Cosmology,

lect. 5

Curved Cosmos
&
Observational Cosmology

Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Cosmological Principle

General Relativity

A crucial aspect of any particular configuration is the geometry of spacetime: because Einstein's General Relativity is a metric theory, knowledge of the geometry is essential.

Einstein Field Equations are notoriously complex, essentially 10 equations. Solving them for general situations is almost impossible.

However, there are some special circumstances that do allow a full solution. The simplest one is also the one that describes our Universe. It is encapsulated in the

Cosmological Principle

On the basis of this principle, we can constrain the geometry of the Universe and hence find its dynamical evolution.

Cosmological Principle: the Universe Simple & Smooth

"God is an infinite sphere whose centre is everywhere and its circumference nowhere"
Empedocles, 5th cent BC

Cosmological Principle:

Describes the symmetries in global appearance of the Universe:

- **Homogeneous** → The Universe is the same everywhere:
  - physical quantities (density, T,p,…)
- **Isotropic** → The Universe looks the same in every direction
- **Universality** → Physical Laws same everywhere
- **Uniformly Expanding** → The Universe "grows" with same rate in
  - every direction
  - at every location

"all places in the Universe are alike"
Einstein, 1931

Geometry of the Universe

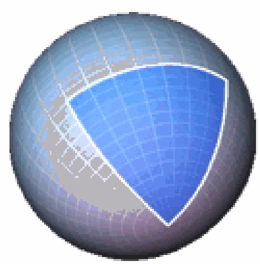
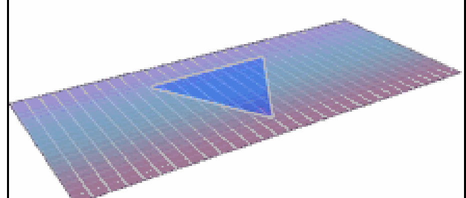
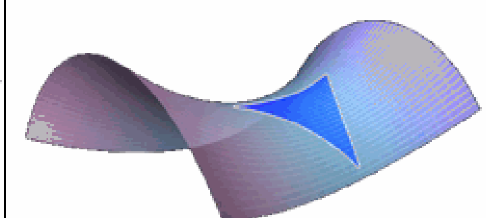
Fundamental Tenet

of (Non-Euclidian = Riemannian) Geometry

There exist no more than THREE uniform spaces:

1)	Euclidian (flat) Geometry	Euclides
2)	Hyperbolic Geometry	Gauß, Lobachevski, Bolyai
3)	Spherical Geometry	Riemann

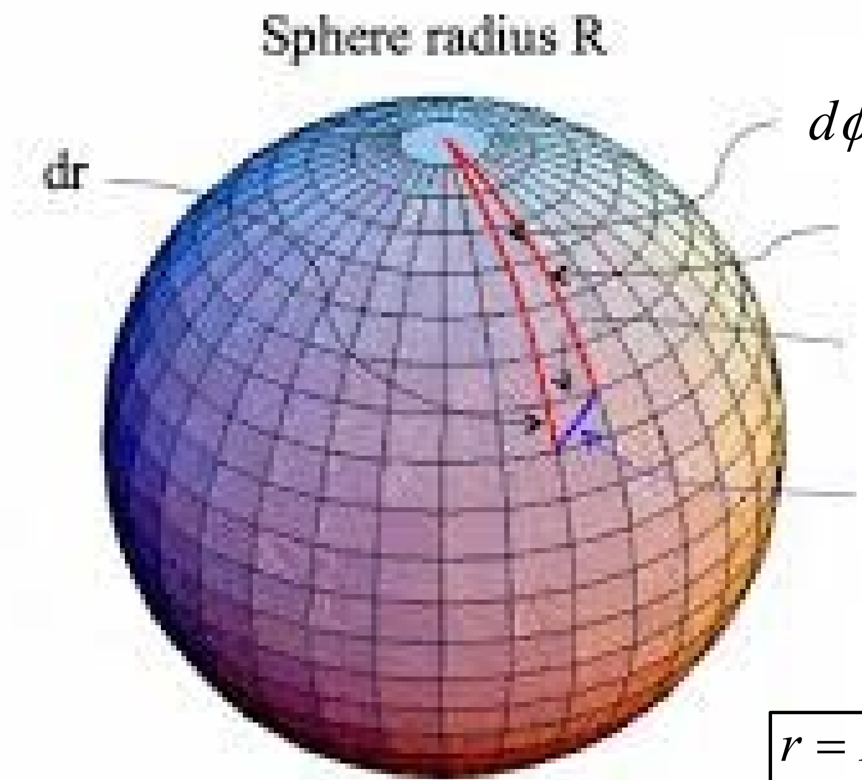
uniform=
homogeneous & isotropic
(cosmological principle)

Property	Closed	Euclidean	Open
Spatial Curvature	Positive	Zero	Negative
Circle Circumference	$< 2\pi R$	$2\pi R$	$> 2\pi R$
Sphere Area	$< 4\pi R^2$	$4\pi R^2$	$> 4\pi R^2$
Sphere Volume	$< {}^4/_3\ \pi R^3$	${}^4/_3\ \pi R^3$	$> {}^4/_3\ \pi R^3$
Triangle Angle Sum	$> 180°$	$180°$	$< 180°$
Total Volume	Finite ($2\pi^2 R^3$)	Infinite	Infinite
Surface Analog	Sphere 	Plane 	Saddle 

Curvature of the Universe:

Robertson-Walker Metric

Spherical Surface Distances

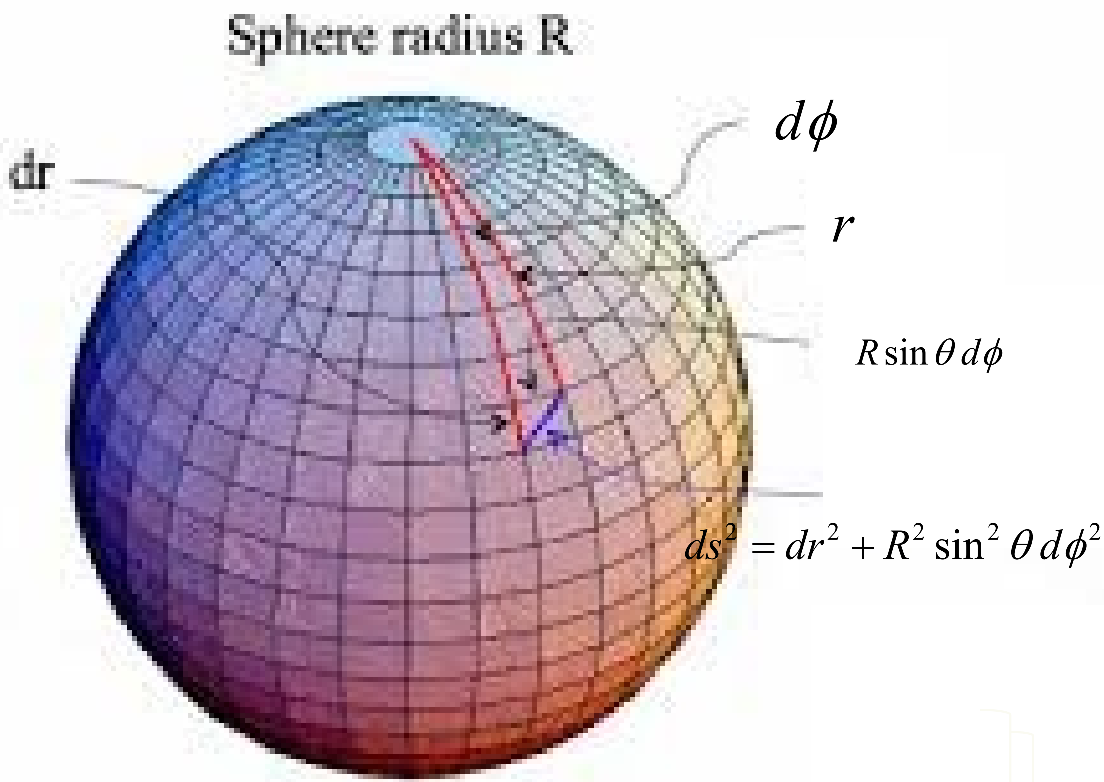


$$R\sin\theta\, d\phi$$

$$ds^2 = dr^2 + R^2 \sin^2\theta\, d\phi^2$$

$$\left.\begin{aligned} r &= R\theta \\ dr &= R\, d\theta \end{aligned}\right\} \quad ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi^2$$

Spherical Surface Distances:
alternative – geodetic distance x



$$ds = R\sin\theta\, d\phi = x\, d\phi$$

$$x \equiv R\sin\left(\frac{r}{R}\right) \quad \Rightarrow \quad dx = \cos\left(\frac{r}{R}\right)dr$$

$$\Rightarrow \quad dx^2 = \left\{1 - \sin^2\left(\frac{r}{R}\right)\right\}dr^2$$

$$\kappa = \frac{1}{R^2} \quad \Rightarrow \quad dr^2 = \frac{dx^2}{1-\kappa x^2}$$

$$ds^2 = dr^2 + R^2\sin^2\left(\frac{r}{R}\right)d\phi^2$$

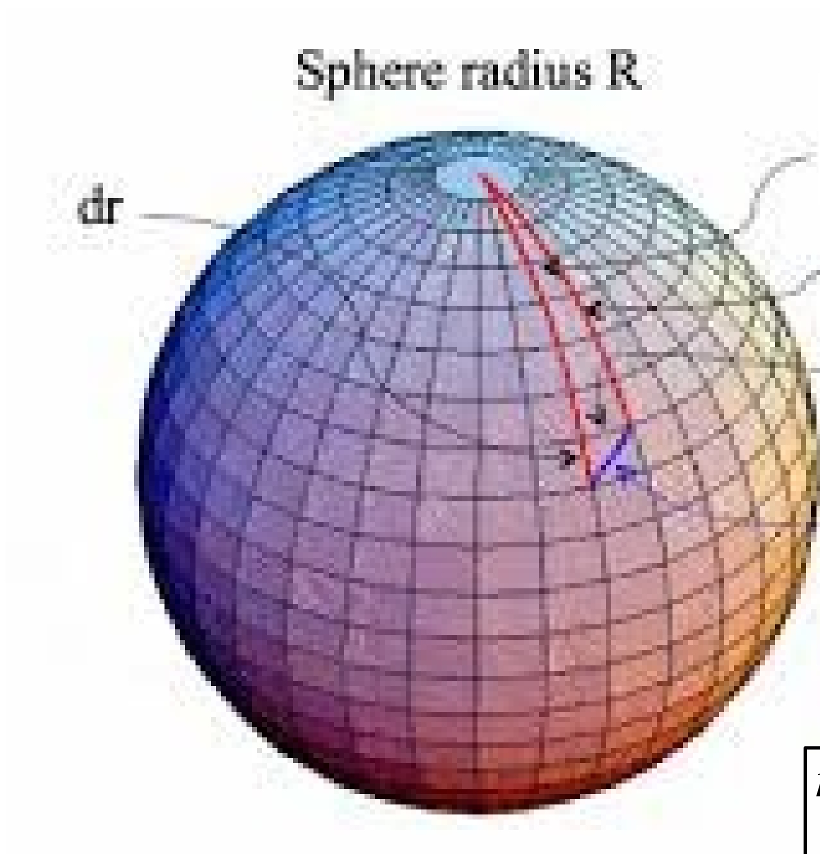
$$\Downarrow$$

$$ds^2 = \frac{dx^2}{1-\kappa x^2} + x^2 d\phi^2$$

Spherical Space Distances

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left\{d\theta^2 + \sin^2\theta\, d\phi^2\right\}$$

Spherical surface is a 2-D section through an isotropic curved 3D space: generalization to 3D solid angle (θ,ϕ)



$d\phi$

r

$R\sin\theta\, d\phi$

$$ds^2 = dr^2 + R^2 \sin^2\theta\, d\phi^2$$

$r = R\theta$

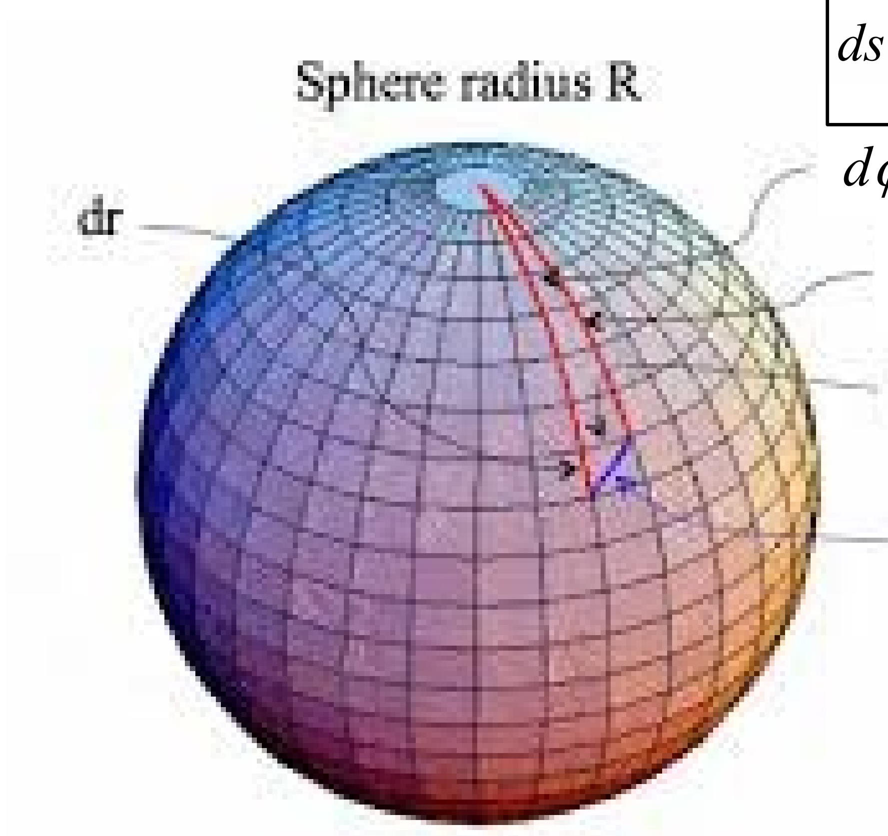
$dr = R\, d\theta$

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi^2$$

Spherical Space Distances

alternative: geodetic distance

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left\{d\theta^2 + \sin^2\theta\, d\phi^2\right\}$$



$d\phi$

r

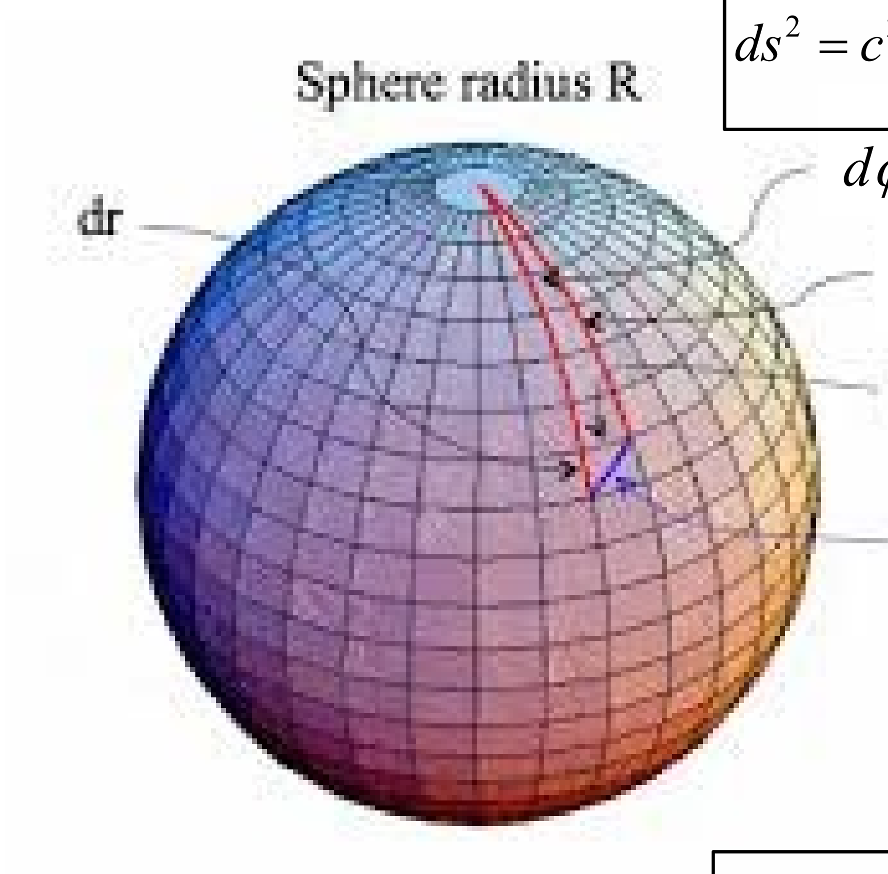
$R \sin\theta\, d\phi$

$$ds^2 = dr^2 + R^2 \sin^2\theta\, d\phi^2$$

$$ds^2 = \frac{dx^2}{1-\kappa x^2} + x^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Minkowski Metric
spherically isotropic 3D space

$$ds^2 = c^2 dt^2 - \left[dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left\{d\theta^2 + \sin^2\theta\, d\phi^2\right\}\right]$$



