Tutorial II
Newtonian Cosmology; Hubble Expansion; Observational Cosmology

Exercise 1:

Newtonian Cosmology

In 1934 – i.e. way after Friedman derived his equations- Milne and McCrea showed that relations of the 'Friedman' form can be derived using **non-relativistic Newtonian dynamics**.

- Write down the field equation for the gravitational force in the non-relativistic limit.
- Imagine you are a particle moving outside a spherically symmetric mass concentration of radius R with a total mass M and a density profile $\rho(r)$. What two essential simplifications can you invoke to derive your equation of motion ?
- Write down the equation of motion (ie. the equation for your acceleration). In addition, derive the corresponding energy equation (conservation of energy).
- We go one step further, and assume you are embedded within the spherically symmetric mass concentration. Imagine you are at a radius r, what will be your equation of motion ?

Subsequently, the situation becomes even more benevolent: we find ourselves in a homogeneous and isotropic medium.

- Write down the equation of motion and the energy equation.
- What three qualitative different situations can you distinguish on the basis of the energy E of a shell ?
- Take a shell of initial radius $r_{1,i}$ and another shell of initial radius $r_{2,i}$, in how far does their evolution differ (or not) ? (assume that there are no non-radial motions). What does this imply for the evolution $r(t)$ for any shell in the mass distribution ?
- What does the latter imply for the evolution of the density $\rho(t)$.

In principle, we are now all set to solve the equation of motion of the system, as a function of E. In fact, it is possible to derive the full solution for any spherically symmetric - not even homogeneous - mass distribution. This is the socalled *Spherical Model*. It would be a good exercise to do so ... however, we go for the real work, solving the equation of motion for a matter-dominated FRW Universe.

Exercise 2:
Hubble Expansion and Bounded Objects

We have seen that galaxies are participating in the uniform Hubble expansion. Question is why we ourselves do not expand along. If this were so, we would not notice anything like expansion. Assume a Hubble parameter of $H_0 = 71\,\mathrm{km/s/Mpc}$. As a thought experiment compute

- the expected Hubble expansion rate between your toes and the tip of your head.
- the expected Hubble expansion rate between the core of the Earth and ourselves ?
- What is the reason behind the Hubble expansion being insignificant under these circumstances ? Suggestion: compute the gravitational binding energy/escape velocity at the surface of the Earth and compare to $v = Hr$.
- Repeat the same exercise for Planet Earth wrt. the Sun and Dwarf Planet Pluto wrt. the Sun. Subsequently, consider the Sun and the Galaxy. Next, consider the Local Group (mainly M31 and the Galaxy). Then, consider the Local Group, or the Galaxy, wrt. the Local Supercluster dominated by the Virgo Cluster. Thus, what is your conclusion with respect to the scale at which the Hubble expansion becomes noticeable ? Note that you are expected to look up the relevant numbers yourself !

Exercise 3:
Cosmology, the Search for Two Numbers

In a famous 1970 Annual Review of Astronomy and Astrophysics review, the observational cosmologist Allan Sandage described all of cosmology as a "Search for Two Numbers". The two numbers were

$$\begin{cases} H_0 \\ \\ q_0 \end{cases}$$

We are going to explore what the reason was behind this stout statement (which by now is far besides reality ...). And in this we do not need any dynamics ... no assumption about a Friedmann Universe needed (yet),

- On the basis of a Taylor series expansion of the expansion factor $a(t)$, with respect to the current cosmic time t_0, show that

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 \tag{1}$$

- By inverting the above expression for $a(t)$ and using the expression for coordinate distance of a source whose radiation was emitted at t_e and has just reached us at t_0,

$$d_p(t_0) = r = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{2}$$

  show that

$$d_p(t_0) \approx c(t_0 - t_e) + \frac{cH_0}{2} (t_0 - t_e)^2 . \tag{3}$$

- Imagine you receive radiation from an object with redshift z, which it radiated at cosmic time t_e. Show that the approximate relationship between z and t_e is given by

$$z \approx H_0(t_0 - t_e) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t_e)^2 \tag{4}$$

