Bernoulli Applications
A Venturi meter is used to measure the flow rate through a tube.

It is based on the use of the Venturi effect, the reduction of fluid pressure that results when a fluid runs through a constricted section of pipe. It is called after Giovanni Battista Venturi (1746-1822), an Italian physicist.

Look at the construction in figure:

- we assume the flow is smooth and effectively inviscid, i.e. friction is negligible.

- the fluid is incompressible, and has density \( \rho \) throughout the pipe.

- downstream we have a flow through a pipe section of area \( A_1 \), with a flow velocity \( v_1 \), and pressure \( p_1 \).

- in the narrow section with area \( A_2 \), the fluid flows with flow speed \( v_2 \), and has accompanying pressure \( p_2 \).

- as a result the two meters indicate the difference in pressure by means of a height difference \( h \).
To find the pressure difference between the downstream flow and the pipe narrow, we invoke 1) the Bernoulli theorem and 2) the continuity equation. The latter assures that the rate of fluid flow through any section remains constant, i.e. mass is preserved.

1) Bernoulli Theorem:

\[
\frac{p_1}{\rho} + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2
\]

as the flow is horizontal, we do not have to take into account the gravity term.

2) Continuity equation:

\[A_1v_1 = A_2v_2\]

Combining both equations, we find for the pressure difference in the two parts of the pipe:

\[p_1 - p_2 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)\]
To read of the pressure difference between the two locations 1 and 2 in the fluid, we use the height difference $h$ between the fluid level in the vertical tubes. To connect this height difference $h$ to the pressure difference $p_1$ and $p_2$, we invoke the Euler equation:

3) Euler equation
(for a static medium):

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

With the outside atmosphere pressure being $P_{atm}$, we then directly infer for the pressure $P_1$ and $P_2$,

$$P_1 = P_{atm} + \rho gh_1$$
$$P_2 = P_{atm} + \rho gh_2$$

From this we may then infer the flow velocity $v_1$ (as well as $v_2$):

$$v_1^2 = \frac{2g(h_1 - h_2)}{\left(\frac{A_1^2}{A_2^2} - 1\right)}$$
Air Flow along Wing

Airflow along the wing of an airplane:

Note the condensation over the upper part of the wing, where the higher flow speeds correspond to a lower pressure and thus lower temperature.
One of the most interesting applications of the Bernoulli equation, is the flight of aeroplanes.

Here we will provide a simplified explanation, based on the Bernoulli equation (reality is somewhat more complex).

Aeroplanes can fly because of the pressure difference between the flow below the wing and the flow over the wing*. This pressure difference results in a lift force that opposes the weight of the aeroplane (note that similar lifting forces work on many different objects, eg. wings of mills or wind turbines, sails on a sailboat, propellors).

Flow around an airfoil: the dots move with the flow. Note that the velocities are much higher at the upper surface than at the lower surface. The black dots are on timelines, which split into two — an upper and lower part — at the leading edge. The part of a timeline below the airfoil does not catch up with the one above! Colors of the dots indicate streamlines.

Jet fighters often are not kept aflight by Bernoulli. Instead, they have the thrust of the jet motor, with vertical component, to keep them in the air.

Forces working on a wing - airfoil.
One of the most interesting applications of the Bernoulli equation, is the flight of aeroplanes. Here we will provide a simplified explanation, based on the Bernoulli equation (reality is somewhat more complex).

Aeroplanes can fly because of the pressure difference between the flow below the wing and the flow over the wing. This pressure difference results in a lift force that opposes the weight of the plane. Lift forces work on many different objects, e.g. wings of mills or wind turbines, sails on a sailboat, propellers.

Air Flow along Wing

According to Bernoulli, the Bernoulli function $B$ is constant along any streamline. Thus, for a horizontally flying plane, we have that

1) $\frac{p_l}{\rho} + \frac{1}{2}v_l^2 = \frac{p_u}{\rho} + \frac{1}{2}v_u^2 = \text{cst.}$

2) $\rho A_l v_l = \rho A_u v_u$ (Continuity eqn.)

$C_L$ is the lift coefficient, dependent on various factors, including the angle of wing wrt. air.