$d\phi$

r

$R \sin\theta\, d\phi$

$$ds^2 = dr^2 + R^2 \sin^2\theta\, d\phi^2$$

$$\Downarrow$$

$$ds^2 = c^2 dt^2 - \left[\frac{dx^2}{1-\kappa x^2} + x^2 (d\theta^2 + \sin^2\theta d\phi^2)\right]$$

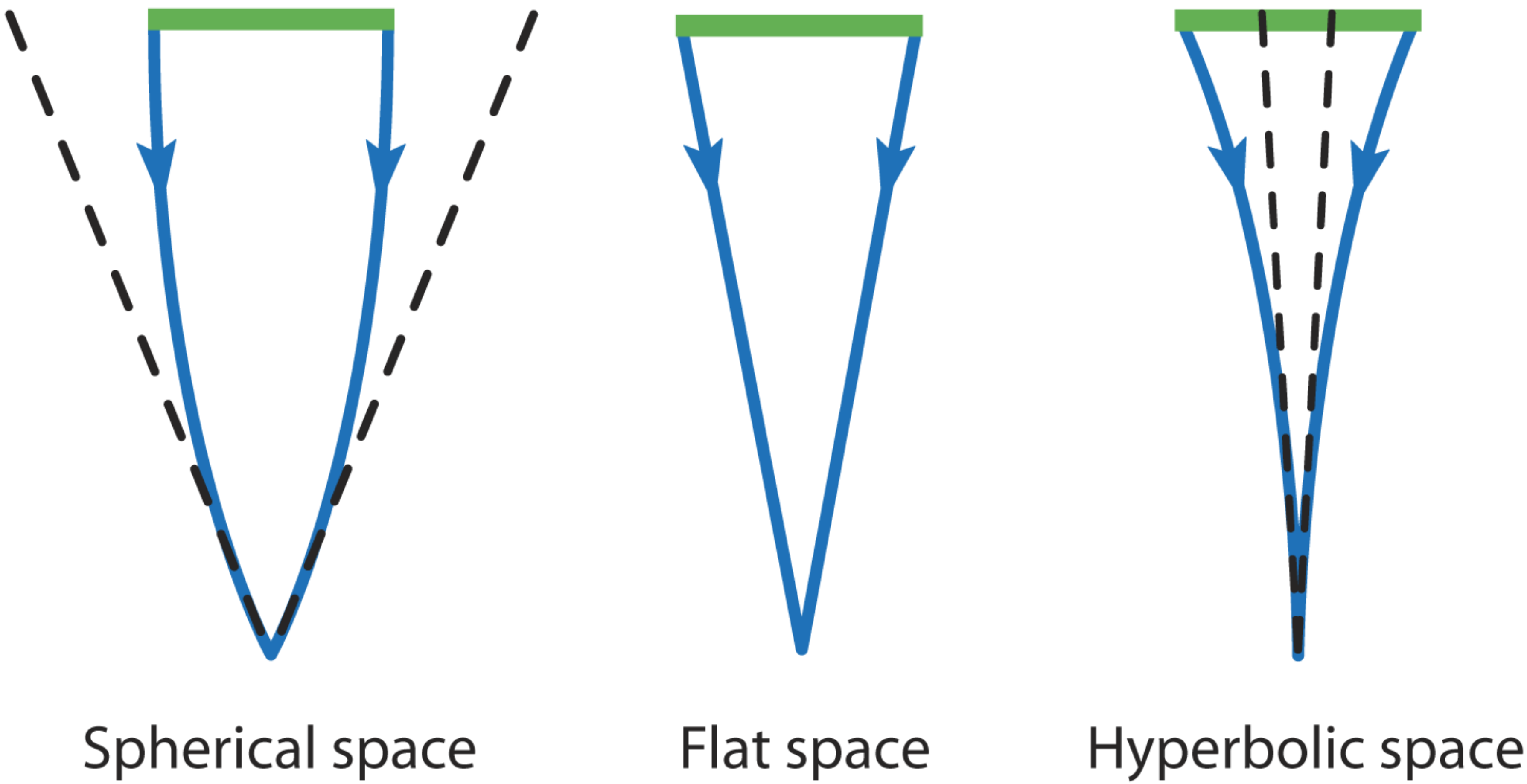
Robertson-Walker Metric

Distances in a uniformly curved spacetime is specified in terms of the Robertson-Walker metric. The spacetime distance of a point at coordinate (r,θ,ϕ) is:

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2 \left(\frac{r}{R_c} \right) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

where the function $S_k(r/R_c)$ specifies the effect of curvature on the distances between points in spacetime

$$S_k\left(\frac{r}{R_c}\right) = \begin{cases} \sin\left(\frac{r}{R_c}\right) & k = +1 \\ \frac{r}{R_c} & k = 0 \\ \sinh\left(\frac{r}{R_c}\right) & k = -1 \end{cases}$$


Spherical space
Flat space
Hyperbolic space

Conformal Time

Conformal Time

Proper time τ

Robertson-Walker metric

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c^{\,2} S_k^{\,2}\left(\frac{r}{R_c}\right) d\psi^2 \right\}$$

$$d\psi^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$\eta(t) = \int_0^t \frac{c\,dt}{R(t)}$$

Conformal Time η(t)

$$d\eta_\tau^{\,2} = \frac{c^2 d\tau^2}{R^2} = \frac{c^2 dt^2}{R^2} - \left\{ d\left(\frac{r}{R_c}\right)^2 + S_k^{\,2}\left(\frac{r}{R_c}\right) d\psi^2 \right\}$$

$$= d\eta^2 - \left\{ dr'^2 + S_k^{\,2}(r')\, d\psi^2 \right\}$$

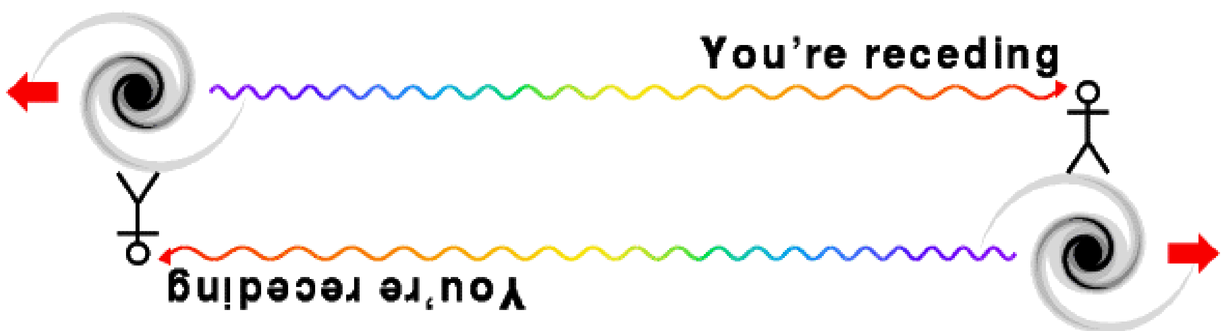
Observational Cosmology
in
FRW Universe

Redshift

Cosmic Redshift

$$1 + z = \frac{1}{a} \quad \Longleftarrow \quad \begin{cases} \lambda_{em} = \lambda_0 \\ \\ \lambda_{obs} = \dfrac{a(t_{obs})}{a(t_{em})} \lambda_0 \end{cases}$$

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$



Cosmic Time Dilation

Cosmic Time Dilation

In an (expanding) space with Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2 \left(\frac{r}{R_c} \right) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

In a RW metric, light travels with

$$ds = 0: \qquad c^2 dt^2 - a(t)^2 dr^2 = 0$$

$$\Downarrow$$

$$cdt = a(t)dr: \qquad \frac{dt}{a(t)} = cst.$$

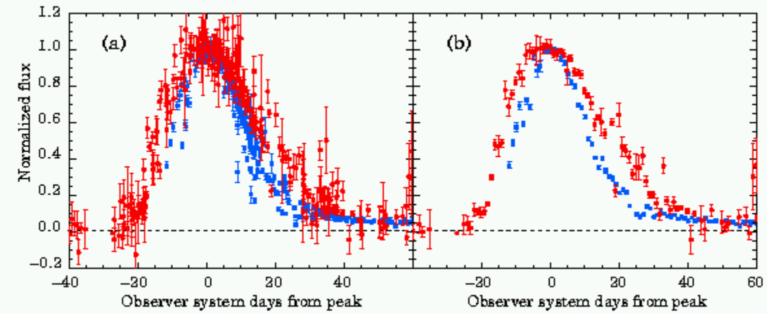
$$\Downarrow$$

Cosmic Time Dilation:

$$\Delta t_{obs} = \frac{\Delta t_e}{a(t_e)}$$

Cosmic Time Dilation

$$\Delta t_{obs} = \frac{\Delta t_e}{a(t_e)}$$

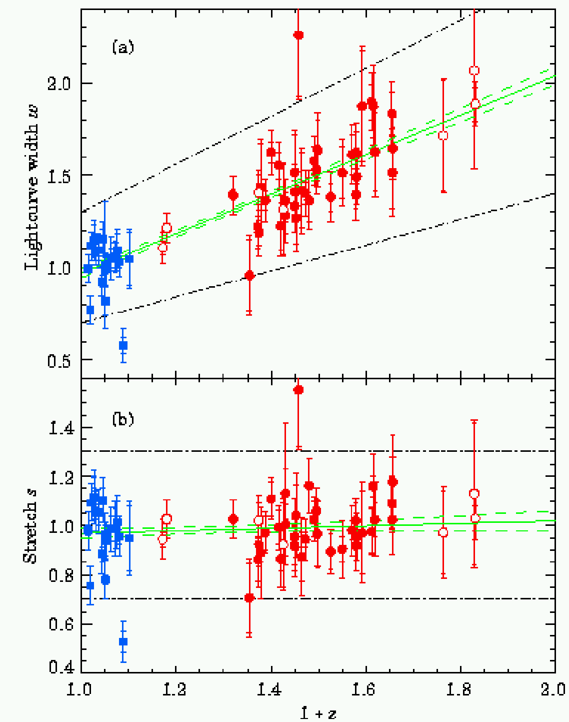


Evidence Cosmic Time Dilation:

light curves supernovae (exploding stars):

characteristic time interval over which the supernova rises and then dims:

systematic shift with redshift (depth)



Hubble Expansion

Expanding Universe

- Einstein, de Sitter, Friedmann and Lemaitre all realized that in General Relativity, there cannot be a stable and static Universe:
- The Universe either expands, or it contracts …

- Expansion Universe encapsulated in a

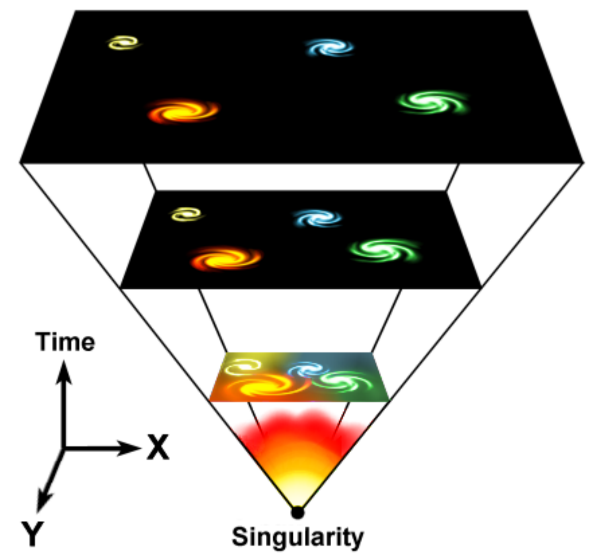
GLOBAL expansion factor a(t)

- All distances/dimensions of objects uniformly increase by a(t):

  at time t, the distance between two objects i and j has increased to

$$\vec{r}_i - \vec{r}_j = a(t)\left(\vec{r}_{i,0} - \vec{r}_{j,0}\right)$$

- Note: by definition we chose $a(t_o)=1$, i.e. the present-day expansion factor



Interpreting Hubble Expansion

- Cosmic Expansion manifests itself in the
  in a recession velocity which linearly increases with distance

- this is the same for any galaxy within the Universe !

- There is no centre of the Universe:
  would be in conflict with the Cosmological Principle

Hubble Expansion

- Cosmic Expansion is a uniform expansion of space
- Objects do not move themselves:
  they are like beacons tied to a uniformly expanding sheet:

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$H(t) = \frac{\dot{a}}{a}$$

Hubble Expansion

- **Cosmic Expansion is a uniform expansion of space**

- **Objects do not move themselves:**
  they are like beacons tied to a uniformly ex

Comoving Position

Hubble Parameter:

Hubble "constant":
$H_o \equiv H(t=t_o)$

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$H(t) = \frac{\dot{a}}{a}$$

Hubble Parameter

- For a long time, the correct value of the Hubble constant H_o was a major unsettled issue:

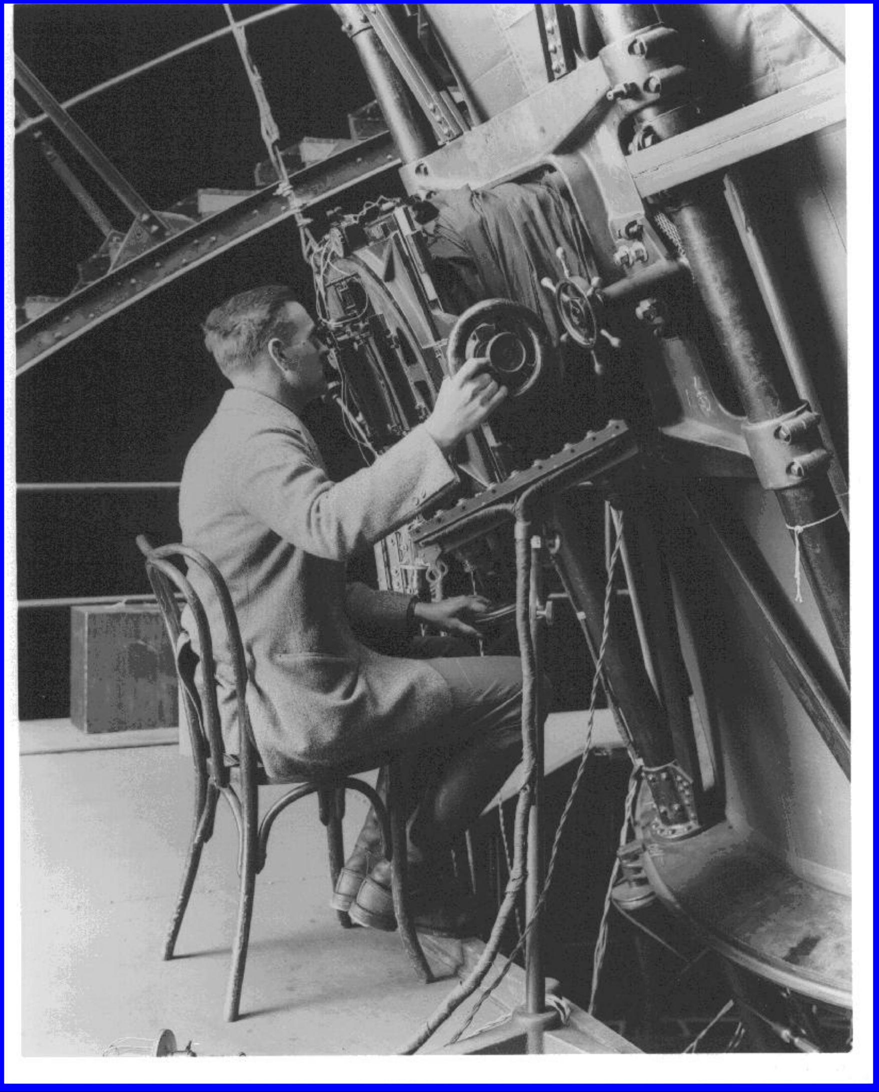
H_o = 50 km s^{-1} Mpc^{-1} ⟷ H_o = 100 km s^{-1} Mpc^{-1}

- This meant distances and timescales in the Universe had to deal with uncertainties of a factor 2 !!!

