DYNAMICS OF GALAXIES

Restricted three-body problem

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Winter 2008/9

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Fundamentals

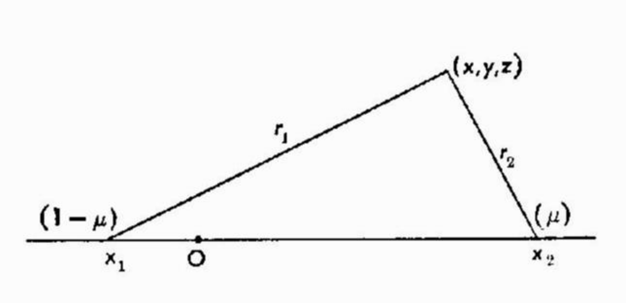
The restricted three-body problem concerns the case where two bodies with masses M_1 and M_2 revolve around each other in circular orbits with a third body with negligible mass in this force field.

Recall that in a rotating frame we have a modified potential

$$\Phi_{\mathrm{eff}} = \Phi + \frac{L_{\mathrm{z}}^2}{2R^2} = \Phi + \tfrac{1}{2}\omega^2 R^2$$

Here ω is the angular velovcity of the rotating coordinate system.

The geometry is as follows.



So we have the center of gravity at the origin and two bodies with masses $M_1 = M(1-\mu)$ and $M_2 = M\mu$, where we assume $\mu < 0.5$, and $x_1 = -X\mu$ and $x_2 = X(1-\mu)$.

For clarity, the distances r_1 and r_2 are those in three-dimensions, not projections onto the (x,y)-plane.

Now the total energy of the third body is

$$E = \frac{1}{2}v^2 - \frac{GM(1-\mu)}{r_1} - \frac{GM\mu}{r_2} - \frac{1}{2}\omega^2 R^2$$

So

$$v^2 = \omega^2(x^2 + y^2) + \frac{2GM(1-\mu)}{r_1} + \frac{2GM\mu}{r_2} - E$$

Now from the two-body problem we know that the objects move in elliptical orbits with Kepler's third law:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M_1 + M_2)}$$

We have circular orbits, so the angular velocity is

$$\omega^2 = \left(\frac{2\pi}{T}\right)^2 = \frac{G(M_1 + M_2)}{r^3}$$

Now let us take unit of distance $r = X = -x_1 + x_2 = 1$, unit of mass $M = M_1 + M_2 = 1$ and unit of time such that $G = 1$. Then $\omega = 1$.

Then we have

$$v^2 = (x^2 + y^2) + \frac{2(1-\mu)}{r_1} + \frac{\mu}{r_2} - E$$

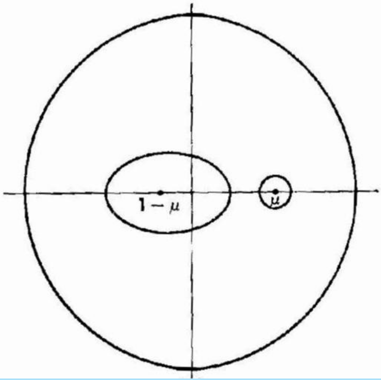
Now look at the surface with $v = 0$, where all energy is potential energy:

$$(x^2 + y^2) + \frac{2(1-\mu)}{r_1} + \frac{\mu}{r_2} = E \geq 0$$

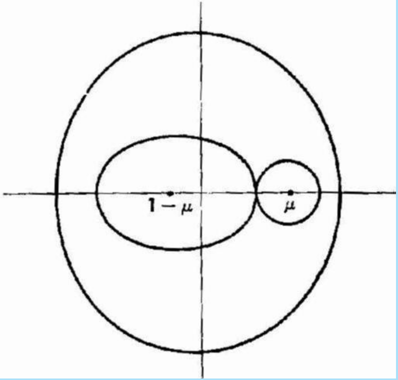
For each E these are curves outside which v^2 becomes negative, so the third body cannot go there with this E.

Now look at these surface first in the (x,y)-plane.

