Cosmology, 
lect. 3

Cosmological Principle 
& 
Friedmann-Lemaitre Equations
Einstein Field Equation

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \]

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \]
Cosmological Principle
A crucial aspect of any particular configuration is the geometry of spacetime: because Einstein’s General Relativity is a metric theory, knowledge of the geometry is essential.

Einstein Field Equations are notoriously complex, essentially 10 equations. Solving them for general situations is almost impossible.

However, there are some special circumstances that do allow a full solution. The simplest one is also the one that describes our Universe. It is encapsulated in the

Cosmological Principle

On the basis of this principle, we can constrain the geometry of the Universe and hence find its dynamical evolution.
Cosmological Principle: the Universe Simple & Smooth

"God is an infinite sphere whose centre is everywhere and its circumference nowhere"
Empedocles, 5th cent BC

Cosmological Principle:
Describes the symmetries in global appearance of the Universe:

- **Homogeneous**
  - The Universe is the same everywhere:
    - physical quantities (density, T,p,...)

- **Isotropic**
  - The Universe looks the same in every direction

- **Universality**
  - Physical Laws same everywhere

- **Uniformly Expanding**
  - The Universe "grows" with same rate in:
    - every direction
    - at every location

"all places in the Universe are alike"
Einstein, 1931
Geometry of the Universe

Fundamental Tenet of (Non-Euclidian = Riemannian) Geometry

There exist no more than THREE uniform spaces:

1) Euclidian (flat) Geometry            Euclides
2) Hyperbolic Geometry                  Gauß, Lobachevski, Bolyai
3) Spherical Geometry                   Riemann
<table>
<thead>
<tr>
<th>Property</th>
<th>Closed</th>
<th>Euclidean</th>
<th>Open</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial Curvature</td>
<td>Positive</td>
<td>Zero</td>
<td>Negative</td>
</tr>
<tr>
<td>Circle Circumference</td>
<td>(&lt; 2\pi R)</td>
<td>2\pi R</td>
<td>(&gt; 2\pi R)</td>
</tr>
<tr>
<td>Sphere Area</td>
<td>(&lt; 4\pi R^2)</td>
<td>4\pi R^2</td>
<td>(&gt; 4\pi R^2)</td>
</tr>
<tr>
<td>Sphere Volume</td>
<td>(&lt; \frac{4}{3} \pi R^3)</td>
<td>(\frac{4}{3} \pi R^3)</td>
<td>(&gt; \frac{4}{3} \pi R^3)</td>
</tr>
<tr>
<td>Triangle Angle Sum</td>
<td>(&gt; 180^\circ)</td>
<td>180°</td>
<td>(&lt; 180^\circ)</td>
</tr>
<tr>
<td>Total Volume</td>
<td>Finite ((2\pi^2 R^3))</td>
<td>Infinite</td>
<td>Infinite</td>
</tr>
</tbody>
</table>

**Surface Analog**
- Sphere
- Plane
- Saddle
Distances in a uniformly curved spacetime is specified in terms of the Robertson-Walker metric. The spacetime distance of a point at coordinate \((r, \theta, \phi)\) is:

\[
ds^2 = c^2 \, dt^2 - a(t)^2 \left\{ dr^2 + R_c^2 S_k^2 \left( \frac{r}{R_c} \right) \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] \right\}
\]

where the function \(S_k(r/R_c)\) specifies the effect of curvature on the distances between points in spacetime:

\[
S_k \left( \frac{r}{R_c} \right) = \begin{cases}
\sin \left( \frac{r}{R_c} \right) & k = +1 \\
\frac{r}{R_c} & k = 0 \\
\sinh \left( \frac{r}{R_c} \right) & k = -1
\end{cases}
\]
Friedmann-Robertson-Walker-Lemaitre (FRLW)

Universe
Einstein Field Equation

\[ G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \]

\[ g_{\mu\nu, RW} \Rightarrow \Gamma^\mu_{\lambda\nu} \Rightarrow R_{\mu\nu}, R \]

\[ T_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) U^\mu U^\nu - pg^{\mu\nu} \]

\[ = \text{diag} \left( \rho c^2, p, p, p \right) \]
Einstein Field Equation

\[ G_{\mu \nu} = -\frac{8\pi G}{c^4} T_{\mu \nu} \]

\[ G^0_0 \rightarrow G^0_0 = 3\left(\dot{R}^2 + kc^2\right) / R^2 = \frac{8\pi G}{c^2} \rho c^2 \]

\[ G^1_1 \rightarrow G^1_1 = \left(2R\ddot{R} + \dot{R}^2 + kc^2\right) / R^2 = -\frac{8\pi G}{c^2} p \]
Friedmann-Robertson-Walker-Lemaître Universe

\[
\ddot{R} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) R + \frac{\Lambda}{3} R
\]

\[
\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{\Lambda}{3} R^2
\]
Cosmic Expansion Factor
Cosmic Expansion Factor

\[
a(t) = \frac{R(t)}{R_0}
\]

