General Relativity:

Einstein Field Equations

Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Metric tensor: $g_{\mu\nu}$

Energy-Momentum tensor: $T_{\mu\nu}$

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^{\mu} U^{\nu} - p g^{\mu\nu}$$

Einstein Field Equation

Einstein Tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$G_{\mu\nu;\nu} = T_{\mu\nu;\nu} = 0$$

⇩ **Einstein Tensor only rank 2 tensor for which this holds:**

$$G_{\mu\nu} \propto T_{\mu\nu}$$

Einstein Field Equation

also: $$g_{\mu\nu;\nu} = 0$$

↓ Freedom to add a multiple of metric tensor to Einstein tensor:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Λ : Cosmological Constant

Einstein Field Equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

↓ Dark Energy

$$G^{\mu\nu} = -\frac{8\pi G}{c^4}\left(T^{\mu\nu} + T^{\mu\nu}{}_{vac}\right)$$

$$T^{\mu\nu}{}_{vac} \equiv \frac{\Lambda c^4}{8\pi G} g^{\mu\nu}$$

Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Einstein Field Equation

curvature side

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

energy-momentum side

Equation of State

$$T^{\mu\nu}{}_{vac} \equiv \frac{\Lambda c^4}{8\pi G} g^{\mu\nu}$$

restframe ⇒

$$T^{\mu\nu}{}_{vac} \equiv \frac{\Lambda c^4}{8\pi G} \eta^{\mu\nu}$$

$$\eta^{00} = 1, \quad \eta^{ii} = -1$$

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^{\mu} U^{\nu} - p g^{\mu\nu}$$

restframe:

$$\left.\begin{array}{l} T^{00}{}_{vac} = \rho_{vac} c^2 \\ T^{ii}{}_{vac} = p \end{array}\right\} \Rightarrow \begin{array}{l} \rho_{vac} c^2 = \dfrac{\Lambda c^4}{8\pi G} \\ p = -\dfrac{\Lambda c^4}{8\pi G} \end{array}$$

Equation of State

$$\rho_{vac} c^2 = \frac{\Lambda c^4}{8\pi G}$$

$$p = -\frac{\Lambda c^4}{8\pi G}$$

$$p_{vac} = -\rho_{vac} c^2$$

Dynamics

Relativistic Poisson Equation:

$$\nabla^2\phi = 4\pi G\left(\rho + \frac{3p}{c^2}\right)$$

$$\rho_{vac} + \frac{3p_{vac}}{c^2} = -2\rho_{vac} < 0; \qquad \rho_{vac} = \frac{\Lambda}{8\pi G}$$

⇓

$$\nabla^2\phi < 0$$ Repulsion !!!

Curved Space:

Friedmann-Robertson Metric

Cosmological Principle: the Universe Simple & Smooth

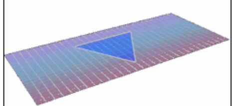
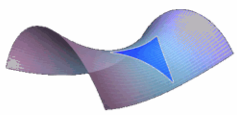
"God is an infinite sphere whose centre is everywhere and its circumference nowhere"
Empedocles, 5th cent BC

Cosmological Principle:

Describes the symmetries in global appearance of the Universe:

- Homogeneous → The Universe is the same everywhere:
  - physical quantities (density, T,p,…)
- Isotropic → The Universe looks the same in every direction
- Universality → Physical Laws same everywhere
- Uniformly Expanding → The Universe "grows" with same rate in
  - every direction
  - at every location

"all places in the Universe are alike"
Einstein, 1931

Property	Closed	Euclidean	Open
Spatial Curvature	Positive	Zero	Negative
Circle Circumference	$< 2\pi R$	$2\pi R$	$> 2\pi R$
Sphere Area	$< 4\pi R^2$	$4\pi R^2$	$> 4\pi R^2$
Sphere Volume	$< {}^4/_3\ \pi R^3$	${}^4/_3\ \pi R^3$	$> {}^4/_3\ \pi R^3$
Triangle Angle Sum	$> 180°$	$180°$	$< 180°$
Total Volume	Finite ($2\pi^2 R^3$)	Infinite	Infinite
Surface Analog	Sphere 	Plane 	Saddle 

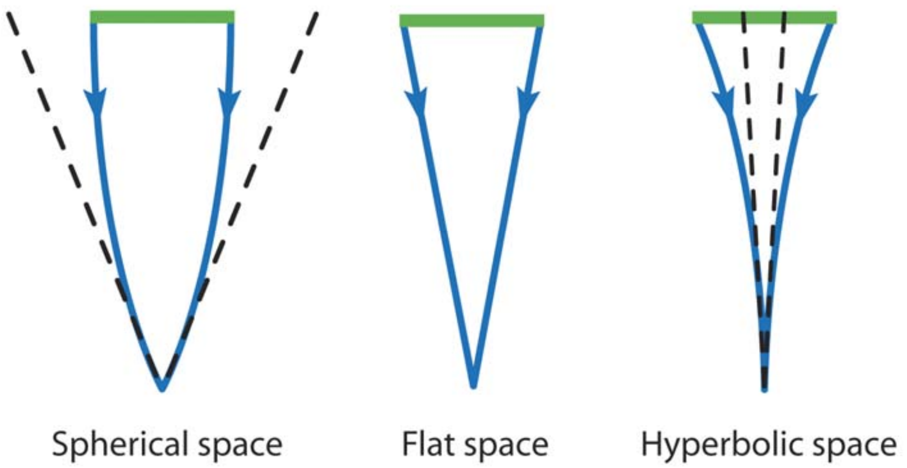
Robertson-Walker Metric

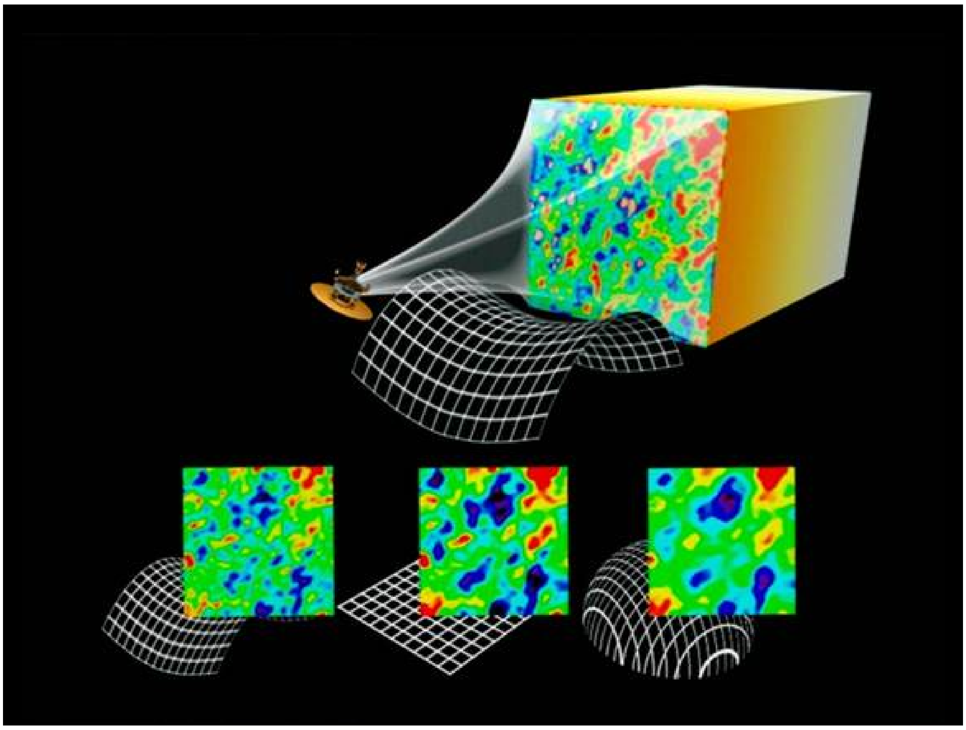
Distances in a uniformly curved spacetime is specified in terms of the Robertson-Walker metric. The spacetime distance of a point at coordinate (r,θ,ϕ) is:

