Constructing Affine and Curved Voronoi Diagrams

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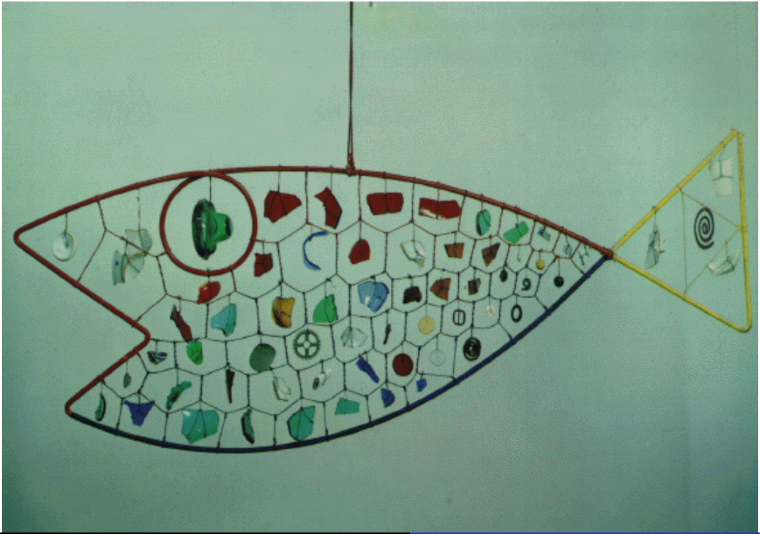
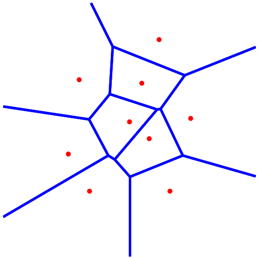
An artistic view of a Voronoi diagram



A gallery of Voronoi diagrams



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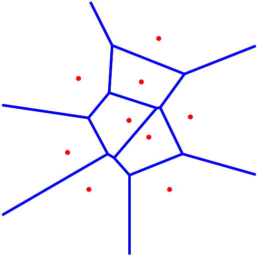

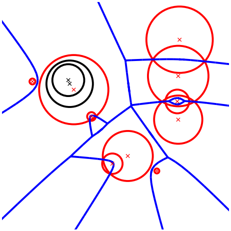

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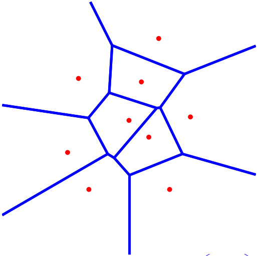

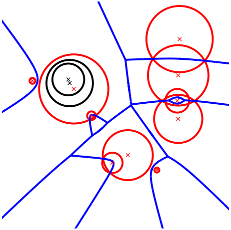

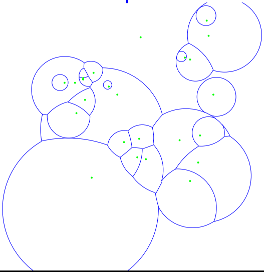

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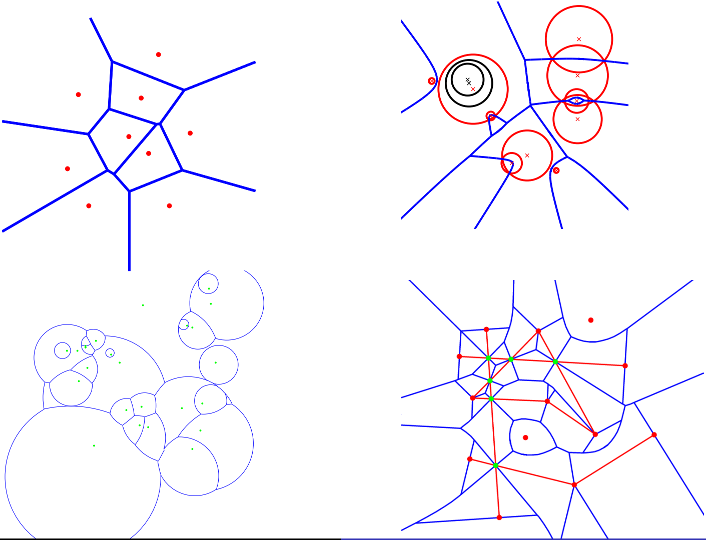
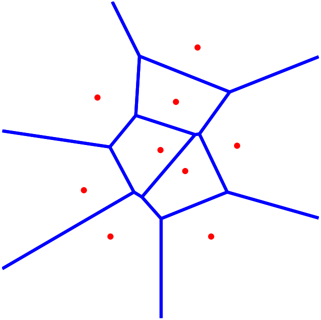