Along the wing, the flow over the upper (longer) edge of the wing has a (considerably) higher velocity $u_u$ than below the wing, $u_l$.

As a result (Bernoulli), the pressure $p_l$ at the lower end of the wing is higher than the pressure $p_u$ at the upper end.

The resulting pressure difference generates a lift force $F_{lift}$

$$ F_{lift} = (p_2 - p_1) A $$

The usual assumption of equal cross-time of upper and lower flow is not correct.
With the pressure difference between lower and upper wing being

\[ p_2 - p_1 = \frac{1}{2} \rho v^2 C_L \]

the total effective lift force

\[ F_{\text{lift}} = (p_2 - p_1) A \]

is, with \( A \) the “effective” planform area, and \( C_L \) the lift coefficient,

\[ F_{\text{lift}} = \frac{1}{2} \rho v^2 AC_L \]

In other words, we need a certain speed \( V \) and wing area \( A \) to get sufficient lift force to lift a plane into the air ... (see right) ...
Air Flow along Wing

Reality, of course, is slightly more complex. The accompanying movie gives an impression...
Bernoulli Equation: compressible fluids.

A very interesting application of the Bernoulli equation, for compressible fluids, concerns the de Laval nozzle.

A de Laval nozzle is a tube that is pinched in the middle, making a carefully balanced, asymmetric hourglass-shape. The nozzle was developed in 1888 by the Swedish inventor Gustaf de Laval for use on a steam turbine. The principle was first used for rocket engines by Robert Goddard.

The de Laval nozzle forms a nice platform to highlight the differences introduced by the compressibility of a gas when applying Bernoulli’s theorem.
The de Laval nozzle is used to accelerate a hot, pressurised gas passing through it to a supersonic speed.

High-pressure gas coming from the combustion chamber enters the nozzle and flows into a region where the nozzle cross section decreases, \( \frac{dA}{dx} < 0 \). The thermal energy is converted into kinetic energy of the flow, and the flow goes through a sonic point at the critical point where the nozzle cross section narrows to its minimum (\( \frac{dA}{dx} = 0 \)). At that point the flow speed reaches the sound velocity. The cross section increases again after the critical point, and the gas is further accelerate to supersonic speeds.

The de Laval nozzle shapes the exhaust flow so that the heat energy propelling the flow is maximally converted into directed kinetic energy.

Because of its properties, the nozzle is widely used in some types of steam turbine, it is an essential part of the modern rocket engine, and it also sees use in supersonic jet engines.

Astrophysically, the flow properties of the de Laval nozzle have been applied towards understanding jet streams, such as observed in AGNs (see figure), the outflow from young stellar objects and likely occur in Gamma Ray Bursts (GRBs).
If we make the approximation of steady, quasi-1-D barotropic flow, we may write Bernoulli’s theorem and the equation of continuity as

\[
\frac{1}{2} u^2 + \int \frac{dP}{\rho} = \text{cst.}
\]

\[
\rho u A = \text{cst.}
\]

where \(A\) is the local sectional area of the nozzle.

Note that because of the compressibility of the gas we no longer assume a constant density, and thus have to keep \(\rho\) in the integral.

Gravitational potential variations are ignored, as for terrestrial applications the fast flow of jet gases is not relevant over the related limited spatial extent.
De Laval Nozzle

The variation of the area $A$ along the axis of the nozzle will introduce spatial variations for each of the other quantities.

To consider the rate of such variations, take the differential of the Bernoulli equation,

$$\frac{1}{2} u^2 + \int \frac{dp}{\rho} = \text{cst.} \quad \Rightarrow \quad u \, du + \frac{1}{\rho} \frac{dp}{d \rho} = 0$$

Taking into account that the sound velocity $c_s$ associated with the barotropic relation is

$$c_s^2 \equiv \frac{dp}{d \rho}$$

we find from the equations above that

$$u \, du + \frac{c_s^2}{\rho} \, d \rho = 0$$

We define the Mach number of the flow as the ratio of the flow velocity to the sound velocity,

$$M = \frac{u}{c_s}$$

Illustration of the run of flow speed $u$, pressure $p$ and temperature $T$, as the gas passes through the nozzle and its sonic point.
De Laval Nozzle

From the relation between velocity and density,

\[ u \, du + \frac{c_s^2}{\rho} \, d\rho = 0 \]

we find that the fractional change of density \( \rho \) is related to the fractional change of the fluid velocity \( u \) via the equation

\[
\frac{d\rho}{\rho} = -M^2 \frac{du}{u}
\]

This equation states that the square of the Mach number provides a measure of the importance of compressibility.

