– Astrophysical Hydrodynamics –
Assignment 1

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In the first two assignments, the aim is to review vector analysis, curvilinear coordinates and some of their applications in classic mechanics and electrodynamics. We will also derive some identities that will be used later in the context of hydrodynamics.

1 Gradient, divergence and curl

An important operator is the differential operator $\boldsymbol{\nabla} = \hat{e}_i \partial_i$. Let's now review some definitions and vector identities. In Cartesian coordinates, the gradient and Laplacian of a scalar field ϕ and the divergence and curl of a vector field $\boldsymbol{A}$ are defined as follows:

$$\mathbf{grad}(\phi) = \boldsymbol{\nabla}\phi = \partial_i \phi \hat{\boldsymbol{e}}_i \tag{1}$$

$$\nabla^2 \phi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \partial_i^2 \phi \tag{2}$$

$$\mathbf{div}(\boldsymbol{A}) = \boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_k A_k \tag{3}$$

$$\mathbf{curl}(\boldsymbol{A}) = \boldsymbol{\nabla} \times \boldsymbol{A} = \epsilon_{ijk} \partial_j A_k \hat{\boldsymbol{e}}_i \,. \tag{4}$$

The Einstein summation convention is used and ϵ_{ijk} is the *Levi-Civita symbol* or *permutation tensor* (to be discussed during tutorial if you aren't familiar). Successive applications of $\boldsymbol{\nabla}$ lead us to the following results:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) \tag{5}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) \tag{6}$$

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{7}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0 \tag{8}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0 \tag{9}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{10}$$

A divergence free vector field is also called as incompressible (in hydrodynamics) or solenoidal (in electrodynamics) vector field and a curl free vector field is known as irrotational (in hydrodynamics) or conservative (in classical mechanics) vector field. See textbooks (Arfken, Spiegel) for derivation of the above identities.

1.1 Maxwell's equations, Wave equation and Gauge transformation

Maxwell's equations of classical electrodynamics (in vacuum i.e. without media) can be written in differential form as follows:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \tag{11}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{12}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{13}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\,. \tag{14}$$

As a short recap, the equations are Gauss's law (11), Gauss's law for magnetism (12), the Maxwell-Faraday equation (13) and Maxwell-Ampère equation (14) (see also: Griffiths).

1. Using the vector identities given above, proof the continuity equation,

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \tag{15}$$

   and try to give a physical interpretation of divergence from the above equation.

2. In a source free region in space ($\rho = 0$, $\boldsymbol{J} = 0$), derive the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\nabla^2 \boldsymbol{E} = \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{16}$$

$$\nabla^2 \boldsymbol{B} = \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}\,. \tag{17}$$

3. Let's now consider the case when the electric charge density and current density are non-zero. From the two homogeneous Maxwell's equations, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ show that, $\boldsymbol{E}$ and $\boldsymbol{B}$ can be written in terms of vector and scalar potentials $\boldsymbol{A}$ and V respectively as follows:

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \tag{18}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}\,. \tag{19}$$

4. Now show that the other two inhomogeneous Maxwell's equations can be expressed in terms of the vector and scalar potential as follows:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{1}{\epsilon_0}\rho \tag{20}$$

$$(\nabla^2 \boldsymbol{A} - \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2}) - \nabla(\nabla \cdot \boldsymbol{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}) = -\mu_0 \boldsymbol{J}\,. \tag{21}$$

5. Now, using Lorenz gauge $\nabla \cdot \boldsymbol{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$, show that vector and scalar potentials ($\boldsymbol{A}$, V) follow the following two inhomogeneous wave equations,

$$\Box V = -\tfrac{1}{\epsilon_0}\rho \tag{22}$$

$$\Box \boldsymbol{A} = -\mu_0 \boldsymbol{J}\,. \tag{23}$$

   Here, $\Box = (\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2})$ is the d'Alembert operator.

1.2 Classical Mechanics: Central Force motion

In classical mechanics, central force is defined as $\boldsymbol{F} = \boldsymbol{r} f(r)$, where $\boldsymbol{r}$ is the position vector of the particle from the origin.

1. Calculate the divergence of this field. Assume a general form $f(r) = r^{n-1}$ and show that the divergence is $(n+2)r^{n-1}$. Think about the inverse square law force field (e.g. gravitational or electrostatic force) in the context of this result.