Outline

Euclidean Voronoi diagrams of points



Voronoi region : $V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram = Cell Complex consisting of the $V(p_i)$ and their faces

Voronoi diagrams and polytopes

In $\mathbb{R}^{d+1}$ (space of spheres)
$h_p : x_{d+1} = 2p \cdot x - p^2$
plane tangent to $\mathcal{P} : x_{d+1} = x^2$ at (p, p^2)

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$$(x - p_i)^2 < (x - p_j)^2$$
$$\iff 2\,p_i \cdot x - p_i^2 > 2\,p_j \cdot x - p_j^2$$
$$\iff h_{p_i} \text{ is above } h_{p_j} \text{ at } x$$

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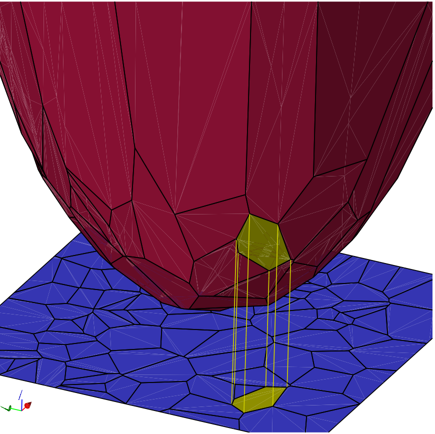

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$V(p_i)$ is the projection of the portion of h_{p_i} that is above all h_{p_j}

The faces of Vor(E) are the projection of the faces of $\mathcal{V}(E) = \bigcap_i h_{p_i}^+$

Duality

hyperplane $h : x_{d+1} = a \cdot x' - b$ of $\mathbb{R}^{d+1}$ $\longleftrightarrow$ point $h^* = (a, b) \in \mathbb{R}^d \times \mathbb{R}$

point $p = (p', p_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$ $\longleftrightarrow$ hyperplane $p^* \subset \mathbb{R}^{d+1}$
$= \{(a, b) \in \mathbb{R}^d \times \mathbb{R} : b = p' \cdot a - p_{d+1}\}$

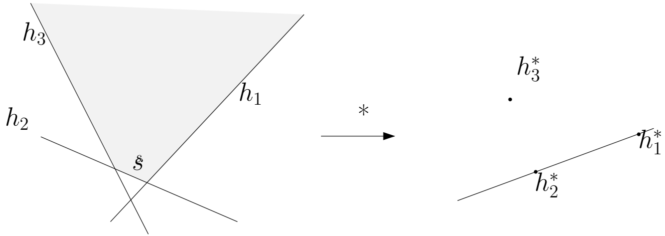
The mapping *

- is an involution and thus is bijective : $h^{**} = h$ and $p^{**} = p$
- preserves incidences :

$$p = (p', p_{d+1}) \in h \iff p_{d+1} = a \cdot p' - b \iff b = p' \cdot a - p_{d+1} \iff h^* \in p^*$$

- reverses inclusions : $p \in h^+ \iff h^* \in p^{*+}$
  where $h^+ = \{(x', x_{d+1}) \in p^{*+} : x_{d+1} > a \cdot x' - b\}$

Let $h_1, \ldots, h_n$ be n hyperplanes and let $P = \cap h_i^+$



A vertex s of P is the intersection point of d hyperplanes $h_1, \ldots, h_d$ lying above all other hyperplanes

$\Longrightarrow s^* = \mathrm{aff}(h_1^*, \ldots, h_d^*)$
s^* is a hyperplane supporting $\mathrm{conv}^-(\{h_i^*\})$

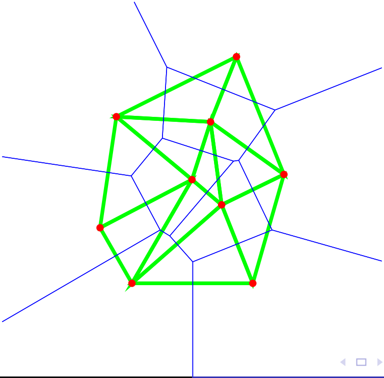
Hence, computing P reduces to computing a lower convex hull

Delaunay triangulation

$$\mathcal{V}(E) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ \quad \longleftrightarrow \quad \mathcal{D}(E) = \text{conv}^-(\{\phi(p_1), \ldots, \phi(p_n)\})$$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

Voronoi Diagram of E $\longleftrightarrow$ Delaunay Triangulation of E



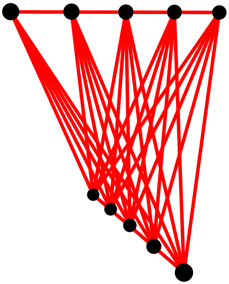
Some properties of the Delaunay triangulation

1. A triangulation T of a finite set of points E such that any simplex of T has a circumscribing sphere that does not enclose any point of E is a Delaunay triangulation of E.

