Diffusion & Viscosity:

Navier-Stokes Equation
Imagine a quantity $C(x,t)$ representing a local property in a fluid, eg.

- thermal energy density
- concentration of a pollutant
- density of photons propagating diffusively through a scattering medium

For a fluid at rest, $V=0$, the diffusive transport of the quantity $C$ in the fluid is described by the Diffusion Equation,

$$ \frac{\partial C}{\partial t} = \vec{V} \cdot D \vec{\nabla} C $$

In this expression, $D$ is the diffusion coefficient,

$$ D = \frac{v_0 \lambda}{3} $$

with $v_0$ the velocity of the diffusing particles, and $\lambda$ the mean free path.
In general, the viscous force $f_{\text{visc}}$ includes 2 different aspects, that of shear viscosity $\eta$ and bulk viscosity $\zeta$.

entailing the following full viscous force

$$f_{\text{visc}} = \eta \nabla^2 \vec{v} + (\zeta + \frac{1}{3} \eta) \nabla \left( \nabla \cdot \vec{v} \right)$$

which for incompressible flow, $\nabla \cdot \vec{v} = 0$, is restricted to

$$f_{\text{visc}} = \eta \nabla^2 \vec{v}$$
Navier-Stokes Equation

- For a fluid with (shear) viscosity $\eta$, the equation of motion is called the Navier-Stokes equation. In its most basic form, i.e., for incompressible media

$$ \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \eta \nabla^2 \vec{v} $$

- Without any discussion, this is THE most important equation of hydrodynamics.

- While the Euler equation did still allow the description of many analytically tractable problems, the nonlinear viscosity term in the Navier-Stokes equation makes the solving of the NS equation very complicated.

- There are only a few situations that allow analytical solutions for the NS equation, the remainder needs to be solved numerically/computationally.

Navier-Stokes Equation

- The general and full Navier–Stokes equation, for a fluid with
  - shear viscosity $\eta$
  - bulk viscosity $\zeta$

  is given by

$$ \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \eta \nabla^2 \vec{v} + (\zeta + \frac{1}{3} \eta) \nabla (\nabla \cdot \vec{v}) $$
Reynolds Number

- The Reynolds number is the measure of the importance of viscous effects of a flow - hereby assuming the bulk viscosity $\zeta = 0$ - and is defined as the ratio of the magnitude of the inertial force - magnitude of the viscous force

$$Re = \frac{\text{magnitude inertial force}}{\text{magnitude viscous force}} = \left| \frac{\rho (\vec{v} \cdot \vec{\nabla})\vec{v}}{\eta \nabla^2 \vec{v}} \right|$$

- For large Reynolds number, the flow gets unstable, and finally becomes turbulent.

Reynolds Number

- The Reynolds number is the ratio of the magnitude of the inertial force to the magnitude of the viscous force

$$Re = \frac{\text{magnitude inertial force}}{\text{magnitude viscous force}} = \left| \frac{\rho (\vec{v} \cdot \vec{\nabla})\vec{v}}{\eta \nabla^2 \vec{v}} \right|$$

- We can find an order of magnitude rough estimate for the Reynolds number. With $U$ the characteristic magnitude of the velocity in a system of characteristic size $L$, we have

$$| (\vec{v} \cdot \vec{\nabla})\vec{v} | \sim \frac{U^2}{L}$$

$$| \eta \nabla^2 \vec{v} | \sim \frac{\rho \nu U}{L^2}$$

$$\left\{ \begin{array}{l}
Re \sim \frac{UL}{\nu}
\end{array} \right.$$
Navier-Stokes Equation: analytical soln’s

- Due to the high level of nonlinearity and complexity of the full compressible Navier-Stokes equations, there are hardly any analytical solutions known of the Navier-Stokes equation.

\[
\frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v}
\]

- One may try to find some specific configurations that would allow an analytical treatment. This involves simplifying the equations by making the following assumptions:
  - about the fluid
  - about the flow
  - geometry of the problem

- Typical assumptions are:
  - laminar flow
  - steady flow
  - incompressible flow

- Examples are:
  - parallel flow in a channel
  - Couette flow
  - Hagen-Poiseuille flow, ie. flow in a cylindrical pipe.