- Following major programs, such as Hubble Key Project, the Supernova key projects and the WMAP CMB measurements,

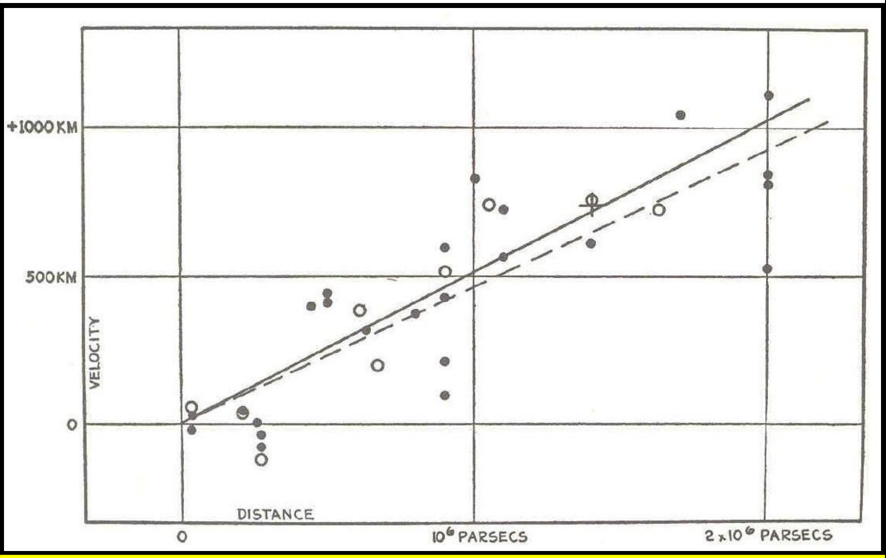
$$H_0 = 71.9^{+2.6}_{-2.7}\, km\, s^{-1} Mpc^{-1}$$

Hubble Expansion



Edwin Hubble

(1889-1953)



$v = H\ r$

Hubble Expansion

Hubble Parameter

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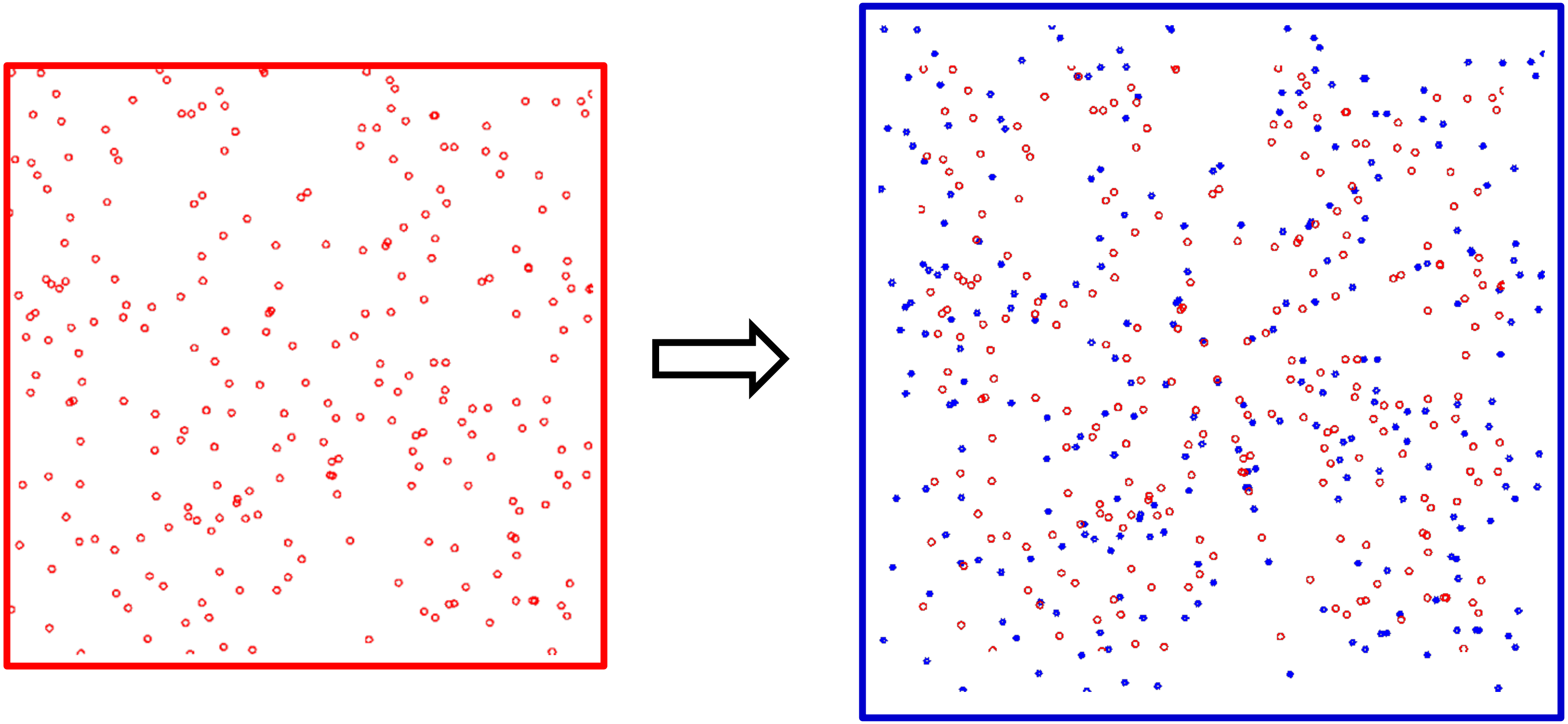
Hubble Expansion

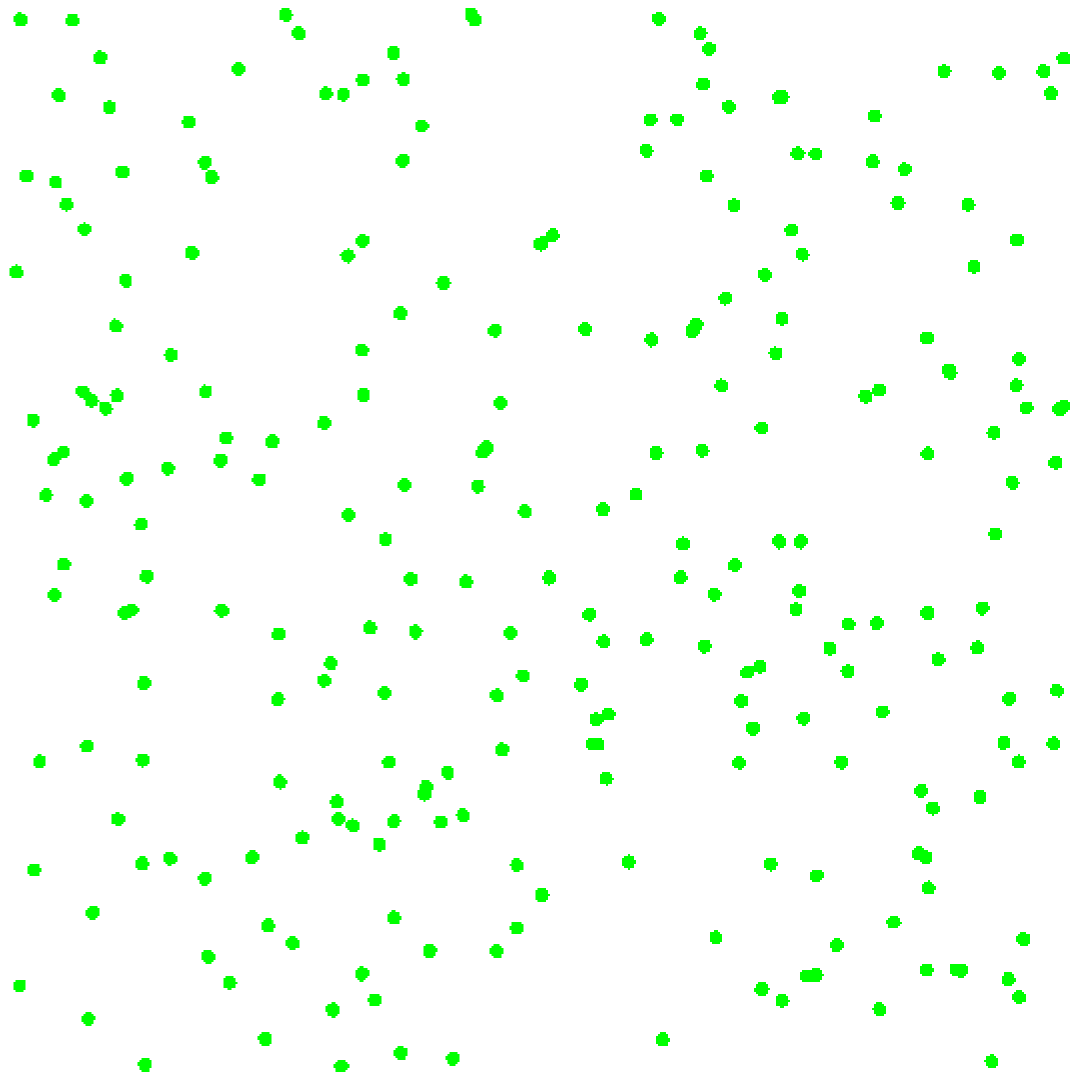
Space expands:

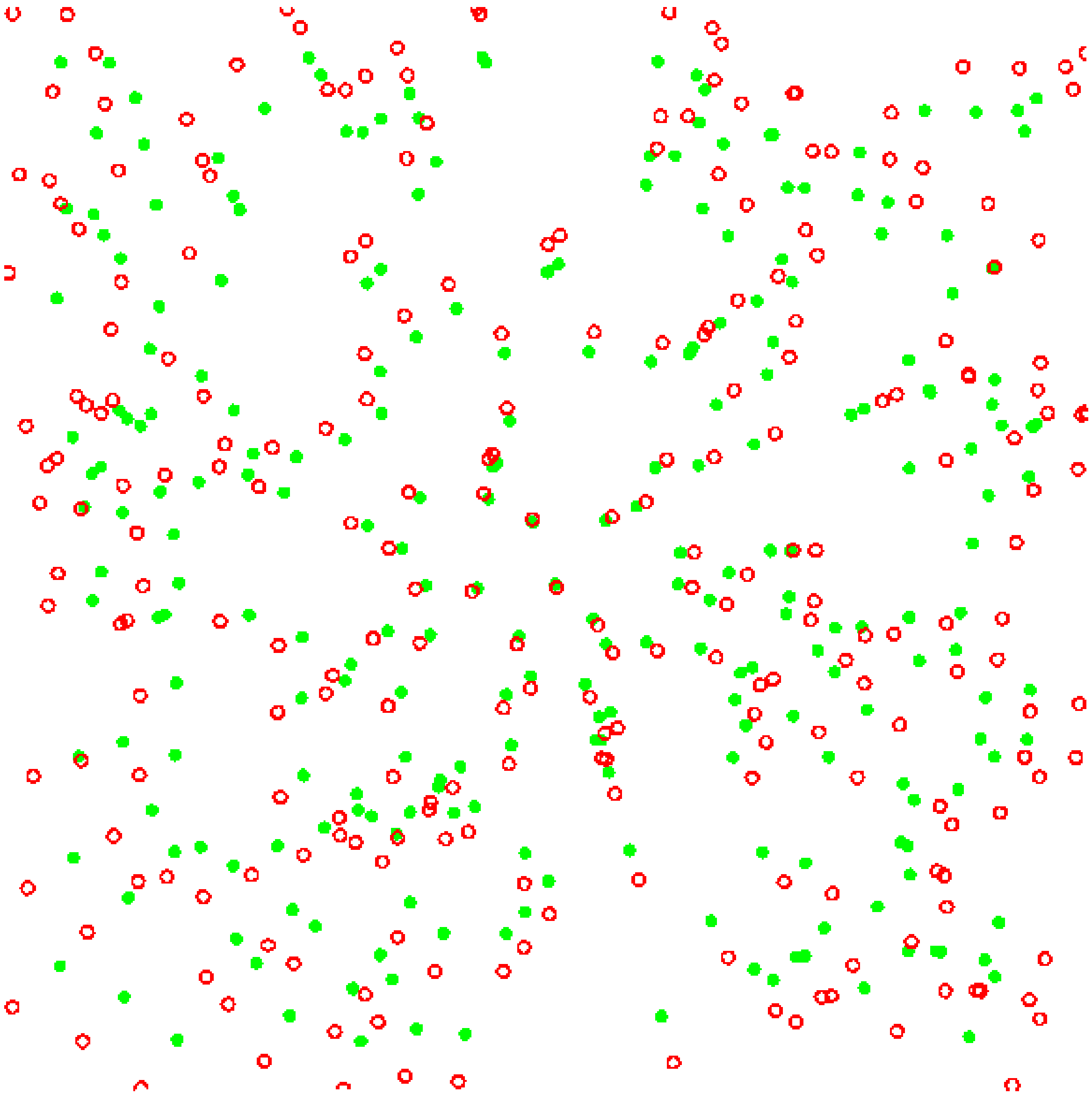
displacement - distance ⟹

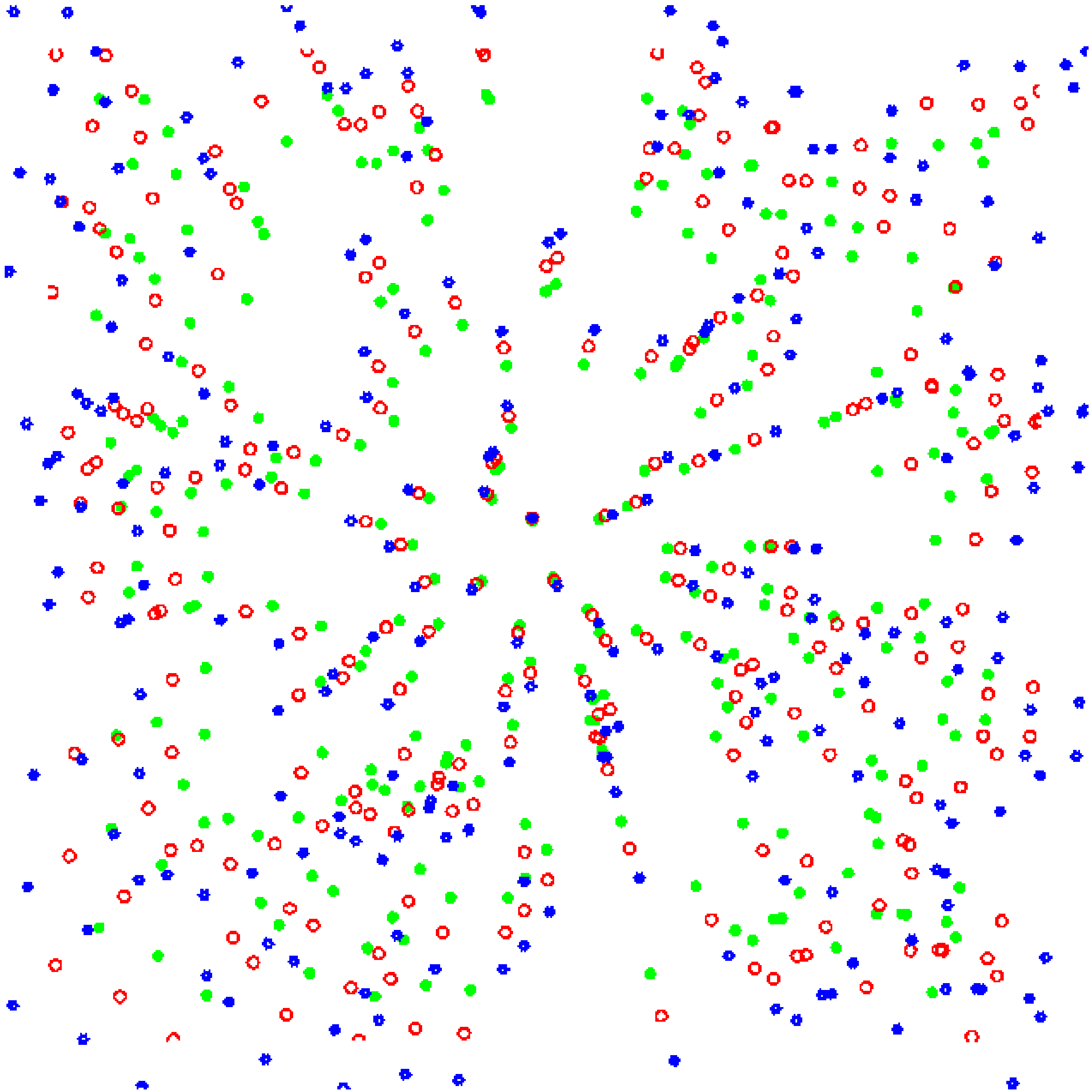
$$v = H\ r$$

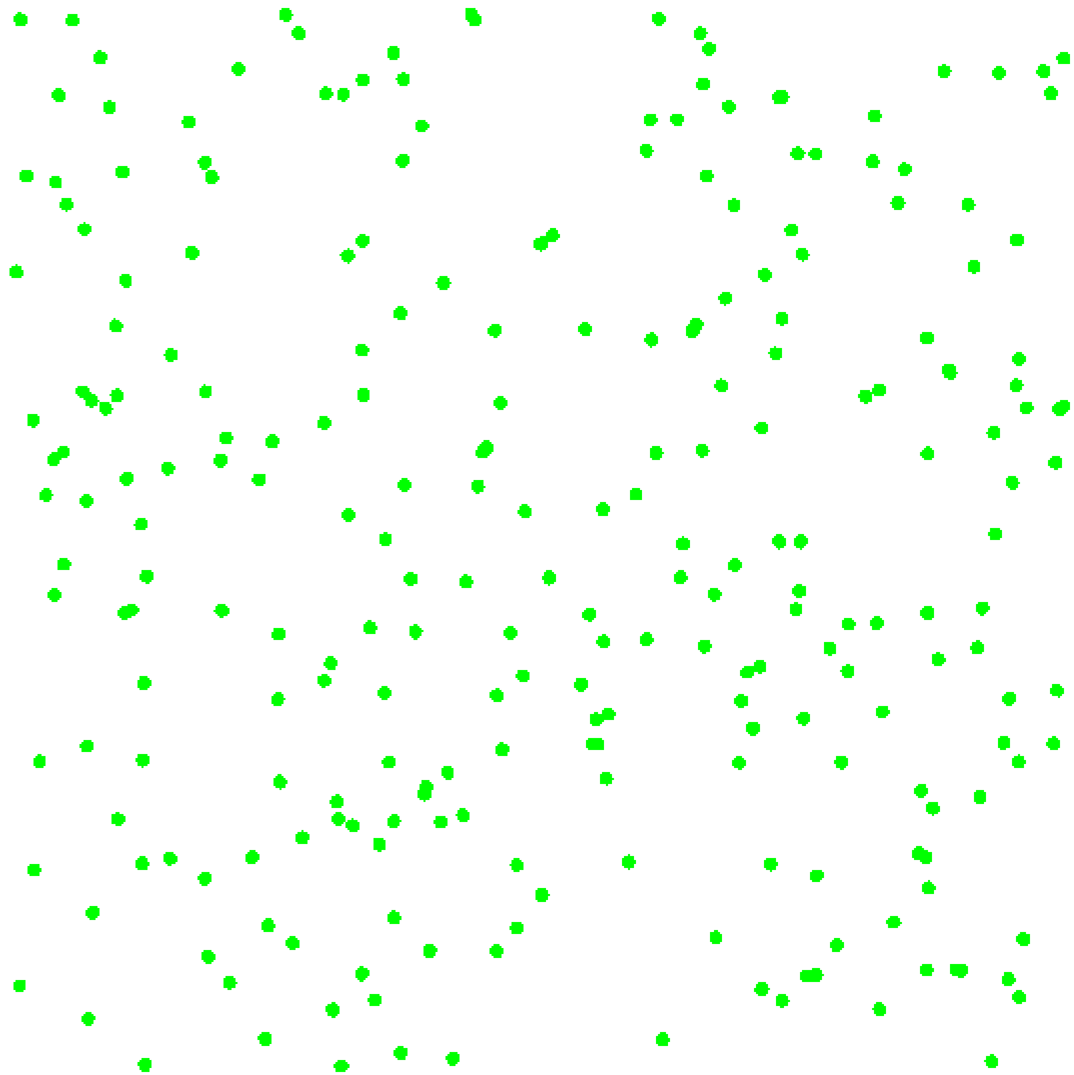
Hubble law: velocity - distance

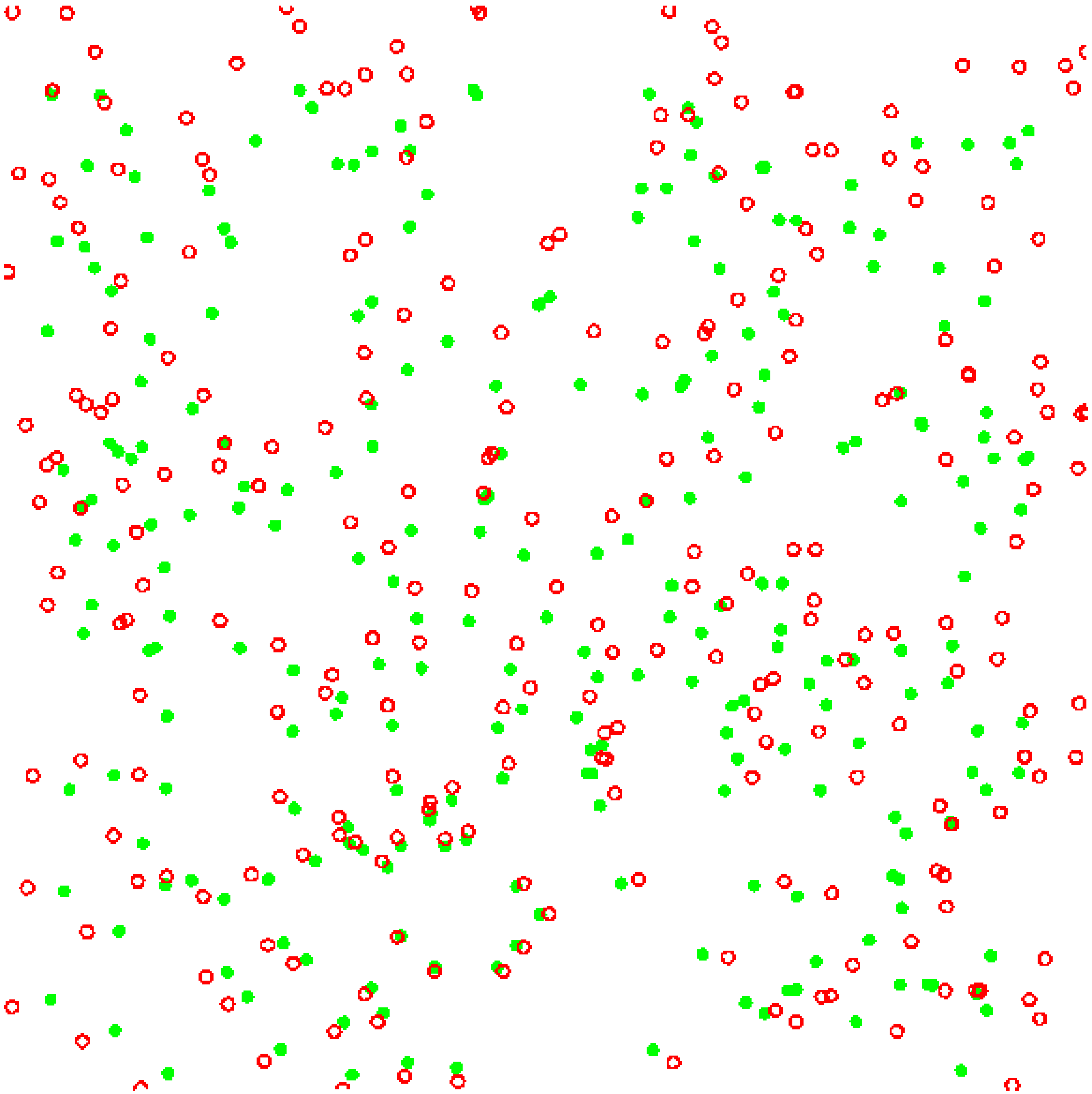


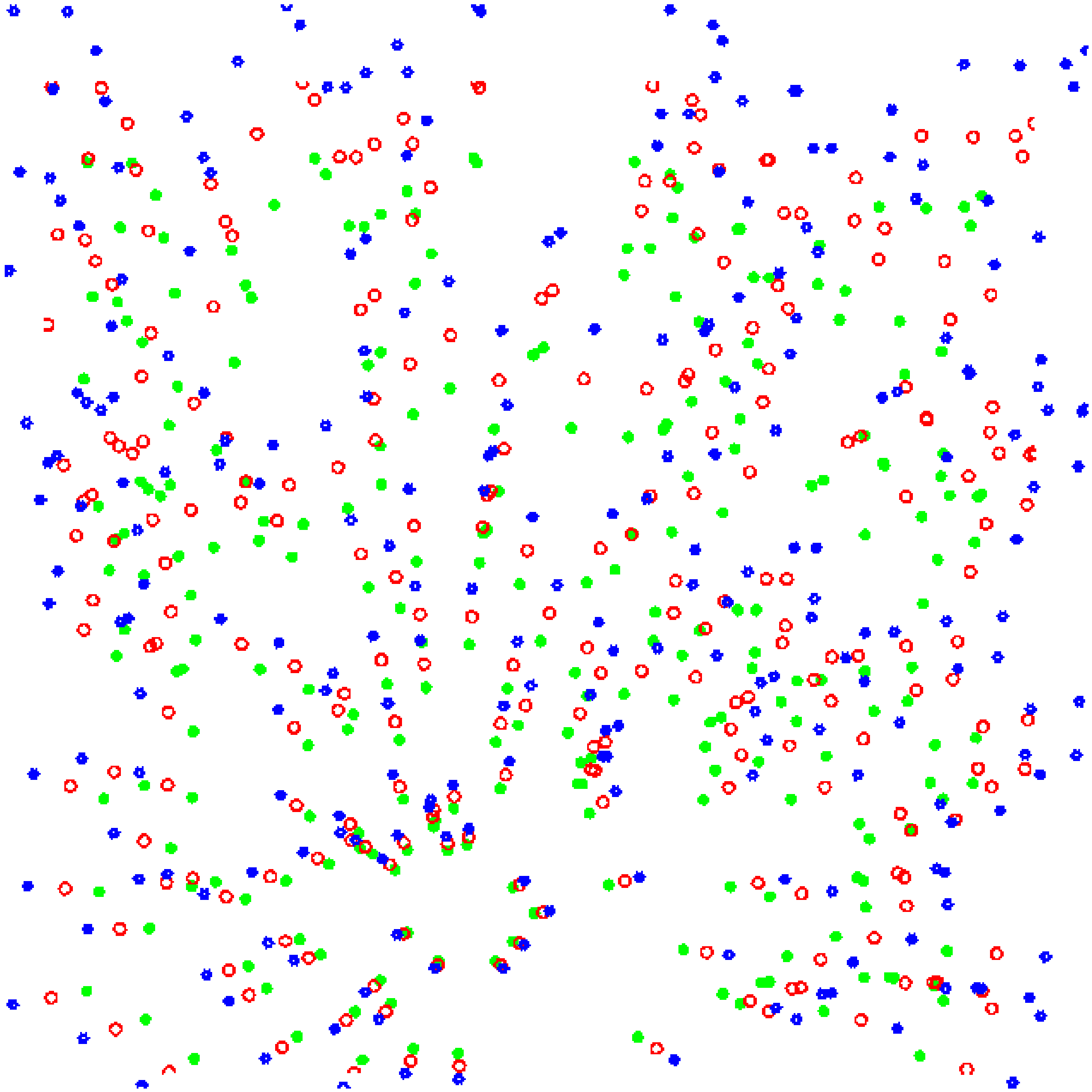




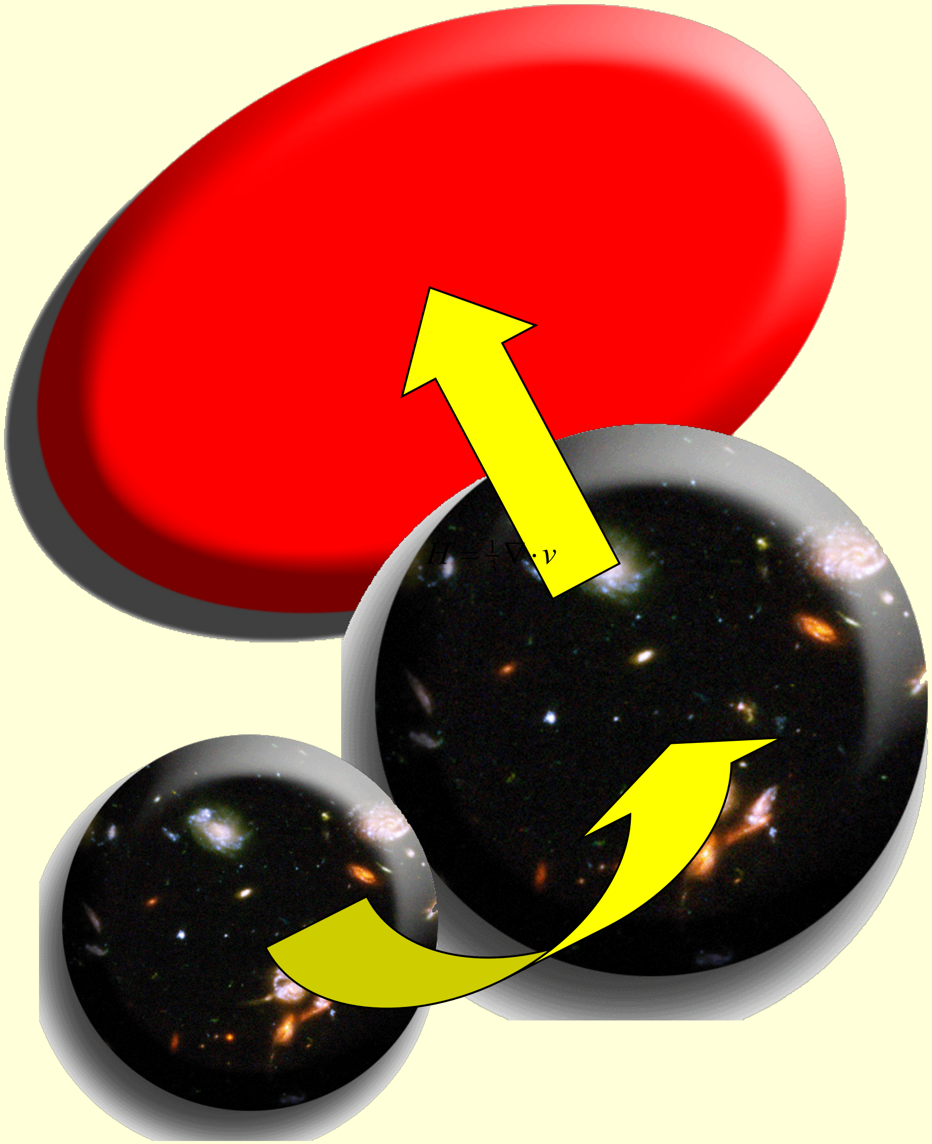








Deformation
Cosmic Volume Element



The evolution of a fluid element on its path through space may be specified by its velocity gradient:

$$\frac{1}{a}\frac{\partial v_i}{\partial x_j} = \frac{1}{3}\theta\,\delta_{ij} + \sigma_{ij} + \omega_{ij}$$