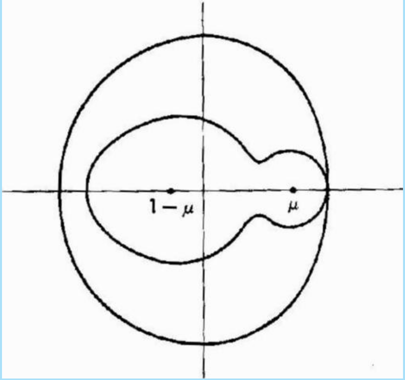
Now first look for very large E; then either $x^2 + y^2$ is very large or r_1 is very small or r_2 is very small.



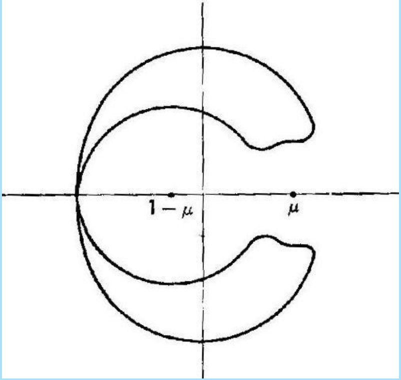
Now we decrease the value of the total energy E. The 'circle' shrinks and the ovoids increase untill they touch.



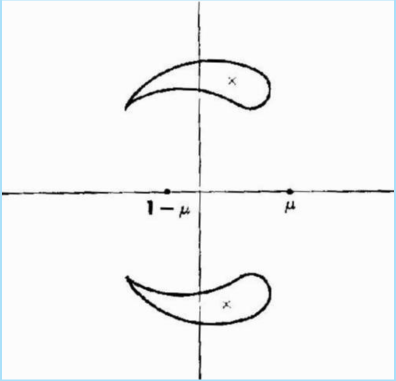
We decrease the value of E further. The 'circle' shrinks further and the ovoids increase until one touches the 'circle'.



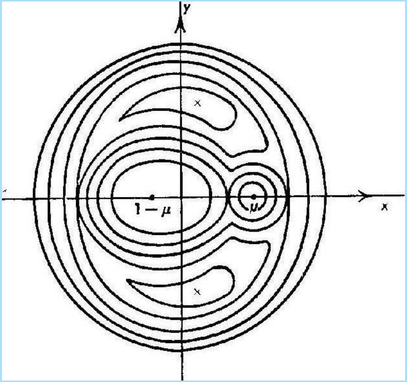
If we decrease the value of E even more, the 'circle' opens on one side while the ovoids touch it on the other.



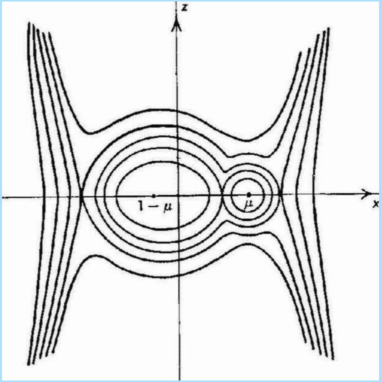
If we decrease the value of E still further we are left with two small areas that eventually shrink to two points.



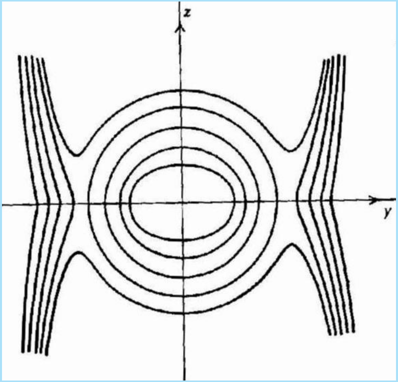
Here the previous figures are collected together. Five 'double points' occur in the (x, y)-plane.



Here we see the surfaces in the (x, z)-plane. Double point only occur on the x-axis.



Here we see the surfaces in the (y, z)-plane. Double points do not occur in this plane (on the y-axis).



Position of the equilibrium points

Put a test particle in this force field with zero velocity.