In particular, flow of air at subsonic speeds past terrestrial obstacles can often be approximated as occurring at incompressibility, because the fractional change of density \( \rho \) is negligible in comparison with the fractional change of \( u \) if \( M \ll 1 \).

In contrast, supersonic flight past obstacles necessarily involves substantial compressions and expansions.

Illustration of the run of flow speed \( u \), pressure \( p \) and temperature \( T \), as the gas passes through the nozzle and its sonic point.
To relate the change of velocity $u$ to the change of sectional area $A$ in the nozzle, we take the logarithmic derivative of the continuity equation,

To consider the rate of such variations, take the differential of the Bernoulli equation,

\[ \rho u A = \text{cst.} \]

\[ \Downarrow \]

\[ \frac{d \rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \]

which, taking into account the relation between density $\rho$ and flow speed $u$, yields the following relation between $u$ and $A$:

\[ (1 - M^2) \frac{du}{u} = - \frac{dA}{A} \]

Illustration of the run of flow speed $u$, pressure $p$ and temperature $T$, as the gas passes through the nozzle and its sonic point.
the relation

\[
(1 - M^2) \frac{du}{u} = -\frac{dA}{A}
\]

has the following implications:

1) Subsonic speeds:

\[
(1 - M^2) > 0
\]

\[
\text{down}
\]

\[
du > 0 \iff dA < 0
\]

this corresponds to normal experience, e.g. the speeding up of a river as the channel narrows.

Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.
2) *Supersonic speeds.*

\[ (1 - M^2) < 0 \]

\[ \downarrow \]

\[ du > 0 \iff dA > 0 !!! \]

In other words, an increase in the velocity requires an *increase* in the area of the nozzle, \( dA > 0 !!! \)

This counterintuitive result has a simple explanation: for \( M > 1 \), the density decreases faster than the area increases, so the velocity must increase to maintain a constant flux of mass.
3) A sonic transition, ie.

\[
\left(1 - M^2\right) = 0
\]

can be made smoothly, ie. With

\[
\frac{du}{dx} \text{ finite}
\]

only at the throat of the nozzle where

\[
dA = 0
\]

To obtain supersonic exhaust, therefore, we must accelerate the reaction gases through a converging-diverging nozzle, a fundamental feature behind the design of jet engines and rockets.

4) Note that the converse does not necessarily hold:

\( M \) does not necessarily equal unity at the throat of the nozzle, where \( dA=0 \). If \( M \neq 1 \), the fluid velocity reaches a local extremum when the area does, i.e. \( du=0 \) where \( dA=0 \).

Whether the extremum corresponds to a local maximum or minimum depends on whether we have subsonic or supersonic flow and whether the nozzle has a converging-diverging shape.
Final Notes:

5) whether supersonic exhaust is actually achieved in nozzle flow also depends on the boundary conditions. In particular, on the pressure of the ambient medium in comparison with the pressure of the reaction chamber.

If a sonic transition does occur, the flow behaviour depends sensitively on the nozzle conditions, since the coefficient $1-M^2$ becomes arbitrarily small near the transition region.

6) When external body forces are present, we do not need to have a throat to achieve the smooth transition of subsonic to supersonic flow. The external forces can provide the requisite acceleration.
Werner von Braun:
father of the Saturn 5 moon rocket
(and the German V2 missiles 
WWII)

Standing near motor exhausts
Saturn V
Apollo 11: mission to the moon

Launch Apollo 11

July 16, 1969

exhausts of motors of the giant Saturn 5 moon rocket

beautiful illustration of workings De Laval nozzle
Potential Flow
Many problems of practical importance, involving a large number of engineering and terrestrial conditions concern incompressible flows.