in which

θ: velocity divergence

→ **contraction/expansion**

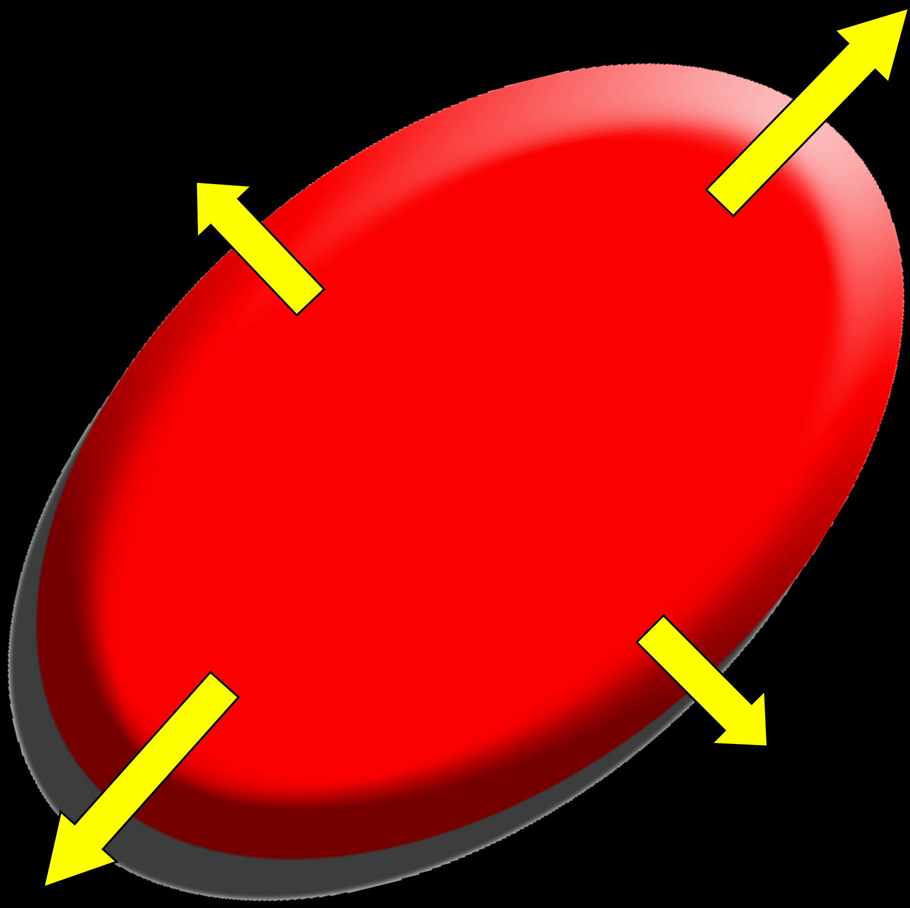
σ: velocity shear

→ **deformation**

ω: vorticity

→ **rotation of element**

Deformation
Cosmic Volume Element

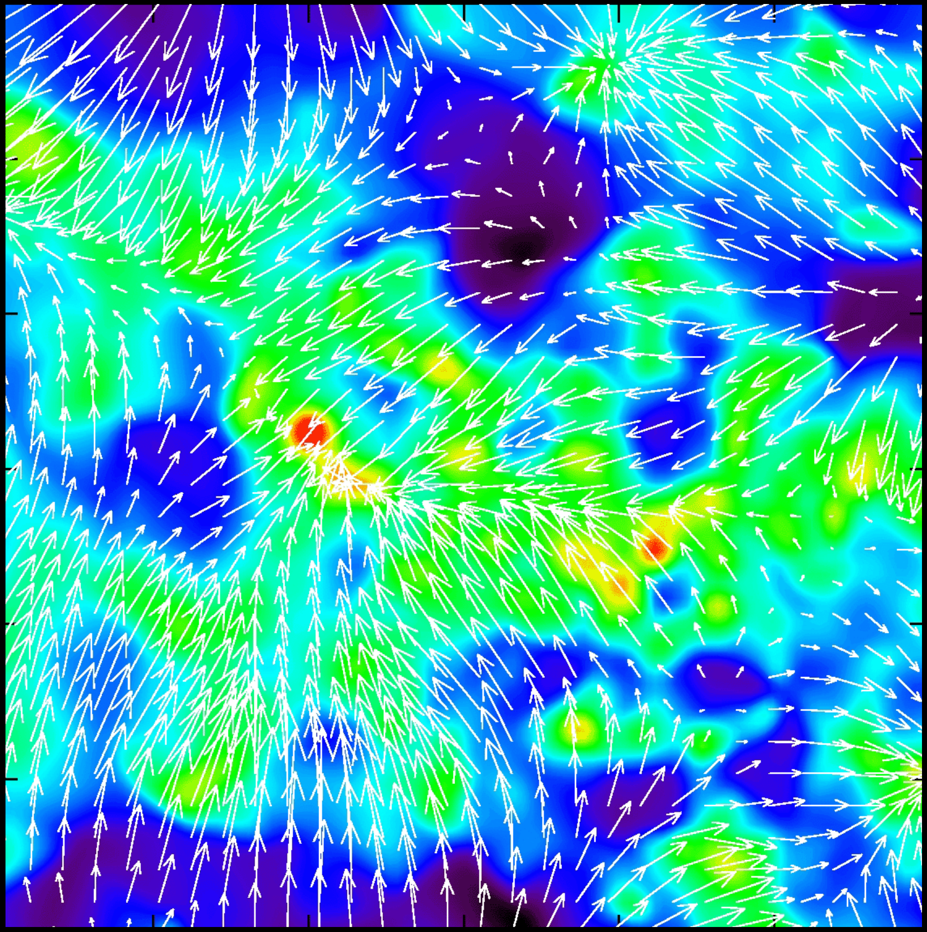


Global Anisotropic expansion/contraction

Anisotropic Relativistic Universe Models:
Bianchi I-IX Universe models

- expand anisotropically
- have to be characterized by at least 3 Hubble parameters (expansion rate different in different directions
- Only marginal claims indicate the possibility on the basis of CMB anisotropies

Deformation
Cosmic Volume Element

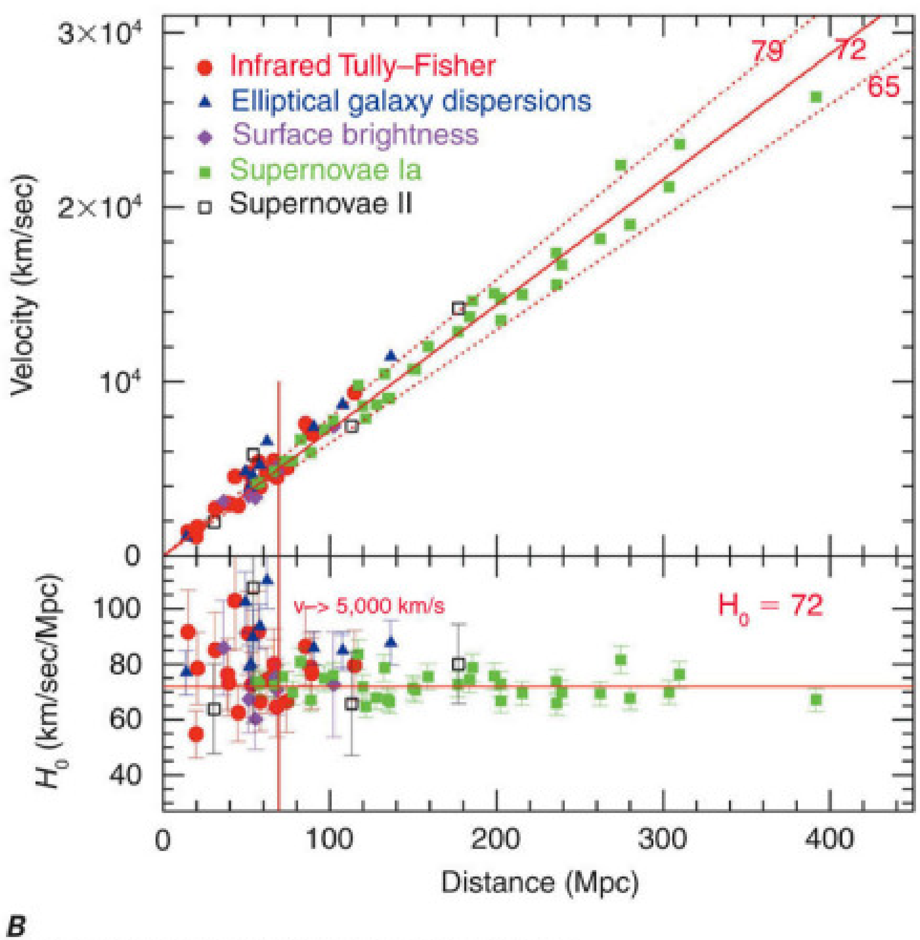


Local Anisotropic Flows:
“fatal” attractions

- **In our local neighbourhood the cosmic flow field has a significant shear**

- **This shear is a manifestation of**
  - infall of our Local Group into the Local Supercluster
  - motion towards the Great Attractor
  - possibly motion towards even larger mass entities: Shapley concentration
    Horologium supercluster

Deformation
Cosmic Volume Element



(Wendy L. Freedman, Observatories of the Carnegie Institution of Washington, and NASA)

Global Hubble Expansion
Observations over large regions
of the sky, out to large cosmic depth:

- **the Hubble expansion offers a very good description of the actual Universe**
- **the Hubble expansion is the same in whatever direction you look: isotropic**

Hubble flow:

$$H = \frac{1}{3}\nabla \cdot v$$

Pure expansion/contraction

Cosmic Distances

Light paths RW space

In an (expanding) space with Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c^{\ 2} S_k^{\ 2}\left(\frac{r}{R_c}\right)\left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

radial comoving distance r travelled by radiation

in a RW space:

$$ds = 0: \qquad c^2 dt^2 - a(t)^2 dr^2 = 0$$

$$\Downarrow$$

$$cdt = a(t)dr$$

$$r = r(t_0) - r(t_e) = \int_{t_e}^{t_0} \frac{cdt}{a(t)}$$

Distance Measure

RW Distance Measure

In an (expanding) space with Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + {R_c}^2 {S_k}^2\left(\frac{r}{R_c}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2 \right] \right\}$$

there are several definitions for distance, dependent on how you measure it.

They all involve the central distance function, the *RW Distance Measure,*

$$D(r) = R_c S_k\left(\frac{r}{R_c}\right)$$

RW Redshift-Distance

Light propagation in a RW metric (curved space):

$$\left.\begin{aligned} ds^2 &= 0 \\ d\psi^2 &= 0 \end{aligned}\right\} \Rightarrow c\,dt = -R(t)\,dr$$

Note: - light propagation is along radial lines

- the "-" sign is an expression for the fact that the light ray propagating towards you moves in opposite direction of radial coordinate r

$$R_0 dr = \frac{c}{H(z)} dz$$

After some simplification and reordering, we find

$$c\,dt = c\frac{dR}{\left(\frac{dR}{dt}\right)} = c\frac{dR}{\dot{R}} = c\frac{dR}{HR}$$

$$R(t) = a(t)R_0 = \frac{R_0}{1+z}$$

$$\Downarrow$$

$$\frac{dR}{R} = -\frac{1}{1+z}dz$$

RW Redshift-Distance

Observing in a FRW Universe, we locate galaxies in terms of their redshift z. To connect this to their true physical distance, we need to know what the coordinate distance r of an object with redshift z,

$$R_0 dr = \frac{c}{H(z)} dz$$

In a FRW Universe, the dependence of the Hubble expansion rate H(z) at any redshift z depends on the content of matter, dark energy and radiation, as well ss its curvature. This leads to the following explicit expression for the redshift-distance relation,

$$R_0 dr = \frac{c}{H_0} \left\{ \left(1-\Omega_0\right)\left(1+z\right)^2 + \Omega_{\Lambda,0} + \Omega_{m,0}\left(1+z\right)^3 + \Omega_{rad,0}\left(1+z\right)^4 \right\}^{-1/2} dz$$

Matter-Dominated FRW Universe

in a matter-dominated Universe, the redshift-distance relation is

$$R_0 dr = \frac{c}{H_0}\left\{\left(1-\Omega_0\right)\left(1+z\right)^2+\Omega_0\left(1+z\right)^3\right\}^{-1/2} dz$$

from which one may find that

$$R_0 r = \frac{c}{H_0}\int_0^z \frac{dz'}{\left(1+z'\right)\sqrt{1+\Omega_0 z'}}$$

Mattig's Formula

The integral expression

$$R_0 r = \frac{c}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_0 z'}}$$

can be evaluated by using the substitution:

$$u^2 = \frac{k(\Omega_0 - 1)}{\Omega_0 (1+z)}$$

This leads to Mattig's formula:

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left\{\sqrt{1+\Omega_0 z} - 1\right\}}{\Omega_0^{\ 2}(1+z)}$$

This is one of the very most important and most useful equations in observational cosmology.

Mattig's Formula

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left\{\sqrt{1+\Omega_0 z} - 1\right\}}{\Omega_0{}^2 (1+z)}$$

In a low-density Universe, it is better to use the following version:

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) = \frac{c}{H_0}\frac{z}{1+z}\frac{1+\sqrt{1+\Omega_0 z}}{1+\sqrt{1+\Omega_0 z} + \Omega_0 z/2}$$

For a Universe with a cosmological constant, there is not an easily tractable analytical expression (a Mattig's formula). The comoving Distance r has to be found through a numerical evaluation of the fundamental dr/dz expression.

Distance-Redshift Relation, 2nd order

For all general FRW Universe, the second-order distance-redshift relation is identical, only depending on the *deceleration parameter* q_0:

$$D(z) = R_c S_k\left(\frac{r}{R_c}\right) \simeq \frac{c}{H_0}\left(z - \frac{1}{2}\left(1+q_0\right)z^2\right)$$

q_0 can be related to Ω_0 once the *equation of state* is known.

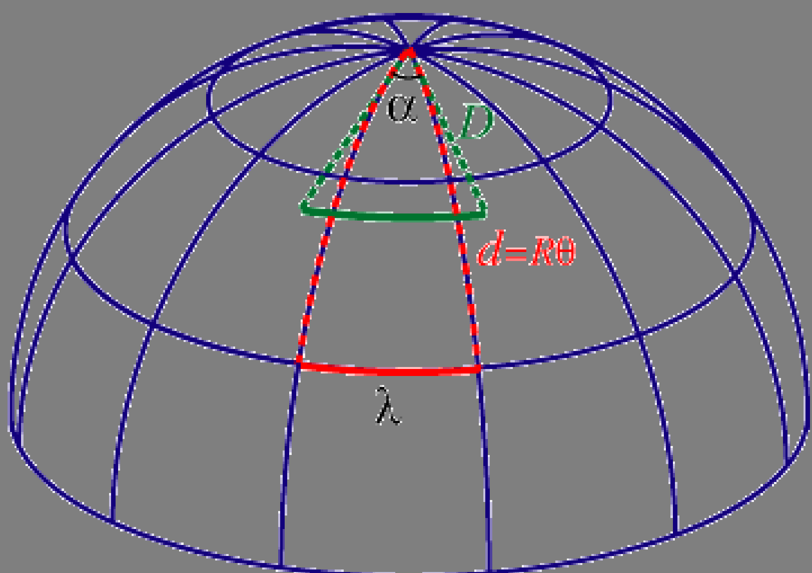
Angular Diameter Distance

Luminosity Distance

Angular Diameter Distance

Imagine an object of *proper size d,* at redshift z, its angular size ▯▯ is given by

$$d = a(t)\, R_c S_k\left(\frac{r}{R_c}\right)\Delta\theta \quad \Rightarrow \quad \Delta\theta = \frac{d(1+z)}{D} = \frac{d}{D_A}$$



Angular Diameter distance:

$$D_A = \frac{D}{1+z}$$

Luminosity Distance

Imagine an object of luminosity $L(\nu_e)$, at redshift z, its flux density at observed frequency ν_o is

$$S(\nu_o) = \frac{L(\nu_e)}{4\pi D^2 (1+z)} \Rightarrow S_{bol} = \frac{L_{bol}}{4\pi D^2 (1+z)^2} = \frac{L_{bol}}{4\pi D_L{}^2}$$

Luminosity distance:

$$D_L = D(1+z)$$

Angular vs. Luminosity Distance

The relation between the Luminosity and the Angular Diameter distance of an object at redshift z is sometimes indicated as

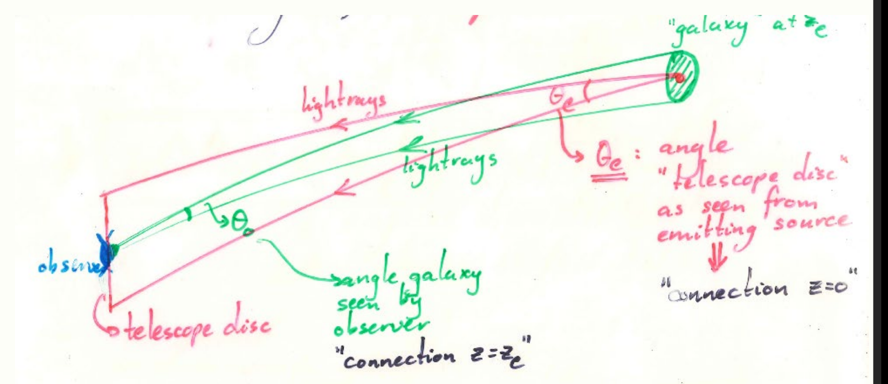
Reciprocity Theorem

The difference between these 2 fundamental cosmological measures stems from the fact that they involve "radial paths" measured in opposite directions along the lightcone, and thus are

forward - luminosity distance
backward - angular diameter distance

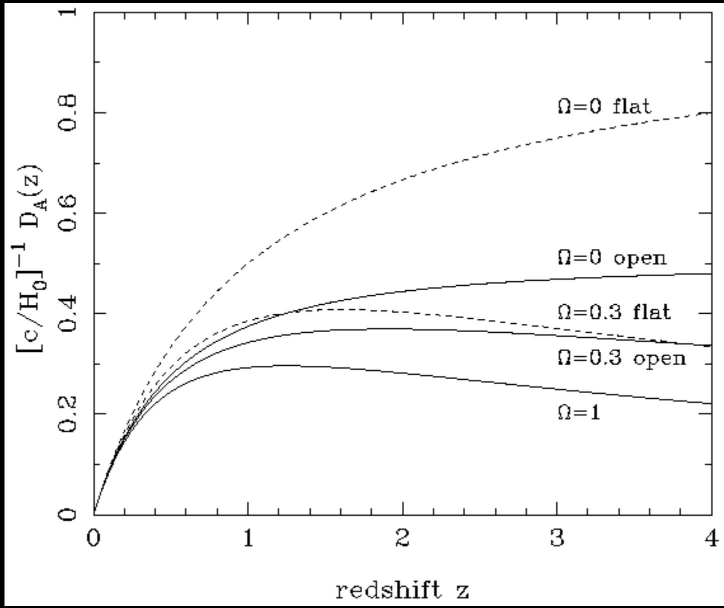
wrt. expansion of the Universe

$$\left.\begin{aligned} D_L &= D(1+z) \\ D_A &= D/(1+z) \end{aligned}\right\} \Rightarrow \frac{D_L}{D_A} = (1+z)^2$$



Angular Diameter Distance
matter-dominated FRW Universe

$$D_A = \frac{D}{1+z} = \frac{1}{1+z} R_c S_k\left(\frac{r}{R_c}\right)$$



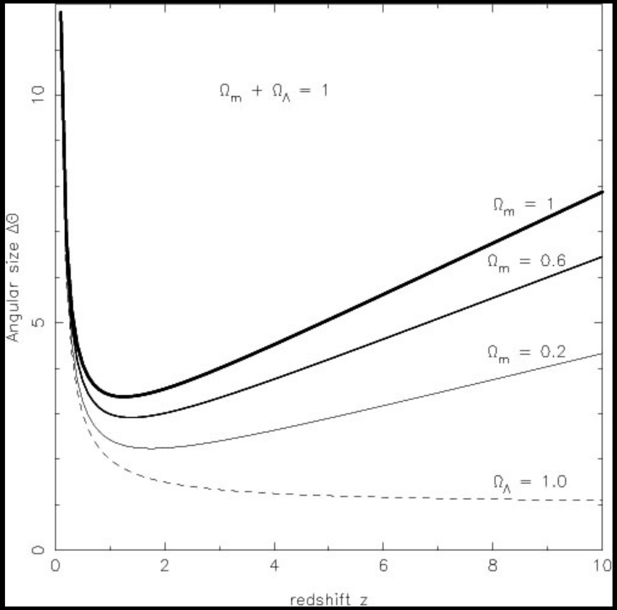
In a matter-dominated Universe, the angular diameter distance as function of redshift is given by:

$$D_A(z) = \frac{1}{1+z} R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{H_0}\frac{1}{\Omega_0^{\,2}\left(1+z\right)^2}\left\{\Omega_0 z + \left(\Omega_0 - 2\right)\left(\sqrt{1+\Omega_0 z} - 1\right)\right\}$$

Angular Size - Redshift
FRW Universe

$$\theta(z) = \frac{\ell}{D_A}$$

The angular size $\theta(z)$ of an object of physical size ℓ at a redshift z displays an interesting behaviour. In most FRW universes is has a minimum at a medium range redshift – z=1.25 in an Ω_m=1 EdS universe – and increases again at higher redshifts.