2. Among all possible triangulations of E, $\mathrm{Del}(E)$
   2.1 maximizes the smallest angle (in the plane) [Lawson]
   2.2 minimizes the radius of the maximal smallest ball enclosing a simplex) [Rajan]

Combinatorial complexity

- in $\mathbb{R}^2$, Euler's formula implies $t \leq 2n - 5, e \leq 3n - 6$
- in $\mathbb{R}^d$, the Upper Bound Theorem [Mc Mullen 1970] implies
  # faces = $\Theta\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$



A variety of results

Probabilistic results

- a ball : $\Theta(n)$ [Dwyer 1993]
- a convex polytope : $\Theta(n)$ [Golin & Na 2000]
- a polytope : $O(n \log^4 n)$ [Golin & Na 2002]

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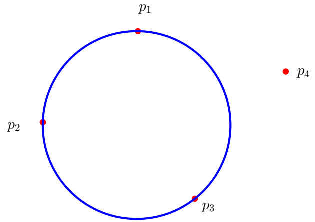
Deterministic results

- points wrt spread: $O(\text{spread}^3)$ [Erickson 2002]
- points on a surface with spread $\sqrt{n}$: $O(n\sqrt{n})$ [Erickson 2002]
- points on a polyhedral surface: $O(n)$ [Attali & B. 2002]
- points on a smooth surface with generic properties: $O(n \log n)$ [Attali, B. & Lieutier 2003]

Algorithms

Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$
Predicate :



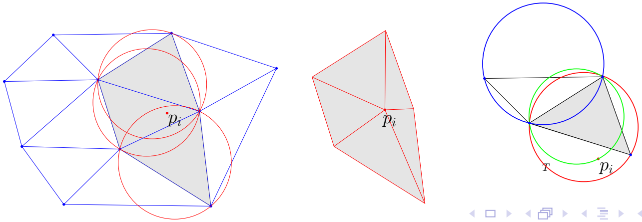
$$\texttt{insphere}(p_0, \ldots, p_{d+1}) = \operatorname{sign} \begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}$$

CGAL code : $d = 3$: 10^6 points/mn (1.7 GHz)

Online algorithm

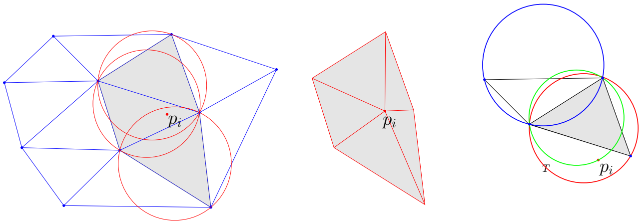
Insertion of a new point p_i:

1. Location : find all the triangles that conflict with p_i
   i.e. whose circumscribing ball contains p_i
2. Update : construct the new triangles

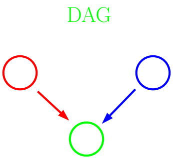


Location : the Delaunay DAG

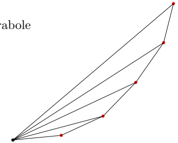
stores the history of the construction



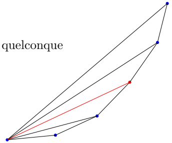
Conflicts are discovered by traversing the Delaunay DAG



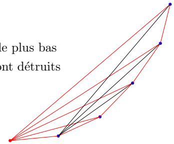
n points sur une parabole



insertion d'un point quelconque
un triangle détruit



insertion du point le plus bas
tous les triangles sont détruits



Bad order $\longrightarrow$ cost= $\Theta(n^2)$

Randomized analysis

- Hypothesis on the insertion order : all the permutations of the points can occur with the same probability
- No hypothesis is made about the spatial distribution of the points
- The algorithm always computes the exact Delaunay triangulation of the given points
- The computing time depends on the random choices of the algorithm and will be analyzed by averaging over all the permutations of the input data

Cost of maintaining the Delaunay DAG

$$\begin{aligned}\text{E(\# triangles created at step } k) &= \sum_{T \in \mathcal{T}_k} \text{proba } (T \text{ is created at step } k) \\ &= \frac{3}{k} |\mathcal{T}_k| \\ &= O(1)\end{aligned}$$