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2 \left(\frac{r}{R_c} \right) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

where the function $S_k(r/R_c)$ specifies the effect of curvature on the distances between points in spacetime

$$S_k\left(\frac{r}{R_c}\right) = \begin{cases} \sin\left(\frac{r}{R_c}\right) & k = +1 \\ \frac{r}{R_c} & k = 0 \\ \sinh\left(\frac{r}{R_c}\right) & k = -1 \end{cases}$$





Friedmann-Robertson-Walker Universe

Einstein Field Equation

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu,RW} \quad \Rightarrow \quad \Gamma^{\mu}{}_{\lambda\nu} \quad \Rightarrow \quad R_{\mu\nu} \quad , \quad R$$

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^{\mu} U^{\nu} - p g^{\mu\nu}$$

$$= diag\left(\rho c^2, p, p, p \right)$$

Einstein Field Equation

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$G^0{}_0 \quad \rightarrow \quad G^0{}_0 = 3\left(\dot{R}^2 + kc^2 \right) / R^2 = \frac{8\pi G}{c^2} \rho c^2$$

$$G^1{}_1 \quad \rightarrow \quad G^1{}_1 = \left(2R\ddot{R} + \dot{R}^2 + kc^2 \right) / R^2 = -\frac{8\pi G}{c^2} p$$

Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda}{3}R$$

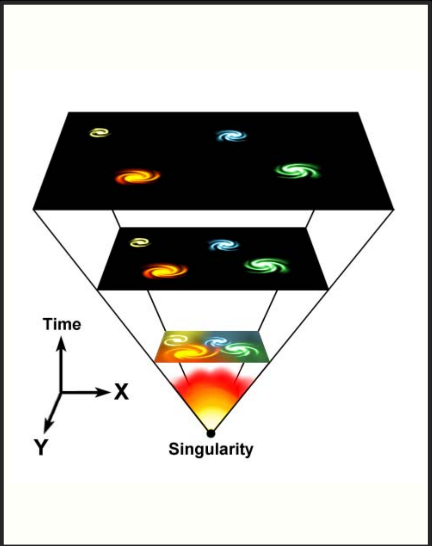
$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{\Lambda}{3}R^2$$

Cosmic Expansion Factor

$$a(t) = \frac{R(t)}{R_0}$$

- Cosmic Expansion is a uniform expansion of space

$$\vec{r}(t) = a(t)\vec{x}$$



Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

density

pressure

cosmological constant

curvature term

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3}a^2$$

Friedmann-Robertson-Walker-Lemaitre Universe

Because of General Relativity, the evolution of the Universe is determined by four factors:

- density $\rho(t)$
- pressure $p(t)$
- curvature kc^2 / R_0^2 — $k = 0, +1, -1$; R_0: present curvature radius
- cosmological constant Λ

• Density & Pressure:
- in relativity, energy & momentum need to be seen as one physical quantity (four-vector)
- pressure = momentum flux

• Curvature:
- gravity is a manifestation of geometry spacetime

• Cosmological Constant:
- free parameter in General Relativity
- Einstein's "biggest blunder"
- mysteriously, since 1998 we know it dominates the Universe

Cosmological Constant & FRW equations

Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3}a^2$$

Dark Energy & Energy Density

$$\tilde{\rho} = \rho + \rho_\Lambda$$

$$\tilde{p} = p + p_\Lambda$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

$$p_\Lambda = -\frac{\Lambda c^2}{8\pi G}$$

Friedmann-Robertson-Walker-Lemaitre Universe

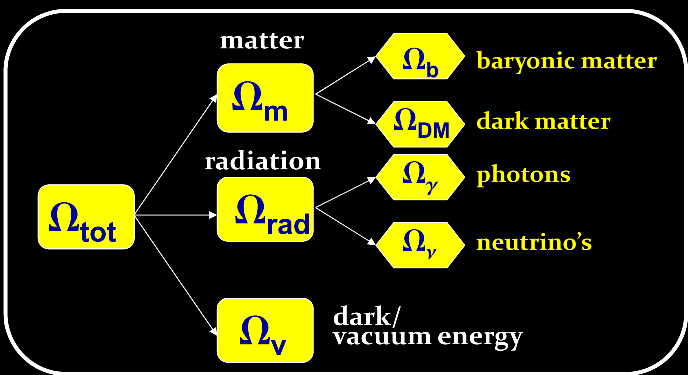
$$\ddot{a} = -\frac{4\pi G}{3}\left(\tilde{\rho} + \frac{3\tilde{p}}{c^2}\right)a$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\tilde{\rho} - \frac{kc^2 / R_0^{\ 2}}{a^2}$$

Cosmic Constituents

Cosmic Constituents

The total energy content of Universe made up by various constituents, principal ones:

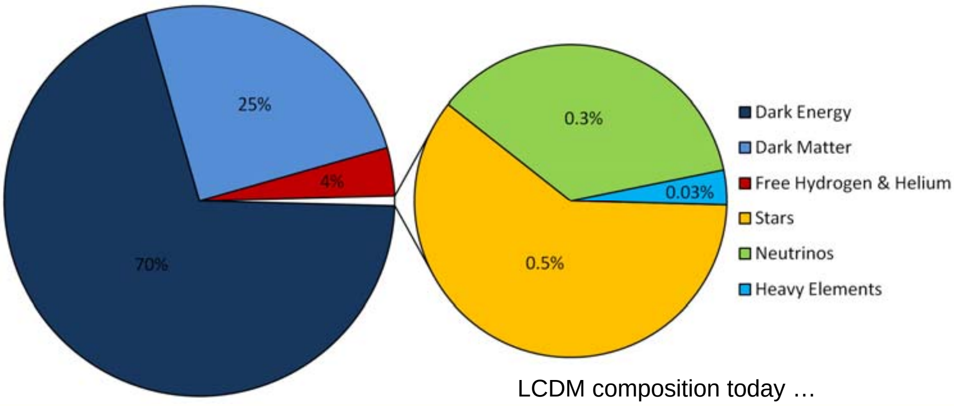


In addition, contributions by
- gravitational waves
- magnetic fields,
- cosmic rays ...

Poor constraints on their contribution: henceforth we will not take them into account !

LCDM Cosmology

- Concordance cosmology
  - model that fits the majority of cosmological observations
  - universe dominated by Dark Matter and Dark Energy



LCDM composition today …

Cosmic Energy Inventory

1	dark sector			0.954 ± 0.003
1.1	dark energy		0.72 ± 0.03	
1.2	dark matter		0.23 ± 0.03	
1.3	primeval gravitational waves		$\lesssim 10^{-10}$	
2	primeval thermal remnants			0.0010 ± 0.0005
2.1	electromagnetic radiation		$10^{-4.3\pm0.0}$	
2.2	neutrinos		$10^{-2.9\pm0.1}$	
2.3	prestellar nuclear binding energy		$-10^{-4.1\pm0.0}$	
3	baryon rest mass			0.045 ± 0.003
3.1	warm intergalactic plasma		0.040 ± 0.003	
3.1a	virialized regions of galaxies	0.024 ± 0.005		
3.1b	intergalactic	0.016 ± 0.005		
3.2	intracluster plasma		0.0018 ± 0.0007	
3.3	main sequence stars	spheroids and bulges	0.0015 ± 0.0004	
3.4		disks and irregulars	0.00055 ± 0.00014	
3.5	white dwarfs		0.00036 ± 0.00008	
3.6	neutron stars		0.00005 ± 0.00002	
3.7	black holes		0.00007 ± 0.00002	
3.8	substellar objects		0.00014 ± 0.00007	
3.9	HI + HeI		0.00062 ± 0.00010	
3.10	molecular gas		0.00016 ± 0.00006	
3.11	planets		10^{-6}	
3.12	condensed matter		$10^{-5.6\pm0.3}$	
3.13	sequestered in massive black holes		$10^{-5.4}(1+\epsilon_n)$	
4	primeval gravitational binding energy			$-10^{-6.1\pm0.1}$
4.1	virialized halos of galaxies		$-10^{-7.2}$	
4.2	clusters		$-10^{-6.9}$	
4.3	large-scale structure		$-10^{-6.2}$	

Fukugita & Peebles 2004

Cosmic Constituents:

Evolving Energy Density

FRW Energy Equation

To infer the evolving energy density ρ(t) of each cosmic component, we refer to the cosmic energy equation. This equation can be directly inferred from the FRW equations

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

The equation forms a direct expression of the adiabatic expansion of the Universe, ie.