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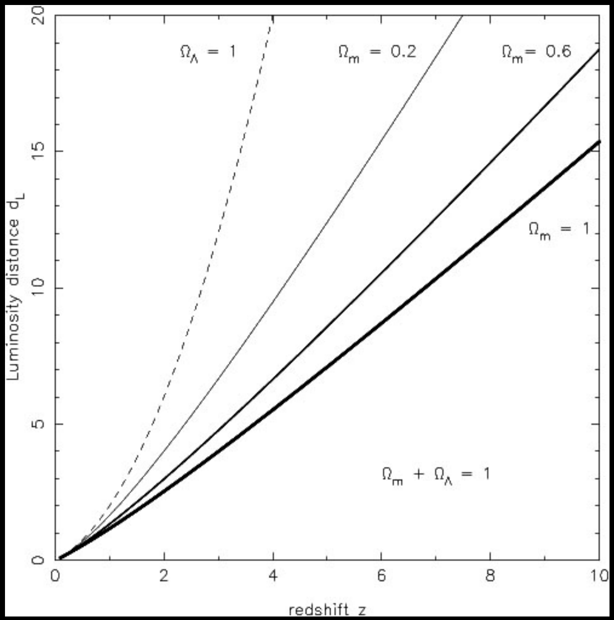
Luminosity Distance
matter-dominated FRW Universe

$$D_L = D(1+z) = (1+z)R_c S_k\left(\frac{r}{R_c}\right)$$



In a matter-dominated Universe, the luminosity distance as function of redshift is given by:

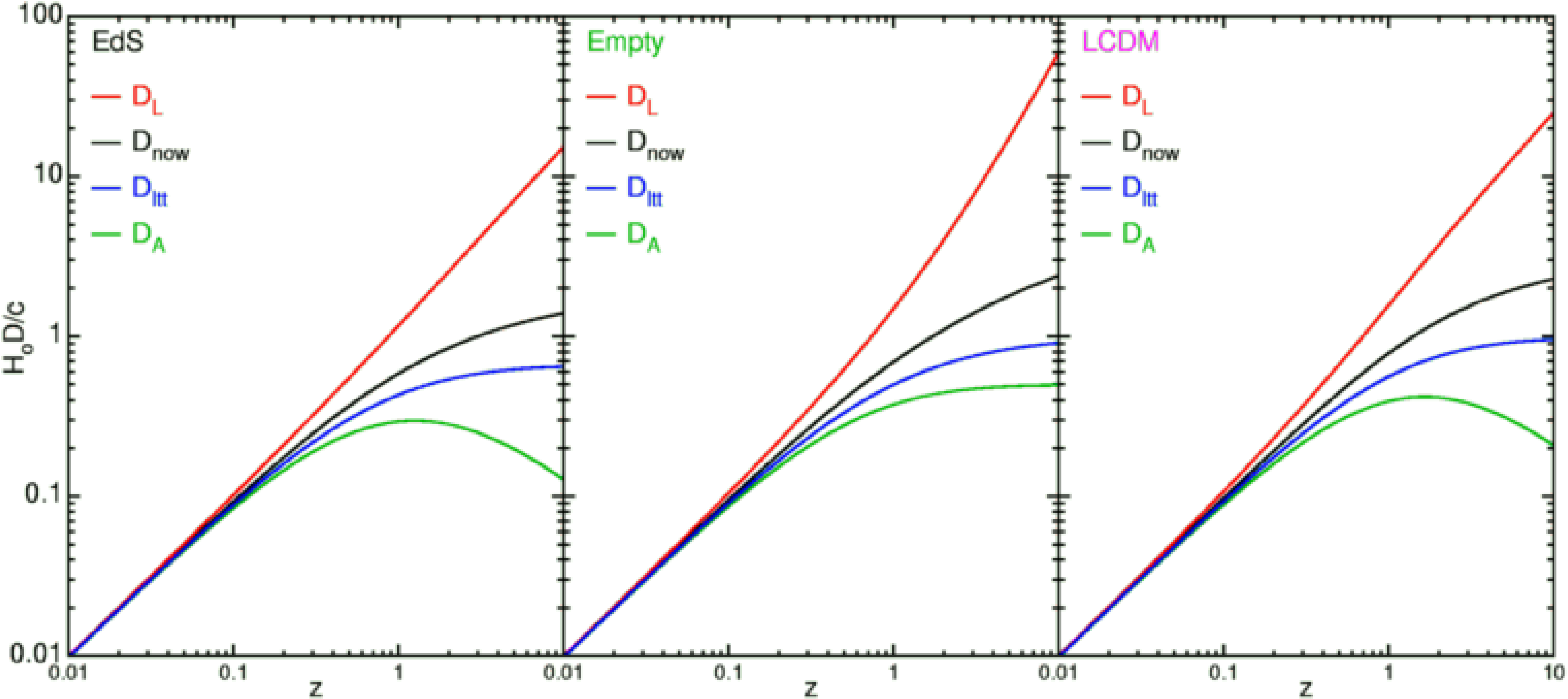
$$D_L(z) = (1+z)R_c S_k\left(\frac{r}{R_c}\right) = \frac{2c}{\Omega_0{}^2 H_0}\left\{\Omega_0 z + \left(\Omega_0 - 2\right)\left(\sqrt{1+\Omega_0 z} - 1\right)\right\}$$

Cosmological Distances:

Comparison

FRW Universe Distances

summary



Cosmology:

the search for 2 numbers

Cosmology, the search for 2 numbers

Sandage, ARAA 1970 :

Cosmology is the "Search for 2 numbers":

H_0 q_0

How to measure the values of H_0 and q_0, without any prior assumption on the dynamics, ie. of the particular FRWL cosmological model ? Ie. how to infer these numbers from observables:

- redshift
- luminosity
- angular size

- Establish relation expansion factor a(t) up to 2nd order (Taylor series):

$$a(t) \approx 1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0{}^2 (t - t_0)^2$$

Time of Emission - Redshift

- The corresponding redshift z of the source that emitted its radiation at time t_e:

$$z \approx H_0 \left(t_0 - t_e\right) + \left(\frac{1+q_0}{2}\right) {H_0}^2 \left(t_0 - t_e\right)^2$$

- whose inversion translates into the expression of the emission time t_e for a given redshift z:

$$t_0 - t_e \approx \frac{1}{H_0} \left\{ z - \left(\frac{1+q_0}{2}\right) z^2 \right\}$$

Coordinate Distance - Redshift

- Coordinate distance $d_P(t_0)$ of source whose radiation is emitted at t_e, and reached us at t_0:

$$d_P(t_0) = r = c\int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$d_P(t_0) \approx c\left(t_0 - t_e\right) + \frac{cH_0}{2}\left(t_0 - t_e\right)^2$$

- Using the relation between (t0-te) and redshift z, establishes the relation between coordinate distance $d_P(t_0)$ of source and z:

$$d_P(t_0) \approx \frac{c}{H_0} z \left\{1 - \frac{1+q_0}{2} z\right\}$$

Luminosity Distance - Redshift

- **Luminosity Distance**

$$d_L(z) \approx (1+z)\, d_P(t_0)$$

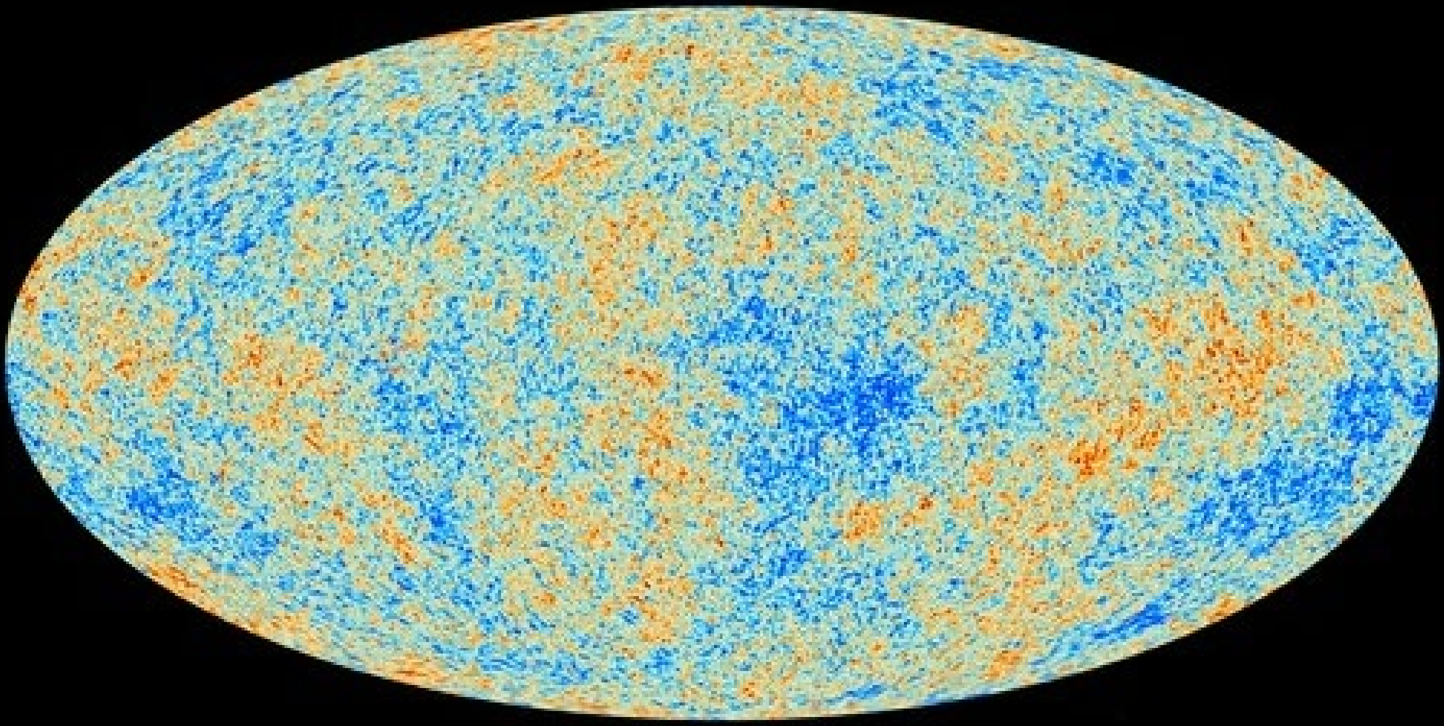
$$= \frac{c}{H_0} z \left\{ 1 + \frac{1-q_0}{2} z \right\}$$

- **In terms of an object at redshift z, with absolute bolometric magnitude M_{bol}, we may infer the acceleration parameter q_0 from:**

$$m_{bol} \approx M_{bol} + 5\log\left[\frac{c}{H_0} / 10\, pc\right] + 5\log z + 1.086(1-q_0)z + O(z^2)$$

Cosmic Curvature Measured

Cosmic Microwave Background



Map of the Universe at Recombination Epoch (Planck, 2013):

- **379,000 years after Big Bang**
- **Subhorizon perturbations: primordial sound waves**
- **$\Delta T/T < 10\text{-}5$**

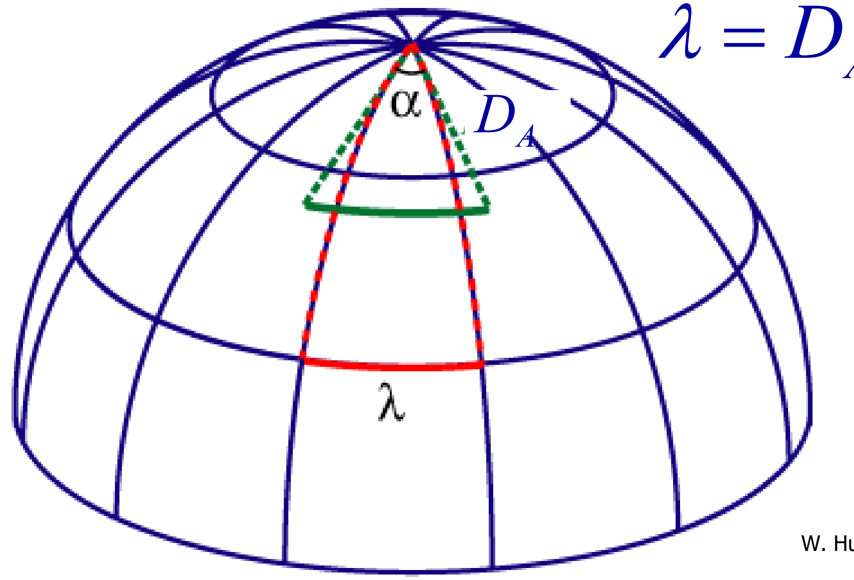
Measuring Curvature

Measuring the Geometry of the Universe:

- **Object with known physical size, at large cosmological distance**
- **Measure angular extent on sky**
- **Comparison yields light path, and from this the curvature of space**

Geometry of Space

$$\lambda = D_A \alpha$$



FRW Universe:
lightpaths described by Robertson-Walker metric

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2\left(\frac{r}{R_c}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right\}$$

Here: angular diameter distance D_A: $\lambda = D_A \alpha$

Angular Size - Redshift
FRW Universe

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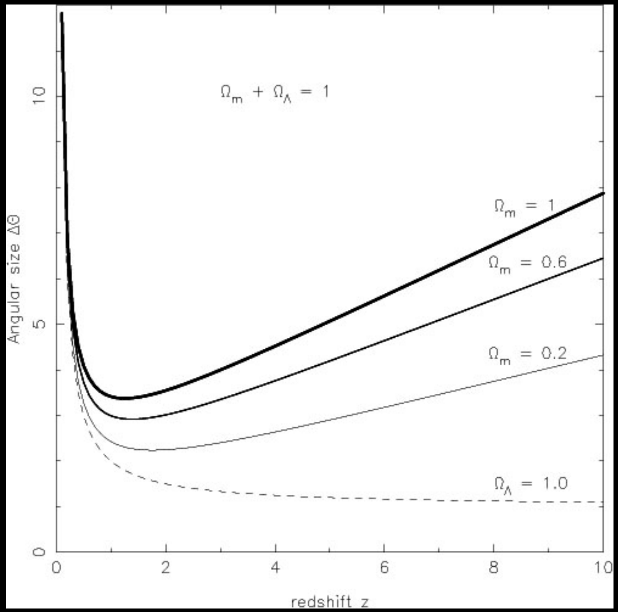

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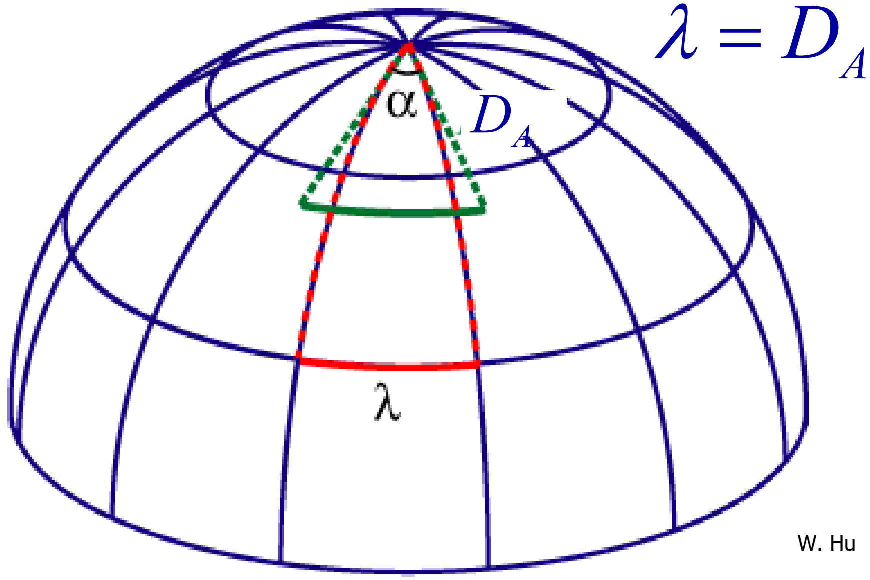
Measuring Curvature

- Object with known physical size, at large cosmological distance:
- Sound Waves in the Early Universe !!!!