It will then start to move perpendicular to the surface it happens to be on.

Unless it is in one of the double points, where there is no unambigious direction to go.

If we use

$$F(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu)}{r_1} + \frac{\mu}{r_2} + \text{constant}$$

the condition that the double points are stationary points is

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

Remember

$$r_1^2 = (x - x_1)^2 + y^2 + z^2 \quad ; \quad r_2^2 = (x_2 - x)^2 + y^2 + z^2$$

$$r_1 = \sqrt{\mathrm{const.} + z^2} \quad \Rightarrow \quad \frac{dr_1}{dz} = \frac{2z}{2\sqrt{\mathrm{const.} + z^2}} = \frac{z}{r_1}$$

$$r_2 = \sqrt{\mathrm{const.} + z^2} \quad \Rightarrow \quad \frac{dr_2}{dz} = \frac{2z}{2\sqrt{\mathrm{const.} + z^2}} = \frac{z}{r_2}$$

Then

$$\frac{\partial F}{\partial z} = -z \left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \right)$$

And this equals zero for $z = 0$ and the equilibrium points are thus in the (x, y)-plane.

Analogously we have the conditions

$$\frac{\partial F}{\partial x} = x - (1-\mu)\frac{x - x_1}{r_1^3} + \mu\frac{x - x_2}{r_2^3} = 0 \qquad (1)$$

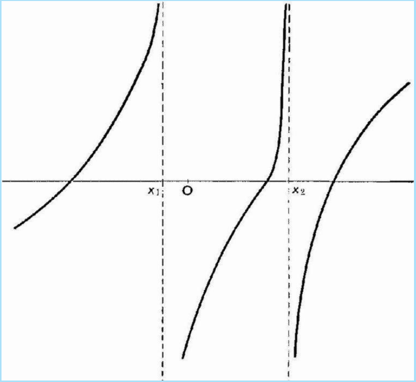
$$\frac{\partial F}{\partial y} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} = 0 \qquad (2)$$

First look at the case $y = 0$; this is allowed according to eqn. (2).

Eqn. (1) then becomes (with $r_1 = |x - x_1|$ and $r_2 - |x - x_2|$)

$$x - (1-\mu)\frac{x - x_1}{|x - x_1|^3} - \mu\frac{x - x_2}{|x - x_2|^3} = 0$$

A graph of this looks as follows:



and we see that there are three solutions (but there is no general analytic form).

Then the case $y \neq 0$. Eqn. (2) then becomes

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0$$

Multiplying with $(x - x_2)$ and $(x - x_1)$ and subtracting from eqn. (1) gives

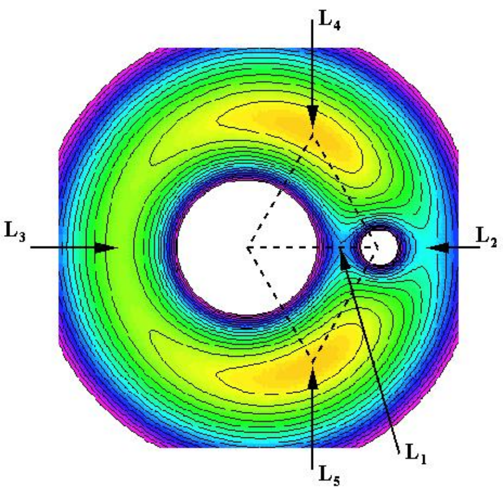
$$x_2 - (1-\mu)\frac{x_2 - x_1}{r_1^3} = o \quad ; \quad x_1 - \mu\frac{x_1 - x_2}{r_2^3} = 0$$

and this implies (remember $x_2 = 1 - \mu$ and $x_1 = -\mu$)

$$r_1 = r_2 = 1$$

These two solutions are on equilateral triangles with the two primary masses.

So we find the five Lagrangian libration points L_1 through L_5.



Stability of the Lagrange points

To test the stability we put the third mass to be at one of the Lagrange points and then give it a small velocity.