2. Show that a central force field is conservative.

1.3 Continuum Mechanics: Convective operator and convective derivative

For a vector field $\boldsymbol{A}$ the convective operator is defined as $(\boldsymbol{A} \cdot \boldsymbol{\nabla})$. In fluid dynamics we often use the Lagrangian description, where we follow the fluid elements when describing macroscopic properties of the fluid. Instead of talking e.g. about a density $\rho_{\text{Euler}}(\boldsymbol{x}, t)$ at a fixed position $\boldsymbol{x}$, we consider the density $\rho_{\text{Lagrangian}}(\boldsymbol{x}(t), t)$ of a specific fluid element that moves in the flow.

When we want to take the time derivative of a macroscopic property of a fluid element, like the density, we need to use the Lagrangian derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \tag{24}$$

where $\boldsymbol{u}$ is the flow velocity.

1. Apply chain rule of partial derivative and derive the result. Interpret in the context of convection of a property of the fluid, e.g. temperature $T(\boldsymbol{x}(t), t)$.

2. Consider an accelerating fluid flow with a velocity field $\boldsymbol{u} = (U+cx)\hat{\boldsymbol{\imath}} + (U-cy)\hat{\boldsymbol{\jmath}}$. Find the acceleration.

2 Vector integration, Gauss and Stokes theorem

Line integral of the tangential component of a vector field $\boldsymbol{A}$ along a curve C or work done (in mechanics):

$$\int_C \boldsymbol{A} \cdot d\boldsymbol{r} \tag{25}$$

Surface integral of the normal component of a vector field $\boldsymbol{A}$ over a surface S or flux or flow:

$$\iint_S \boldsymbol{A} \cdot d\boldsymbol{S} \tag{26}$$

Volume integral of a vector field $\boldsymbol{A}$ over a volume V:

$$\iiint_V \boldsymbol{A} dV \tag{27}$$

In fluid dynamics, line integral of velocity field along a simple closed curve is denoted as **circulation** Γ:

$$\Gamma = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r} \tag{28}$$

Gauss' Theorem:

$$\oiint_S \boldsymbol{A} \cdot d\boldsymbol{s} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{A} dV \tag{29}$$

Stokes' Theorem:

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{r} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot d\boldsymbol{S} \tag{30}$$

In fluid dynamics, circulation Γ can be related to **vorticity** $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ using Stokes' theorem:

$$\Gamma = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \iint_S \boldsymbol{\omega} \cdot d\boldsymbol{S} \tag{31}$$

2.1 Electrostatics, Magnetostatics, Electromagnetic Induction

Using Gauss' and Stokes' theorem Maxwell's equations can be written in integral form as follows,

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{\epsilon_0} \iiint_V \rho dV \tag{32}$$

$$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0 \tag{33}$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{d}{dt} \iint_S \boldsymbol{B} \cdot d\boldsymbol{S} \tag{34}$$

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 \iint_S \boldsymbol{J} \cdot d\boldsymbol{S} + \epsilon_0 \mu_0 \frac{d}{dt} \iint_S \boldsymbol{E} \cdot d\boldsymbol{S}\,. \tag{35}$$

1. Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and its base $x^2 + y^2 \leq 1, z = 0$. Let $\boldsymbol{E}(x, y, z) = 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}}$ be an electric field. Find the electric flux across S.

2. Consider an infinitely long straight wire, carrying a steady current I. Find out the magnetic field as a function of distance from the wire.

3. Also calculate the magnetic vector potential $\boldsymbol{A}$ outside the wire, which is defined as $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

4. Now consider the same wire but this time the current is a slowly varying function of time, $I(t)$. Consider a rectangular Amperian loop and using quasi static approximation (i.e. $\frac{\partial \boldsymbol{E}}{\partial t} \approx 0$), determine the induced electric field as a function of distance from the wire.

2.2 Classical Mechanics: Work done, Conservative force field, Potential

From Stokes' theorem we get that the necessary and sufficient condition for $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for any vector field $\boldsymbol{F}$ for every simple closed curve C is $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$. So, $\boldsymbol{F}$ can be written as a gradient of a scalar potential function, $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

1. If $\boldsymbol{F} = 2xy^3z^4\hat{\boldsymbol{\imath}} + 3x^2y^2z^4\hat{\boldsymbol{\jmath}} + 4x^2y^3z^3\hat{\boldsymbol{k}}$, calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the path $C : x = t, y = t^2, z = t^3$.

2. Show that $\boldsymbol{F}$ is a conservative force field and find the corresponding potential function ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

References

1. Arfken G. B. & Weber H. J., Mathematical Methods for Physicists, Elsevier
2. Spiegel M. R., Schaum's Outline of Vector Analysis, McGraw-Hill
3. Griffiths D. J., Introduction to Electrodynamics.