# Life and Times of Georgy Voronoï (1868-1908)

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**Summary.** Georgy Theodosiyovych Voronoï (1868-1908) is famous for his seminal contributions to number theory, perhaps mostly those involving quadratic forms and Voronoi tessellations. He was born and grew up in the town of Zhuravka in the Ukraine, at the time part of the Russian Empire. Having studied at St. Petersburg University under the supervision of Andrey Markov, in 1894 he became a professor of pure mathematics at the University of Warsaw. In his career he published six large memoirs and six short papers, each of which were so profound and significant that they left a deep trace in modern number theory. Together with Minkowski, he can be considered as the founder of the Geometry of Numbers. In this contribution, a brief sketch will be given of his life, work and legacy.

## 1 Ukrainian Origins

Georgy Theodosiyovych Voronoï [1] was born on April 28, 1868,[2] in the small town of Zhuravka in the Poltava Gubernia, in Russia (now the village Zhuravka belongs to the Varva District of the Chernihiv Region in Ukraine). A picturesque hill over

[1] G.Voronoï himself used this transliteration of his name, that is "ï" at the end of the word, in the papers written by him in French, whereas the experts of Voronoi diagrams used to write "Voronoi" in accordance with the term "Voronoi diagram". As it was accepted in Russia, the second name "Theodosiyovych" is his patronymic. The first letter of his father's name was "Fita", the letter used in Russia in the nineteenth century, we reproduce it here as "Th". In Ukrainian the name of the scientist is pronounced as: Heorhii Voronyi.

[2] According to the Julian calendar used in Russia in the nineteenth century, in Russia this date was April, 16. Also the dates specifically mentioned by G. Voronoï himself concern the Julian calendar. This concerns the dates mentioned on pp. 8, 9, 10, 11, 20, 21, 22 and 25.

**Fig. 1.** Georgy Voronoï, portrait.

the Udai river in Zhuravka appealled to one of Georgy's ancestors, perhaps his great-grandfather, who started out as *chumak* (an ox-cart driver; in medieval times these carts were used for carrying the salt from Crimea to Ukraine). After earning enough money, he bought a patch of land and settled there with his family [19].

According to the family legend, the Voronyis have acquired their family name from a remote ancestor, a cossack captain – a Ukrainian military called *esaul* at the time of Hetmanate, a commander of the Voronivka Fortress. There are several settlements that carry such a name. On the map of De Beauplan of the middle of seventeenth century, the fortified town of Voronivka was on the right Dnipro bank, near Chyhyryn. And, according to the description by M. Tkachenko of the Uman land on the right bank of the Dnipro river [21], the town of Vorone was near present-day Buky. There were mentions on Vorone and of its fortress from 1545 to 1674, when the advance of the Turks devastated much of the Uman area and forced the local population to migrate en masse to the left bank. Perhaps some of the mathematician's ancestors began as chumaks from the time of this migration ...

**Fig. 2.** The Voronoï family house in Zhuravka. Picture of around 1953. Regretfully, the house does not exist anymore. The street at which the house stood is now called "Voronoï Street".

Georgy's father, Theodosii Yakovych Voronyi (1837-1910), a son of ober-officer Yakov Tarasovych Voronyi, was educated at Kyiv University at the history and philology department (1857-1861). After his studies he worked at the Nemyriv gymnasium and the Nizhyn Lycee as a professor of Russian literature (1864-1872). Subsequently, he was a director of the gymnasium in Kyshyniv, then the gymnasium in Berdyansk and finally the gymnasium in Pryluky, some 18 kilometers from Zhu-

ravka. Georgy was born while his father worked at Nizhyn, and spent his years as a child in the towns of Nizhyn, Kyshyniv, Berdyansk, Pryluky, and Zhuravka. Theodosii was a man of progressive convictions and favoured efforts at popular education. Even in his student years at Kyiv University, Theodosii Voronyi was involved in organizing free Sunday schools for working youth and taught history in the Kyiv-Podil Sunday school (1859-1861) [12]. He also hosted student literary soirees. His initiative to create a Sunday school was supported by the progressive circles amongst the Ukrainian intelligentsia. The famous poet Taras Shevchenko [3] visited this school in 1859 and donated to the school 50 copies of his "Kobzar". Later on, describing events of national importance at Kyiv University in the late 1850s and early 1860s, Olena Pchilka [4] put a special emphasis on the honorable actions of student Theodosii Voronyi [15].

Theodosii Voronyi left us a manuscript of 15 pages (1861) in which he expressed his views on teaching and education at the Sunday School[5]. In particular, he believed that "any success in political and social life is impossible without people being enlightened by moral sciences." He emphasized the need to spread historical knowledge among the people. He believed that it "clarifies man's intellect, provides him with a better understanding of his social status in life and frequently points out the best ways of using one's abilities and to achieve prosperity for oneself and wellbeing to others" [12].

In 1887, Theodosii Voronyi retired. He got himself engaged in gardening in Zhuravka and took an active part in the work of the Pryluky Agricultural Society. Theodosii implemented his ideas of popular education in Zhuravka. At his own expense he built a school for the village children. The school also became the place where lectures, concerts and performances were held. The money collected was spent to enlarge the public library in Zhuravka, opened with the help of the Voronyi family.

Theodosii had a daughter and three sons. His elder son, Leonid, became a doctor, his son Mykhailo was an agronomist. Georgy's fate was to become a well-known scholar.

Mykhailo inherited his father's passion for gardening and took part in creating a small company for cultivation and herb processing in Zhuravka (1885). This was one of the first companies (if not even the first) of such kind in Russia. Thanks to the Voronyi family, Zhuravka played a role as a kind of regional cultural centre.

## 2 Schooldays: Zhuravka & Pryluky

Georgy Voronoï first followed school at Berdyansk, and later at the Pryluky gymnasium from which he graduated in 1885. While a student at the Pryluky gymnasium,

[3] Taras Shevchenko (1814–1861) is a great Ukrainian poet and artist. Many Ukrainian people worship him almost as their prophet. "Kobzar" is the famous book of poetry written by him.

[4] Olena Pchilka is a pen-name of Olena Kosach Drahomanova (1849-1930), a well-known writer, ethnographer and a public person. Her daughter, Larysa Kosach-Kvitka (1871-1913) became the famous Ukrainian poetess Lesia Ukrainka.

[5] It is kept at Kyiv City State archives, fund 16, discr. 471, 1861, file 105.

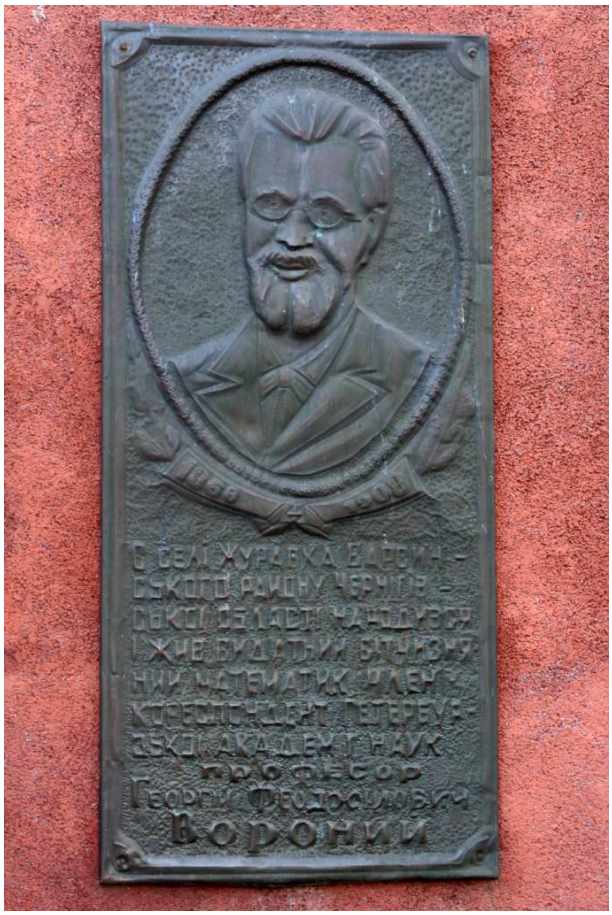



**Fig. 3.** The Voronoï school in Zhuravka: the commemoration plaquette of Voronoï at the entrance wall was made by local inhabitants. The school was built next to the location of Voronoï's old elementary school. It has a room with exhibition on Georgy Voronoï.

Georgy already distinguished himself by his deep interest in science, and by his diligence and punctuality.

His pupil's testimonial report stated that ".. with excellent talents, in spite of his young age, he has reached a considerable mental maturity and displays a very serious interest and love for learning. He acquired himself a very good knowledge in all subjects taught at high school. He has a particular inclination and calling for mathematics, in which he has considerably surpassed the average student's progress" [19].

He was particularly attracted to mathematics, which he studied very thoroughly and in which he revealed his brilliance, being especially fond of algebra. Georgy's interest in mathematics was stimulated by his favourite teacher of mathematics, Ivan Volodymyrovych Bogoslovskii, who greatly influenced his outlook and attitude to-

wards the world. Later on, in his student years, Voronoï appealed several times to his beloved teacher and his father as his moral and mental guides. Characteristically, in these years Georgy also kept evaluating himself again and again to see if he would have enough talent and persistence to be a professional scientist.