The point is stable if that results in an oscillation with a small amplitude.

The equations of motion are

$$\ddot{x} - 2\dot{y} = -\frac{\partial \phi}{\partial x}$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial \phi}{\partial y}$$

$$\ddot{z} = -\frac{\partial \phi}{\partial z}$$

Take coordinates (ξ, η, ζ) so that

$$x = x_\circ + \xi \quad ; \quad y = y_\circ + \eta \quad ; \quad z = z_\circ + \zeta$$

Do a Taylor expansion, neglect squares and products and use that at (x,y,z) $\partial\Phi/\partial x = \partial\Phi/\partial y = \partial\Phi/\partial z = 0$, etc. Then

$$\ddot{\xi} - 2\dot{\eta} = -\xi\Phi_{\rm xx} - \eta\Phi_{\rm xy} - \zeta\Phi_{\rm xz}$$

$$\ddot{\eta} + 2\dot{\xi} = -\xi\Phi_{\rm yx} - \eta\Phi_{\rm yy} - \zeta\Phi_{\rm yz}$$

$$\ddot{\zeta} = -\xi\Phi_{\rm zx} - \eta\Phi_{\rm zy} - \zeta\Phi_{\rm zz}$$

with $\Phi_{\rm xy} = \partial^2\Phi/\partial x\partial y$, etc.

We have

$$\Phi = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\sqrt{(x-x_1)^2+y^2+z^2}} + \frac{\mu}{\sqrt{(x-x_2)^2+y^2+z^2}}$$

Now define

$$\alpha = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}$$

$$\beta = \frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}$$

This gives

$$\Phi_{\rm xx} = -1 + \alpha - 3(1-\mu)\frac{(x-x_1)^2}{r_1^5} - 3\mu\frac{(x-x_2)^2}{r_2^5}$$

$$\Phi_{\rm yy} = -1 + \alpha - 3y^2\beta \quad ; \quad \Phi_{\rm zz} = \alpha - 3z^2\beta$$

$$\Phi_{\rm xy} = \Phi_{\rm yx} = -3xy\beta \;\; ; \;\; \Phi_{\rm xz} = \Phi_{\rm zx} = -3zx\beta \;\; ; \;\; \Phi_{\rm yz} = \Phi_{\rm zy} = -3yz\beta$$

First look at the Lagrangian points on the x-axis.

So $y = z = 0$. Write $x = x_\circ$ so that $r_1^2 = (x_\circ - x_1)^2$ and $r_2^2 = (x_\circ - x_2)^2$, then

$$\Phi_{\rm xx} = -1 - 2\alpha \quad ; \quad \Phi_{\rm yy} = -1 + \alpha \quad ; \quad \Phi_{\rm zz} = \alpha$$

$$\Phi_{\rm xy} = \Phi_{\rm yx} = \Phi_{\rm xz} = \Phi_{\rm zx} = \Phi_{\rm yz} = \Phi_{\rm zy} = 0$$

Then the equations of motion are

$$\ddot{\xi} - 2\dot{\eta} = \xi(1 + 2\alpha) \tag{3}$$

$$\ddot{\eta} + 2\dot{\xi} = \eta(1 - \alpha) \tag{4}$$

$$\ddot{\zeta} = -\zeta\alpha \tag{5}$$

Eqn. (5) is easily solved; it gives

$$\zeta \propto e^{\sqrt{-\alpha}t} = e^{i\sqrt{\alpha}t}$$

Now $\alpha > 0$, so $\sqrt{\alpha}$ is imaginary.

Remembering that

$$e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$$

we see that if and only if the exponent is fully imaginary (or $a = 0$) we will have an oscillating solution.

This is the case, so we have a harmonic oscillation and these libration points are **stable** in the z-direction.

Say the solutions in the (x,y)-plane are $\xi = Ke^{\lambda t}$ and $\eta = Le^{\lambda t}$.

When λ has a real component, ξ and η can assume arbitrary values and the point is unstable.

So the libration point is stable only if λ is fully imaginary.

