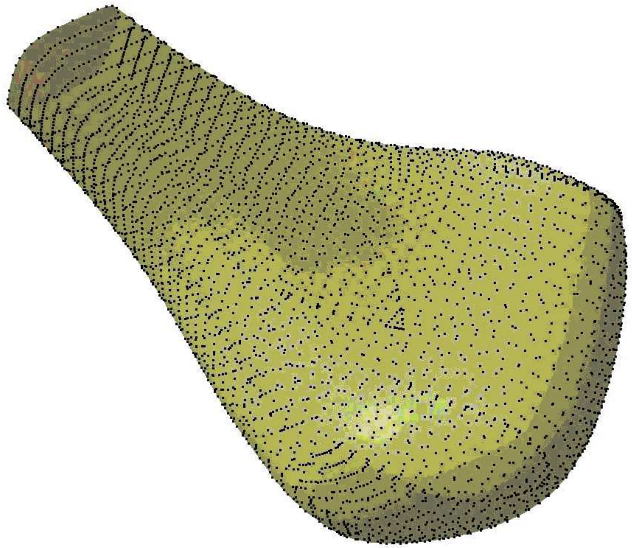
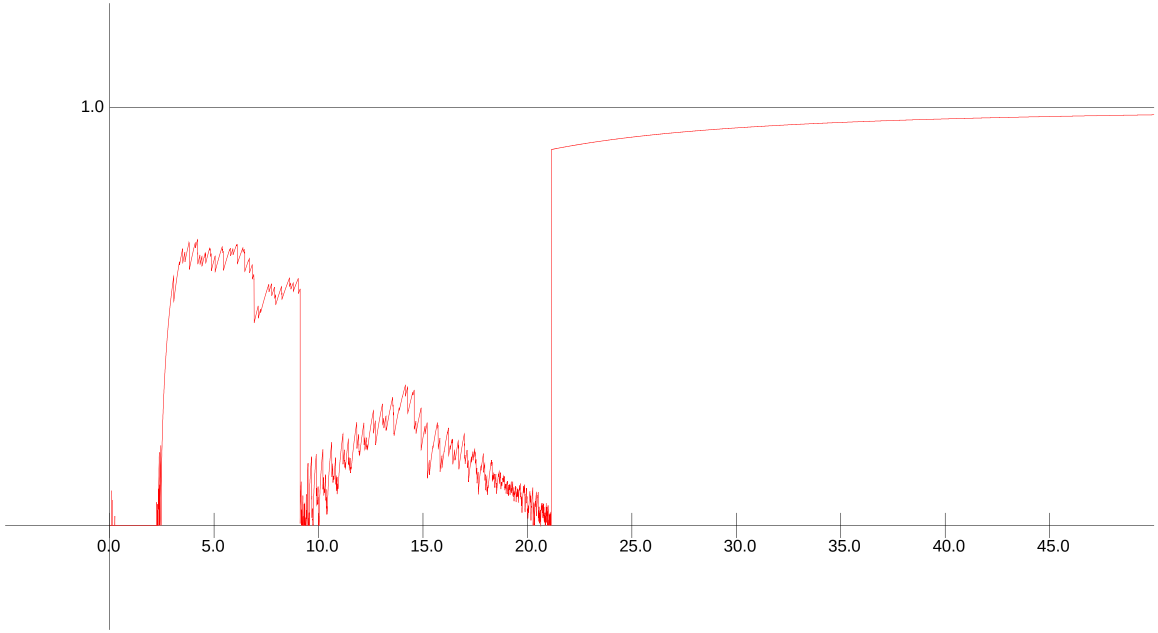
S = $3D$-square of edge length 50, K= a sampling of S

The critical function of S: $\chi_S : (0, +\infty) \to \mathbb{R}_+$ defined by

$$\chi_S(d) = \inf_{R_S^{-1}(d)} ||\nabla_S||$$

stability of μ-critical points $\Rightarrow$ stability of critical functions

An example of critical function

A sampled gearshift and its critical function.

Topological stability of offsets

Let $S \subset \mathbb{R}^3$ be a compact - non necessarily smooth - surface, let $0 < \mu \leq 1$ and let r_μ be s.t. $\chi_S > \mu$ on the interval $(0, r_\mu)$.

Thm: let $\kappa > 0$ be such that

$$\kappa < \frac{\mu^2}{5\mu^2 + 12}$$

If K is a compact set such that $d_H(S, K) < \kappa r_\mu$, then the components of $R_K^{-1}(\alpha)$ are surfaces isotopic to S provided that

$$\frac{4 d_H(K, S)}{\mu^2} \leq \alpha < r_\mu - 3 d_H(K, S)$$

Rmk: Similar results exist in any dimension and more general setting.

Informal conclusion

- Idea: "if the critical function of S remains greater than some $\mu > 0$ on a sufficiently large interval, then the offsets of any sufficiently near approximation K of S have the same topology as the ones of S."
- In another way: "given a compact K, large intervals where χ_K is sufficiently big are good candidates to represent a "stable topology" of the offsets of K at some scale level...."
- Analogy with a wave-front propagating from a compact.

For precise statements and results

- F. Chazal, A. Lieutier, The λ-medial axis, in Graphical Models, Volume 67, Issue 4 , July 2005, Pages 304-331.
- F. Chazal, D. Cohen-Steiner, A. Lieutier, A Sampling Theory for Compacts in Euclidean Spaces, to appear in SoCG'06.