**Fig. 4.** Pryluky, view of the market place. The picture has been taken around the time Georgy Voronoï attended the gymnasium of Pryluky, around the year 1885.

While at gymnasium, he solved a problem posed by Professor Ermakov of Kiev University on factorising polynomials, which had appeared in the Journal of Elementary Mathematics (Kyiv). In 1885 the young Georgy Voronoï presented his solution in the paper *"Decomposition of the polynomials on factors based on the properties of the roots of quadratic equation"* [25]. Although still a school pupil, this led to his first publication, in the same journal. It got published in the same year he entered St. Petersburg University.

## 3 University Student: St. Petersburg

In 1885 Georgy Voronoï entered the Faculty of Physics and Mathematics of the University of St. Petersburg, where he studied mathematics during the period 1885-1889. For mathematical education this was at that time the best university in Russia: instruction in mathematical disciplines was being given there at the high level to which it had been raised by the genius of P.L. Chebyshev and the staff of the celebrated mathematical school of which he was the head (also see [17]).

At the university Voronoï attended the lectures of A.A. Markov, A.N. Korkin and Yu.V. Sokhotsky. He was particularly attracted to algebra and found Sokhotsky's course the most enjoyable of all. In his diary he wrote: "The pure mathematics lectures captivate me more and more. I prefer Professor Sokhotsky's lectures in the

special course on higher algebra to all the others. ... The main thing that concerns me is whether I have enough talent."

There is no doubt that Voronoï had ample talent and his subsequent accomplishments show this very clearly. Nonetheless, his student years were very difficult for him. In particular financially Georgy did not have an easy time. Even though his father had held posts of considerable standing, while retired he was not able to fully support his son. Georgy had to earn his living by private tutoring of mathematics, often for rather small fees. Taking such work very seriously and putting much effort into it, he found giving lessons exhausting [7]. As he wished to spend as much time as possible on his own investigations and mathematical research, he had to restrict the tutoring to the level of providing him with the bare essentials. He also found life in a student residence difficult, as he had to live in a cold, damp room. In his diary he frequently writes about the difficulties in accustoming himself to working under straining conditions. The constant amount of loud noise made by his roommates, the frequent student disturbances, and working at night did not make his life easy. It was only when he was in his last year, that Voronoï was offered a very meagre student bursary.

**Fig. 5.** The central building of St. Petersburg University, the Twelve Collegium, at the end of the 19th century.

All these sad self-revelations do not prevent Georgy from studying his favorite science with fervour and perseverance. He developed a strong self-discipline, marked by the habit of regular and diligent work and the ability of concentrating his intellect on the problems at hand.

### 3.1 Georgy's student diary

It says something about the personality of Georgy Voronoï that in these student years he confided his doubts to his diary. Fortunately, this diary has been partially preserved. Along with his descriptions of everyday experiences and events, it is a sincere self-confession of a young man. It discloses his character, his inner world, the process of his creative growth and self-consciousness. The author is active and sensitive and cannot remain indifferent to the events around him. He also tries to help when necessary. At times he is hot-tempered, for which he later expresses regret. He states "I am merrily gazing at God's world and to everything I touch I submit myself with rapture". Georgy aims "to reach everything by heart, and not just by intellect" and tries to look at himself from the outside. In this, he displays a rather low self-esteem, while also trying to grasp his own feelings and inclinations:

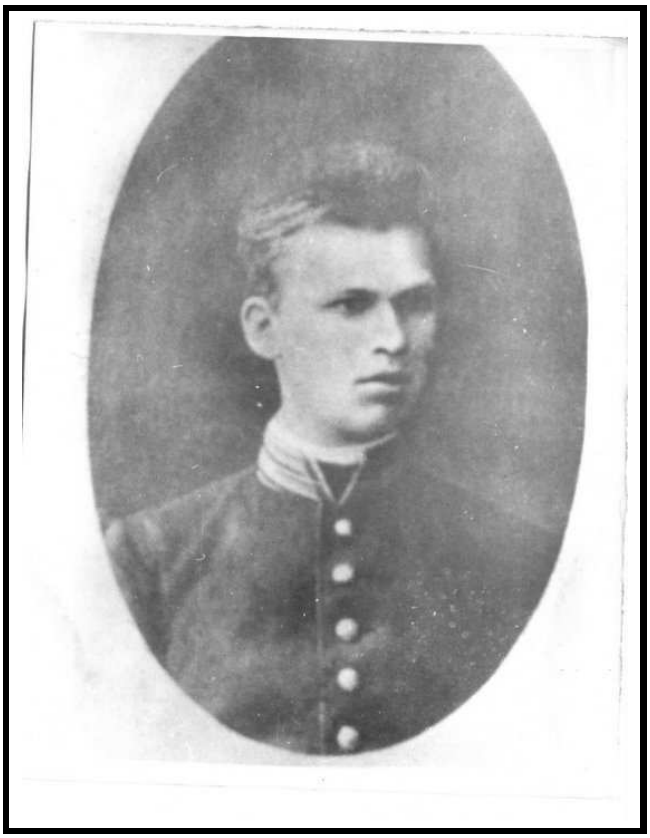


**Fig. 6.** Georgy Voronoï in his student years, 1885-1889.

"What am I after all? I am fond of playing cards. I do not have any noble pride. That is, if I am mocked I do not get angry and do not quarrel with the offender. I feel my weakness in front of the powerful of this world"...

The author has a rather melancholic view of contemporary society: ”Our times are hard, we are victims of a terrible regime. The most innocent things cannot be said. Otherwise one gets into the hands of the *custodians of hearts*. We are characterized by mistrust. We distrust each other.” The false feelings and hypocrisy that reigned around him led him to seclude himself from social life.

We may obtain some insight into his lifestyle during these student days from remarks like those on his study of astronomy: ”For the second day I have been busy calculating the solar eclipse of August 7. Yesterday I was working for 10 hours, today for 7 hours, the work has progressed considerably but I am feeling quite exhausted, the more so because I have not had a breath of fresh air for two days. Figures, figures ... Yesterday I so crammed my head with them, that they pestered me all the night and I had to get up and throw cold water on my head. Maybe today I shall have to resort to the same means.” Ten days later he wrote in his diary: ”I have not been writing my diary for ten days. It was not because of being in a lazy mood, but just because I am short of time. All the Holy Week I was busy at calculating the solar and lunar eclipses. Concerning the former I seem to obtain a wrong result. So I’ll have to redo some parts of the calculation.”

In spite of his studiousness, Georgy failed the astronomy exam. It happened for the first time in his life. ”This failure made a terrible impression on me, but I hid it; as usual it provoked in my mind a lot of questions: am I really a failure even here ? No, I shall not agree with it. I can call to witness my comrades, they will confirm that I possess enough knowledge. But the main thing was that it all awoke in me a passionate desire for study”.

## 4 Research Student

The main subject that Georgy Voronoï chose for his specialization was number theory. In the second half of the 19th century, the leading professors of St. Petersburg University, – P.L. Chebyshev, Yu.V. Sokhotsky, A.A. Markov, Ye.I. Zolotaryov, – actively worked in this field. The mathematical school later called the St. Petersburg School of Number Theory arose as a result of their activities. It was Professor Andrey Markov who became Georgy Voronoï’s main mentor and supervisor [6].

### 4.1 Scientific debut: Bernoulli numbers and the Staudt Theorem

Georgy made his debut in the mathematical circle on December, 2, 1888 with a presentation on his own investigation on some properties of Bernoulli numbers. Professor Markov received Voronoï’s presentation with great approval. This first success increased Georgy Voronoï’s enthusiasm, encouraging him to continue and persist in his research work with even greater intensity.

[6] Andrey Andreyevich Markov (1856-1922) was a famous Russian mathematician. He is best known for his work on the theory of stochastic processes, resulting in well-known and important concepts such as Markov chains and Markov process.

In a timespan of only three weeks Voronoï succeeded in proving the assertion stated earlier by Adams. The latter had done so without proof, posing the following proposition in his paper, "If $p$ is a prime divisor of $n$ and it is not a factor of the denominator of the $n$-th Bernoulli number, then the numerator of $B_n$ is divisible by $p$" [1]. In this connection Adams had noted: "I have not succeeded however, in obtaining a general proof of this proposition, though I have no doubt of its truth".

Voronoï continued his research and by following the same methods, he obtained the next result a couple of days later: the Staudt theorem. He also found several new generalizations. He handed his new results to professor Markov.

**Fig. 7.** Andrey Markov (1856-1922), master and doctoral thesis supervisor of Georgy Voronoï. Markov is famous for his contributions to the theory of stochastic processes, resulting in well-known and important concepts such as Markov chains and Markov processes.

On the eve of the new year 1889, Georgy wrote a brief report and evaluation of the past year, as he had become accustomed to do:

"Well, this year has not passed in vain for me. I have been working a lot, very much, and made certain that I can work and, it seems to me, succeeded in making others to be convinced of this.