$U = \rho c^2 V$ Internal energy

$V \propto a^3$ Expanding volume

$$dU = -pdV$$

FRW Energy Equation

To infer $\rho(t)$ from the energy equation, we need to know the pressure p(t) for that particular medium/ingredient of the Universe.

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

To infer p(t), we need to know the nature of the medium, which provides us with the equation of state,

$$p = p(\rho, S)$$

Cosmic Constituents:

Evolution of Energy Density

• **Matter:**	$\rho_m(t) \propto a(t)^{-3}$
• **Radiation:**	$\rho_{rad}(t) \propto a(t)^{-4}$
• **Dark Energy:**	$\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$ $\Downarrow \quad w = -1$ $\rho_\Lambda(t) = cst.$

Dark Energy: Equation of State

Dark Energy & Cosmic Acceleration

Nature Dark Energy:

(Parameterized) Equation of State

$$p(\rho) = w\rho c^2$$

Cosmic Acceleration:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a$$

Gravitational Repulsion:

$$p = w\rho c^2 \quad \Leftrightarrow \quad w < -\frac{1}{3} \quad \Rightarrow \quad \ddot{a} > 0$$

Dark Energy & Cosmic Acceleration

DE equation of State

$$p(\rho) = w\rho c^2$$

$$\rho_w(a) = \rho_w(a_0)\, a^{-3(1+w)}$$

Cosmological Constant:

$\Lambda :$ $w = -1$ $\rho_w = cst.$

-1/3 > w > -1:

$\rho_w \propto a^{-3(1+w)}$ $1 + w > 0$ **decreases with time**

Phantom Energy:

$\rho_w \propto a^{-3(1+w)}$ $1 + w < 0$ **increases with time**

Dynamic Dark Energy

DE equation of State

$$p(\rho) = w\rho c^2$$

Dynamically evolving dark energy, parameterization:

$$w(a) = w_0 + (1-a)w_a \approx w_\phi(a)$$

$$\rho_w(a) = \rho_w(a_0)\ \exp\left\{-3\int_1^a \frac{1 + w_\phi(a')}{a'}\, da'\right\}$$

Critical Density & Omega

FRW Dynamics

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^{\ 2}}$$

Critical Density:

- For a Universe with Λ=o
- Given a particular expansion rate H(t)
- Density corresponding to a flat Universe (k=o)

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

FRW Dynamics

In a FRW Universe,
densities are in the order of the critical density,

$$\rho_{crit} = \frac{3H_0^{\,2}}{8\pi G} = 1.8791 h^2 \times 10^{-29}\, g\, cm^{-3}$$

$$\rho_0 = 1.8791 \times 10^{-29}\, \Omega h^2 \; g\, cm^{-3}$$
$$= 2.78 \times 10^{11}\, \Omega h^2 \quad M_{\odot} Mpc^{-3}$$

FRW Dynamics

In a matter-dominated Universe,
the evolution and fate of the Universe entirely determined
by the (energy) density in units of critical density:

$$\Omega \equiv \frac{\rho}{\rho_{crit}} = \frac{8\pi G \rho}{3H^2}$$

Arguably, Ω is the most important parameter of cosmology !!!

Present-day
Cosmic Density:

$$\rho_0 = 1.8791 \times 10^{-29}\, \Omega h^2 \; g\, cm^{-3}$$
$$= 2.78 \times 10^{11}\, \Omega h^2 \quad M_{\odot} Mpc^{-3}$$

FRW Dynamics

- The individual contributions to the energy density of the Universe can be figured into the Ω parameter:

- radiation $\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$

- matter $\Omega_m = \Omega_{dm} + \Omega_b$

- dark energy/ cosmological constant $\Omega_\Lambda = \frac{\Lambda}{3H^2}$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

Critical Density

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

$\Omega < 1$	$k = -1$	*Hyperbolic*	*Open Universe*
$\Omega = 1$	$k = 0$	*Flat*	*Critical Universe*
$\Omega > 1$	$k = +1$	*Spherical*	*Close Universe*

FRW Universe: Curvature

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1) \qquad \Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

$\Omega < 1$	$k = -1$	*Hyperbolic*	*Open Universe*
$\Omega = 1$	$k = 0$	*Flat*	*Critical Universe*
$\Omega > 1$	$k = +1$	*Spherical*	*Close Universe*

Radiation, Matter & Dark Energy

The individual contributions to the energy density of the Universe can be figured into the Ω parameter:

- radiation $\quad \Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$
- matter $\quad \Omega_m = \Omega_{dm} + \Omega_b$
- dark energy/ cosmological constant $\quad \Omega_\Lambda = \frac{\Lambda}{3H^2}$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

Hubble Expansion

Hubble Expansion

- Cosmic Expansion is a uniform expansion of space
- Objects do not move themselves:
  they are like beacons tied to a uniformly expanding sheet:

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$H(t) = \frac{\dot{a}}{a}$$

Hubble Expansion

- Cosmic Expansion is a uniform expansion of space
- Objects do not move themselves:
  they are like beacons tied to a uniformly ex[panding]

Comoving Position

Hubble Parameter:

Hubble "constant":
$H_o \equiv H(t=t_o)$

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$H(t) = \frac{\dot{a}}{a}$$

Hubble Parameter

- For a long time, the correct value of the Hubble constant H_o was a major unsettled issue:

  $H_o = 50$ km s^{-1} Mpc^{-1} ⟷ $H_o = 100$ km s^{-1} Mpc^{-1}

- This meant distances and timescales in the Universe had to deal with uncertainties of a factor 2 !!!
- Following major programs, such as Hubble Key Project, the Supernova key projects and the WMAP CMB measurements,

$$H_0 = 71.9^{+2.6}_{-2.7}\, km\, s^{-1} Mpc^{-1}$$

Hubble Time

$$t_H = \frac{1}{H}$$

$$H_0 = 100h\ km\,s^{-1}Mpc^{-1}$$
$$\Downarrow$$
$$t_0 = 9.78h^{-1}\ Gyr$$

Hubble Distance

Just as the Hubble time sets a natural time scale for the universe,
one may also infer a natural distance scale of the universe, the

Hubble Distance

$$R_H = \frac{c}{H_0} \approx 2997.9h^{-1}Mpc$$

Acceleration Parameter

FRW Dynamics: Cosmic Acceleration

Cosmic acceleration quantified by means of dimensionless deceleration parameter q(t):

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

$$q = \frac{\Omega_m}{2} + \Omega_{rad} - \Omega_\Lambda$$

$$q \approx \frac{\Omega_m}{2} - \Omega_\Lambda$$

Examples:

$\Omega_m = 1; \quad \Omega_\Lambda = 0;$
$q = 0.5$

$\Omega_m = 0.3; \quad \Omega_\Lambda = 0.7;$
$q = -0.65$

Dynamics

FRW Universe

General Solution
Expanding FRW Universe

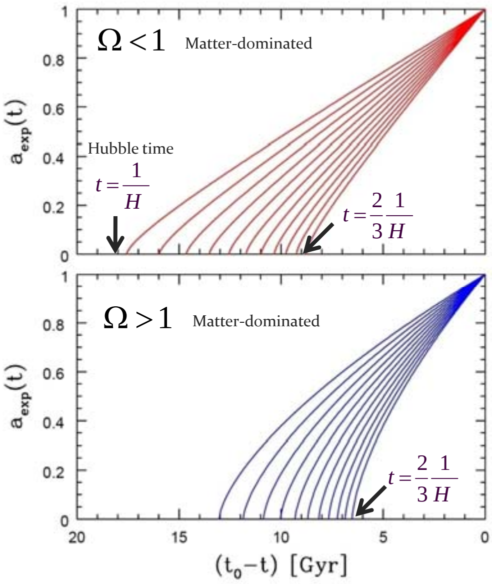
From the FRW equations:

$$\frac{H(t)^2}{H_0^{\ 2}} = \frac{\Omega_{rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

$a(t)$ Expansion history Universe

$$H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)}}$$

Age of the Universe



Age of a FRW universe at
Expansion factor a(t)

$$H\,t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad}}{a^2} + \frac{\Omega_m}{a} + \Omega_\Lambda a^2 + (1-\Omega\ )}}$$