Substitution, using $\dot{\xi} = \lambda K e^{\lambda t}$, $\ddot{\xi} = \lambda^2 K e^{\lambda t}$, etc. in eqn. (3) and (4) gives

$$K\lambda^2 - 2L\lambda = K(1 + 2\alpha)$$

$$L\lambda^2 + 2K\lambda = L(1 - \alpha)$$

Eliminate K and L:

$$\frac{K}{L} = \frac{2\lambda}{\lambda^2 - (1+2\alpha)} = \frac{\lambda^2 - (1-\alpha)}{-2\lambda}$$

Or

$$\lambda^4 + (2-\alpha)\lambda^2 + (1+2\alpha)(1-\alpha) = 0$$

Regard this as a quadratic polynomial equation in λ^2.

We need for stability that λ is purely imaginary so the two roots for λ^2 should both be real and negative.

Then for their product[1] we should have $(1+2\alpha)(1-\alpha) > 0$, or $(1-\alpha) > 0$.

[1]For $ax^2 + bx + c = 0$ the product of the roots is c/a.

We had according to eqn. (1)

$$x_\circ - (1-\mu)\frac{x - x_1}{|x - x_1|^3} - \mu\frac{x - x_2}{|x - x_2|^3} = 0$$

With the definition of α we can write

$$x_\circ(1-\alpha) + (1+\mu)\frac{x_1}{r_1^3} + \mu\frac{x_2}{r_2^3} = 0$$

With $x_1 = \mu$ and $x_2 = 1 - \mu$, this becomes

$$(1-\alpha) = \frac{\mu(1-\mu)}{x_\circ}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

Now we have in the cases of the three points on the x-axis

$$L_1: \quad x_\circ > x_2 > 0 \quad \text{and} \quad r_1 > r_2 \quad \Rightarrow \quad (1-\alpha) < 0$$
$$L_2: \quad 0 < x_\circ < x_2 \quad \text{and} \quad r_1 > r_2 \quad \Rightarrow \quad (1-\alpha) < 0$$
$$L_3: \quad x_\circ < x_1 < 0 \quad \text{and} \quad r_1 < r_2 \quad \Rightarrow \quad (1-\alpha) < 0$$

Then we only have real solutions for λ.

So all three Lagrangian points on the x-axis are **unstable**.

Now turn to the triangular points.

$r_1 = r_2 = 1$ and therefore

$$x = \tfrac{1}{2}(1 - 2\mu) \quad ; \quad y = \pm\frac{\sqrt{3}}{2} \quad ; z = 0$$

We do here the solution for the positive value of y. Then

$$\Phi_{xx} = -\frac{3}{4} \quad ; \quad \Phi_{yy} = -\frac{9}{4} \quad ; \quad \Phi_{zz} = 1$$

$$\Phi_{xy} = \Phi_{yx} = -\frac{3\sqrt{3}}{4}(1 - 2\mu)$$

And the others are equal to zero.

The equations of motions now are

$$\ddot{\xi} - 2\dot{\eta} = \frac{3}{4}\xi + \frac{3\sqrt{3}}{4}(1-2\mu)\eta$$

$$\ddot{\eta} + 2\dot{\xi} = \frac{9}{4}\eta + \frac{3\sqrt{3}}{4}(1-2\mu)\xi$$

$$\ddot{\zeta} = -\zeta$$

From the last one we get $\zeta \propto e^{it}$, so the motion is harmonic with a period equal to that of the primary masses and the point is **stable** in the z-direction.

Try again the solutions $\xi = Ke^{\lambda t}$ and $\eta = Le^{\lambda t}$ and find out whether or not λ is fully imaginary or not.

The result is

$$\left(\lambda^2 - \frac{3}{4}\right) K - \left\{ \frac{3\sqrt{3}}{4}(1 - 2\mu) + 2\lambda \right\} L = 0$$

$$- \left\{ \frac{3\sqrt{3}}{4}(1 - 2\mu) - 2\lambda \right\} K + \left(\lambda^2 - \frac{9}{4}\right) L = 0$$

Eliminate again K and L, then

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

Again regard this as a quadratic equation in λ^2.