At this time last year I had expressed a timid wish in the diary and now I see it has come true. The thing, which I was afraid of, does not exist. I know, I do believe, that on the basis of science work, and only that, I shall find my good luck [...] I am not a poet and I do not know the inspiration which poets feel, but I know minutes not of complacency, not of pride – they all come later – but moments when the mind completely grips the idea which before kept slipping off like a small ball. Then I forget about my existence.

I firmly believe that in this respect the new year will bring me still more joy, because I noticed that for my latest successes I am obliged to a habit of thinking without a pen and paper. All assertions, which I proved, occurred to me quite independently, and I only had to verify them. I hope this habit to think in such way will stand me in good stead".

On January 9, 1889 Georgy got a positive response from Markov on his new results. Markov proposed to him to prepare the paper for publication, but Georgy Voronoï, demanding to himself, first wanted to refine the result obtained and to generalize it. Markov told that he had looked through the table of Bernoulli numbers and doubted whether it would be possible to find the law of the numerator, because it involves a very large number of simple quantities. In response (as Georgy wrote in the diary) Voronoï presented to him the theorem which introduced these quantities.

"Markov was greatly interested by it, as the theorem which I had presented to him before gave nothing of the kind. I told him I had so far no full proof of that theorem, but that I would hope to prove it soon...

I came home feeling quite jaded, unwilling to do anything, and I felt at this time the whole burden of solitude. But the habit insisted on its own, I went to bed soon and today from 6 o'clock I have been working to prove that proposition. If anybody asked me where I had taken that theorem from I should certainly find difficulty in replying. I did not invent it, only proved it, but in leaps, and filled gaps between them by imagination. But then I checked it on very big examples and never got a contradicting result. Even today when I felt some doubt I checked its correctness with the help of the 44th Bernoulli number, which has 70 figures in the numerator. Though probability of a mistake was only $1/17$, just on the basis of selected conditions I got to doubt. Again I got a confirmation of my theorem. Had there been any misprint in the 70-figure number, it would have grieved me so deeply, but fortunately it did not happen."

Georgy was so much carried away by his new research that he could not make himself do anything else. For example, to prepare for the exams...

"An idea flashed across my mind: why shouldn't I occupy myself with some problem in mathematics – it seemed as if a ray of light fell on me, but then went out at once: I cannot, no time (even though I do nothing at all)... Yesterday I wrote down these lines, but once I sat to work within a few hours I revised all my former work. All that was vague may be understood in such a simple way that one cannot have wished for a better one."

That year Voronoï continued to work persistently during his summer vacation in Zhuravka. Only when his exposition became sufficiently clear and precise he sent the paper, *"On Bernoulli numbers"* to the Communications of the Kharkov Mathematical Society [26]. In the autumn of 1889, Georgy Voronoï passed his final exams brilliantly and defended his candidate thesis[7]. The theme was Bernoulli numbers.

In November 1889 he stayed at the University to prepare for his candidate degree exams. The representation necessary for his degree was signed by all the leading mathematics professors: A.A. Markov, O.N. Korkin, Yu.V. Sokhotsky and K.A. Posse. At the suggestion of these he was allowed to remain at the university for further studies leading to the title of professor. While gaining a scholarship to enable the continuation of his studies, Georgy Voronoï was also appointed as a supernumerary teacher at the Peterhof Progymnasium. From 1889 to 1894 he worked at this school as a civilian instructor in mathematics.

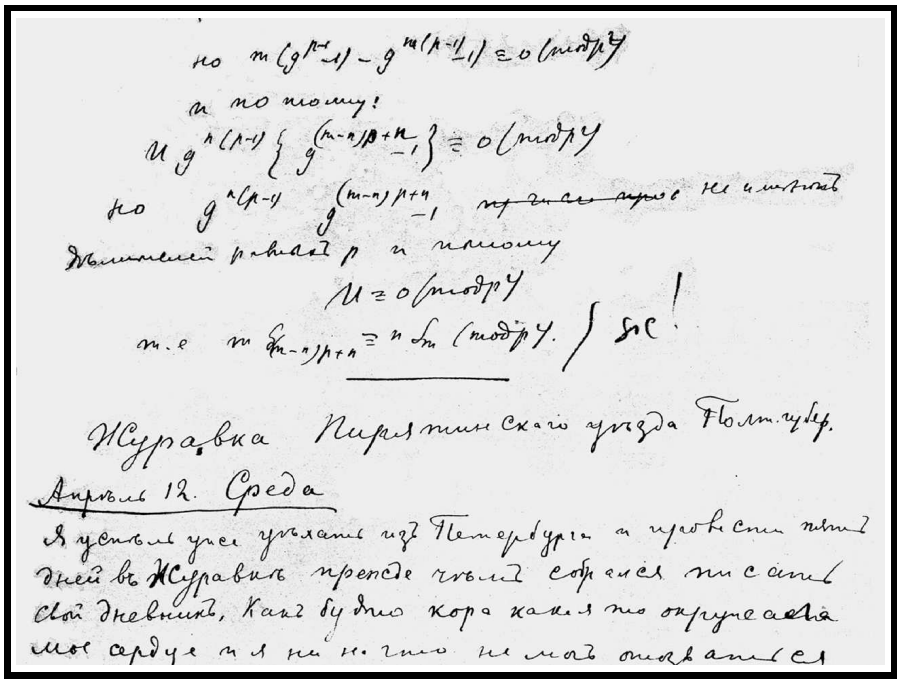



**Fig. 8.** Excerpts from the diary of Georgy Voronoï. Just below the centre of the image we can distinguish the phrase *Zhuravka of Pyryatyn uyezd of Polatava gubernia* ("uyezd" means district, in tsarist Russia "gubernia" was the word for province).

[7] The term "candidate thesis" corresponds to the present term "degree work".

### 4.2 New Year 1890: promise and prospects

On the eve of the new year 1890 Georgy, following his habit, sums up his accomplishments of the past year:
".. Everything that I did not even dare to dream of came true: I passed the exams, having been awarded fives for all of them. I could stay at the University with the help of the grant. In a word, to a great extent my future has already taken shape, even though it happened at the last moment. Until then I had only been suffering, enduring my pupil, who vexed me with his laziness. When I came home, I fell exhausted on the bed and thought: when will there be an end to it? Alas! I had to get up and swot, swot on and on!

If I add that at this time I had influenza, was without a copeck of money, had no prospect except for the awful private lessons, was living in a damp cold room, which I remember now as a nightmare, it is strange indeed that I did not collapse.

Passion for research, for finding new properties and relations between quantities has developed in me to an inconceivable extent; I can hardly put down my pen. The most urgent things and obligations – everything steps to the background, and I go on writing and writing...

I often compare myself with an alchemist, because, like him, I have no guiding star and have only a passion. And this passion has developed to such extent that I am losing my sleep as soon as it seems to me I have touched anything of importance. But alas, so far after a sleepless night I could see only that I had run against a solid wall and just was nourishing my illusions.

I am not embarassed by it at all, as I have already become firmly confident that I can easily take a simpler problem and get some result – not an essential one, only formal; but it does not tempt me, these laurels I have already reaped and now can say like Themistocles: "Gauss' laurels keep me awake!"

## 5 Olia

Georgy usually spent his vacations in Zhuravka, in his "native palestines", as he wrote. He visited the nearby village of Bohdany quite often to see his beloved girl Olia Krytska, his future wife.

Recollections about his acquaintance and the development of his relations with Olia Krytska occupy a particular place in the diary. Georgy writes so sincerely about his feelings, with such virtue and temperament – (events are almost ignored, only his feelings are recorded) – that these pages read like a real novel. He determined once and forever for himself that his destiny was in Bohdany, but he concealed his feelings for the time being because he had no financial basis for his own family. His father insisted on this decision. Such a vagueness in relations brought him many sufferings, but he patiently waited for his hour and did not permit any other passion to find the way to his heart. In 1889, on the eve of his departure, Georgy wrote about his last visit to Bohdany:

”Once more I am writing down my last visit to Krytskis... I am mounting the horse, once more saying goodbye to everybody, that is the end to everything which filled my life during the four months and which will cause me to behave stern and cool during the whole stretch of the Petersburg year.

Only mathematics as a bright star is shining afore me, in it I trust all my hopes... The experience of the last year has strengthened my endurance, and my creative eagerness, suppressed before, is bursting into action, and I am certain that Petersburg will bring me much that is new in this respect.

So goodbye, Olia, goodbye, Zhuravka! Till the new spring I shall cover myself with my armour. And, as if dreaming I shall see this summer, which gave me so much strength and health and those grains of happiness, which I know I shall so often experience when reading my diary in Petersburg, picking them from those talks with Olia, which I wrote down, along with everything which so often made my heart beat.”