For an incompressible flow, we have

\[ \nabla \cdot \vec{u} = 0 \]

which follows directly from the continuity equation on the basis of the conditions

\[ \frac{\partial \rho}{\partial t} = 0; \quad \nabla \rho = 0 \]

In other words, for an incompressible fluid (a liquid) the variation of pressure \( p \) in the force (Euler) equation equals whatever it needs so that \( \nabla \times \vec{u} = 0 \).

Note that for an incompressible medium, Kelvin's circulation theorem is valid independent of the barotropic assumption: because \( \nabla \rho = 0 \) the vorticity equation is true independent of
Many problems in hydrodynamics involve the motion of a solid body (eg., a ship) through water that is stationary at infinity.

From the point of view of an observer fixed on the ship, the water flowing past the ship originates from a steady region of uniform conditions.

- Uniform flow has no vorticity: Kelvin's circulation then guarantees that no vorticity will be generated in the flow around the ship.

  Note: this is only true as long as the effect of viscosity can be ignored.

- If $\vec{\omega} = \nabla \times \vec{u} = 0$ everywhere, the flow field can be derived from the gradient of a scalar, the velocity potential $\Phi_v$.

\[ \vec{u} = \nabla \Phi_v \]
If we substitute the velocity potential definition

\[ \vec{u} = \nabla \Phi_v \]

into the continuity equation

\[ \nabla \cdot \vec{u} = 0 \]

we obtain the condition for potential flow,

\[ \nabla^2 \Phi_v = 0 \]

This is nothing else than the Laplace equation.

Thus, the solution of many problems in hydrodynamics boils down to a solution of the Laplace equation. The problem is well-posed, and there is a vast body of work on its solution.

To solution of the Laplace equation is dictated by the boundary (and initial) conditions that are imposed.
To solve the Laplace equation, we need to specify the boundary conditions. There are a variety of boundary conditions. Usually, these involve one (or more) of the following:

- **a) The value of \( \Phi \)** on the bounding surface of the fluid → Dirichlet boundary condition

- **b) The value of its normal derivatives on the boundary** → Neumann boundary condition

For the problem of a flow past a solid object like a ship or a sphere, we have the important condition that the water should not penetrate the object, i.e., there should be no flow normal to its surface. This translates into the **Neumann boundary condition**:

- **a) for the object at rest:**
  \[
  \vec{u} \cdot \hat{n} = 0 \quad \Rightarrow \quad \hat{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n} = 0
  \]

- **b) if the object moves with velocity \( \vec{U} \), then**
  \[
  \vec{u} \cdot \hat{n} = \vec{U} \cdot \hat{n} \quad \Rightarrow \quad \hat{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n}
  \]
To find the solutions to the Laplace equation,

$$\nabla^2 \Phi_v = 0$$

one can apply the mathematical machinery of potential theory. To provide an idea of the solutions we concentrate on solutions for Spherical geometry (of object) and Axisymmetric flow.

The general axisymmetric solution of Laplace’s equation

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi_v}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi_v}{\partial \theta} \right) = 0$$

is obtained by the separation-of-variables method in spherical polar coordinates, i.e.

$$\Phi_v = R(r) \Theta(\theta)$$

which yields

$$\Phi_v = \sum_{n=0}^{\infty} \left\{ A_n r^n + B_n r^{-(n+1)} \right\} P_n(\cos \theta)$$

$P_n$ is the Legendre Polynomial of degree $n$:

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_2(\mu) = \frac{3}{2} \mu^2 - \frac{1}{2}$$

$r$ is the spherical radius
$	heta$ the co-latitude (see figure):

$$z = r \cos \theta$$

$A_n$ and $B_n$ are arbitrary constants, whose value is determined by the boundary conditions.
In the case of potential flow, we find from the fact that it is irrotational,

$$\omega = \nabla \times \vec{u} = 0$$

and the velocity can be written as the gradient of a potential \( \Phi_v \) that the Euler equation for barotropic flow and potential external forces can be written as

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \phi - \nabla h$$

\[\Downarrow\]