Specific Solutions
FRW Universe

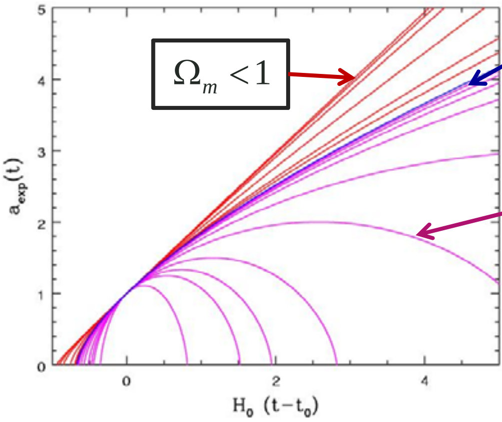
While general solutions to the FRW equations is only possible by numerical integration, analytical solutions may be found for particular classes of cosmologies:

- **Single-component Universes:**
  - empty Universe
  - flat Universes, with only radiation, matter or dark energy
- **Matter-dominated Universes**
- **Matter+Dark Energy flat Universe**

Matter-Dominated Universes

- Assume radiation contribution is negligible:
- Zero cosmological constant:
- Matter-dominated, including curvature

$$\Omega_{rad,0} \approx 5\times10^{-5}$$
$$\Omega_{\Lambda} = 0$$



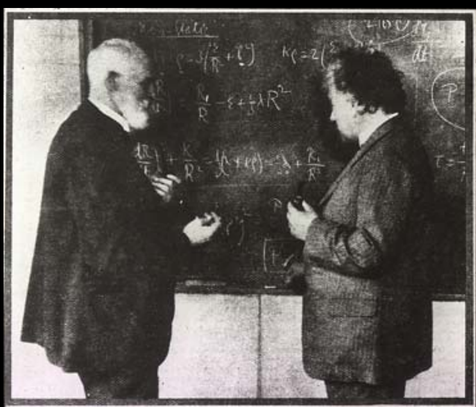
Einstein-de Sitter Universe

$$\left.\begin{matrix}\Omega_m = 1 \\ \Omega_\Lambda = 0\end{matrix}\right\} \; k = 0$$

FRW: $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G \rho_0}{3}\frac{1}{a}$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

Age EdS Universe: $t_0 = \frac{2}{3}\frac{1}{H_0}$



Albert Einstein and Willem de Sitter discussing the Universe. In 1932 they published a paper together on the Einstein-de Sitter universe, which is a model with flat geometry containing matter as the only significant substance.

Free Expanding "Milne" Universe

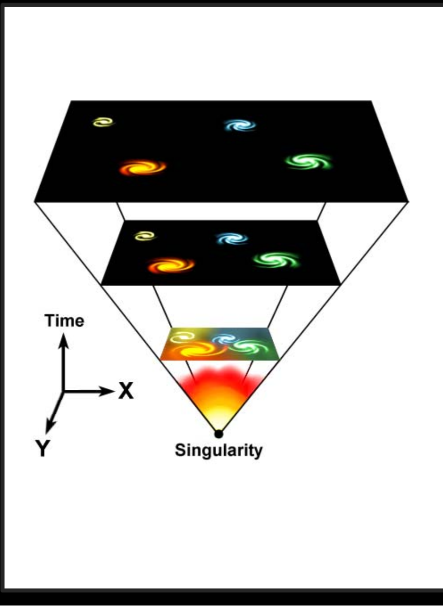
$$\left.\begin{array}{l}\Omega_m = 0 \\ \Omega_\Lambda = 0\end{array}\right\} k = -1$$

Empty space is curved

FRW: $\dot{a}^2 = -\frac{kc^2}{R_0^{\,2}} = cst.$

$$a(t) = \left(\frac{t}{t_0}\right)$$

Age
Empty Universe: $t_0 = \frac{1}{H_0}$



De Sitter Expansion

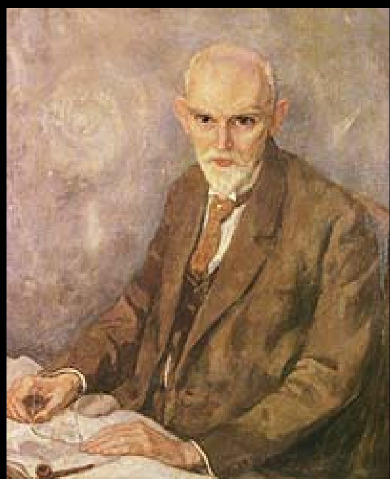
$$\left.\begin{array}{l}\Omega_m = 0 \\ \Omega_\Lambda = 1\end{array}\right\} k = 0$$

$\Omega_\Lambda = \frac{\Lambda}{3H_0^{\,2}} \quad \Rightarrow \quad H_0 = \sqrt{\frac{\Lambda}{3}}$

FRW: $\dot{a}^2 = \frac{\Lambda}{3}a^2 \quad \Rightarrow \quad \dot{a} = H_0 a$

$$a(t) = e^{H_0(t-t_0)}$$

Age
De Sitter Universe: infinitely old



Willem de Sitter (1872-1934; Sneek-Leiden)
director Leiden Observatory
alma mater: Groningen University

Expansion
Radiation-dominated Universe

$$\left.\begin{array}{l}\Omega_{rad} = 1 \\ \Omega_m = 0 \\ \Omega_\Lambda = 0\end{array}\right\} k = 0$$

In the very early Universe, the energy density is completely dominated by radiation. The dynamics of the very early Universe is therefore fully determined by the evolution of the radiation energy density:

FRW: $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G \rho_0}{3}\frac{1}{a^2}$ ⟵ $\rho_{rad}(a) \propto \frac{1}{a^4}$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

Age Radiation Universe: $t_0 = \frac{1}{2}\frac{1}{H_0}$

General Flat FRW Universe

$$k = 0$$

$$\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$$

FRW:

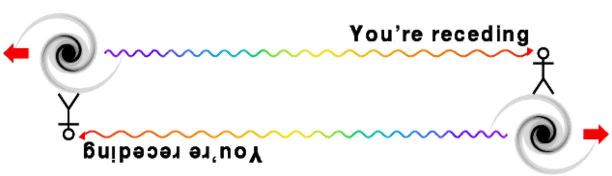
$$a(t) \propto t^{\frac{2}{3+3w}}$$

Observational Cosmology in FRW Universe

Cosmic Redshift

$$1 + z = \frac{1}{a} \Longleftarrow \begin{cases} \lambda_{em} = \lambda_0 \\ \lambda_{obs} = \dfrac{a(t_{obs})}{a(t_{em})} \lambda_0 \end{cases}$$

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$



RW Distance Measure

In an (expanding) space with Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2 \left(\frac{r}{R_c} \right) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

there are several definitions for distance, dependent on how you measure it.

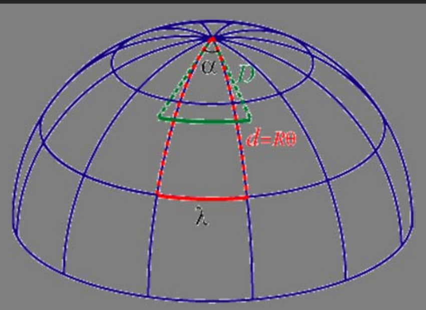
They all involve the central distance function, the *RW Distance Measure,*

$$D(r) = R_c S_k \left(\frac{r}{R_c} \right)$$

Angular Diameter Distance

Imagine an object of *proper size d,* at redshift z, its angular size $\Delta\theta$ is given by

$$d = a(t) R_c S_k \left(\frac{r}{R_c} \right) \Delta\theta \quad \Rightarrow \quad \Delta\theta = \frac{d\,(1+z)}{D} = \frac{d}{D_A}$$



Angular Diameter distance:

$$D_A = \frac{D}{1+z}$$

Luminosity Distance

Imagine an object of luminosity $L(\nu_e)$, at redshift z, its flux density at observed frequency ν_o is

$$S(\nu_o) = \frac{L(\nu_e)}{4\pi D^2(1+z)} \Rightarrow S_{bol} = \frac{L_{bol}}{4\pi D^2(1+z)^2} = \frac{L_{bol}}{4\pi D_L^2}$$