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$$\mathrm{E}(\text{ \# created triangles}) = O(n)$$

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E(# created triangles) $= O(n)$

since each node has one parent
E(# arcs in the Delaunay DAG) $= O(n)$

Cost of the location queries in the DAG

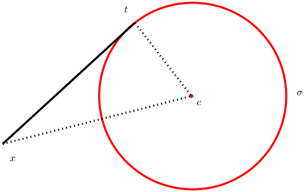
E(# triangles created at step k and in conflict with p_n

$$= \sum_{T \in \mathcal{T}_k} \text{proba } (T \text{ is created at step } k) \times \text{proba } (p_n \in \sigma_T)$$
$$= \sum_{T \in \mathcal{T}_k} \text{proba } (T \text{ is created at step } k) \times \text{proba } (p_{k+1} \in \sigma_T)$$
$$= \frac{3}{k} \text{ E(\# triangles removed at step } k+1)$$
$$= \frac{3}{k} \left(\frac{3}{k+1}|\mathcal{T}_{k+1}| - 2\right)$$
$$= O\left(\frac{1}{k}\right)$$

Cost of locating the n-th site $= O(\log n)$

Power diagrams of spheres

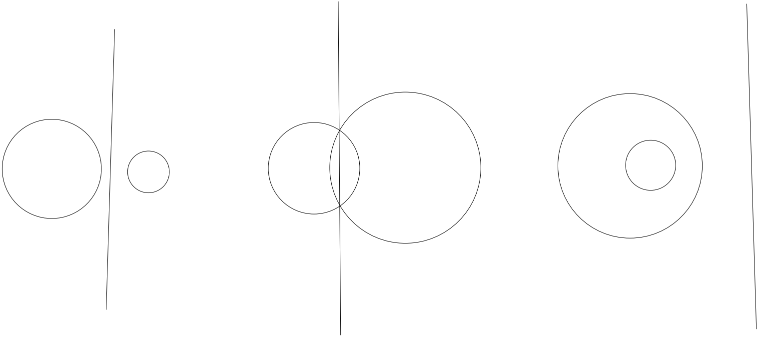
Power of a point to a sphere



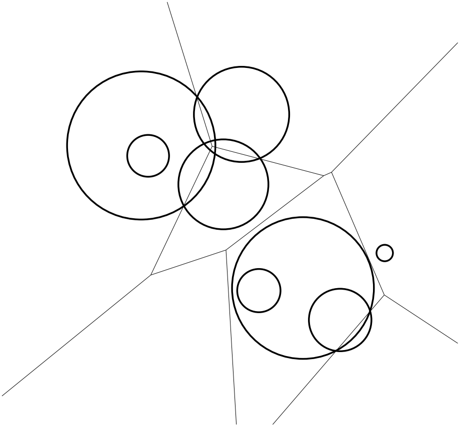
$$\sigma(x) = (x - t)^2 = (x - c)^2 - r^2$$
$$\sigma(x) < 0 \iff x \in \mathrm{int}(\sigma)$$

Bisector of two sites = hyperplane

$$\sigma_i(x) = \sigma_j(x) \iff \cancel{x}^2 - 2c_i \cdot x + s_i = \cancel{x}^2 - 2c_j \cdot x + s_j$$



Power diagram



Sites : n spheres $\sigma_1, \ldots, \sigma_n$

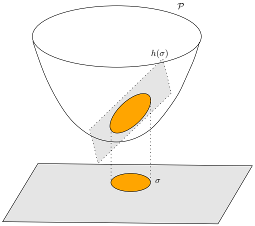
Distance of a point x to σ_i

$$\sigma_i(x) = (x - c_i)^2 - r_i^2$$

$$\mathrm{Pow}(\sigma_i) = \{x : \sigma_i(x) \leq \sigma_j(x), \ \forall j\}$$

$\mathrm{Pow}(\sigma_i)$ may be empty

Space of spheres



$\sigma \rightarrow$ the polar hyperplane h_σ of $\mathbb{R}^{d+1}$: $x_{d+1} = 2c \cdot x - s$

1. If $\sigma_i = p_i$, h_{σ_i} is the hyperplane h_{p_i} tangent to the paraboloid $\mathcal{P}$
2. The vertical projection of $h_{\sigma_i} \cap \mathcal{P}$ onto $x_{d+1} = 0$ is σ_i
3. $$\sigma_i(x) < \sigma_j(x) \Longleftrightarrow 2c_i \cdot x - s_i > 2c_j \cdot x - s_j$$ $$\Longleftrightarrow \text{at point } x,\ h_{\sigma_i} \text{ is above } h_{\sigma_j}$$