For stability it must have only real, negative roots.

First their sum must be negative and their product positive. This is true, since the sum of the roots[2] is -1 and their product $\frac{27}{4}\mu(1-\mu) > 0$.

[2] For $ax^2 + bx + c = 0$ the sum of the roots is $-b/a$ and the product c/a.

Then the roots λ^2 should be real[3], which is the case for

$$1 - 27\mu(1-\mu) = 27\mu^2 - 27\mu + 1 > 0$$

This is a quadratic inequality in μ.

Since $\mu \leq 1/2$ this corresponds to the root

$$\mu < \frac{1}{2} - \sqrt{\frac{23}{108}} = 0.0285.$$

So these Lagrangian points are **stable** only in the case of a large difference in the two primary masses.

[3]The roots are real if $(b^2 - 4ac) > 0$.

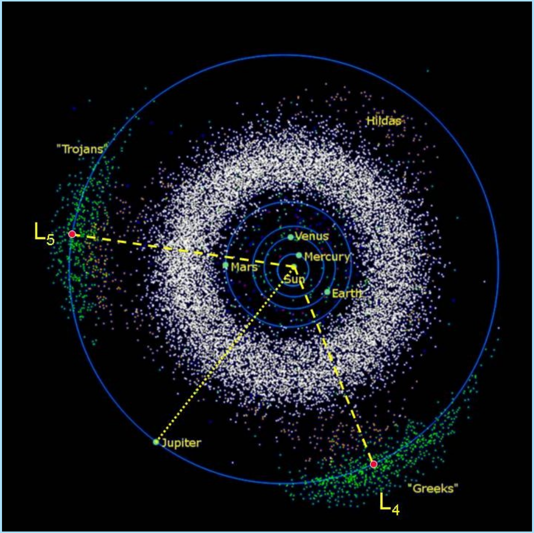
Applications

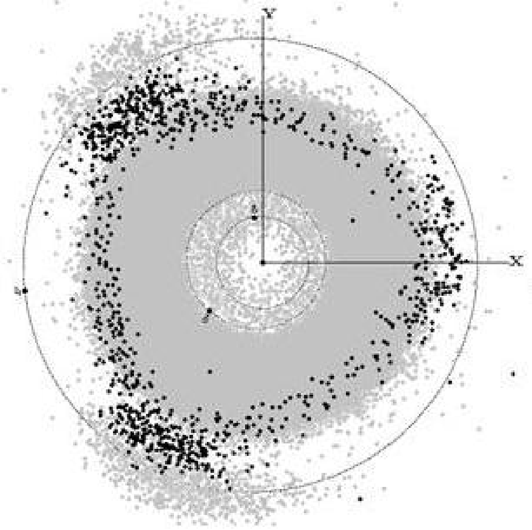
For the Jupiter - Sun system $\mu = 0.0010$ so the Lagrangian points L_4 and L_5 are stable.

Indeed that is where we find the Trojans, a dynamical group of asteroids.

In principle these can cross from L_5 to L_4 through L_3 and vice versa.

Another group are the Hildas, that are in a 2–3 orbital resonance with Jupiter. They are also affected by L_3, L_4 and L_5.


Hildas
"Trojans"
L_5
Venus
Mercury
Mars
Sun
Earth
Jupiter
"Greeks"
L_4


Y
X

For the Earth-Sun system $\mu = 0.0123$, so the Langragian points L_4 and L_5 are stable.

Only L_1 and L_2 are useful and are exploited to "park" satellites. The triangular points and L_3 are too far away (and there are too many distortions).

In L_1 we find satellites for solar observations , such as SOHO (SOlar and Heliospheric Observatory).

Point L_2 was used for WMAP (Wilkinson Microwave Anisotropy Probe) and will be used for Herschel with HIFI and JWST (James Webb Space Telescope).

Satellites need to be actively kept at these unstable points.

Sun – Earth