Luminosity distance:

$$D_L = D(1+z)$$

FRW Redshift-Distance

Observing in a FRW Universe, we locate galaxies in terms of their redshift z. To connect this to their true physical distance, we need to know what the coordinate distance r of an object with redshift z,

$$R_0 dr = \frac{c}{H(z)} dz$$

In a FRW Universe, the dependence of the Hubble expansion rate H(z) at any redshift z depends on the content of matter, dark energy and radiation, as well ss its curvature. This leads to the following explicit expression for the redshift-distance relation,

$$R_0 dr = \frac{c}{H_0}\left\{(1-\Omega_0)(1+z)^2 + \Omega_{\Lambda,0} + \Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4\right\}^{-1/2} dz$$

Matter-Dominated FRW Universe

in a matter-dominated Universe, the redshift-distance relation is

$$R_0 dr = \frac{c}{H_0}\left\{(1-\Omega_0)(1+z)^2 + \Omega_0(1+z)^3\right\}^{-1/2} dz$$

from which one may find that

$$R_0 r = \frac{c}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_0 z'}}$$

Mattig's Formula

The integral expression

$$R_0 r = \frac{c}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_0 z'}}$$

can be evaluated by using the substitution: $u^2 = \frac{k(\Omega_0 - 1)}{\Omega_0(1+z)}$

This leads to Mattig's formula:

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left\{\sqrt{1+\Omega_0 z} - 1\right\}}{\Omega_0^{\,2}(1+z)}$$

This is one of the very most important and most useful equations in observational cosmology.

Mattig's Formula

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left\{\sqrt{1+\Omega_0 z} - 1\right\}}{\Omega_0^{\,2}(1+z)}$$

In a low-density Universe, it is better to use the following version:

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{c}{H_0}\frac{z}{1+z}\frac{1+\sqrt{1+\Omega_0 z}}{1+\sqrt{1+\Omega_0 z}+\Omega_0 z/2}$$

For a Universe with a cosmological constant, there is not an easily tractable analytical expression (a Mattig's formula). The comoving Distance r has to be found through a numerical evaluation of the fundamental dr/dz expression.

Distance-Redshift Relation, 2nd order

For all general FRW Universe, the second-order distance-redshift relation is identical, only depending on the *deceleration parameter* q_0:

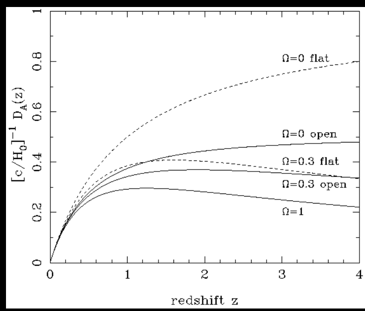
$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) \simeq \frac{c}{H_0}\left(z - \frac{1}{2}(1+q_0)z^2\right)$$

q_0 can be related to Ω_0 once the *equation of state* is known.

Angular Diameter Distance

matter-dominated FRW Universe

$$D_A = \frac{D}{1+z} = \frac{1}{1+z} R_c S_k \left(\frac{r}{R_c} \right)$$



In a matter-dominated Universe, the angular diameter distance as function of redshift is given by:

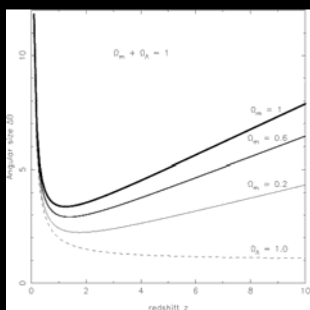
$$D_A(z) = \frac{1}{1+z} R_c S_k \left(\frac{r}{R_c} \right) = \frac{2c}{H_0} \frac{1}{\Omega_0^2 (1+z)^2} \left\{ \Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1 \right) \right\}$$

Angular Size - Redshift

FRW Universe

$$\theta(z) = \frac{\ell}{D_A}$$

The angular size $\theta(z)$ of an object of physical size l at a redshift z displays an interesting behaviour. In most FRW universes is has a minimum at a medium range redshift – z=1.25 in an Ω_m=1 EdS universe – and increases again at higher redshifts.



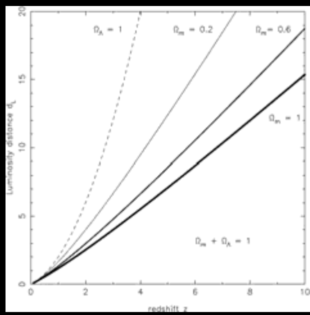
In a matter-dominated Universe, the angular diameter distance as function of redshift is given by:

$$D_A(z) = \frac{1}{1+z} R_c S_k \left(\frac{r}{R_c} \right) = \frac{2c}{H_0} \frac{1}{\Omega_0^2 (1+z)^2} \left\{ \Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1 \right) \right\}$$

Luminosity Distance

matter-dominated FRW Universe

$$D_L = D(1+z) = (1+z) R_c S_k\left(\frac{r}{R_c}\right)$$

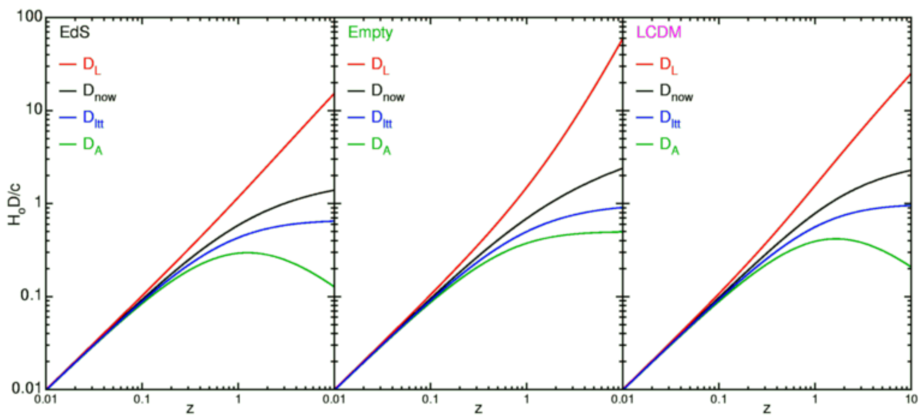


In a matter-dominated Universe, the luminosity distance as function of redshift is given by:

$$D_A(z) = (1+z) R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{\Omega_0{}^2 H_0}\left\{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)\right\}$$

FRW Universe Distances

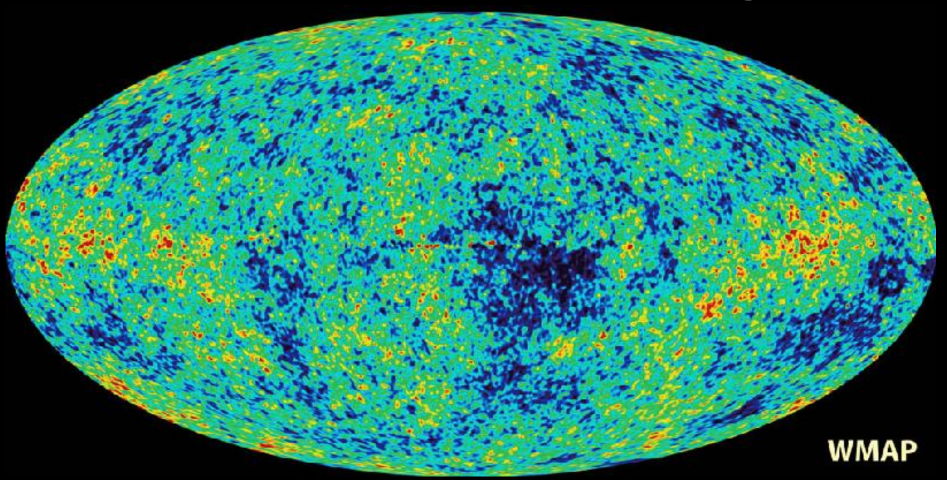
summary



How Much ?