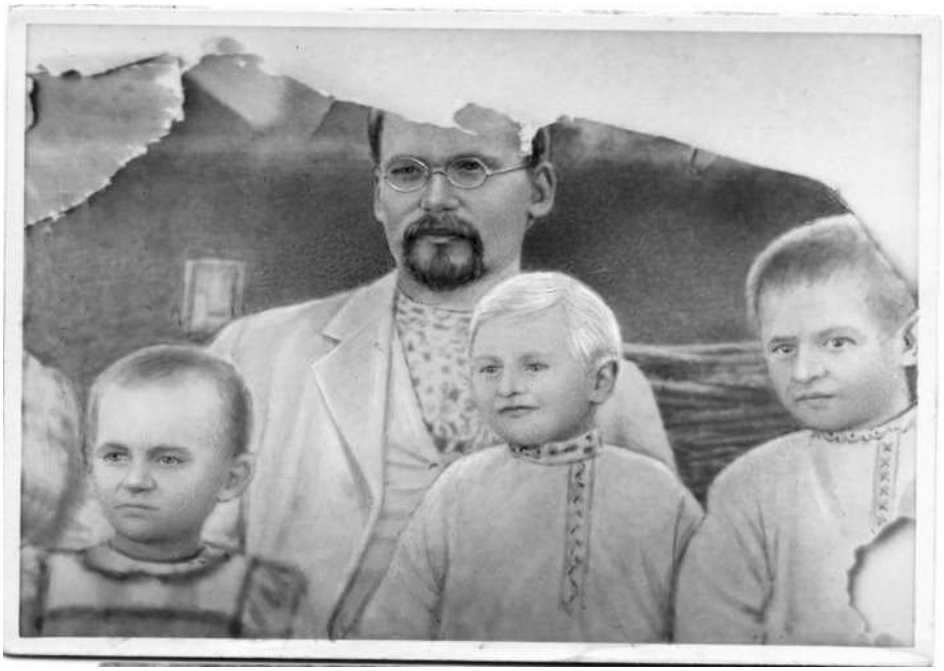


**Fig. 9.** Georgy Voronoï with his three eldest children, Oleksandra, Yuri and Oleksander. Zhuravka, 1904-1905.

As Georgy Voronoï gained his scholarship following his candidate’s degree in 1889, and got appointed as supernumerary teacher at the Peterhof Progymnasia, his future was finally ensured. At last, he became Olia’s fiancé. On the eve of 1891, he wrote in his diary:

**Fig. 10.** Photo of the large Georgy Voronoï family (approx. 1905-06, Zhuravka). Central row, sitting from left to right: Olha Mytrophanivna Vorona Krytska (Georgy Voronoï's wife); Georgy Voronoï with his daughter Tetyana (she died in childhood); Kleopatra Mykhailivna Vorona Lychkova (Georgy Voronoï's mother); Theodosii Yakovych Voronyi (Georgy Voronoï's father), with his grandson Igor (son of Mykhailo Voronyi, Georgy Voronoï's brother); Evfrosiniya Ivanivna Vorona (Mykhailo Voronyi's wife) with her daughter Halyna; Nadiya Theodosiivna Ermakova Vorona (Georgy Voronoï's sister) with her son Peter – Nadiya became a widow quite early, and had a large family of seven children, she was helped by Georgy Voronoï in her family problems; Vira Petrivna Prosvirlina-Ermakova (N.Th.Ermakova's daughter). Other persons around are their children.

December 31, 1890, Peterhof

> True to the old custom, today, on the eve of New Year, I cast a glance at how I have lived through and deeply felt the Old Year. The first thing which I gladly note and which has become a harbinger of my future happiness is: *Olia loves me*. I know it now for certain ! How happy I am ! So long I had been silently suffering from doubts, and at last it has been clarified, and I have already become Olia's fiancé ! ...

> Yes, now I know well that Olia loves me, but nevertheless lasting doubts and expectations have brought some bitterness. I seem to have become hardened in my permanent solitude. Ever growing passion for Mathematics has developed in me an egotism of no small degree. I am afraid I cannot feel strongly and surrender fully to my feelings.
>
> As for me the mind comes ahead always and everywhere. And the worldly wisdom, known from books, is saying that mind and love can scarcely be reconciled. That is what makes me fear sometimes that Olia probably will not be happy with me. As for me, I shall probably always take refuge in Mathematics.

Georgy and Olha married in 1891. They got six children, two sons and four daughters: Oleksander (1892), Oleksandra (1894), Yuri (1895), Maria (1900), Tetyana (1904) and Olena (1906). One of his daughters, Tetyana, died in childhood.

**Fig. 11.** End of summer: Voronoï and his family leave the Zhuravka house for Warsaw, to take up his post at Warsaw University.

His family followed him to his posts. While throughout his short life he kept spending his summers in Zhuravka, Olha and the children moved back with him to St. Petersburg and Warsaw to stay with him throughout the rest of the year.

## 6 Master and Doctoral Thesis: the Voronoï Algorithm

Having decided to stay at St. Petersburg University, the scientific interests of the young researcher concentrated more and more on the theory of irrationalities of the third degree. He wrote two large studies on this subject, one which formed the basis of his master dissertation and the second one being his doctoral thesis work. Both Voronoï's master thesis and doctoral thesis were of such high quality that they were awarded the Bunyakovsky prize by the St. Petersburg Academy of Sciences [7].

### 6.1 Master Thesis: integral algebraic numbers

His master's dissertation concerned the issue of algebraic integers associated with the roots of an irreducible cubic equation, "On algebraic integers dependent on a root of an equation of the third degree". He defended the master thesis in 1894. In the preface he wrote [7]:

"In the essay I am now presenting, results from the general theory of algebraic integers are applied to the particular case of numbers depending on the root of an irreducible equation $x^3 = rx + s$. The results obtained turn out to be very graphic. All integers of the field in question have a certain form. Using the form taken by the integers, it is not difficult to find a form embracing all the integral numbers divisible by a given ideal number, or, in other words, to find the ideal corresponding to that given ideal number. In our exposition the resolution of these questions is based on a detailed study of the solutions of third-degree equations relative to a prime and a composite modulus."

### 6.2 Doctoral Thesis: the Voronoï Algorithm

The other study concerned his doctoral thesis, titled "On a generalization of the algorithm for continued fractions", which was published in 1896 and which he defended in 1897.

With the help of continued fractions it became possible to approximate the quadratic irrational numbers, because each of these numbers may be represented by the periodic continued fraction. Generalization of this algorithm to the cubic irrational numbers appeared to be so difficult that it could not be solved during the nineteenth century. G.F. Voronoï was the first mathematician who managed to find a generalization.

It took another 42 years until the Voronoï algorithm was rediscovered by G. Bullig [6]. In 1976 I.O. Angell [2] used it in his tabulation of totally real cubic fields with a positive discriminant $\leq 100000$ [18]. A generalization of the Voronoï algorithm has been proposed by J. Buchmann in 1985 [4, 5].

In the preface to his thesis Voronoï writes:

"In his article 'De relatione inter ternas pluresve quantitates instituenda' Euler gave the first generalisation of the continued fraction algorithm. ... Jacobi changed

Euler's algorithm somewhat in order to unify the calculations". Voronoï then discussed Jacobi's generalisation, after which he went on to look at generalisations due to Poincaré and Hurwitz. He then looked at contributions by Dirichlet and Hermite, showing that none of the above provided a satisfactory generalisation. He then writes: "Taking as the basis of our investigations a special way of viewing the continued fraction algorithm, we propose a new generalisation of it." [7].

Within the context of this work, Georgy Voronoï addressed the issue of the analysis of cubic fields and the ring of integer algebraic numbers of such fields. The units of the ring are the invertible elements $a \in K$ for which there exist elements $b \in K$ such that $ab = 1$. The result obtained was so striking that prof. Andrey Markov could not believe the correctness of Voronoï proofs and did not dare to approve his work. In this connection D. Grave asserted [11]:

"Markov asked Voronoï by telegraph to come from Warsaw to Petrograd (this fact is known to me from Markov himself) [8]. Markov invited Voronoï to his office and proposed him to calculate the unit for the cubic equation $r^3 = 23$. By artificial means, Markov had found for this example the unit

$$e = 2166673601 + 761875860r + 267901370r^2\,.$$

Voronoï calculated for three hours. The period had 21 terms and in order to find the main unit it was necessary to multiply 21 expressions

$$-2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{-3-\rho+\rho^2}{4},\ \frac{-9+5\rho+\rho^2}{17},\ \frac{4-3\rho+\rho^2}{10},$$
$$\frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},\ \frac{1+3\rho-\rho^2}{10},\ \frac{-5-\rho+\rho^2}{3},\ \frac{-1+\rho}{2},$$
$$\frac{-10+\rho+\rho^2}{11},\ -2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},$$
$$\frac{1+3\rho-\rho^3}{5},\ \frac{-1+\rho}{2},\ \frac{-1+10\rho-\rho^2}{33},\ \frac{-11+7\rho+\rho^2}{20},$$
$$\frac{9-7\rho+2\rho}{31},\ \frac{-5-\rho+\rho^2}{6}.$$

Following this analysis, he found the unit

$$E = -41399 - 3160r + 6230r^2\,.$$

It turned out that $Ee = 1$. So, it was verified that the algorithm really worked." [10].