- Invert this equation to obtain

$$t_0 - t_e \approx \frac{1}{H_0} \left\{ z - \left(1 + \frac{q_0}{2}\right) z^2 \right\} \tag{5}$$

- On the basis of results above show that for $z \ll 1$ we have the following approximate relation between z and coordinate distance $d_p(t_0)$,

$$d_p(t_0) = \frac{c}{H_0} z \left\{ 1 - \frac{1 + q_0}{2} z \right\} \tag{6}$$

- Taking along this approximation for $z \ll 1$, we may find an approximation for the luminosity distance of any universe, $d_L \approx (1+z)d_p(t_0)$. Show that an approximate relation for d_L is therefore

$$d_L \approx \frac{c}{H_0} z \left\{ 1 + \frac{1 - q_0}{2} z \right\} \tag{7}$$

- When we observe an object with absolute bolometric magnitude M_{bol} at a redshift z, show that in a Universe with acceleration parameter q_0,

$$m_{bol} \approx M_{bol} + 5 \log \left[\frac{c}{H_0} / 10pc \right] + 5 \log z + 1.086(1 - q_0) z + \mathcal{O}(z^2) \tag{8}$$

- Take an empty $\Omega_0 = 0$ matter-dominated Universe (i.e. $q_0 = 0$) as your reference for $(m - M)_)$. Infer what the relation $(m - M)(z) - (m - M)_0(z)$ is for three Universes. One should have $q_0 = 0.5$ (EdS), one $q_0 = 0.15$ ($\Omega_0 = 0.3$ matter-dominated) and one $q_0 = -0.55$ (concordance Universe). Draw a graph of the various predictions.

Note: the latter is exactly what the Supernova Ia experiments have found: an accelerating Universe.

Exercise 4 (Computer Task):
Hubble Expansion and Anisotropic Velocities

Kinematically speaking, the isotropic and uniform Hubble expansion of the Universe is a rather special circumstance. We may appreciate this when looking at the general flow of a fluid around a position $\mathbf{r}_0$. The kth component $v_k(\mathbf{r}$ of the velocity at location $\mathbf{r}$ is given by

$$v_k(\mathbf{r}) \;=\; v_k(\mathbf{r}_0) \,+\, \frac{1}{3}\nabla\cdot\mathbf{v}\,(r_k - r_{k,0}) \,+\, \sum_j \sigma_{kj}(r_j - r_{j,0}) \,+\, \sum_j \omega_{kj}(r_j - r_{j,0}) \quad (9)$$

in which the divergence $\nabla \cdot \mathbf{v}$ encapsulates the expansion or contraction of a volume element, the shear σ_{ij} its shape deformation and ω_{ij} the vorticity,

$$\begin{aligned} \nabla\cdot\mathbf{v} &\equiv \left(\frac{\partial v_1}{\partial r_1} + \frac{\partial v_2}{\partial r_2} + \frac{\partial v_3}{\partial r_3}\right) \\ \sigma_{ij} &\equiv \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) - \frac{1}{3}(\nabla\cdot\mathbf{v})\,\delta_{ij} \\ \omega_{ij} &\equiv \left(\frac{\partial v_i}{\partial r_j} - \frac{\partial v_j}{\partial r_i}\right) \end{aligned} \quad (10)$$

The Hubble expansion is unique in that it does not have any anisotropic terms, both shear and vorticity are equal to zero. In other words, the Hubble parameter is equal to

$$H(t) \;=\; \frac{1}{3}\nabla\cdot\mathbf{v} \quad (11)$$

In the lecture the Hubble expansion illustrated by means of a two-dimensional cartoon involving hundred randomly distributed points within a square. In this computer task you will need to follow up on this experiment for the generic case involving both an anisotropic shear term and a vorticity term. You may need whatever computer program (Matlab, Python, IDL) you feel most at ease with.