Cosmic Curvature

Cosmic Microwave Background



Map of the Universe at Recombination Epoch (WMAP, 2003):

- **379,000 years after Big Bang**
- **Subhorizon perturbations: primordial sound waves**
- **$\Delta T/T$ < 10-5**

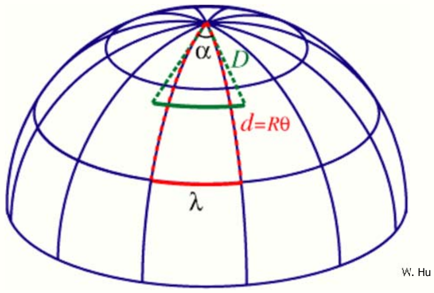
Measuring Curvature

Measuring the Geometry of the Universe:

- Object with known physical size, at large cosmological distance
- Measure angular extent on sky
- Comparison yields light path, and from this the curvature of space

⬇

Geometry of Space



In a FRW Universe: lightpaths described by Robertson-Walker metric

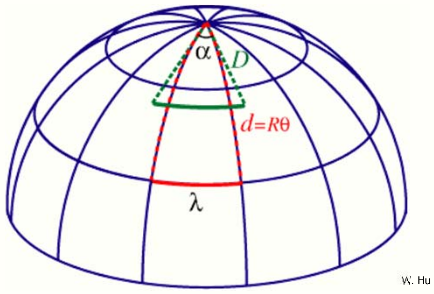
$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c^{\,2} S_k^{\,2}\left(\frac{r}{R_c}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right\}$$

Measuring Curvature

- Object with known physical size, at large cosmological distance:
- Sound Waves in the Early Universe !!!!

⬇

Temperature Fluctuations CMB



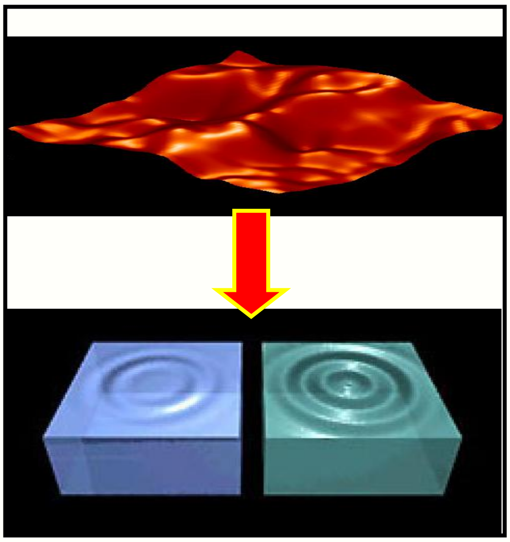
In a FRW Universe: lightpaths described by Robertson-Walker metric

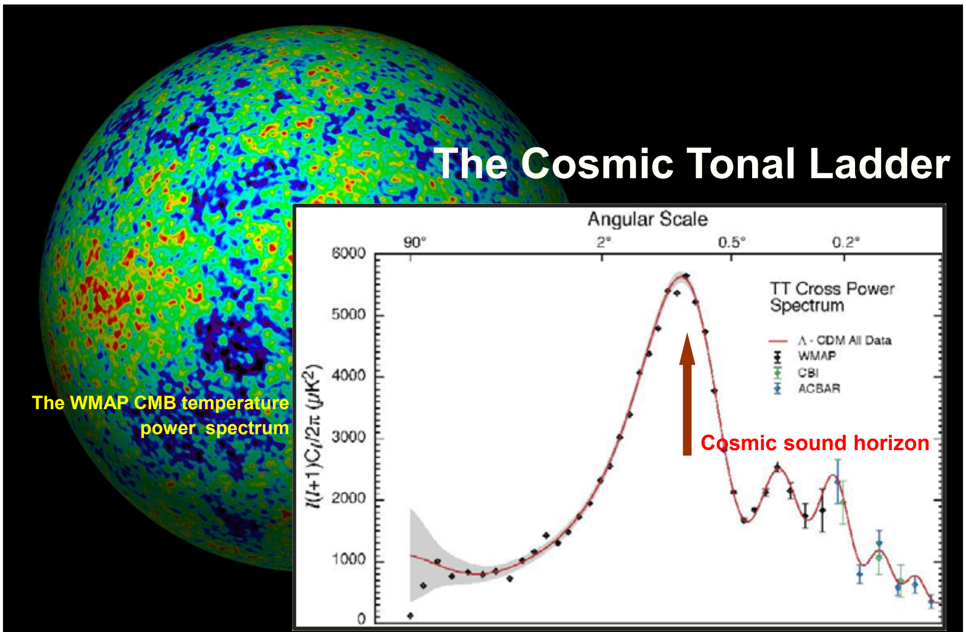
$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c^{\,2} S_k^{\,2}\left(\frac{r}{R_c}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right\}$$

Music of the Spheres

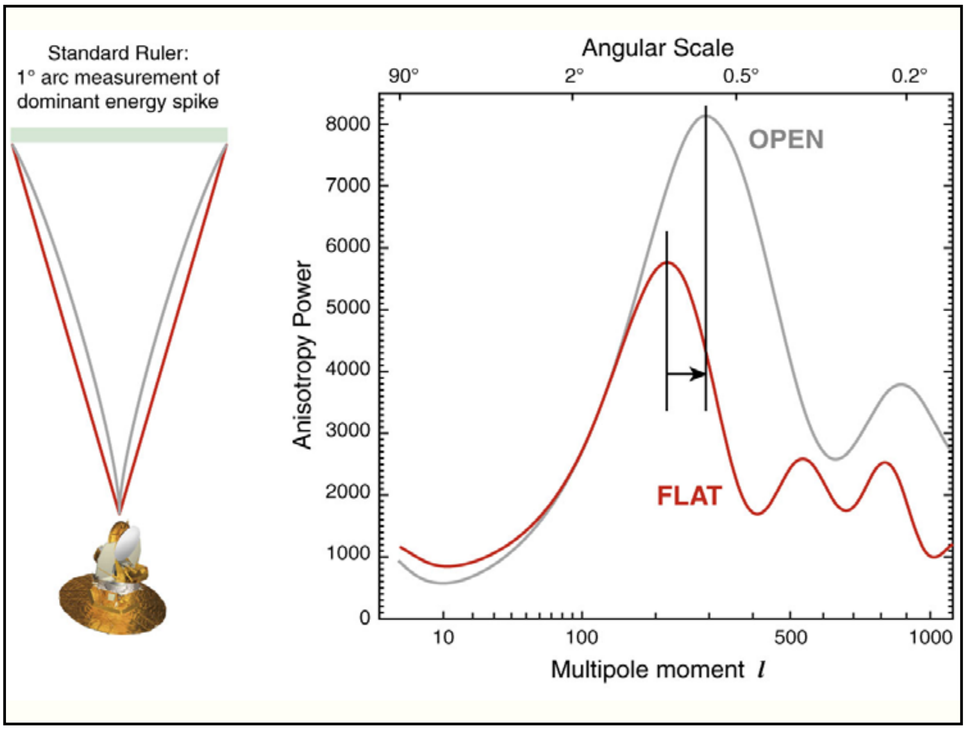
- small ripples in primordial matter & photon distribution
- gravity:
  - compression primordial photon gas
  - photon pressure resists
- compressions and rarefactions in photon gas: sound waves
- sound waves not heard, but seen:
  - compressions: (photon) T higher
  - rarefactions: lower
- fundamental mode sound spectrum
  - size of "instrument":
  - (sound) horizon size last scattering
- Observed, angular size: θ~1º
  - exact scale maximum compression, the "cosmic fundamental mode of music"