- Cosmic Expansion is a uniform expansion of space

\[
\vec{r}(t) = a(t) \vec{x}
\]
Friedmann-Robertson-Walker-Lemaitre Universe

\[
\ddot{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{\Lambda}{3} a
\]

\[
\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2
\]
\[
\ddot{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{\Lambda}{a}
\]

\[
\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{k c^2}{R_0^2} a^2 + \frac{\Lambda}{a^2}
\]
Because of General Relativity, the evolution of the Universe is fully determined by four factors:

- **Density** \( \rho(t) \)
- **Pressure** \( p(t) \)
- **Curvature** \( \frac{k c^2}{R_0^2} \) \( k = 0, +1, -1 \) \( R_0 \): present curvature radius
- **Cosmological Constant** \( \Lambda \)

**Density & Pressure:**
- In relativity, energy & momentum need to be seen as one physical quantity (four-vector)
- Pressure = momentum flux

**Curvature:**
- Gravity is a manifestation of geometry spacetime

**Cosmological Constant:**
- Free parameter in General Relativity
- Einstein’s “biggest blunder”
- Mysteriously, since 1998 we know it dominates the Universe
Friedmann-Robertson-Walker-Lemaitre Universe

<table>
<thead>
<tr>
<th>Relativistic Cosmology</th>
<th>Newtonian Cosmology</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \ddot{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{\Lambda}{3} a ]</td>
<td>[ \ddot{a} = -\frac{4\pi G}{3} \rho a ]</td>
</tr>
<tr>
<td>[ \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{k c^2}{R_0^2} + \frac{\Lambda}{3} a^2 ]</td>
<td>[ \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 + E ]</td>
</tr>
</tbody>
</table>

- \( -k c^2 / R_0^2 \)  
- \( \Lambda \)  
- \( \rho \)  

Curvature  
Cosmological Constant  
Pressure  

\[ E \]

Energy
Hubble Parameter
Hubble Expansion

- Cosmic Expansion is a uniform expansion of space

- Objects do not move themselves:
  they are like beacons tied to a uniformly expanding sheet:

\[
\begin{align*}
    \ddot{r}(t) &= a(t) \dddot{x} \\
    \dot{r}(t) &= \dot{a}(t) \ddot{x} = \frac{\dot{a}}{a} a \dddot{x} = H(t) \dddot{r}
\end{align*}
\]

\[ H(t) = \frac{\dot{a}}{a} \]
Cosmic Expansion is a uniform expansion of space.

Objects do not move themselves: they are like beacons tied to a uniformly expanding sheet.

Comoving Position

\[ \ddot{r}(t) = a(t) \ddot{x} \]

\[ \dot{r}(t) = \dot{a}(t) \dot{x} = \frac{\dot{a}}{a} a \dot{x} = H(t) \dot{r} \]

Hubble Parameter:

Hubble “constant”:

\[ H_0 \equiv H(t = t_0) \]

\[ H(t) = \frac{\dot{a}}{a} \]
For a long time, the correct value of the Hubble constant $H_0$ was a major unsettled issue:

$$H_0 = 50 \text{ km \text{s}^{-1} \text{Mpc}^{-1}} \quad \leftrightarrow \quad H_0 = 100 \text{ km \text{s}^{-1} \text{Mpc}^{-1}}$$

This meant distances and timescales in the Universe had to deal with uncertainties of a factor 2!!!

