Tutorial VI.
The Inflationary Universe

Question 1. Horizons

The *particle horizon* of the Universe is the distance over which light can have travelled since the Big Bang. We will be exploring some of its properties.

a) Show, on the basis of the RW metric, that the expression for the horizon scale is

$$R_H(t) = a(t) \int_0^t \frac{c\,dt'}{a(t')} \,. \tag{1}$$

b) Given that the conformal time $\eta(t)$ is given by

$$\eta(t) = \int_0^t \frac{c\,dt'}{a(t')} \,. \tag{2}$$

show that the horizon grows linearly in terms of conformal time $\eta(t)$. Make a sketch of the evolution of the horizonscale as function of conformal time $\eta(t)$.

c) Show that the horizon scale $R_h(t)$ for an Einstein-de Sitter Universe is

$$R_H(t) = 3ct = \frac{2c}{H(t)} \,. \tag{3}$$

d) Equivalently, infer the horizon scale $R_H(t)$ for a radiation-dominated Universe.

e) Show that for a single-component flat Universe with dark energy with equation of state

$$p = w\rho c^2 \,, \tag{4}$$

which yields an expansion scale

$$a(t) \propto t^{\frac{2}{3+3w}} \,, \tag{5}$$

that the hoirzon scale $R(t)$ evolves as

$$R_H(t) \propto t^{\frac{1+3w}{3+3w}} \tag{6}$$

Discuss the difference between the particle horizon $R_H(t)$ for dark energy with $w < -1/3$ and for a cosmology in which the dark energy has equation of state $w > -1/3$.

f) The *event horizon* is the distance over which light/radiation can propagate when emitted at a particular time t. In other words, it specifies with which part of the universe you would be able to communicate,

$$R_{eH}(t) = a(t) \int_t^{\infty} \frac{c\, dt'}{a(t')} \,. \tag{7}$$

Infer the value of the event horizon $R_{eH}(t)$ for cosmologies with dark energy whose equation of state is $p = w\rho c^2$. Discuss the outcome between $w < -1/3$ and $w > -1/3$. What does this mean for dark energy cosmologies with $w < -1/3$.

Question 2. Horizon at Recombination

Let us consider a major problem in standard cosmology, the *horizon problem.* We will get to talk in detail about the CMB later during the course. Here we do some simple calculation in preparation for that. We'll work out the angular size of the particle horizon of the Universe at recombination/decoupling. In order to keep things manageable we make the simplifying assumption that we life in a matter-dominated Universe. The horizon scale at recombination is given by:

$$R_{\text{rec}} = 3ct_{\text{rec}} \tag{8}$$

a) Show that in the limit $z \gg 1$ the angular diameter distance $D_A(z)$ can be approximated as:

$$D_A \approx \frac{2c}{\Omega_0 H_0} \frac{1}{z} \tag{9}$$

b) Now combine eq. 8 and eq. 9 to determine the angular size θ_{rec} of a patch on the sky of the size of the horizon at recombination. Your expression should depend on H_{rec}, z_{rec}, Ω_0 and H_0.

c) Show that in the limit $z \gg 1$ the Hubble parameter can be approximated as:

$$H^2(z) \approx \Omega_0 H_0^2 z^3 \tag{10}$$

d) Finally put everything together to find:

$$\theta_{\text{rec}} \approx 1.74° \Omega_0^{1/2} \left(\frac{z_{\text{dec}}}{1089} \right)^{-1/2} \tag{11}$$

e) Given the high degree of isotropy in the CMB sky, what conclusion do you have to draw?

Question 3. Flatness problem

In this exercise we are going to look at what is called the *flatness problem* or *finetuning problem.* It was one of the main reasons for introducing the inflation mechanism. Let's see if we can understand this.

a) Starting from the FRWL equations, show that the evolution of $\Omega(z)$ in a matter-dominated Universe is given by the following equivalent expressions:

$$\begin{aligned} \left(\frac{1}{\Omega} - 1\right) &= \frac{1}{z+1}\left(\frac{1}{\Omega_0} - 1\right) \\ \Omega(z) &= \frac{\Omega_0(z+1)}{1+\Omega_0 z} \end{aligned} \tag{12}$$

b) Create plots of Ω versus a for some different values of Ω_0 between 0.1 and 10.0. Make sure to include $\Omega_0 = 1$. How does the behaviour depend on the value of Ω_0?

c) Within (relatively small) measurement errors, WMAP and Planck have shown us that $\Omega_0 = 1$. Can you explain in your own words what we mean by the flatness problem?

d) Inflation was suggested as a possible solution to the flatness problem. Let's see how that could work. Take a general single component Universe with $p = w\rho c^2$ and ignore the curvature contribution. Show:

$$a(t) \propto t^{\frac{2}{3+3w}} \tag{13}$$

1. Then derive:

$$|\Omega(t) - 1| \propto t^{\frac{2(1+3w)}{3+3w}} \tag{14}$$

Hint: in the previous set you already derived an expression for $(\Omega - 1)$.

e) Find an expression for $|\Omega(t) - 1|$ for $w = -1$. Note that the general result is not valid in this case!

f) What happens for different values of w? What are the restrictions on w for inflation to work?

g) We define $N \equiv \sqrt{\Omega H^2}\Delta t$, with Δt the duration of inflation. Use the following Universe model:

- the Universe starts out radiation dominated
- prior to inflation the Universe is curved: $|\Omega - 1| = 1.0$
- inflation starts at $t_i = t_{\mathrm{GUT}} = 10^{-36}\, s$ and ends at $t_f = 10^{-34}\, s$. Inflation is exponential.
- after inflation the Universe is radiation-dominated until radiation-matter equality at $t_{\mathrm{rm}} = 4.7 \cdot 10^4\, yr$.
- the Universe is matter dominated until matter-Λ equality at $t_{\mathrm{m}\Lambda} = 9.8\, Gyr$
- the dark energy has $w = -1$
- we are now at $t_0 = 13.8\, Gyr$.

We demand a Universe that is flat within 10%:

$$|\Omega_0 - 1| \leq 0.1 \tag{15}$$

How many e-foldings N do you need during the epoch of inflation in order to satisfy this demand? Note that the data tells us $|\Omega - 1| \ll 0.1$, so the actual number will be larger than the one you find!