Navier-Stokes Equation: Channel flow

- Consider the following configuration:
  - flow of a fluid through a channel
  - steady flow
  - incompressible flow
  - axisymmetric geometry (2-D problem)

- the 2-D flow field is represented by a 2-D velocity field, \( \mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix} \)

- with \( u \) the component in the x-direction, \( v \) in the y-direction
Navier-Stokes Equation: Channel flow

- the 2-D flow field is represented by a 2-D velocity field, with $u$ the component in the $x$-direction, $v$ in the $y$-direction

- the flow of the system is then described by the
  (a) continuity equation
  (b) Navier-Stokes equation

\[
\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \eta \nabla^2 \vec{v}
\]

- which for the system at hand simplify to:

  continuity equation: \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \)
  (notice: incompressibility)

  x-momentum (NS): \( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \eta \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \)

  y-momentum (NS): \( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \eta \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \)

Navier-Stokes Equation: Channel flow

- Boundary condition:
  the flow is constrained by flat parallel walls of the channel,
  \( v_y = v = 0 \)

  \[ \downarrow \]

  \( \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial y} = 0 \)

  - Continuity equation:
    \( \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = 0; \quad \frac{\partial^2 u}{\partial x^2} = 0 \)

  - Using these relations, we end up with the Navier-Stokes equations:

\[
- \frac{1}{\rho} \frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} = 0
\]

\[
- \frac{1}{\rho} \frac{\partial p}{\partial y} = 0
\]
Navier-Stokes Equation: Channel flow

- Given that
  \[ \frac{\partial u}{\partial x} = 0 \]

  we immediately infer that \(u(x,y)\) must be independent of \(x\). Hence
  \[ \eta \frac{\partial^2 u}{\partial y^2} \]
  can only be a function of \(y\), i.e. \(u(x,y) = u(y)\). This implies, via the relation,
  \[ -\frac{1}{\rho} \frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} = 0 \]
  that,
  \[ \frac{\partial p}{\partial x} = \frac{dp}{dx} = \text{cst.} \]

  and that the general solution for \(u(y)\) is given by

  \[ u(y) = \frac{1}{2} \frac{1}{\rho \eta} \frac{\partial p}{\partial x} y^2 + Ay + B \]

Navier-Stokes Equation: Channel flow

- The general solution for \(u(y)\) is given by

  \[ u(y) = \frac{1}{2} \frac{1}{\rho \eta} \frac{\partial p}{\partial x} y^2 + Ay + B \]

- Using the boundary conditions that the velocity \(u=0\) at the border of the channel, i.e. \(u(\pm R) = 0\), the constants \(A\) and \(B\) get fixed

  \[ A = 0; \quad B = -\frac{1}{2} \frac{R^2}{\rho \eta} \frac{dp}{dx} \]

  which yields the complete solution for the flow velocity \(u(y)\) through the channel:

  \[ u(y) = -\frac{1}{2} \frac{R^2}{\rho \eta} \frac{dp}{dx} \left[ 1 - \left( \frac{y}{R} \right)^2 \right] \]
Navier-Stokes Equation: Channel flow

\[ u(y) = -\frac{1}{2} \frac{R^2}{\rho \eta} \left[ 1 - \left( \frac{y}{R} \right)^2 \right] \]

Flow through a channel thus displays a parabolic velocity distribution, symmetric about the central axis. The maximum velocity \( u_{\text{max}} \) is attained along the central axis.

\[ u_{\text{max}} = -\frac{1}{2} \frac{R^2}{\rho \eta} \frac{dp}{dx} \]

Because of friction along the banks, flow velocity in a straight channel is highest near the surface in the middle of the stream.