Following major programs, such as Hubble Key Project, the Supernova key projects and the WMAP CMB measurements,

$$H_0 = 71.9^{+2.6}_{-2.7} \text{ km} \text{s}^{-1} \text{Mpc}^{-1}$$
Hubble Constant: Tension

- As more accurate measurements of $H_0$ become available, gradual rising tension
- CMB determination much lower than "local values"
- Latest value: strong grav. lensing

$H_0 = 82.4 \pm 8.3$ km/s/Mpc
Hubble Time

\[ t_H = \frac{1}{H} \]

\[ H_0 = 100h \text{ km s}^{-1}\text{Mpc}^{-1} \]

\[ t_0 = 9.78h^{-1}\text{Gyr} \]
Cosmological Constant & FRW equations
Friedmann-Robertson-Walker-Lemaitre Universe

\[ \ddot{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{\Lambda}{3} a \]

\[ \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2 \]
Dark Energy & Energy Density

\[ \tilde{\rho} = \rho + \rho_\Lambda \]

\[ \tilde{p} = p + p_\Lambda \]

\[ \rho_\Lambda = \frac{\Lambda}{8\pi G} \]

\[ p_\Lambda = -\frac{\Lambda c^2}{8\pi G} \]
\[ \ddot{a} = -\frac{4\pi G}{3} \left( \tilde{\rho} + \frac{3 \tilde{p}}{c^2} \right) a \]

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \tilde{\rho} - \frac{k c^2 / R_0^2}{a^2} \]
Cosmic Constituents
Cosmic Constituents

The total energy content of the Universe is made up by various constituents, the principal ones:

$$\Omega_{\text{tot}}$$

- **matter**
  - $\Omega_m$
  - $\Omega_b$ baryonic matter
  - $\Omega_{DM}$ dark matter
  - $\Omega_\gamma$ photons
  - $\Omega_\nu$ neutrino's

- **radiation**
  - $\Omega_{\text{rad}}$
  - $\Omega_\nu$ dark/vacuum energy

In addition, contributions by:
- gravitational waves
- magnetic fields,
- cosmic rays ...

Poor constraints on their contribution: henceforth we will not take them into account!
LCDM Cosmology

- Concordance cosmology
  - model that fits the majority of cosmological observations
  - universe dominated by Dark Matter and Dark Energy

LCDM composition today …
<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
<th>Energy Content</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong> dark sector</td>
<td></td>
<td>0.954 ± 0.003</td>
</tr>
<tr>
<td>1.1 dark energy</td>
<td></td>
<td>0.72 ± 0.03</td>
</tr>
<tr>
<td>1.2 dark matter</td>
<td></td>
<td>0.23 ± 0.03</td>
</tr>
<tr>
<td>1.3 primeval gravitational waves</td>
<td></td>
<td>&lt; 10^{-10}</td>
</tr>
<tr>
<td><strong>2</strong> primeval thermal remnants</td>
<td></td>
<td>0.0010 ± 0.0005</td>
</tr>
<tr>
<td>2.1 electromagnetic radiation</td>
<td></td>
<td>10^{-4.3±0.0}</td>
</tr>
<tr>
<td>2.2 neutrinos</td>
<td></td>
<td>10^{-2.9±0.1}</td>
</tr>
<tr>
<td>2.3 prestellar nuclear binding energy</td>
<td></td>
<td>-10^{-4.1±0.0}</td>
</tr>
<tr>
<td><strong>3</strong> baryon rest mass</td>
<td></td>
<td>0.045 ± 0.003</td>
</tr>
<tr>
<td>3.1 warm intergalactic plasma</td>
<td></td>
<td>0.040 ± 0.003</td>
</tr>
<tr>
<td>3.1a virialized regions of galaxies</td>
<td></td>
<td>0.024 ± 0.005</td>
</tr>
<tr>
<td>3.1b intergalactic</td>
<td></td>
<td>0.016 ± 0.005</td>
</tr>
<tr>
<td>3.2 intracluster plasma</td>
<td></td>
<td>0.0018 ± 0.0007</td>
</tr>
<tr>
<td>3.3 main sequence stars</td>
<td>spheroids and bulges</td>
<td>0.0015 ± 0.0004</td>
</tr>
<tr>
<td>3.4 disks and irregulars</td>
<td></td>
<td>0.00055 ± 0.00014</td>
</tr>
<tr>
<td>3.5 white dwarfs</td>
<td></td>
<td>0.00036 ± 0.00008</td>
</tr>
<tr>
<td>3.6 neutron stars</td>
<td></td>
<td>0.00005 ± 0.00002</td>
</tr>
<tr>
<td>3.7 black holes</td>
<td></td>
<td>0.00007 ± 0.00002</td>
</tr>
<tr>
<td>3.8 substellar objects</td>
<td></td>
<td>0.00014 ± 0.00007</td>
</tr>
<tr>
<td>3.9 HI + HeI</td>
<td></td>
<td>0.00062 ± 0.00010</td>
</tr>
<tr>
<td>3.10 molecular gas</td>
<td></td>
<td>0.00016 ± 0.00006</td>
</tr>
<tr>
<td>3.11 planets</td>
<td></td>
<td>10^{-6}</td>
</tr>
<tr>
<td>3.12 condensed matter</td>
<td></td>
<td>10^{-5.6±0.3}</td>
</tr>
<tr>
<td>3.13 sequestered in massive black holes</td>
<td></td>
<td>10^{-5.4} (1 + \epsilon_n)</td>
</tr>
<tr>
<td><strong>4</strong> primeval gravitational binding energy</td>
<td></td>
<td>-10^{-6.1±0.1}</td>
</tr>
<tr>
<td>4.1 virialized halos of galaxies</td>
<td></td>
<td>-10^{-7.2}</td>
</tr>
<tr>
<td>4.2 clusters</td>
<td></td>
<td>-10^{-6.9}</td>
</tr>
<tr>
<td>4.3 large-scale structure</td>
<td></td>
<td>-10^{-6.2}</td>
</tr>
</tbody>
</table>
Critical Density & Omega
FRW Dynamics