Space of spheres

the faces of the power diagram are the vertical projections of the faces of $\mathcal{P}(\mathcal{S}) = \bigcap_i h^+_{\sigma_i}$

The vertical projection of the dual complex $\mathcal{R}(\mathcal{S})$ of $\mathcal{P}(\mathcal{S})$ is called the regular triangulation of $\mathcal{S}$

$$\mathcal{P}(\mathcal{S}) = h^+_{\sigma_1} \cap \ldots \cap h^+_{\sigma_n} \quad \longleftrightarrow \quad \mathcal{R}(\mathcal{S}) = \text{conv}^-(\{\phi(\sigma_1), \ldots, \phi(\sigma_n)\})$$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

power diagram of $\mathcal{S}$ $\longleftrightarrow$ Regular triangulation of $\mathcal{S}$

Complexity and algorithm

nb of faces $= \Theta\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$ (Upper Bound Th.)

can be computed in time $\Theta\left(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor}\right)$

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Main predicate

$$\texttt{power\_test}(\sigma_0, \ldots, \sigma_{d+1}) = \operatorname{sign} \begin{vmatrix} 1 & \ldots & 1 \\ c_0 & \ldots & c_{d+1} \\ c_0^2 - r_0^2 & \ldots & c_{d+1}^2 - r_{d+1}^2 \end{vmatrix}$$

Affine Voronoi diagrams

Definition

Diagrams defined for objects and a distance function s.t. bisectors are hyperplanes

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Theorem [Aurenhammer]

Any affine Voronoi diagram of $\mathbb{R}^d$ *is the power diagram of a set of spheres of* $\mathbb{R}^d$.

Examples of affine diagrams

1. *The vertical projection of the faces of any polyhedron that is the intersection of upper half-spaces of* $\mathbb{R}^{d+1}$

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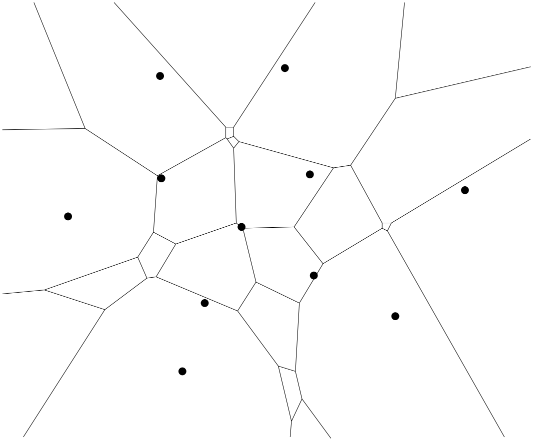
4. k-order Voronoi diagrams

Order k Voronoi Diagrams



Order 2 Voronoi Diagram

A k-order Voronoi diagram is a power diagram

Let $E_1, E_2, \ldots$ denote the subsets of k points of E

$$\sigma_i(x) = \frac{1}{k} \sum_{j \in E_i} (x - p_j)^2 = x^2 - \frac{2}{k} \sum_{j \in E_i} p_j \cdot x + \frac{1}{k} \sum_{j \in E_i} p_j^2$$

The k nearest neighbors of x are the points of E_i iff

$$\forall j, \quad \sigma_i(x) \leq \sigma_j(x)$$

σ_i is the sphere centered at $\frac{1}{k} \sum_{j=1}^{k} p_{i_j}$

$$\sigma_k(0) = \frac{1}{k} \sum_{j=1}^{k} p_{i_j}^2$$

Möbius Diagrams

- Weighted points : $W_i = (p_i, \lambda_i, \mu_i)$, $p_i \in \mathbb{R}^d$, $\lambda_i \in \mathbb{R} \setminus \{0\}$, $\mu_i \in \mathbb{R}$
- Distance function :

$$\delta_M(x, W_i) = \lambda_i \|x - p_i\|^2 - \mu_i$$

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Generalization of

- Voronoï diagrams ($\lambda_i = \lambda > 0$ et $\mu_i = 0$)
- Power diagrams ($\lambda_i = \lambda > 0$)
- multiplicatively weighted Voronoi diagrams ($\mu_i = 0$)