Temperature Fluctuations
CMB

$\lambda = D_A \alpha$



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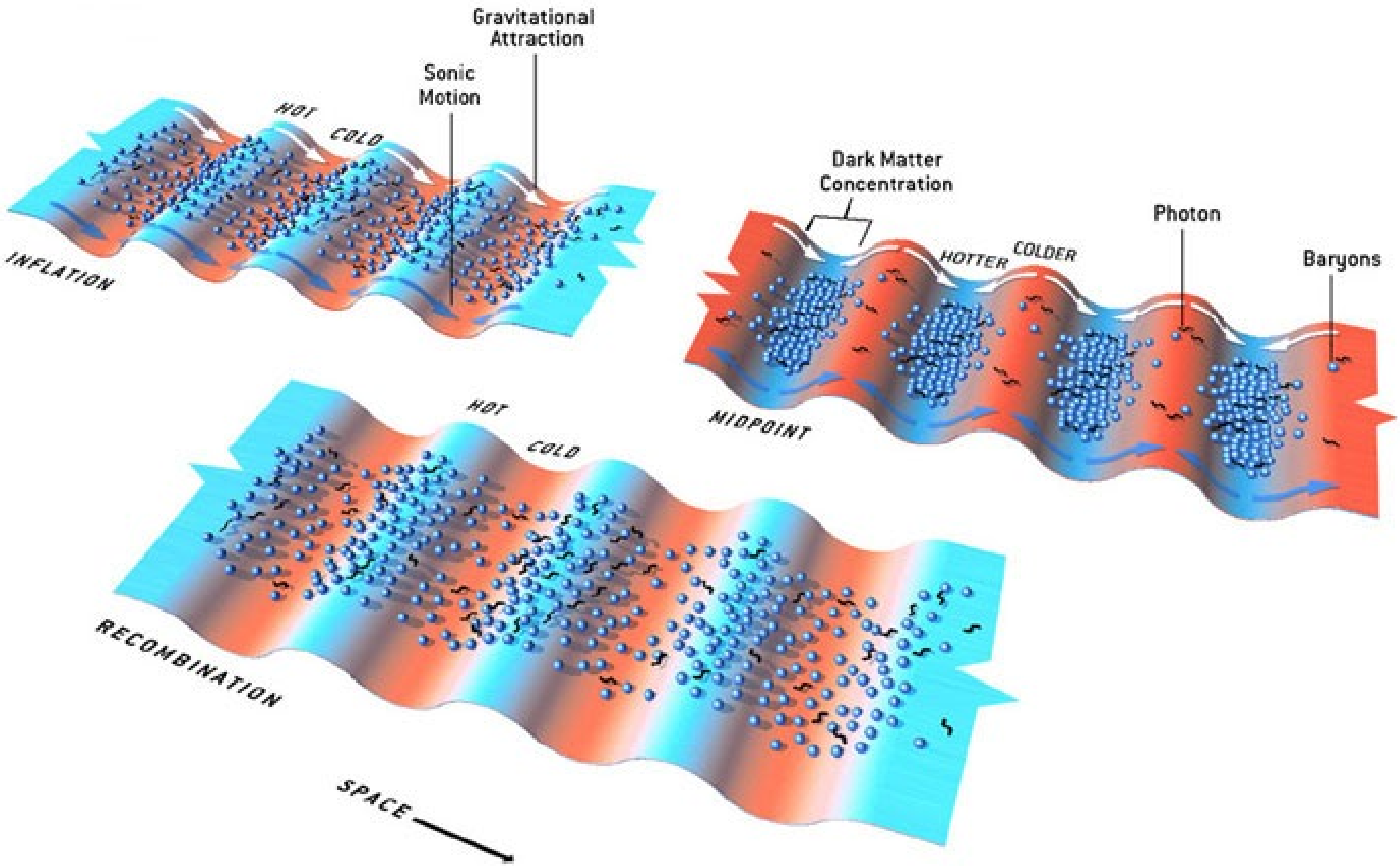
Here: angular diameter distance D_A: $\lambda = D_A \alpha$

Fluctuations-Origin



Music of the Spheres

- small ripples in
  primordial matter & photon distribution
- gravity:
  - compression primordial photon gas
  - photon pressure resists

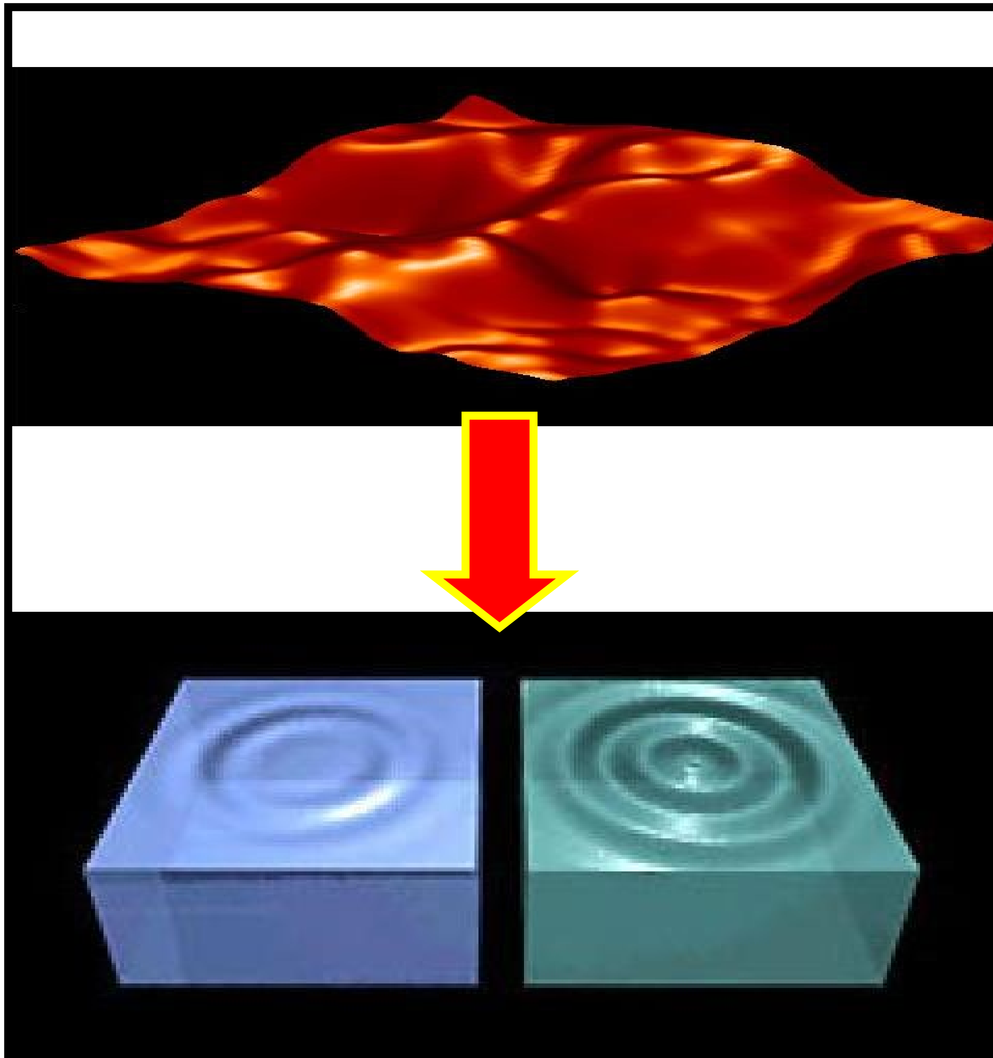
- compressions and rarefactions
  in photon gas: sound waves

- sound waves not heard, but seen:
  - compressions: (photon) T higher
  - rarefactions: lower

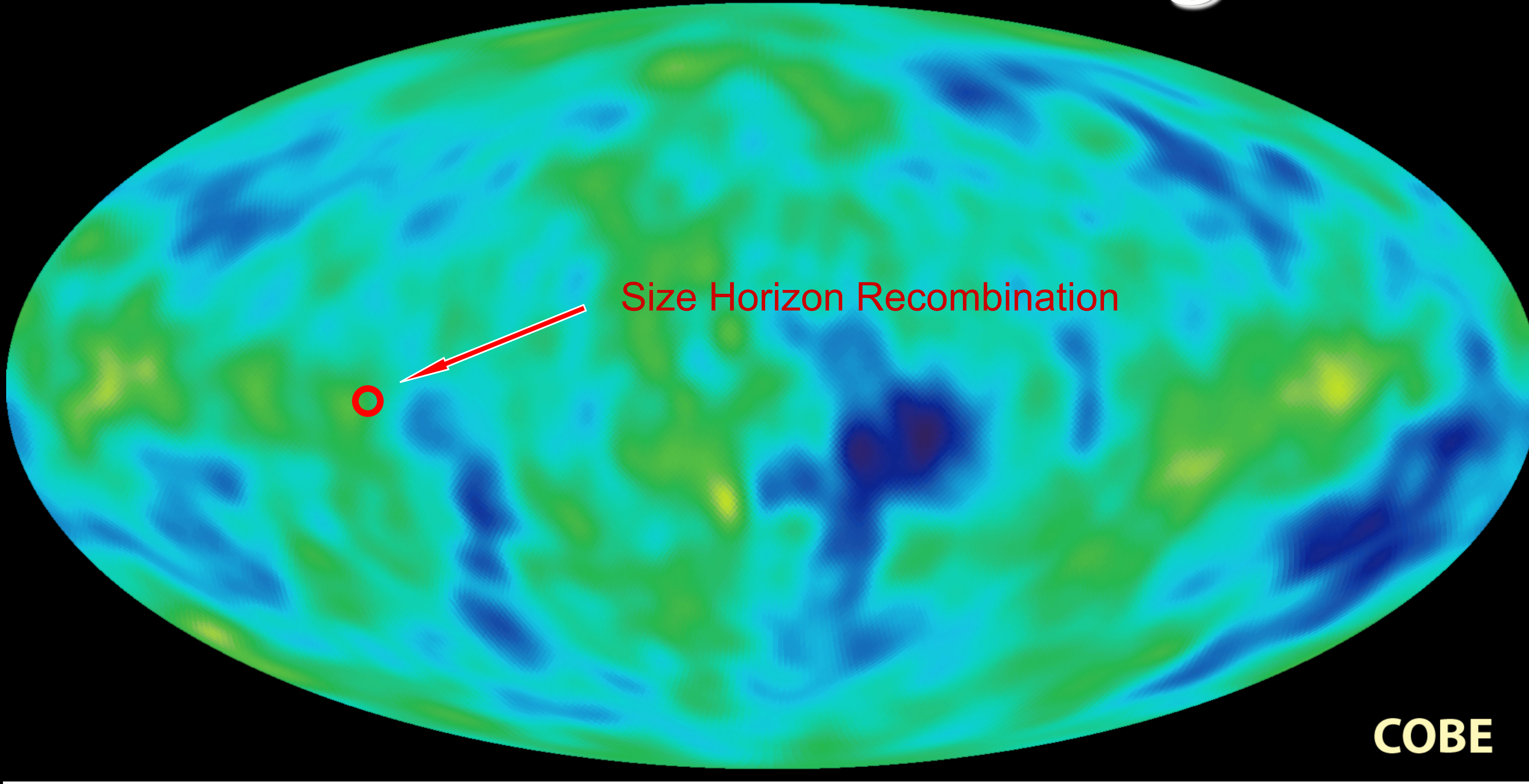
- fundamental mode sound spectrum
  - size of “instrument”:
  - (sound) horizon size last scattering

- Observed, angular size: θ~1º
  - exact scale maximum compression, the
    “cosmic fundamental mode of music”

W. Hu



Cosmic Microwave Background



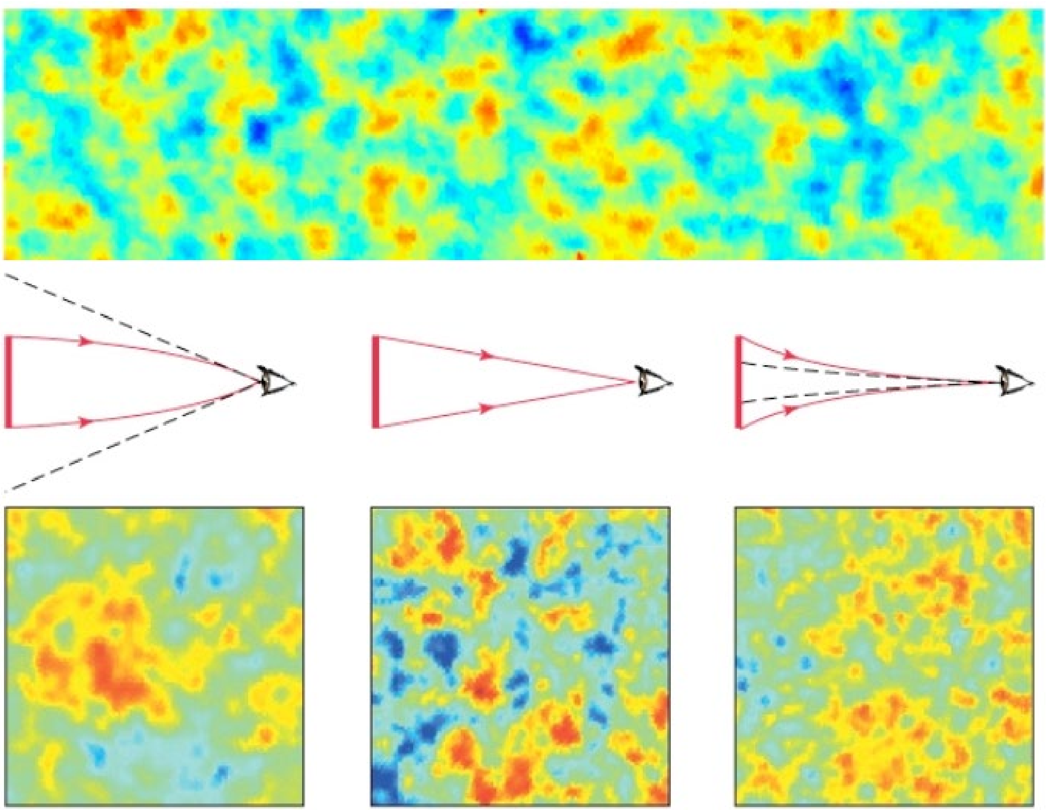
COBE measured fluctuations: > 7°

Size Horizon at Recombination spans angle ~ 1°

COBE proved that superhorizon fluctuations do exist: prediction Inflation !!!!!