\[ \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{kc^2}{R_0^2} \]

Critical Density:
- For a Universe with \( \Omega = 0 \)
- Given a particular expansion rate \( H(t) \)
- Density corresponding to a flat Universe (k=0)

\[ \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \]
In a FRW Universe, densities are in the order of the critical density,

\[
\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G} = 1.8791h^2 \times 10^{-29} \text{ g cm}^{-3}
\]

\[
\rho_0 = 1.8791\times10^{-29} \Omega h^2 \text{ g cm}^{-3}
\]

\[
= 2.78\times10^{11} \Omega h^2 \text{ M}_\odot \text{Mpc}^{-3}
\]
In a matter-dominated Universe, the evolution and fate of the Universe entirely determined by the (energy) density in units of critical density:

$$\Omega \equiv \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi G \rho}{3H^2}$$

Arguably, $\Omega$ is the most important parameter of cosmology !!!

Present-day Cosmic Density:

$$\rho_0 = 1.8791 \times 10^{-29} \Omega h^2 \ g \ cm^{-3}$$

$$= 2.78 \times 10^{11} \Omega h^2 \ M_\odot \ Mpc^{-3}$$
FRWL Dynamics & Cosmological Density
• The individual contributions to the energy density of the Universe can be figured into the $\Omega$ parameter:

- radiation

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4}{c^2} = \frac{8\pi G \sigma T^4}{3H^2c^2}$$

- matter

$$\Omega_{m} = \Omega_{dm} + \Omega_{b}$$

- dark energy / cosmological constant

$$\Omega_{\Lambda} = \frac{\Lambda}{3H^2}$$

$$\Omega = \Omega_{rad} + \Omega_{m} + \Omega_{\Lambda}$$
## Critical Density

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

\[
k = \frac{H^2 R^2}{c^2} (\Omega - 1)
\]

\[
\Omega = \Omega_{rad} + \Omega_m + \Omega_{\Lambda}
\]

<table>
<thead>
<tr>
<th>(\Omega)</th>
<th>(k)</th>
<th>Shape</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt;1)</td>
<td>(-1)</td>
<td>Hyperbolic</td>
<td>Open Universe</td>
</tr>
<tr>
<td>(=1)</td>
<td>(0)</td>
<td>Flat</td>
<td>Critical Universe</td>
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The individual contributions to the energy density of the Universe can be figured into the $\Omega$ parameter:

- **radiation**

  \[
  \Omega_{\text{rad}} = \frac{\rho_{\text{rad}}}{\rho_{\text{crit}}} = \frac{\sigma T^4 / c^2}{\rho_{\text{crit}}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}
  \]

- **matter**

  \[
  \Omega_m = \Omega_{dm} + \Omega_b
  \]

- **dark energy/cosmological constant**

  \[
  \Omega_{\Lambda} = \frac{\Lambda}{3H^2}
  \]

\[
\Omega = \Omega_{\text{rad}} + \Omega_m + \Omega_{\Lambda}
\]
Cosmic Constituents:
Evolving Energy Density
To infer the evolving energy density $\rho(t)$ of each cosmic component, we refer to the cosmic energy equation. This equation can be directly inferred from the FRW equations

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

The equation forms a direct expression of the adiabatic expansion of the Universe, ie.

$$U = \rho c^2 V \quad \text{Internal energy}$$
$$V \propto a^3 \quad \text{Expanding volume}$$

$$dU = -pdV$$
To infer $\dot{p}(t)$ from the energy equation, we need to know the pressure $p(t)$ for that particular medium/ingredient of the Universe.