Bisectors are *hyperspheres*, hyperplanes or $\emptyset$

$$\lambda_i(x - p_i)^2 - \mu_i = \lambda_j(x - p_j)^2 - \mu_j$$
$$\iff \quad (\lambda_i - \lambda_j)x^2 - 2(\lambda_i p_i - \lambda_j p_j) \cdot x + \lambda_i p_i^2 - \mu_i - \lambda_j p_j^2 + \mu_j = 0$$

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$$\lambda_i(x - p_i)^2 - \mu_i = \lambda_j(x - p_j)^2 - \mu_j$$
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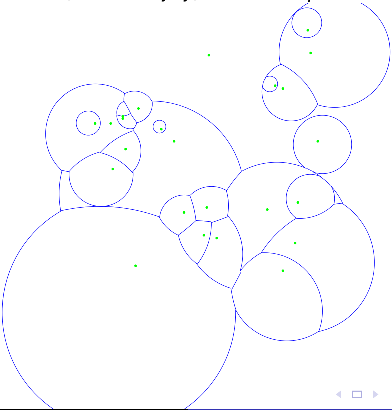

Linearization Lemma

We can associate to each weighted point W_i a hypersphere Σ_i of $\mathbb{R}^{d+1}$ so that

the faces of the Möbius diagram of the W_i are obtained by projecting vertically the faces of the restriction of the Power Diagram of the Σ_i to the paraboloid $\mathcal{P} : x_{d+1} = x^2$

Proof

$$\lambda_i(x-p_i)^2-\mu_i \leq \lambda_j(x-p_j)^2-\mu_j$$

$$\iff (x-\lambda_i p_i)^2+(x^2+\frac{\lambda_i}{2})^2-\lambda_i^2 p_i^2-\frac{\lambda_i^2}{4}+\lambda_i p_i^2-\mu_i$$

$$\leq (x-\lambda_j p_j)^2+(x^2+\frac{\lambda_j}{2})^2-\lambda_j^2 p_j^2-\frac{\lambda_j^2}{4}+\lambda_j p_j^2-\mu_j$$

$$\iff (X-C_i)^2-\rho_i^2 \leq (X-C_j)^2-\rho_j^2$$

where $X=(x,x^2)\in\mathbb{R}^{d+1}$,

$C_i=(\lambda_i p_i,-\frac{\lambda_i}{2})\in\mathbb{R}^{d+1}$ and $\rho_i^2=\lambda_i^2 p_i^2+\frac{\lambda_i^2}{4}-\lambda_i p_i^2+\mu_i$

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1. *Inversion and Möbius transforms map a spherical diagram to another spherical diagram*
2. *The intersection of a spherical diagram with an affine subspace is a a spherical diagram*
3. *Using stereographic projection, one can define spherical diagrams on* S^d
4. *The class of Möbius diagrams is identical to the class of spherical diagrams, i.e.diagrams whose bisectors are hyperspheres*

Constructing Möbius diagrams

The complexity of the Möbius diagram of n doubly weighted points in $\mathbb{R}^d$ is $\Theta(n^{\lfloor \frac{d}{2} \rfloor + 1})$

It can be constructed in time $\Theta(n \log n + n^{\lfloor \frac{d}{2} \rfloor + 1})$

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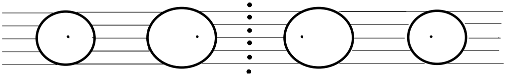

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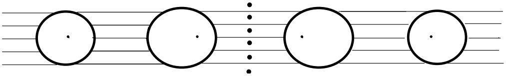


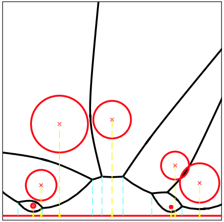
Predicates :

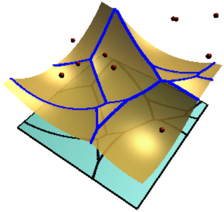
power_test

decide whether a face of Power$(\{\Sigma_i\}_{i=1}^n)$ intersects $\mathcal{P}$

An Euclidean model

σ_0 a hyperplane of $\mathbb{R}^d$ $(x_d = 0)$
a finite set of hyperspheres $\{\sigma_i = (p_i, \omega_i)\}_{i=1}^n$
$V(\sigma_0) = \{x \in \mathbb{R}^d : d(x, \sigma_0) \leq d(x, \sigma_i), \forall i\}$