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} |\vec{u}|^2 \right) + (\nabla \times \vec{u}) \times \vec{u} = -\nabla \phi - \nabla h$$

\[\Downarrow\]

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} |\vec{u}|^2 \right) = -\nabla \phi - \nabla h$$

for barotropic flow and potential external forces can be written as
for barotropic flow and potential external forces can be written as

\[ \nabla \left( \frac{\partial \Phi}{\partial t} \right)_v + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) = -\nabla \phi - \nabla h \]

\[ \Downarrow \]

\[ \nabla \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \phi + h \right) = 0 \]

from which we can immediately infer that the Bernoulli function is a function of time:

\[ \nabla \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \phi + h \right) = 0 \]

\[ \Downarrow \]

\[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \phi + h = \frac{\partial \Phi}{\partial t} + B(t) = T(t) \]
Hydrostatics
Systems where motion is absent altogether, or at least has no dynamic effects, are in hydrostatic equilibrium $\ddot{u} = 0$.

In those situations, the fluid equations reduce to simple equilibrium equations.

1) Continuity equation:
\[
\frac{\partial \rho}{\partial t} = 0
\]

2) Euler equation:
\[
\frac{1}{\rho} \nabla p = \vec{f} = -\nabla \Phi
\]

(the latter identity in the Euler equation is for the body force being the gravitational force).

We will shortly address four typical examples of hydrostatic equilibrium, all of major astrophysical interest:

1) Archimedes' Principle, buoyancy forces

2) Isothermal sphere

3) Stellar Structure equations

4) Mass determination of clusters from their X-ray emission.
Archimedes' Principle
Archimedes

Syracuse, 287-211/212 BC,
Greatest mathematician &
scientist of antiquity (all time?):

- Probably studied in Alexandria,
  under followers Euclides
- Killed by Roman soldier, upon
  Roman conquest Syracuse
- Family Hieron II, king Syracuse?
- Inventions:
  - war machines ...
  - water screw
  - water organ (?)
  - burning mirrors (???)
  - planetarium !!!!!!!
In the situation where an object is (partially) immersed in a fluid (see figure), Archimedes’ principle states, shortly, that

\[
\text{Buoyancy} = \text{Weight of displaced fluid}
\]

Pressure by water on displaced volume:

\[
\frac{1}{\rho} \nabla p = f = g
\]

\[
F = -\int_S p \, dS = -\int_V \nabla p \, dV = -\rho V g
\]

This is called the buoyancy force, and underlies a large amount of practical applications - starting from ships floating on water.
Archimedes Principle

'...the principle is called after
Archimedes of Syracuse (287-212 BC),
Antiquities’ greatest genius.

He got the idea when ordered by
King Hieron II of Syracuse to
investigate whether the
golden crown he had ordered to be
manufactured contained the pure
gold he had provided the goldsmith or
whether the smith had been dishonest
and included silver ...

Immersing the crown in water, Archimedes
determined the volume. Comparing its weight
by a balance containing similar amount of
pure gold, he found the density of the the
crown ... which turned out not to be pure

Buoyancy  =  Weight of displaced fluid
Archimedes Principle

1
Volume of aluminum = 100 cm\(^3\)
Density of aluminum = 2.7 g/cm\(^3\)
Mass of aluminum = 270 g
Weight of aluminum = 2.7 N

Volume of water displaced = 100 cm\(^3\)
Density of water = 1.0 g/cm\(^3\)
Mass of water displaced = 100 g
Weight of water displaced = 1.0 N

2
Volume of wood = 100 cm\(^3\)
Density of wood = 0.6 g/cm\(^3\)
Mass of wood = 60 g
Weight of wood = 0.6 N

Volume of water displaced = 60 cm\(^3\)
Density of water = 1.0 g/cm\(^3\)
Mass of water displaced = 60 g
Weight of water displaced = 0.6 N
Archimedes Principle: Iceberg

A telling example of how Archimedes’ principle works is the floating of icebergs.