- Distribute $N = 1000$ points randomly within a box of size 100×100. This defines timestep t_1. The particles have an initial location $\vec{r}_j(t_1) = (r_{1,j}, r_{2,j})$, $j = (1, \ldots, N)$.

Within the box, take an arbitrary central position $\vec{s}$, from where we observe the displacement of the surrounding Universe.

We are going to follow the evolution of the initial particle distribution at 2 subsequent timesteps t_2 and t_3. The initial timestep is t_1. At t_1 the particles have an initial position $\vec{r}_j(t_1) = (r_{1,j}, r_{2,j})$, $j = (1, \ldots, N)$.

In a time interval $\Delta t = (t - t_1)$ each particle m gets displaced by an amount $\Delta\vec{r}_m(t)$,

$$\vec{r}_m(t) \;=\; \vec{r}_m(t_1) \,+\, \Delta\vec{r}_m(t) \quad (12)$$

where the displacement is a product of the deformation $\vec{\mathcal{D}}$ with the time interval Δt,

$$\Delta\vec{r}(t) \;=\; \vec{\mathcal{D}}\,\Delta t\,, \tag{13}$$

The deformation is the sum of an expansion/contraction H, shear σ and vorticity ω term.

Assume we have a particle m with initial coordinates $\vec{r}(t_1) = (r_1, r_2)$. then the k-th coordinate of its deformation $\vec{\mathcal{D}}$, with $\vec{s}$ as deformation center, is equal to

$$\mathcal{D}_k(t) \;=\; H\,(r_k - s_k) \;+\; \sigma_{kj}(r_j - s_j) \;+\; \omega_{kj}(r_j - s_j)\,. \tag{14}$$

where we use the Einstein summation convention (summing over $j = 1, \ldots, 3$)!

For our experiment the resulting location $\vec{r}(t) = (r_1(t), r_2(t))$ at time t of the particle with initial location $\vec{r}(t_1) = (r_1, r_2)$ is

$$r_k(t) \;=\; r_k \;+\; \mathcal{H}(r_k - s_k) \;+ \Sigma_{kj}(r_j - s_j) \;+\; \Omega_{kj}(r_j - s_j)\,, \tag{15}$$

where $\mathcal{H} \equiv H\Delta t$, $\Sigma_{kj} \equiv \sigma_{kj}\Delta t$ and $\Omega_{kj} \equiv \omega_{kj}\Delta t$. Thus, for the initial time

$$t_1: \qquad \mathcal{H} = 0, \quad \Sigma_{kj} = 0, \quad \Omega_{kj} = 0 \tag{16}$$

Note that strictly speaking this expression is only valid for small displacements (small timesteps): keep it moderate ... (but not too small either, otherwise it is not too clarifying).

For the traceless shear tensor we have the following conditions:

$$\Sigma_{ij} \;=\; \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{17}$$

for which $\sigma_{11} + \sigma_{22} = 0$ (so that Σ_{ij} is effectively specified by 2 numbers, σ_{11} and σ_{21}) while the vorticity tensor is specified via one number, ω,

$$\Omega_{ij} \;=\; \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{18}$$

Repeat the following for two different deformation centres:

- $\vec{s} = (50, 50)$ as centre of the plotbox of size 50×50.
- $\vec{s} = (75, 75)$ as centre of the plotbox of size 50×50.

Generate 6 different configurations:

- pure expansion only: $\Sigma_{ij} = \Omega_{ij} = 0$
- pure shear only: $\mathcal{H} = \Omega_{ij} = 0$
- pure vorticity only: $\mathcal{H} = \Sigma_{ij} = 0$
- expansion + shear: $\Omega_{ij} = 0$
- expansion + vorticity: $\Sigma_{ij} = 0$
- expansion + shear + vorticity

For each timestep plot particle distribution within the central box of size 50×50. You should decide yourself on the values for Σ_{ij}, Ω_{ij} and H. You have some freedom in choice, but do not assume values which are too radical.