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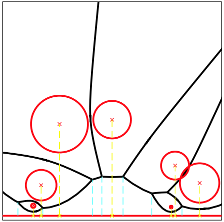
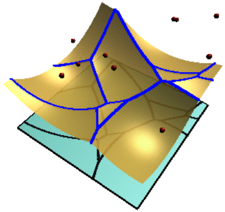
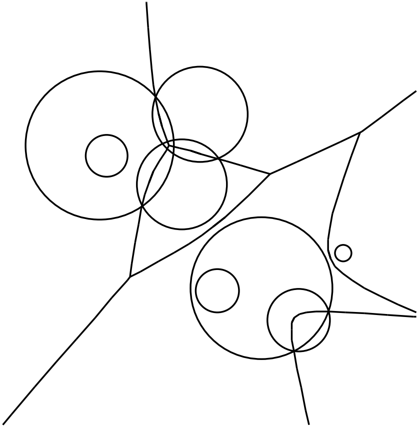




Projection Lemma

The vertical projection of $\partial V(\sigma_0)$ on σ_0 is a Möbius diagram

Apollonius diagrams of spheres



$$\sigma_i = (p_i, r_i)$$
$$\delta(x, \sigma_i) = \|x - p_i\| - r_i$$
$$\text{Apo}(\sigma_i) = \{x, \delta(x, \sigma_i) \leq \delta(x, \sigma_j)\}$$

The Projection Lemma extends to any set of spheres

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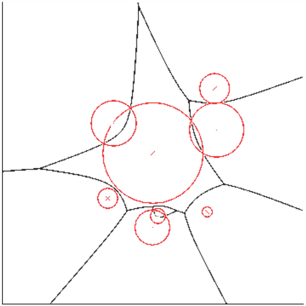

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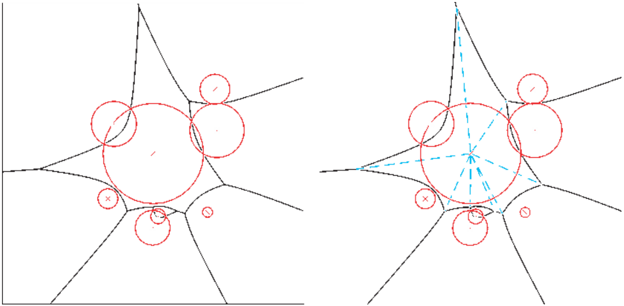

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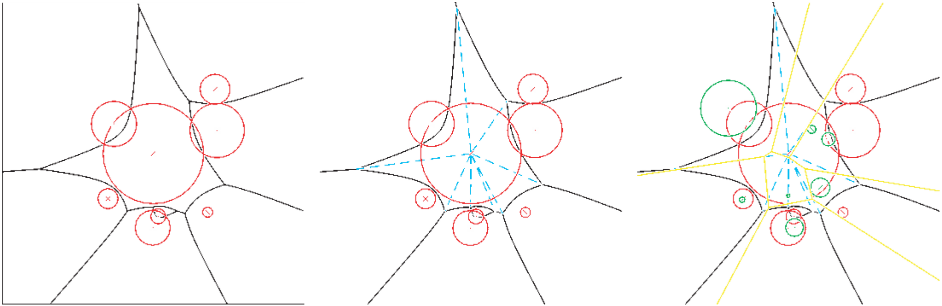

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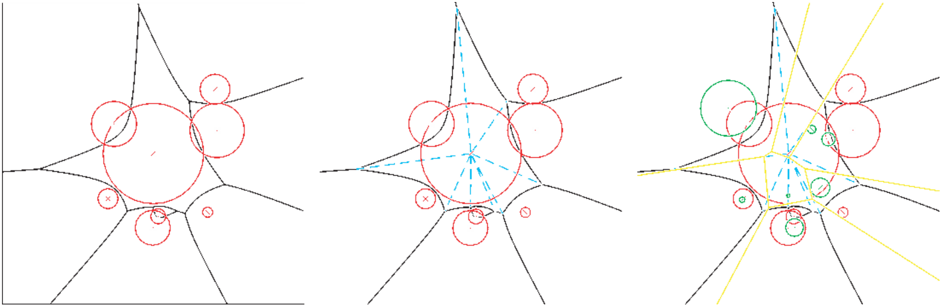

Theorem: *The combinatorial complexity of a single cell in the Apollonius diagram of n spheres of* $\mathbb{R}^d$ *is* $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor})$

CGAL implementations

CGAL planar Apollonius diagrams [M. Karavelas]

100k circles : 40s (Pentium III, 1 GHz)

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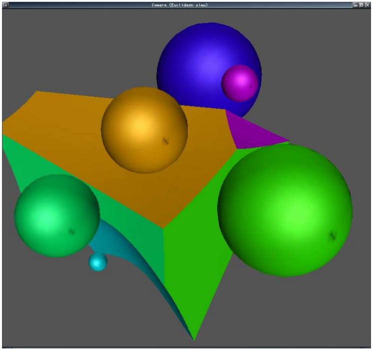
A prototype implementation [C. Delage]