Flat universe from CMB

- **First peak: flat universe**



Closed:
hot spots
appear larger

Flat:
appear as big
as they are

Open:
spots appear
smaller

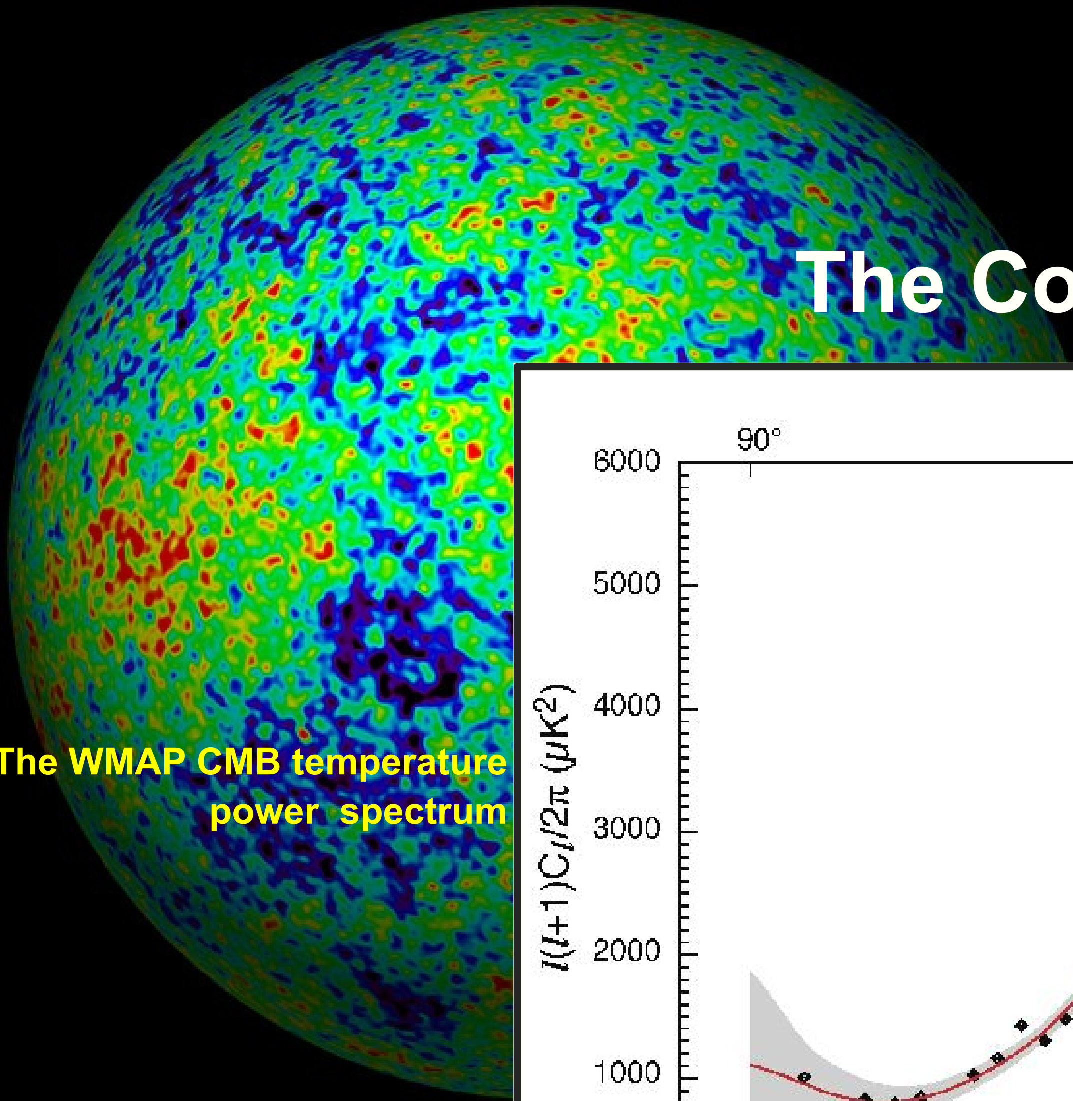
We know the redshift and the time it took for the light to reach us:

from this we know the
- length of the legs of the triangle
- the angle at which we are measuring the sound horizon.

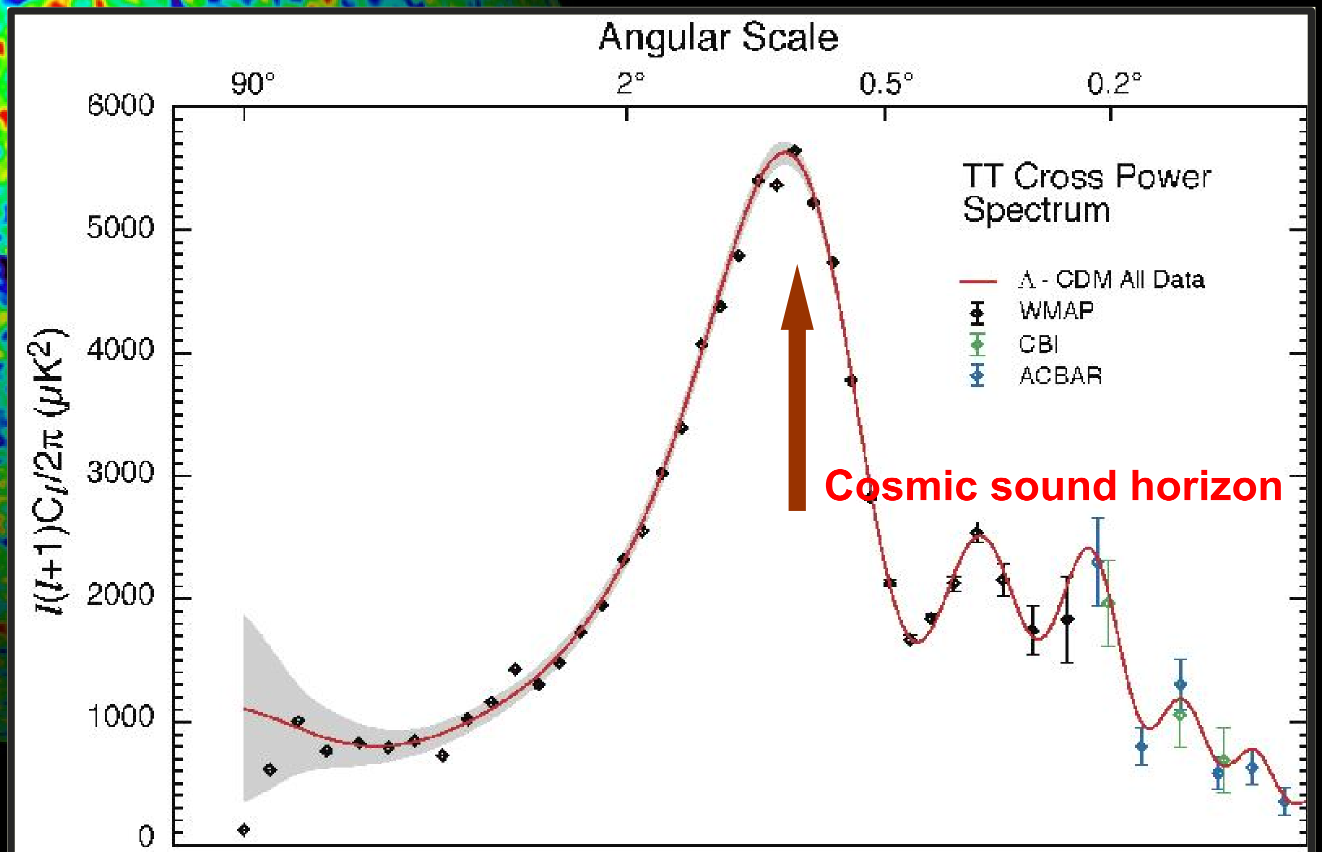
$$v \approx \frac{c}{\sqrt{3}}$$

$$\ell \approx 200/\sqrt{1-\Omega_k}$$

The Cosmic Tonal Ladder

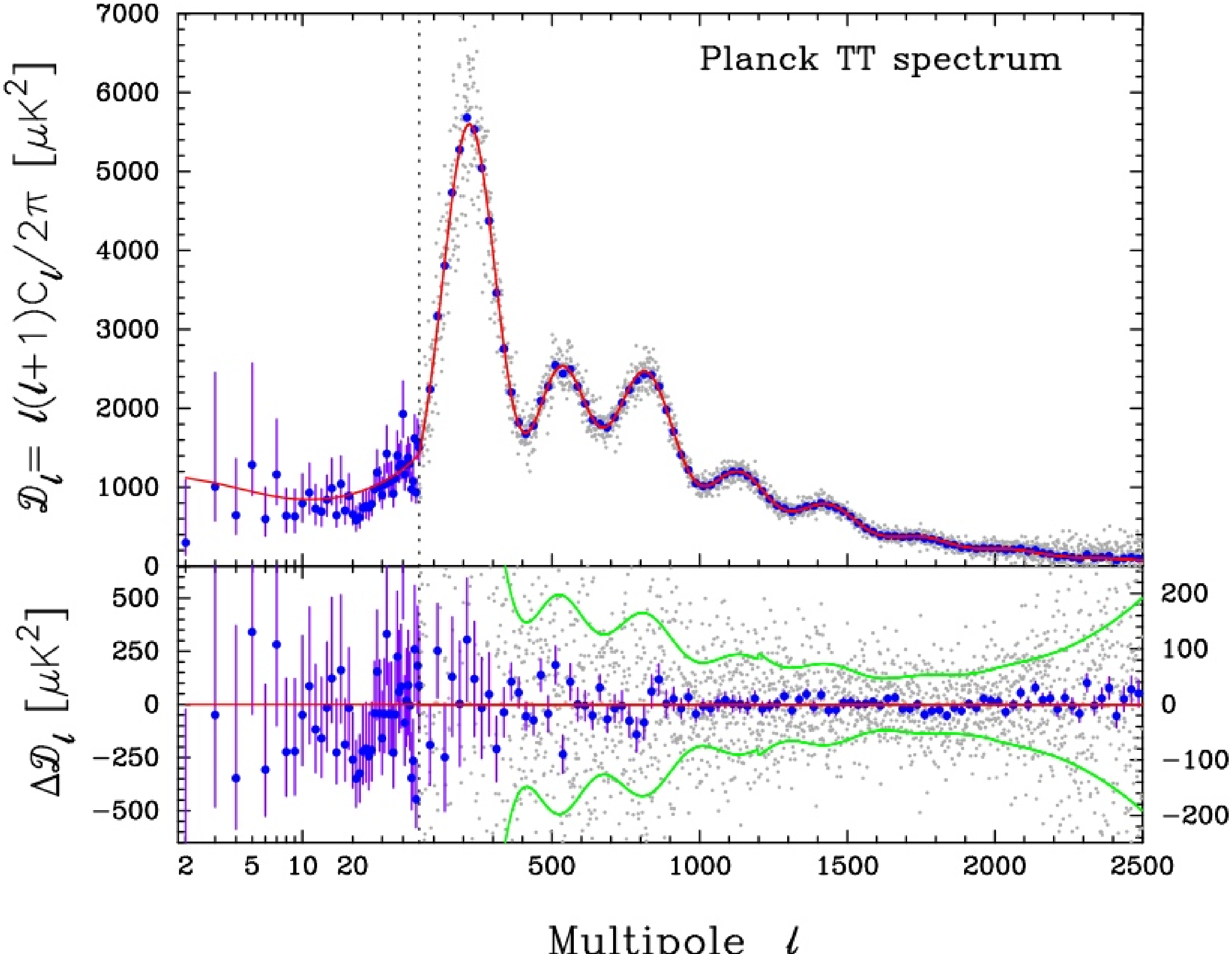


The WMAP CMB temperature power spectrum

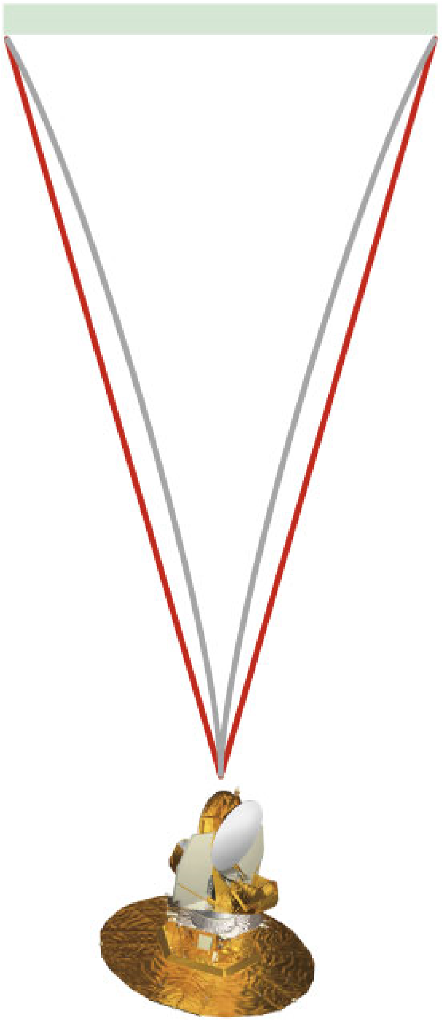


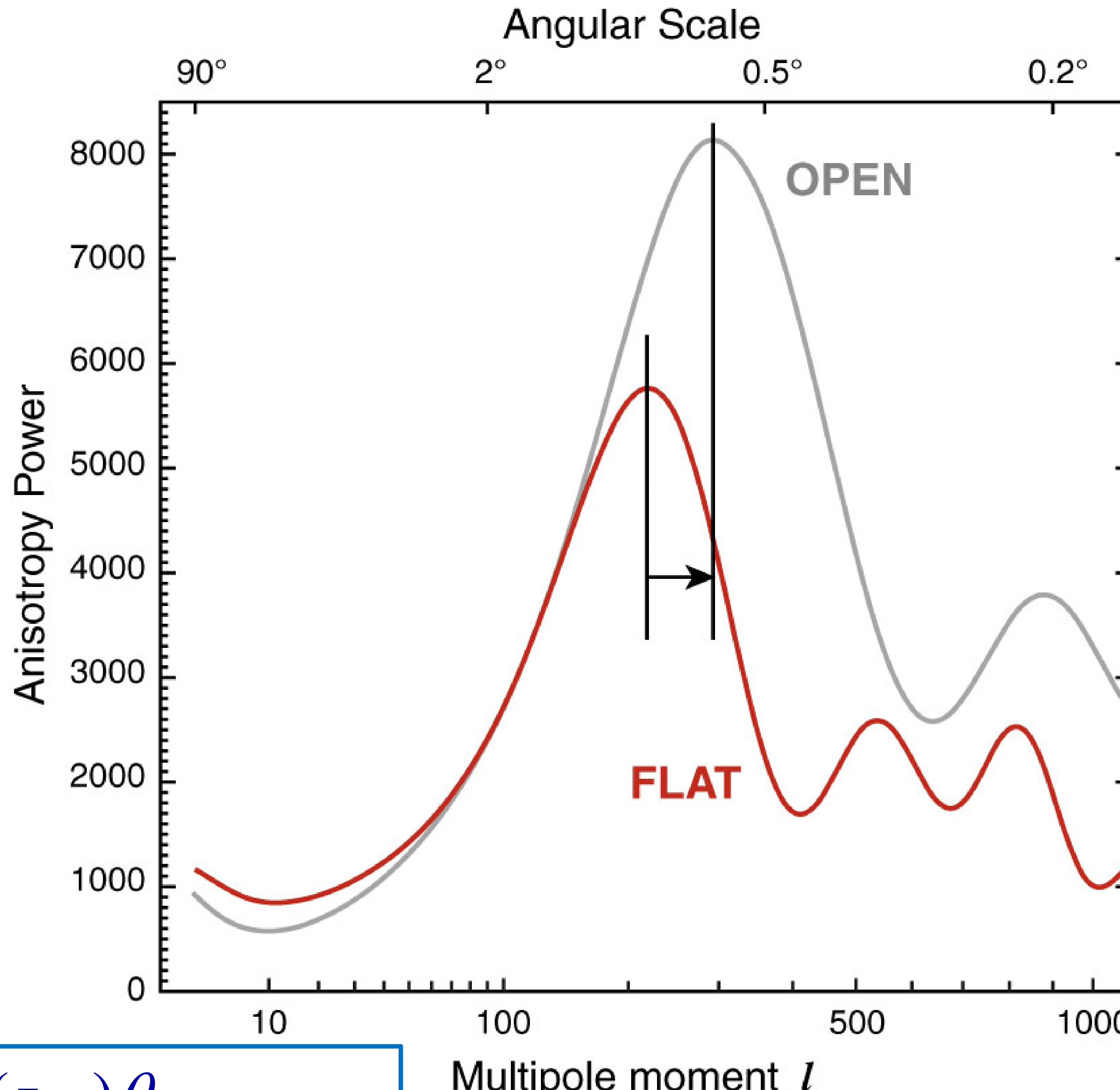
The Cosmic Microwave Background Temperature Anisotropies: Universe is almost perfectly FLAT !!!!

Planck CMB Temperature Fluctuations



Standard Ruler:
1° arc measurement of
dominant energy spike





$$\lambda_{sound\ horizon} = D_A(z_{rec})\,\theta_{CMB,\,1st\ \ peak}$$

FRW Universe: Curvature

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

$\Omega < 1$	$k = -1$	*Hyperbolic*	*Open Universe*
$\Omega = 1$	$k = 0$	*Flat*	*Critical Universe*
$\Omega > 1$	$k = +1$	*Spherical*	*Close Universe*

Cosmic Curvature
&
Cosmic Density

$$q \approx \frac{\Omega_m}{2} - \Omega_\Lambda$$

$$k = \frac{H^2 R^2}{c^2}(\Omega_m + \Omega_\Lambda - 1)$$

SCP Union2 constraints (2010)

on values of matter density Ω_m
dark energy density Ω_Λ