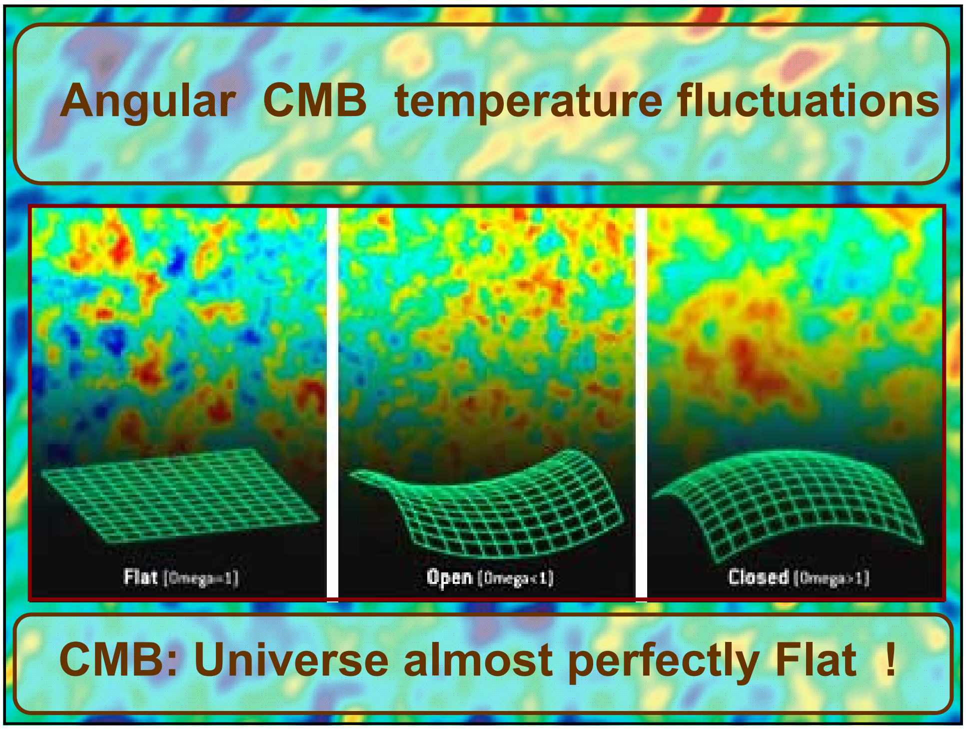
W. Hu





The Cosmic Microwave Background Temperature Anisotropies: Universe is almost perfectly flat


Standard Ruler:
1° arc measurement of
dominant energy spike
Angular Scale
90°
2°
0.5°
0.2°
OPEN
FLAT
Anisotropy Power
8000
7000
6000
5000
4000
3000
2000
1000
0
10
100
500
1000
Multipole moment l


Angular CMB temperature fluctuations
Flat (Omega=1)
Open (Omega<1)
Closed (Omega>1)
CMB: Universe almost perfectly Flat !

Cosmic Constraints

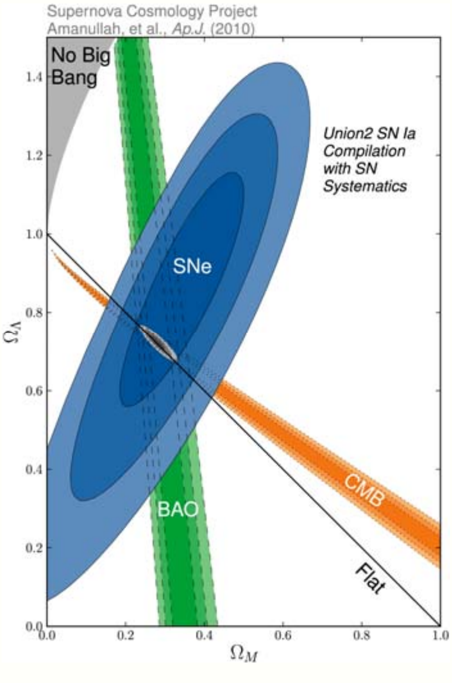
Ω_m vs. Ω_Λ

$$q \approx \frac{\Omega_m}{2} - \Omega_\Lambda$$

$$k = \frac{H^2 R^2}{c^2}(\Omega_m + \Omega_\Lambda - 1)$$

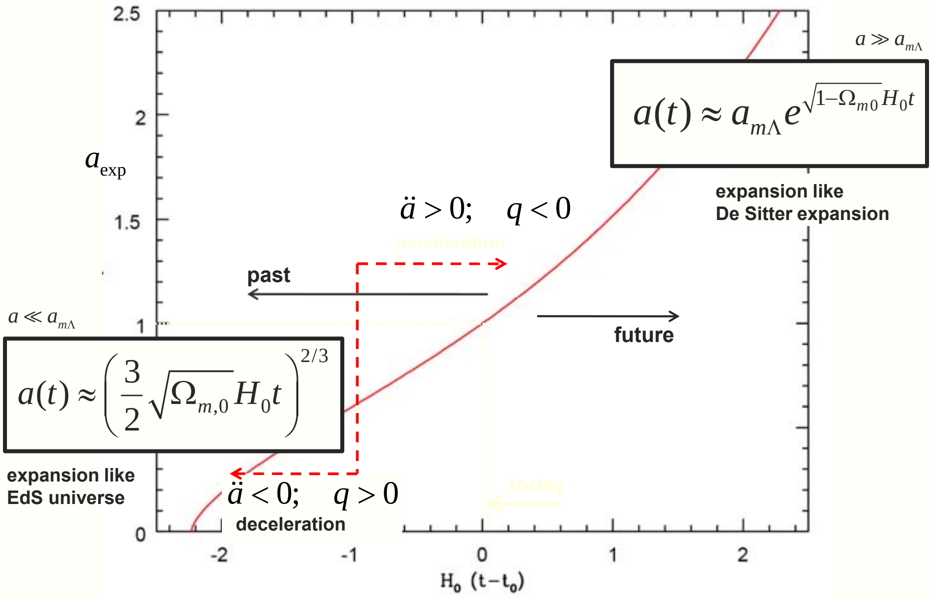
SCP Union2 constraints (2010)

on values of matter density Ω_m
dark energy density Ω_Λ



Concordance Universe

Concordance Expansion

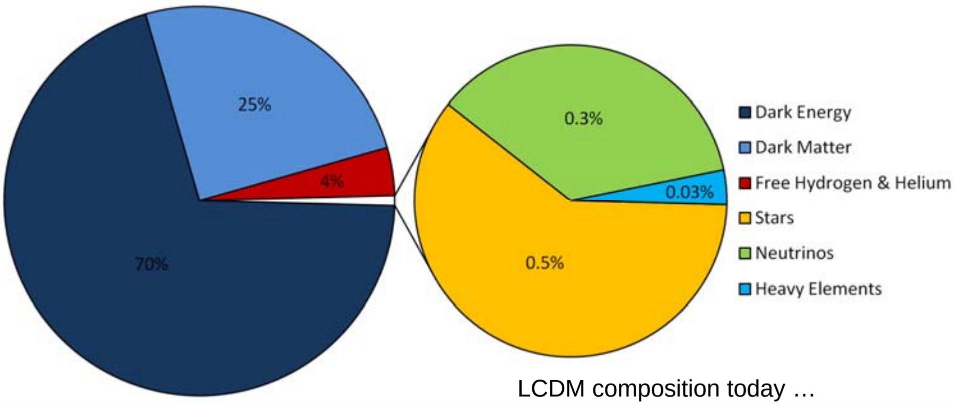


Concordance Universe Parameters

Hubble Parameter		$H_0 = 71.9 \pm 2.6\ km\ s^{-1}\ Mpc^{-1}$	
Age of the Universe		$t_0 = 13.7 \pm 0.12\ Gyr$	
Temperature CMB		$T_0 = 2.725 \pm 0.001\ K$	
Matter	Baryonic Matter Dark Matter	$\Omega_m = 0.27$	$\Omega_b = 0.0456 \pm 0.0015$ $\Omega_{dm} = 0.228 \pm 0.013$
Radiation	Photons (CMB) Neutrinos (Cosmic)	$\Omega_{rad} = 8.4 \times 10^{-5}$	$\Omega_\gamma = 5 \times 10^{-5}$ $\Omega_\nu = 3.4 \times 10^{-5}$
Dark Energy		$\Omega_\Lambda = 0.726 \pm 0.015$	
Total		$\Omega_{tot} = 1.0050 \pm 0.0061$	

LCDM Cosmology

- Concordance cosmology
  - model that fits the majority of cosmological observations
  - universe dominated by Dark Matter and Dark Energy