Anisotropic Voronoi diagrams

Labelle & Shewchuk

Weighted point : (p_i, M_i, r_i) where $p_i \in \mathbb{R}^d$, M_i is a $d \times d$ symmetric positive definite matrix and $r_i \in \mathbb{R}$

Distance to a weighted point : $d_i(x) = (x - p_i)^t \, M_i \, (x - p_i) - r_i$

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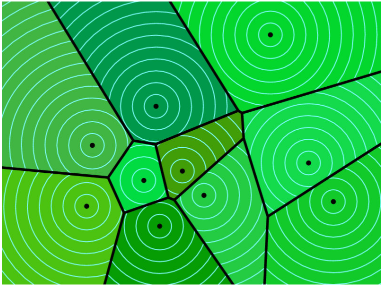
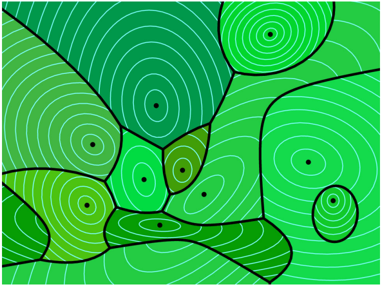


Standard diagram



Anisotropic diagram

Linearization Lemma

In $\mathbb{R}^{\frac{d(d+3)}{2}}$, one can define a set Σ of n hyperspheres so that the anisotropic Voronoi diagram of the n given weighted sites is the projection of the restriction of $\text{Pow}(\Sigma)$ to a d-manifold

Linearization Lemma

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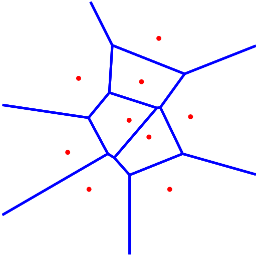
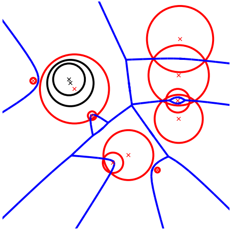
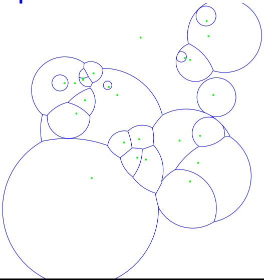
Universality Lemma

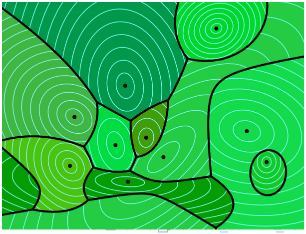
Any quadratic Voronoi diagram (i.e. with quadratic bisectors) is an anisotropic diagram

Conclusion

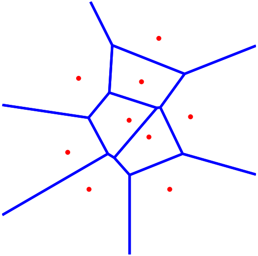


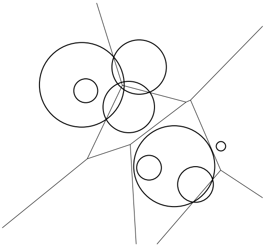


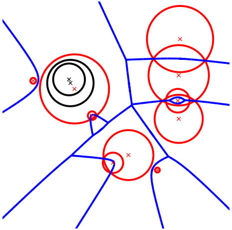


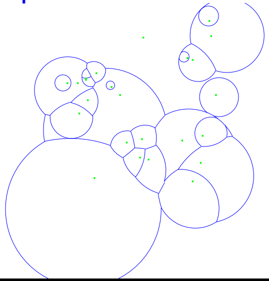


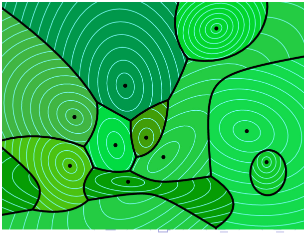
Conclusion











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- Provides a framework for many Voronoi diagrams
- Leads to rather simple data structures and algorithms
- Robust and efficient implementations exist for simple cases

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Further questions

- Does not directly provide good combinatorial bounds
- How to compute the restriction of an affine diagram to a manifold efficiently ?
- Approximation algorithms ?

Acknowledgments

Menelaos Karavelas

Christophe Delage

Camille Wormser

Mariette Yvinec