In connection with his doctoral thesis, the academician D. Grave wrote in 1934 [11]: "Georgy Voronoï is a Ukrainian mathematician of genius. While being at Petersburg University he was busy with an investigation of the cubic sphere which he

[8] At the time of these events the city was called St. Petersburg. Grave used here the name Petrograd, in accordance with the city's name in the 1930s.

performed with remarkable success leading to a great discovery in this field. He extended the algorithm of continued fractions for the cubic area which gives algebraic unities in the quadratic area. This extention had been sought in vain by all of the most outstanding mathematicians throughout the 19th century. This is how the Voronoï algorithm was found".

## 7 Professor in Warsaw

After defence of his master dissertation in 1894, Voronoï was appointed as professor of pure mathematics at the Russian Imperial University of Warsaw (today's Warsaw University), where he worked almost all the rest of his life. While continuing his scientific research on his doctoral thesis work, his lecturer's duties took too much time because the staff of the University included, in addition to Voronoï, only two professors. For this reason, Voronoï was forced to lecture for the students of different terms gathering them in one class. For example, Voronoï lectured the courses of Number Theory and the Probability Theory to students of the third and fourth years. These courses were given by Voronoï once every two years.

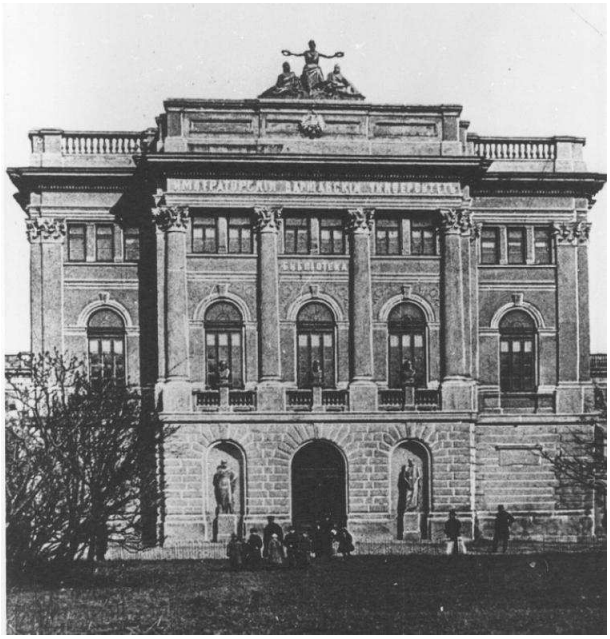


**Fig. 12.** The University of Warsaw around 1900, at the time Georgy Voronoï was professor at the university. The photo shows central library building.

Voronoï treated his lectures very seriously. He tried to acquaint his students with recent achievements of science and with his own new results. To ensure better under-

standing of the course by the students, Voronoï repeatedly asked for a permission to give additional lectures on analytic geometry[9]. In the autumn of 1898, Voronoï also became a Professor at the Warsaw Polytechnical Institute, where he became a Dean of the Faculty of Mechanics. In 1898, Voronoï got elected a member of the Moscow Mathematical Society.

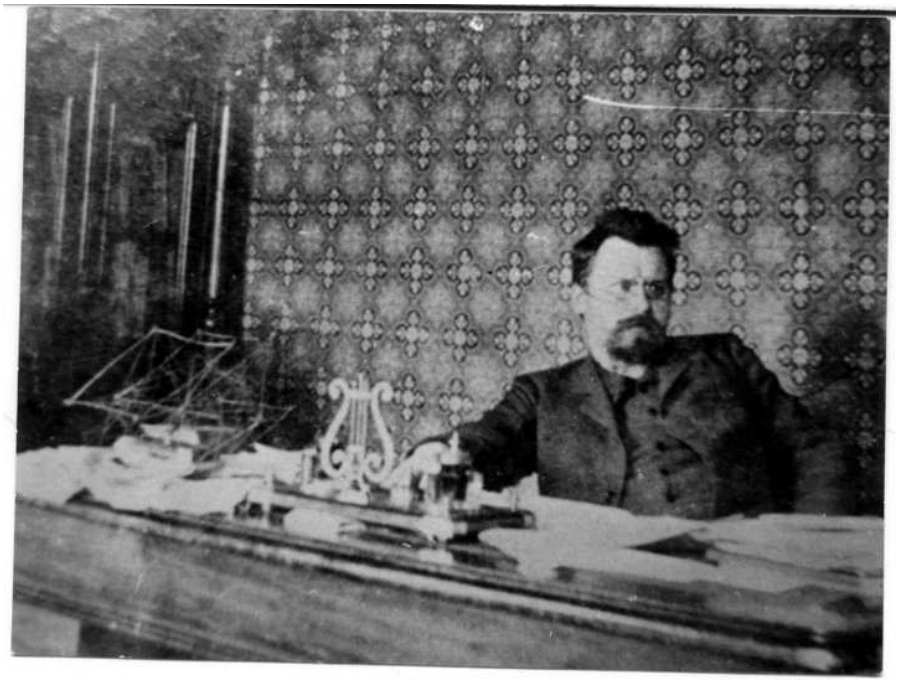


**Fig. 13.** Georgy Voronoï behind his desk. The photo was made around 1902-1903.

In August of 1898, Voronoï took part in the tenth Conference of Russian naturalists and physicians in Kyiv. In 1901, Voronoï was also a participant of the next, eleventh, Conference of Russian naturalists and physicians in St. Petersburg and presented three reports. In one of them, he proposed an original method for generalized summation of divergent series. Later, in 1919, the same technique was independently proposed by the Dutch mathematician N.E. Nörlund, so that for a long time it was known as Nörlund's method. In 1904, Voronoï presented two reports at the International Mathematical Congress held in Heidelberg. Here he met H. Minkowski, and they discovered they were both working on similar topics.

In 1903 and 1904, Voronoï submitted two large papers on Analytic and Algebraic Number Theory, *"Sur un problème du calcul des fonctions asymptotiques"* [32] and *"Sur une fonction transcendente et ses applications à la sommation de quelques*

[9] Though such supplementary lectures were unpaid for, a lecturer had to obtain the permission from not only Dean and Rector, but also from Trustee of the Educational District.

*séries"* [33]. This was a new direction of his studies. In these he established the existence of an explicit formula for sums of the form

$$\sum_{n\geq 1} d(n)\, f(n)$$

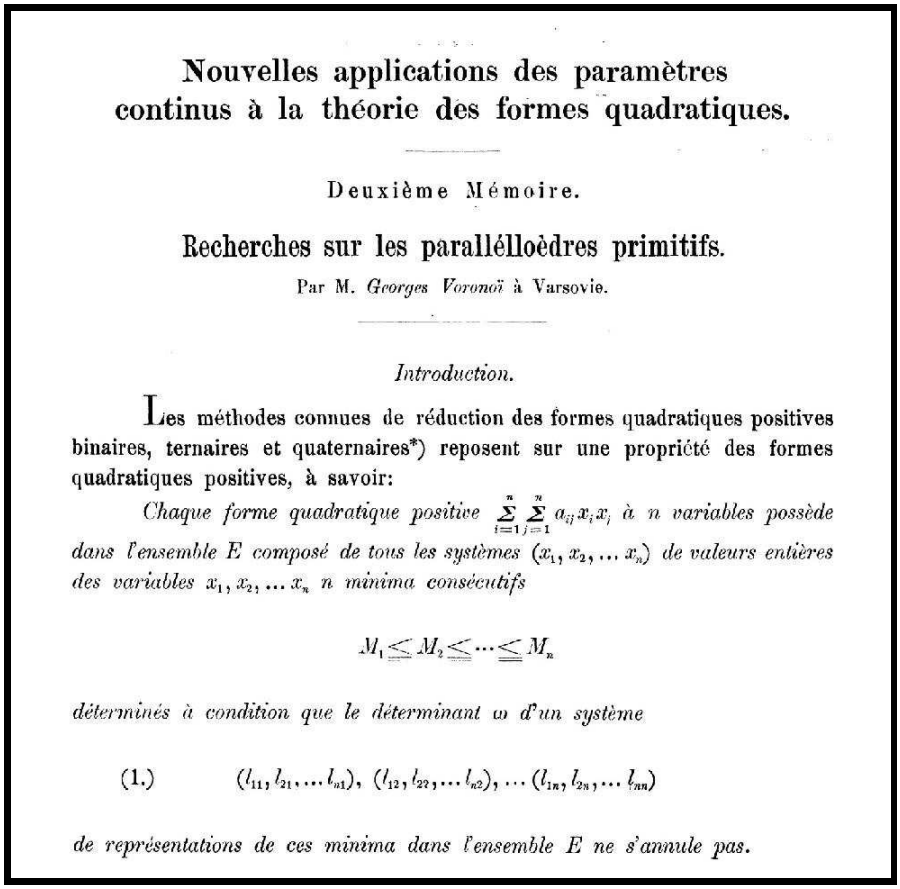
for a large class of functions $f(n)$, including characteristic functions of bounded intervals, and $d(n)$ is the number of positive divisors of $n$ [13, 14]. By allowing these sums over $f(n)$ to be weighted, the *Voronoï summations* introduce more general integral operations on $f$ than the Fourier transform. These results obtained by Voronoï were highly appreciated by mathematicians, and as a result of this he was elected Corresponding Member of the Russian Academy of Sciences in 1907.