How much of an iceberg is visible over the water level depends on the density of ice wrt. the density of fluid water?

\[ \rho_{\text{ice}} = 0.9167 \text{ g/cm}^3 \text{ at } T=0^\circ \text{C} \]

\[ \rho_{\text{water}} = 0.9998 \text{ g/cm}^3 \text{ at } T=0^\circ \text{C} \]

With the volume of the iceberg \( V_{\text{ice}} \), and the volume of the iceberg immersed in the water \( V_{\text{water}} \):

Determine the fraction of the iceberg’s volume immersed in the water ...
A telling example of how Archimedes’ principle works is the floating of icebergs.

How much of an iceberg is visible over the water level depends on the density of ice wrt. the density of fluid water?

\[
\rho_{\text{ice}} = 0.9167 \text{ g/cm}^3 \quad \text{at} \quad T=0^\circ \text{C}
\]

\[
\rho_{\text{water}} = 0.9998 \text{ g/cm}^3 \quad \text{at} \quad T=0^\circ \text{C}
\]

With the volume of the iceberg \(V_{\text{ice}}\) and the volume of the iceberg immersed in the water \(V_{\text{water}}\):

\[
\rho_{\text{water}} V_{\text{water}} g = \rho_{\text{ice}} V_{\text{ice}} g \quad \downarrow
\]

\[
\frac{V_{\text{water}}}{V_{\text{ice}}} = \frac{\rho_{\text{ice}}}{\rho_{\text{water}}} \approx 0.92
\]

I.e., only 8% of the iceberg is visible above the water, hence ...
Isothermal Sphere
What is the equilibrium configuration of a spherically symmetric gravitating body?

The two equations governing the system are the hydrostatic equilibrium (Euler) equation and the Poisson equation:

\[ \nabla p = -\rho \nabla \phi \]

\[ \nabla^2 \phi = 4\pi G \rho \]

Because of spherical symmetry, we write the Laplacian in spherical coordinates:

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \]

Therefore, in spherical coordinates the hydrostatic and Poisson equation become:

\[ \frac{dp}{dr} = -\rho \frac{d\phi}{dr} \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho \]
Integration of the second equation gives:

\[ r^2 \frac{d\phi}{dr} = Gm(r) \]

where \( m(r) \) is the mass contained within the shell of radius \( r \),

\[ m(r) = \int_0^r 4\pi x^2 \rho(x) \, dx \]

To solve this equation, we have to invoke the nature of the gas, ie. the equation of state \( p(\rho) \). We assume an ideal gas, for which

\[ p = \frac{R}{\mu} \rho T = \rho c_s^2 \]

We make the assumption that it concerns a gas with constant molecular weight \( \mu \) and a constant temperature \( T \) (an isothermal sphere). This yields the following equation:

\[ \frac{d}{dr} \left( \frac{r^2 c_s^2}{\rho} \frac{d\rho}{dr} \right) = -4\pi G r^2 \rho \]
The equation for isothermal sphere hydrostatic equilibrium,

\[
\frac{d}{dr} \left( \frac{r^2 c_s^2}{\rho} \frac{d\rho}{dr} \right) = -4\pi Gr^2 \rho
\]

has the solution

\[
\rho = \frac{c_s^2}{2Gr^2}; \quad p = \frac{c_s^4}{2Gr^2}
\]

This is the well-known isothermal sphere solution.

Notice that the isothermal sphere solution is singular at the center. Nevertheless, it provides a useful analytic approximation for various astronomical problems (sometimes a core is added).

Note: in real stars the temperature and, with it, the pressure increases with depth, which provide enough support against self collapse without the need for a singularity at r=0.
Cluster X-ray Emission: Hydrostatic Equilibrium
Clusters of Galaxies

- Assemblies of up to 1000s of galaxies within a radius of only $1.5-2h^{-1}$ Mpc.
- Representing overdensities of $\delta \sim 1000$
- Galaxies move around with velocities $\sim 1000$ km/s
- They are the most massive, and most recently, fully collapsed structures in our Universe.
Clusters of Galaxies

Coma Cluster

HST/ACS
Cluster X-ray Emission
Cluster X-ray Emission

CL 0016+1609
Abell 665
MACS J1149.5+223

CL J1226.9+3332
Abell 1689
Abell 1914
Clusters: X-ray emitting Hot Gas Spheres

- $T \sim 10$-100 million Kelvin
- In Hydrostatic Equilibrium:
  