\[
\dot{\rho} + 3 \left( \rho + \frac{p}{c^2} \right) \frac{\dot{a}}{a} = 0
\]

To infer $p(t)$, we need to know the nature of the medium, which provides us with the equation of state,

\[p = p(\rho, S)\]
## Cosmic Constituents:
### Evolution of Energy Density

<table>
<thead>
<tr>
<th>Matter:</th>
<th>Radiation:</th>
<th>Dark Energy:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_m(t) \propto a(t)^{-3}$</td>
<td>$\rho_{rad}(t) \propto a(t)^{-4}$</td>
<td>$\rho_{\Lambda}(t) = \text{cst.}$</td>
</tr>
<tr>
<td>$\rho_v(t) \propto a(t)^{-3(1+w)}$</td>
<td></td>
<td>$p = w \rho_v c^2$</td>
</tr>
<tr>
<td>$\Downarrow w = -1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\Downarrow \begin{align*} w &= -1 \end{align*}$
Dark Energy:

Equation of State
Einstein Field Equation

\[ \begin{align*}
R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= -\frac{8\pi G}{c^4} T_{\mu\nu}, \\
R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} &= -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}
\end{align*} \]
Equation of State

\[ T_{\mu\nu}^{\text{vac}} \equiv \frac{\Lambda c^4}{8\pi G} g_{\mu\nu} \]

restframe

\[ T_{\mu\nu}^{\text{vac}} \equiv \frac{\Lambda c^4}{8\pi G} \eta_{\mu\nu} \]

\[ \eta^{00} = 1, \quad \eta^{ii} = -1 \]

\[ T_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) U^\mu U^\nu - pg_{\mu\nu} \]

restframe:

\[ T^{00}_{\text{vac}} = \rho_{\text{vac}} c^2 \]

\[ T^{ii}_{\text{vac}} = p \]

\[ \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} \]

\[ p = -\frac{\Lambda c^4}{8\pi G} \]
Equation of State

\[ \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} \]

\[ p = -\frac{\Lambda c^4}{8\pi G} \]

\[ p_{\text{vac}} = -\rho_{\text{vac}} c^2 \]
Dynamics

Relativistic Poisson Equation:

$$\nabla^2 \phi = 4\pi G \left( \rho + \frac{3p}{c^2} \right)$$

$$\rho_{vac} + \frac{3p_{vac}}{c^2} = -2\rho_{vac} < 0; \quad \rho_{vac} = \frac{\Lambda}{8\pi G}$$

$$\nabla^2 \phi < 0$$

Repulsion !!!
Dark Energy & Cosmic Acceleration

Nature Dark Energy:

(Parameterized) Equation of State

\[ p(\rho) = w \rho c^2 \]

Cosmic Acceleration:

\[ \ddot{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a \]

Gravitational Repulsion:

\[ p = w \rho c^2 \quad \Leftrightarrow \quad w < -\frac{1}{3} \quad \Rightarrow \quad \ddot{a} > 0 \]
## Dark Energy & Cosmic Acceleration

### DE equation of State

\[ p(\rho) = w \rho c^2 \]

\[ \rho_w(a) = \rho_w(a_0) a^{-3(1+w)} \]

### Cosmological Constant:

\[ \Lambda : \quad w = -1 \]

\[ \rho_w = \text{cst.} \]

\(-1/3 > w > -1:\)

\[ \rho_w \propto a^{-3(1+w)} \quad 1 + w > 0 \]

decreases with time

### Phantom Energy:

\[ \rho_w \propto a^{-3(1+w)} \quad 1 + w < 0 \]

increases with time
### Dynamic Dark Energy

<table>
<thead>
<tr>
<th>DE equation of State</th>
<th>Dynamically evolving dark energy, parameterization:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(\rho) = w\rho c^2$</td>
<td>$w(a) = w_0 + (1-a)w_a \approx w_\phi(a)$</td>
</tr>
</tbody>
</table>

\[
\rho_w(a) = \rho_w(a_0) \exp \left\{ -3 \int_1^{a} \frac{1 + w_\phi(a')}{a'} \, da' \right\}
\]
General Flat FRW Universe

\( k = 0 \)

\( \rho_v(t) \propto a(t)^{-3(1+w)} \quad \iff \quad p = w \rho_v c^2 \)

FRW:

\[ a(t) \propto t^{\frac{2}{3+3w}} \]
Cosmic acceleration quantified by means of dimensionless deceleration parameter $q(t)$:

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

$$q = \frac{\Omega_m}{2} + \Omega_{rad} - \Omega_\Lambda$$

$$q \approx \frac{\Omega_m}{2} - \Omega_\Lambda$$

Examples:

- $\Omega_m = 1; \; \Omega_\Lambda = 0; \; q = 0.5$
- $\Omega_m = 0.3; \; \Omega_\Lambda = 0.7; \; q = -0.65$
Dynamical Evolution
FRWL Universe
General Solution
Expanding FRW Universe

From the FRW equations:

\[
\frac{H(t)^2}{H_0^2} = \frac{\Omega_{\text{rad}, 0}}{a^4} + \frac{\Omega_{m, 0}}{a^3} + \Omega_{\Lambda, 0} + \frac{1 - \Omega_0}{a^2}
\]

\[
H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{\text{rad}, 0}}{a^2} + \frac{\Omega_{m, 0}}{a} + \Omega_{\Lambda, 0} a^2 + (1 - \Omega_0)}}
\]
Age of the Universe

\[ \Omega < 1 \quad \text{Matter-dominated} \]

\[ t = \frac{1}{H} \]

\[ t = \frac{2}{3} \frac{1}{H} \]

\[ \Omega > 1 \quad \text{Matter-dominated} \]

\[ t = \frac{2}{3} \frac{1}{H} \]

Age of a FRW universe at Expansion factor \( a(t) \)

\[ H t = \int_{0}^{a} \frac{da}{\sqrt{\frac{\Omega_{\text{rad}}}{a^2} + \frac{\Omega_{m}}{a} + \Omega_{\Lambda} a^2 + (1 - \Omega)}} \]
While general solutions to the FRW equations is only possible by numerical integration, analytical solutions may be found for particular classes of cosmologies:

- **Single-component Universes:**
  - empty Universe
  - flat Universes, with only radiation, matter or dark energy

- **Matter-dominated Universes**

- **Matter+Dark Energy flat Universe**
Matter-Dominated Universes

- Assume radiation contribution is negligible:
  \[ \Omega_{rad,0} \approx 5 \times 10^{-5} \]
  \[ \Omega_\Lambda = 0 \]

- Zero cosmological constant:

- Matter-dominated, including curvature
  \[ \Omega_m < 1 \]
  \[ \Omega_m = 1 \]
  \[ \Omega_m > 1 \]
Einstein-de Sitter Universe

\[ \Omega_m = 1 \]
\[ \Omega_\Lambda = 0 \]
\[ k = 0 \]

FRW:
\[ \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 = \frac{8\pi G \rho_0}{3} \frac{1}{a} \]

[Equation]

\[ a(t) = \left( \frac{t}{t_0} \right)^{2/3} \]

Age
EdS Universe:
\[ t_0 = \frac{2}{3} \frac{1}{H_0} \]

Albert Einstein and Willem de Sitter discussing the Universe. In 1932 they published a paper together on the Einstein-de Sitter universe, which is a model with flat geometry containing matter as the only significant substance.
Free Expanding "Milne" Universe

\[ \Omega_m = 0 \quad \Omega_\Lambda = 0 \quad \begin{cases} k = -1 \\ \text{Empty space is curved} \end{cases} \]

FRW:
\[ a^2 = -\frac{kc^2}{R_0^2} = \text{cst.} \]

\[ a(t) = \begin{pmatrix} t \\ t_0 \end{pmatrix} \]

Age Empty Universe:
\[ t_0 = \frac{1}{H_0} \]
In the very early Universe, the energy density is completely dominated by radiation. The dynamics of the very early Universe is therefore fully determined by the evolution of the radiation energy density:

$$\rho_{\text{rad}}(a) \propto \frac{1}{a^4}$$
De Sitter Expansion

\[ \begin{align*}
\Omega_m &= 0 \\
\Omega_\Lambda &= 1
\end{align*} \quad \left\{ \begin{array}{c}
k = 0
\end{array} \right. \]

\[ \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \quad \Rightarrow \quad H_0 = \sqrt{\frac{\Lambda}{3}} \]

FRW: \quad \dot{a}^2 = \frac{\Lambda}{3}a^2 \quad \Rightarrow \quad \dot{a} = H_0a

\[ a(t) = e^{H_0(t-t_0)} \]

Willem de Sitter (1872-1934; Sneek-Leiden)
director Leiden Observatory
alma mater: Groningen University

Age
De Sitter Universe: infinitely old