In between 1905 and 1907, both Warsaw University and the Warsaw Polytechnical Institute were closed because of revolutionary events. A group of their staff, amongst whom Voronoï, were sent to Novocherkassk (Southern part of Russia), where the Donskoi Polytechnical Institute was founded. Voronoï worked there for about a year as a Dean of the Faculty of Mechanics. In autumn 1908, studies at the Warsaw University were resumed and Voronoï returned to Warsaw. His teaching load was enormous because for some time he remained the only professor. Later, Voronoï handed part of his courses to Prof. I.R. Braitsev, who was transferred from Warsaw Polytechnical Institute to assist him. At the time Voronoï gave a new course on mathematical analysis and wrote a textbook on the basis of his lectures. This textbook got published in 1909-1911 by the Warsaw University press, of which Braitsev was editor. Separately, the book got published in Russian in Kyiv in 1914.

## 8 Quadratic Forms & Voronoi Tessellations: 1908

For many years, Georgy Voronoï was working on the development of the arithmetic theory of quadratic forms and the geometry of numbers. He had the particular ability

**Fig. 14.** Georgy Voronoï contemplating crystal forms.



# Nouvelles applications des paramètres continus à la théorie des formes quadratiques.

Deuxième Mémoire.

## Recherches sur les parallélloèdres primitifs.

Par M. *Georges Voronoï* à Varsovie.

*Introduction.*

Les méthodes connues de réduction des formes quadratiques positives binaires, ternaires et quaternaires*) reposent sur une propriété des formes quadratiques positives, à savoir:

*Chaque forme quadratique positive* $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ *à n variables possède dans l'ensemble E composé de tous les systèmes* $(x_1, x_2, \ldots x_n)$ *de valeurs entières des variables* $x_1, x_2, \ldots x_n$ *n minima consécutifs*

$$M_1 \leqq M_2 \leqq \cdots \leqq M_n$$

*déterminés à condition que le déterminant* $\omega$ *d'un système*

$$(1.) \qquad (l_{11}, l_{21}, \ldots l_{n1}),\ (l_{12}, l_{22}, \ldots l_{n2}), \ldots (l_{1n}, l_{2n}, \ldots l_{nn})$$

*de représentations de ces minima dans l'ensemble E ne s'annule pas.*

**Fig. 15.** First page of Voronoï's 2nd seminal 1908 paper ... Reproduced with permission of Journal für die reine und angewandte Mathematik.

to think his ideas over and over again until they matured and acquired the appropriate perfect form. Once he reached this point, he was able to write down the obtained results very fast.

Preceded by the related study *"Sur quelques propriétés des formes quadratiques positives parfaites"* published in 1908 [36], Voronoï started to write his manuscript on the theory of parallelohedra on March 25, 1907. In ten days he completed the work of 106 pages, large size pages filled with a small letter font. Yet, barely two weeks later, on April 5, he rewrote the text, inserting numerous corrections. Only after the third revision, Voronoï sent his manuscript *"Recherches sur les paralléloèdres primitifs"* [37, 38] to the journal, edited by A.L. Krelle, with the following covering letter [10]:

[10] The letter (written in French) is kept at the Voronoï's archive of the Institute of Manuscripts of the National Library of Ukraine.

”For twelve years I have been studying properties of parallelohedra. I can say it is a thorny field for investigation, and the results which I obtained and set forth in this memoir cost me dear...

Three-dimensional parallelohedra are now playing an important role in the theory of crystalline bodies, and crystallographers have already paid attention to properties of these strange polyhedra, but till now crystallographers were satisfied with the description of parallelohedra from a purely geometrical point of view. I noticed already long ago that the task of dividing the n-dimensional analytical space into convex congruent polyhedra is closely related to the arithmetic theory of positive quadratic forms.”

Published in 1908-09, this paper, which was certainly the highest manifestation of his great intelligence, became his Swan Song ... and perhaps the work for which he has received greatest acclaim among many different branches of science, Voronoi Tessellations (for an example, see fig. 10).

Today Voronoi tessellations have wide applications to the analysis of spatially distributed data, ranging from fields such as geophysics, astrophysics and cosmology to biology and archaeology. Under different names, such as Dirichlet cells and Wigner-Seitz cells the notion is also found in condensed matter physics and in the study of Lie groups.

## 9 Failing Health

The intense intellectual work took a lot of his energy. However, Voronoï lacked a good and strong health. Some years before physicians had found a disease of the gallbladder which caused him a lot of physical and psychological agony. At the time, he worked in the new field of indefinite quadratic forms, he spoke about his ailments in a rather emotional way.

¿From his mathematical diary we find that Voronoï first recorded his results on the theory of indefinite quadratic forms on February 20, 1908, while in Novocherkassk. Adverse conditions of his life in Novocherkassk had significantly worsened his disease. In view of this situation, Voronoï hurried to write down his thoughts in his mathematical diary.

”I am advancing very well with respect to the problem on which I am working; meanwhile my health is getting ever worse. Yesterday for the first time I got a clear idea of the algorithm which is to solve all problems concerning the theory of forms which I am working on, but just yesterday I had a strong attack of a bilious colic which prevented me from working in the evening and was keeping me awake almost all night. I am so afraid lest the results of my long efforts, which I got with a great deal of hard work, should perish with me, while it is so difficult to put them in order. Many things I can only guess by some feeling, which just now, while I am ill, became sharper ...”

Physicians believed that Georgy Voronoï needed a long-term holiday. But he could not live without his favourite work. Once Voronoï confessed to one of his friends:

”The doctors have forbidden me to work... Moreover, I noticed myself that a strong intellectual strain always has an effect on my disease. But they do not know what not being occupied with mathematics means to me... For me mathematics is life, everything ...”

The doctors advised him to go to Carlsbad[11] for treatment. But he decided to spend all summer, as in previous years, in Zhuravka, where he had always restored his health. Indeed, Voronoï felt fine but at the end of October in Warsaw his disease passed on to the acute form, and Georgy Voronoï died on the 7(20)th of November.

The early death of Georgy Voronoï struck everybody who knew him. In his obituary professor I. Braitsev wrote [3]: ”Nobody could believe that Georgy Theodosiyovych, who was highly respected and loved by everybody, had died. Something extraordinary had happened – such was the general feeling. Everybody realized that they had prematurely lost an outstanding scientist, renowned professor who had been a pride and an adornment of two higher schools in Warsaw... Seeing off the deceased’s remains to the railway station for carrying them to the place of burial in the village of Zhuravka, everybody was grieved the more so because they had forever lost a truthful, sympathetic and warm-hearted man...”

Georgy Voronoï’s body, according to his last will, was transported to Zhuravka, to his native land. Voronoï was embalmed and interred in a specially built crypt. Later on, in 1910, his father, Theodosii Voronyi, was also buried in this crypt. In 1932, at the period of the collectivization and the communist terror and struggle against the kulaky (the well-to-do peasants) in the Soviet Union, this crypt was destroyed. The Voronyi family had to leave the village that very night – it was dangerous to remain there. They did not return to their native place anymore... The peasants reburied the remains of Georgy Voronoï and his father in the common grave nearby.

## 10 Aftermath: the Voronoï family

The further destiny of the family was rather sad and grievous in the years of the communist great terror [12].

His wife, Olha, was a midwife. When in Zhuravka she treated peasants free of charge[13]. When the family had to leave Zhuravka in 1932, Olha Mytrophanivna went to her eldest son, Oleksander, who lived in Yahotyn, a small town near Pryluky. There she lived and died in 1939 and was buried on the bank of the local lake the Supoi. Oleksander was a physician. He believed that an electrical current in a human body, operated in a specific way, could destroy cancer cells. He succeeded in curing some kinds of tumours, mainly in the stomach. In 1938 he was arrested and did not return from one of Stalin’s hard labour camps.

[11] Now it is Karlovy Vary in the Czech Republic.

[12] These data were communicated to H. Syta by Voronoï’s daughter Maria Vorona-Vasylenko (1900-1984) and confirmed by other relatives.

[13] Her brother, Borys Mytrophanovych Krytskii, was a well-known physician in the Pryluky region.

The eldest daughter, Oleksandra, had been educated at the Women's courses of higher level, and taught at the special institute for advanced studies of Ukrainian leaders. [14] She also published articles on Ukrainian literature. Her husband, Kostyantyn Polubotko, was a research worker at the Ukrainian Academy of Sciences. In 1937 he was arrested and exiled to the North, to Ohotske Sea, and had to work as a fisherman. He had a poor health and within three years he died there. Oleksandra died in a house for aged people.

The youngest son, Yuri, was a surgeon. He was the first person in the world who accomplished transplantation of a human organ, in 1933 in Kharkiv. The transplanted kidney was functioning for two days. He could not continue his work in the oppressive political atmosphere which prevailed in the country. Yuri's wife, Vira Nechayivska, was actively participating in Ukrainian political life at the times of the Ukrainian Central Rada headed by M. Hrushevsky. In 1917, she was a member of the government as a representative of the Women's Council [23].