LCDM composition today …

Matter-Dark Energy Transition

$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}}$$

$$\Omega_{\Lambda,0} = 0.27 \qquad a_{m\Lambda} = 0.72$$

$$\Omega_{m,0} = 0.73 \qquad a_{m\Lambda}{}^{\dagger} = 0.57$$

Flat Universe

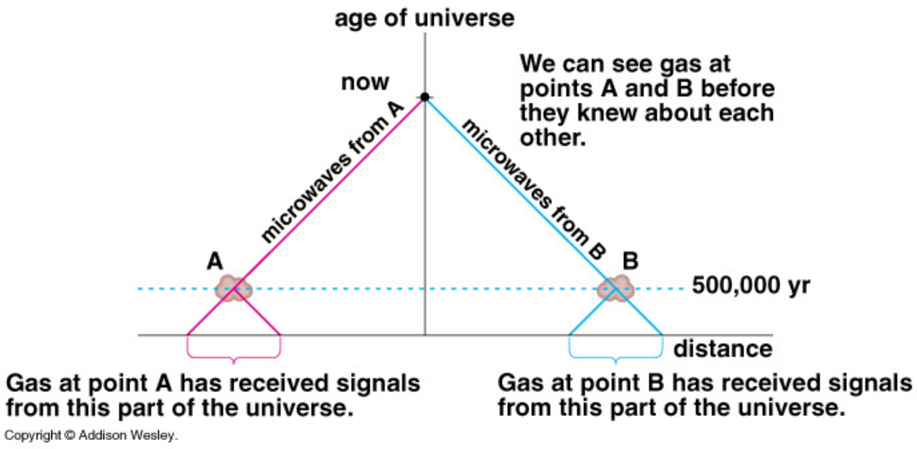
$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}}$$

Note: a more appropriate characteristic transition is that at which the deceleration turns into acceleration:

$$a_{m\Lambda}{}^{\dagger} = \sqrt[3]{\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}} = \sqrt[3]{\frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}}$$

Cosmic Future

Cosmic Horizons



Particle Horizon of the Universe:
distance that light travelled since the Big Bang

Cosmic Particle Horizon

Light travel in an expanding Universe:

- **Robertson-Walker metric:** $ds^2 = c^2 dt^2 - a(t)^2 dr^2$
- **Light:** $ds^2 = 0$

$$d_{Hor} = \int_0^t \frac{c\,dt'}{a(t')}$$

Horizon distance in comoving space

⬄

$$R_{Hor} = a(t)\int_0^t \frac{c\,dt'}{a(t')}$$

Horizon distance in physical space

Particle Horizon of the Universe:
distance that light travelled since the Big Bang

Cosmic Particle Horizon

Particle Horizon of the Universe:
distance that light travelled since the Big Bang

$$d_{Hor} = \int_0^t \frac{c\,dt'}{a(t')}$$

Horizon distance in comoving space

$$R_{Hor} = a(t)\int_0^t \frac{c\,dt'}{a(t')}$$

Horizon distance in physical space

In a spatially flat Universe, the horizon distance has a finite value for w>-1/3

$$a(t) \propto t^{\frac{2}{3+3w}} \quad \Rightarrow \quad d_{Hor}(t_0) = ct_0 \frac{2}{1+3w}$$

Cosmic Particle Horizon

Particle Horizon of the Universe:
distance that light travelled since the Big Bang

In a spatially flat Universe, the horizon distance has a finite value for w>-1/3

$$a(t) \propto t^{\frac{2}{3+3w}} \quad \Rightarrow \quad d_{Hor}(t_0) = ct_0 \frac{2}{1+3w}$$

$d_{Hor}(t_0) = 3ct_0$ flat, matter-dominated universe

$d_{Hor}(t_0) = 2ct_0$ flat, radiation-dominated universe

Infinite Particle Horizon

Particle Horizon of the Universe:
distance that light travelled since the Big Bang

In a spatially flat Universe, the horizon distance is infinite for $w<-11/3$

$$a(t) \propto t^{\frac{2}{3+3w}} \Rightarrow d_{Hor}(t_0) = ct_0 \frac{2}{1+3w}$$

In such a universe, all of space is causally connected to observer:

In such a universe, you could see every point in space.

Cosmic Event Horizon

Light travel in an expanding Universe:

- **Robertson-Walker metric:** $ds^2 = c^2 dt^2 - a(t)^2 dr^2$
- **Light:** $ds^2 = 0$

$$d_{event} = \int_t^{\infty} \frac{c\,dt'}{a(t')}$$

Event Horizon distance in comoving space

$$\Leftrightarrow$$

$$R_{event} = a(t) \int_t^{\infty} \frac{c\,dt'}{a(t')}$$

Event Horizon distance in physical space

Event Horizon of the Universe:
the distance over which one may still communicate …

Cosmic Event Horizon

Event Horizon of the Universe:
distance light may still travel in Universe.

$$d_{EHor} = \int_t^{\infty} \frac{c\,dt'}{a(t')}$$

Event Horizon distance comoving space

⇔

$$R_{EHor} = a(t)\int_t^{\infty} \frac{c\,dt'}{a(t')}$$

Event Horizon distance physical space

In a spatially flat Universe, the event horizon is:

$$a(t) \propto t^{\frac{2}{3+3w}}$$

⇒

$$d_{EHor}(t_0) \propto \left[t^{\frac{1+3w}{3+3w}} \right]_t^{\infty}$$

Cosmic Event Horizon

Event Horizon of the Universe:
distance light may still travel in Universe.

$$a(t) \propto t^{\frac{2}{3+3w}}$$

⇒

$$d_{EHor}(t_0) \propto \left[t^{\frac{1+3w}{3+3w}} \right]_t^{\infty}$$

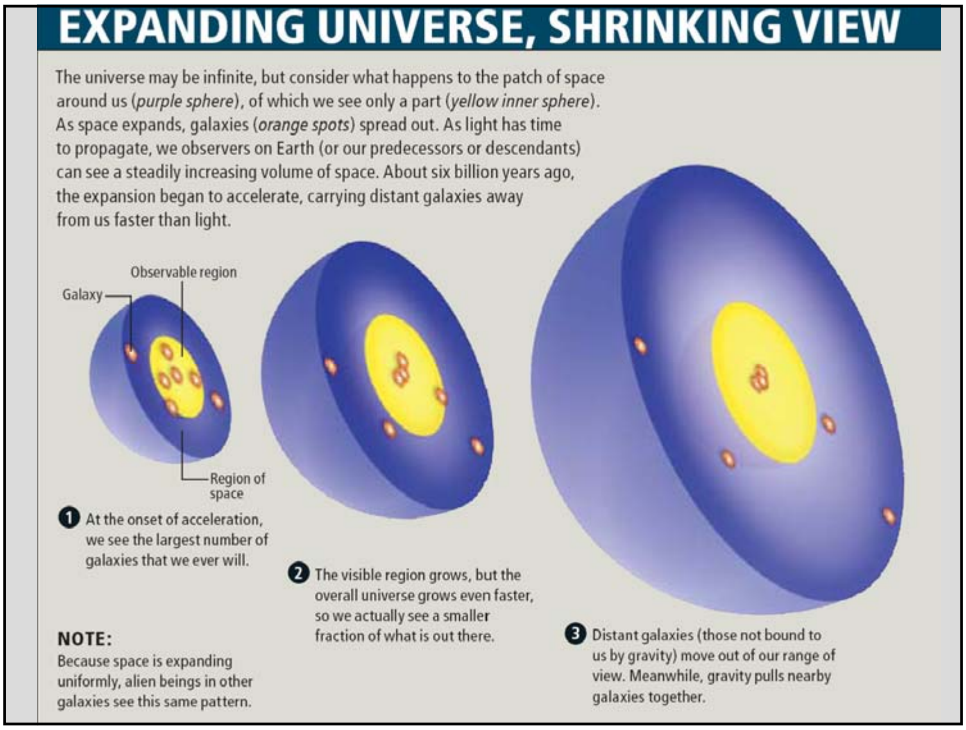
In a spatially flat Universe, the event horizon is:

$w > -1/3 \quad \rightarrow \quad d_{EHor} = \infty$

$w < -1/3 \quad \rightarrow \quad d_{EHor}$ finite

$$d_{EHor}(t_0) \propto t^{\frac{1+3w}{3+3w}}$$

shrinking event horizon:



Cosmic Fate

100 Gigayears: the end of Cosmology