  Gravity = Pressure

- Assume perfect spherically symmetric gas sphere:

$$F_{\text{pressure}} = F_{\text{grav}}$$

$$\downarrow$$

$$4\pi r^2 dP = -\frac{GM(r)}{r^2} \rho(r) 4\pi r^2 dr$$

ROSAT X-ray image Coma Cluster
Clusters: X-ray emitting Hot Gas Spheres

Hydrostatic Equilibrium:

Pressure = Gravity

\[ F_{\text{pressure}} = F_{\text{grav}} \]

\[ \Downarrow \]

\[ 4\pi r^2 dP = -\frac{GM(r)}{r^2} \rho(r) 4\pi r^2 dr \]

where \( M(r) \) is the mass within radius \( r \):

\[ M(r) = \int_{0}^{r} 4\pi x^2 \rho(x) dx \]

Hence,

\[ \frac{1}{\rho} \frac{dp}{dr} = -\frac{GM(r)}{r^2} \]
Clusters: X-ray emitting Hot Gas Spheres

For an ideal gas:

\[ p = k n T = \frac{\rho}{\mu m_p} k T \]

Hence,

\[ \frac{k}{\mu m_p} \frac{1}{\rho} \int d \left\{ \rho(r) T(r) \right\} = -\frac{GM(r)}{r^2} \]

Which, after some algebraic manipulation, leads to ...
Clusters: X-ray emitting Hot Gas Spheres

\[
\frac{k}{\mu m_p} \frac{1}{\rho} \frac{d}{dr} \left\{ \rho(r)T(r) \right\} = -\frac{GM(r)}{r^2}
\]

Which, after some algebraic manipulation, leads to ...

\[
M(r) = -\frac{k}{\mu m_p} T(r)r \frac{r}{\rho T} \frac{d}{dr} \left( \rho T \right)
\]

\[
= -\frac{k}{\mu m_p} T(r)r \frac{d \ln(\rho T)}{d \ln r}
\]

\[
= -\frac{k}{\mu m_p} T(r)r \left\{ \frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right\}
\]
Cluster Mass: X-ray emitting Hot Gas Spheres

Hydrostatic Equilibrium:

\[
\frac{GM(r)}{r^2} = -\frac{k_B T}{\mu m_H} \left[ \frac{d \log \rho}{dr} + \frac{d \log T}{dr} \right]
\]

Determination Mass from X-ray observations:

- assumption:
  - Isothermal: \( T(r) = T_0 \)
- temperature: \( T \sim 10-100 \) million Kelvin
- density profile:
  - X-ray emission Bremsstrahlung: \( L(r) \sim \rho(r)^2 \)

ROSAT X-ray image Coma Cluster
Clusters not only contain galaxies:
in fact, galaxies & stars are a minor component:

I. Clusters are Halos of Dark Matter:
   \[ M_{DM}/M_{total} \sim 82\% \]
II. Clusters are Hot Balls of (highly ionized) Gas
   \[ M_{ICM}/M_{total} \sim 16-17\% \]
III. Galaxies are mainly raisins in a sea of
dark matter & hot gas
   \[ M_{stars}/M_{total} \sim 2\% \]
Stellar Structure:

Hydrostatic Equilibrium
Overview of the solar processes

- Turbulent convection
- Convection zone
- Photosphere
- Core
- Sunspot
- Prominence

Stellar Structure

1.000.000 K

Radiative zone

Thermonuclear reactions

6.000 K

14.000.000 K
Stellar Structure Equations

Continuity equation:

- conservation of mass in shell \((r, r+dr)\)

Hydrostatic Equilibrium:

- Pressure = Gravity
- \(dP = \text{pressure difference over shell mass } dm_r\)

Energy conservation & generation:

- Energy generated by shell \(dm_r\):
  - nuclear energy \(\epsilon_n\)
  - thermodynamic energy \(\epsilon_g\)
  - energy loss neutrinos \(\epsilon_v\)

Energy transport:

- radiative & conductive energy transport, shell opacity \(\kappa\)