The younger daughter, Maria, was a school teacher. Her husband, Vasyl Vasylenko, was a veterinary surgeon. In 1937, he was asked by the secret police to become an informer and to inform about his colleagues. He rejected the proposal and was sentenced to death. Later on Maria was arrested too and sentenced to ten years of the camp of special regime in the Far East. The youngest daughter, Olena, became a dentist.

Old people in Zhuravka recall some episodes of the sad events during the Holodomor, the Soviet famine of 1932-1933, and the Great Terror. The villagers of Zhuravka remember that the trees of a luxuriant garden, which had been grown by Theodosii Voronyi, were cut. "There were apple trees of many kinds, no more such trees grew in Zhuravka thereafter", one old resident relates, "When blossoming they were so fragrant that one could have a headache. Voronyis were very diligent people. G. Voronoï's wife was a small woman and very quick. One could always see her hurry along a village street – to provide medical care to people. On Christmas eve they put up a large Christmas tree for all the children in Zhuravka..."

The Voronyi's house in Zhuravka existed up to 1993, and was used as a building for the small pupils' school. In 1993, when the new school building was erected, it began to go to ruin and gradually was razed to the ground by Zhuravka inhabitants. Now the street on which this house was is called "Voronoï Street" and the local school named after G. Voronoï. There is a small exhibition about the Voronyis family in the local school museum.

## 11 Legacy

Georgy Voronoï's children saved the manuscripts, note-books of the mathematical diary, and other documents that had belonged to him. They transfered them to the possession of the National Library of Ukraine, to its Institute of Manuscripts where at present they are kept. There is a description of Voronoï archives in [16, 22]. In

[14] So-called "faculty of a special purpose".

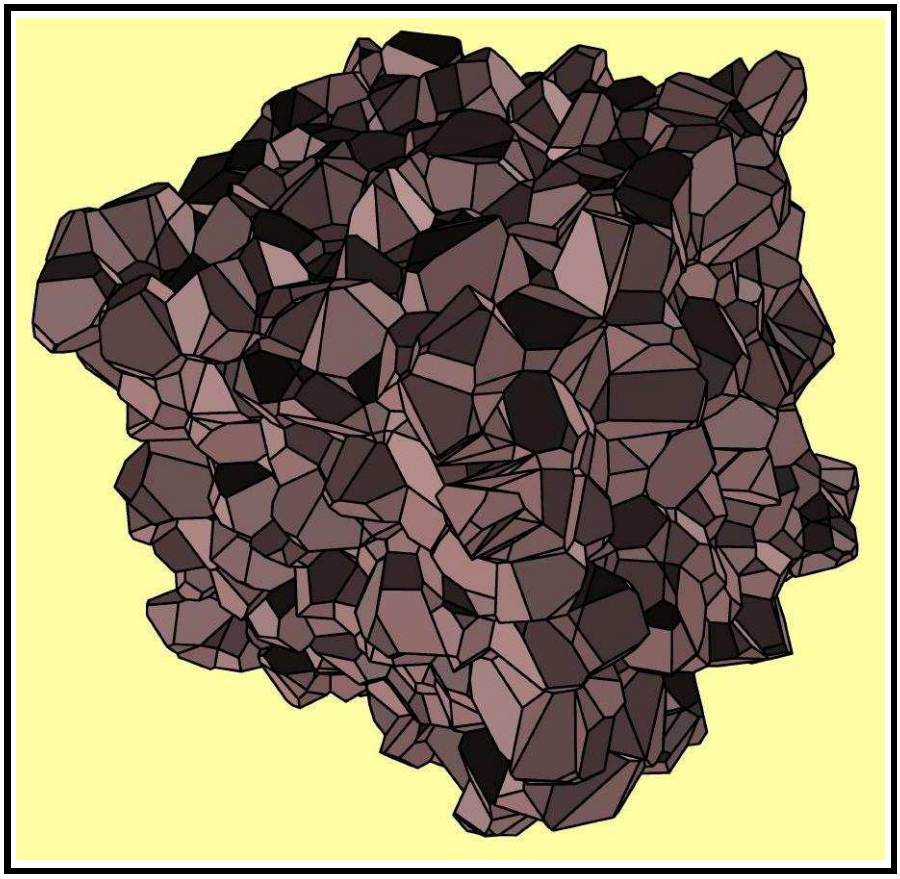


**Fig. 16.** An example of a 3-D Voronoi tessellation, generated by a set of 1000 Poissonian distributed points within a periodic box. Image courtesy of Jacco Dankers.

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1952–1953, the Complete Works of G.Voronoï (in three volumes) were published by the Institute of Mathematics of Ukraine with the detailed comments by B.N. Delone, B.A. Venkov, Yu.V. Linnyk, I.B. Pogrebyssky, I.Z. Shtokalo.

In the middle of the previous century Boris Delone, one of the well-known followers of Voronoï's research, wrote [9]:

"Voronoï investigated mainly the problems of Number Theory. During his short life he published not so many works – six large memoirs and six short papers. But the profundity and the significance of his vast investigations left a deep trace in modern Number Theory. Along with Minkowski, Voronoï is a founder of the Geometry of Numbers.

Voronoï's work of 1903 on the number of points under a hyperbola has to be considered as the landmark from which modern Analytic Number Theory begins.

In Voronoï's work on the algorithm for the calculation of the cubic units, he put a series of problems on the distribution of relative minimum. One of these problems,

amongst the most difficult ones, Voronoï has solved himself. Most of them are still waiting for their solution."

Following his death, I. Braitsev expressed his deep regret that Voronoï only succeeded to write down a very small part of his extensive work on the theory of indefinite quadratic forms on paper.

"Hardly can we hope to recreate from them at least in part those sophisticated trains of geometrical thought which were mentioned in the diary algorithm, and of which the deceased always spoke with great inspiration and enthusiasm. It is not enough to know in general outline those directions which the deceased followed. One should be such a brilliant expert in the theory of quadratic forms with n-variables as he was. It would be necessary to wield the marvelous technnique which the deceased mastered at the end of his life, and it is also necessary to be so staunchly devoted to this branch of mathematics as he was."

On this issue of the geometry of quadratic forms, Boris Delone wrote that "...Voronoï's investigation on quadratic forms and space packing (to which he had added such fundamental results) has not yet exhausted the important questions raised by him ... " [9]. Indeed, it made him make the following remark to characterize the impact of Voronoï short career:

> *" Voronoï's memoir on parallelohedra represents one of the deepest investigations in the geometry of numbers in the world's literature, and the originality of the methods employed in the purely geometrical first part stamps the memoir with the imprint of genius."*

It shows that the research of G.Voronoï's was already recognized as remarkable by his contemporaries. In our time, it remains the source of inspiration for a large diversity of applications ...

**Acknowledgements** The authors are grateful to Vincent Icke and Gert Vegter for many helpful suggestions for improving the manuscript. We also wish to acknowledge prof. O. Ganiyushkin from Kyiv University for advice on algebraic number theory, and to Jacco Dankers for producing fig. 10.

## References

1. Adams J.C.: Table of the values of the first sixty-two numbers of Bernoulli, J. reine angew. Math., **85**, 269-272 (1878)
2. Angell O.: A table of totally real cubic fields, Math. Comp., **30**, 184-187 (1976)
3. Braitsev I.R.: Obituary, Soobshcheniya kharkov. mat. obshchestva (Communications of the Kharkov Mathem. Soc.), 2nd Series, vol. **XI**, No. 6, 197-210 (in Russian) (1910)
4. Buchmann J.: A generalization of Voronoï's algorithm I, J. Number Theory, **20**, 177-191 (1985)
5. Buchmann J.: A generalization of Voronoï's algorithm II, J. Number Theory, **20**, 192-209 (1985)

6. Bullig G.: Ein periodisches Verfahren zur Berechnung eines System von Grundeinheiten in den total reellen kubischen Körpern, Abh. Math. Sem. Univ. Hamburg, **12**, 369-414 (1938)
7. O'Connor J.J., Robertson E.F.: Georgy Fedoseevich Voronoy, MacTutor History of Mathematics, website (2007)
8. Delone (Delaunay) B.: Yuri Thodosiyovych Voronyi, Zhourn. Matem. Tsyklu Vseukrain. Akad. Nauk (Journ. of Mathem. Cycle of the All-Ukrainian Acad. Sci.), vol. **1**, No. 2, 15-16 (in Ukrainian) (1934)
9. Delone (Delaunay) B.: Georgy Fedoseevich Voronoï. In: *The St. Petersburg School of Number Theory*, Acad. Sci. USSR Publ., Moscow-Leningrad, 195-318 (in Russian) (1947), English transl.: AMS (2005)
10. Grave D.O.: On a generalization of Voronoï's algorithm, Zhourn. Matem. Tsyklu Vseukrain. Akad. Nauk (Journ. of Mathem. Cycle of the All-Ukrainian Acad. Sci.), vol. **1**, No. 2, 17-23 (in Ukrainian) (1933)
11. Grave D.O.: Diary, manuscript, kept at the Library of Mathem. Inst. of Ukraine, Kyiv, 43 pp. (in Russian) (1934)
12. Marakhov G.: T. Shevchenko among his contemporaries, Dnipro Publ., 25-26 (in Ukrainian) (1976)
13. Miller S.D., Schmid W.: Summation formulas, from Poisson and Voronoi to the present. In: Noncommutative Harmonic Analysis, in honor of Jacques Carmona, eds. P. Delorme, M. Vergne, Birkhäuser, 419-440 (2003)
14. Miller S.D., Schmid W.: Automorphic distributions, L-functions, and Voronoi summation for GL(3), Ann. Math., **164**, 423-488 (2006)
15. Pchilka, Olena: Mykola Lysenko, in Zhurn: Ridnyi kray (The Native Land Journ.), No. 16-17, 3-42 (in Ukrainian) (1912)
16. Pogrebyssky I.B.: Manuscript archives of G.F.Voronoï. In: G.F.Voronoï, Polnoye sobraniye sochineniy (Complete works), Publ. Inst. Matem. AN UkrSSR, vol. **3**, 248-250 (in Russian) (1953)
17. Sakovich G.N., Slobodenyuk N.P.: Georgii Feodos'evich Voronoi (On the Centenary of his birth and sixtieth anniversery of his death), Ukrain. matem. zhurn., **20**, no. 6, 826-829 (in Russian) (1968), English transl.: Ukrain. Math. J., **20**, 713-715 (1968)
18. Schinzel A.: The Warsaw period of Voronoï's creative work. In: Voronoï's Impact on Modern Science, Book I. eds. P. Engel, H. Syta, Proc. Inst. Math. Nat. Acad. Sci. Ukr., Math. & Appl., **21**, 29-33 (1998)
19. Shtokalo I.Z., Pogrebyssky I.B.: Life and scientific activity of G.F. Voronoï. In: G.F.Voronoï, Polnoye sobraniye sochineniy (Complete Works), Publ. Inst. Matem. AN UkrSSR, vol. **3**, Kiev, 253-302 (in Russian) (1953)
20. Syta H.: Short biography of G. Voronoï. In: Voronoï's Impact on Modern Science, Book I, eds. P. Engel, H. Syta, Proc. Inst. Math. Nat. Acad. Sci. Ukr., Math. & Appl., **21**, 11-24 (1998)
21. Tkachenko M.: Uman Region of XV-XXVII centuries, Kyiv, Ukr.Acad.Sci.Publ., 1-32 (in Russian) (1927)
22. Venkov B.A.: On the scientific diary of G.F. Voronoï, Ukrain. matem. zhurn. **3**, No. 3, 279-289 (in Russian) (1951),
23. Verstiuk V., Ostashko T.: Statesmen of the Ukrainian Tsentralna Rada, Nation. Acad. Sci. Ukr. Publ., 207-209 (in Ukrainian) (1998)

## Publications by Georgy Voronoï


24. Voronoï G.: Diary, manuscript, kept at the Manuscript Institute of the V. Vernadsky National Library of Ukraine, Kyiv, fund **280**, No. 91, 105 double sheets (in Russian) (1885-1890).
25. Voronoï G.: Decomposition of the polynomials on factors based on the properties of the roots of quadratic equation, Zhurn. Elementarnoy Matem. (Elementary Math. J.), Kiev, 11-17 (in Russian) (1885)
    Zhurn.: U sviti matematyky (Mathematical World J.), **14**, issue 3, 24-30 (in Ukrainian) (2008)
26. Voronoï G.: On Bernoulli numbers, Soobshchenia kharkov. matem. obshchestva (Commun. Charkov. Math. Soc.), **2**, 129-148 (in Russian) (1890)
27. Voronoï G.: On algebraic integers dependent on a root of an equation of the third degree, St. Petersburg (in Russian) (1894).
28. Voronoï G.: On a generalization of the algorithm of continued fractions, Doctoral dissertation, Warsaw (in Russian) (1896).
29. Voronoï G.: On the number of roots of congruence of the third degree for a prime modulus. In: Dnevnik X syezda russkikh estestvoispytatelei, i vrachei (Proc. of the X Congress of Russian Naturalists and Physicians), Kyiv, p.329 (in Russian) (1898).
30. Voronoï G.: On determination of the sum of quadratic residues of a prime number $p$ in the form $4m+3$ with the help of Bernoulli numbers, Protokoly S-Peterb. Matem. obshchestva (Records of the St. Petersburg Math. Soc.), St. Petersburg (in Russian) (1899).
31. Voronoï G.: Extension of the notion of the limit of the sum of terms of an infinite series. In: Dnevnik XI syezda russkikh estestvoispytatelei, i vrachei (Proc. of the XI Congress of Russian Naturalists and Physicians), St. Petersburg, 60-61 (in Russian) (1902). Republished: Annals of Mathematics, **33**, 422-423 (1932).
32. Voronoï G.: Sur un problème du calcul des fonctions asymptotiques, J. reine angew. Math., **CXXVI**, 241-282 (1903)
33. Voronoï G.: Sur une fonction transcendente et ses applications à la sommation de quelques séries, Annales Scientifiques de l'École Normale Supérieure, **XXI**, 203-267, 459-533 (1904)
34. Voronoï G.: Sur une propriété du discriminant des fonctions entières. In: Verhandlungen des III internationalen Mathematiker-Kongresses in Heidelberg, Leipzig, 1905, 186-189 (1905)
35. Voronoï G.: Sur le développement à l'aide des fonctions cylindriques, des sommes doubles $\sum f(pm^2 + 2qmn + 2n^2)$, où $pm^2 + 2qmn + 2n^2$ est une forme positive à coefficients entiers. In: Verhandlungen des III internationalen Mathematiker-Kongresses in Heidelberg, Leipzig, 241-245 (1905)
36. Voronoï G.: Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier mémoire. Sur quelques propriétés des formes quadratiques positives parfaites, J. reine angew. Math., **133**, 97-156 (Heft 2), 157-178 (Heft 3) (1907-1908)
37. Voronoï G.: Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire. Recherches sur les paralléllоèdres primitifs, J. reine angew. Math., **134**, 198-246 (Heft 3), 247-287 (Heft 4) (1908-09)
    Introduction (198–211);
    Première partie. Partition uniforme de l'espaceanalytique à $n$ dimensions à l'aide des translations d'un meme polyèdre convexe (212–287).

38. Voronoï G.: Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire. Recherches sur les paralléllоèdres primitifs, J. reine angew. Math., **136**, 67-178 (1909)
Seconde partie. Domaines de formes quadratiques correspondant aux différents types de paralléloèdres primitifs (67-178).
39. Voronoï G.: Notes on indefinite quadratic forms, Ukrain. matem zhurn. (Ukrain. math. J.), **3**, 240-271 (from work over G. Voronoï's archives) (1951).
40. Voronoï G.: On indefinite quadratic forms, Ukrain. matem zhurn. (Ukrain. math. J.), **3**, 272-278 (from work over G. Voronoï's archives) (1951).
41. Voronoï G.: Complete Works in Three Volumes: Polnoye sobraniye sochineniy, Publ. Inst. Matem. AN UkrSSR (Publ. House of the Ukrainian Acad. Sci), 3 vols., Kiev (in Russian) (vol. 1,2: 1952; vol. 3: 1953)

**Publications in Voronoï's Complete Works, from the G. Voronoï archives:**

42. Voronoï G.: Fragments from the manuscript on analytic number theory. In: Voronoï's Complete Works, vol. 3, Kiev, 19-44 (in Russian) (1953)
43. Voronoï G.: Mathematical diary. On prime numbers. In: Voronoï's Complete Works, vol. 3, Kiev, 45-51 (in Russian) (1953)
44. Voronoï G.: Notes on indefinite quadratic forms, I. In: Voronoï's Complete Works, vol. 3, Kiev, 52-67 (in Russian) (1953)
45. Voronoï G.: Notes to the work "Thèorie générale de réduction des formes quadratiques à coefficients réels". In: Voronoï's Complete Works, vol. 3, Kiev, 68-83 (in French) (1953)
46. Voronoï G.: On some variation of the Jacobi algorithm. In: Voronoï's Complete Works, vol. 3, Kiev, 121-180 (in French) (1953)
47. Voronoï G.: Notes on the last Theorem by Fermat on impossibility of the equation $x^\lambda + e^\lambda = z^\lambda$ under $x, y, z$ being integers and $\lambda$ being a prime odd number. In: Voronoï's Complete Works, vol. 3, Kiev, 181-191 (in Russian) (1953)

**Manuscripts**
(library Institute of Mathematics, Ukraine)

48. Voronoï G.: Analytic Geometry. Manuscript of Lectures delivered at Russian Imperial University of Warsaw, 838 pp. (in Russian) (1899-1900), 838 pp. (in Russian).
49. Voronoï G.: Differential and integral calculus. Manuscript of student manual. Warsaw Polytechnical Institute, 535 pp. (in Russian) (1900)