## Lecture 4: Evolution of Density Inhomogeneities

We assume that the early Universe is *smooth* with *small* irregularities produced by inflation. These irregularities are produced by a Zel'dovich spectrum of fluctuations with roughly constant amplitudes coming through the horizon ($\sim 10^{-4}$). We also assume that *only gravity is important* and therefore the irregularities grow in time due to gravitational instability. Finally, we assume that dark matter, radiation, and matter are *well-mixed* in the early Universe.

Let's consider the growth of small perturbations when $\Omega = 1$ and $\rho_{\text{matter}} > \rho_\gamma$ so that $\rho = \rho_{\text{matter}}$. We define

$$\delta \equiv \frac{\delta\rho}{\rho}. \tag{1}$$

In the *linear regime,* $\delta$ is small and the pattern of fluctuations just scales in $\delta$. Thus $\delta$ is *amplified* but the pattern of fluctuations remains the same. The growth of $\delta$ is accomplished by flows of matter from underdense to overdense regions.

The rate at which fluctuations grow depends on $\Omega$:

$\mathbf{\Omega \geq 1}$**:** fluctuations always grow

$\mathbf{\Omega < 1}$**:** fluctuations stop growing at redshift $(1 + z_{\text{stop}}) \sim 1/\Omega$ and therefore the pattern of clustering is "frozen in" at that time.

We'll show that

$$\delta \equiv \frac{\delta\rho}{\rho} \propto a(t)$$

and therefore that $\delta$ grows *in proportion to the scale factor* in the linear regime.

*Growth of spherical perturbations*

Here, we assume (like usual) that $\Omega = 1$ and $\rho = \rho_{\text{matter}}$.

First, let's calculate the *turn-around time* for a spherical perturbation, the time at which a growing perturbation begins to collapse. Define

$r$: radius of our sphere

$M$: mass inside sphere

$v$: velocity at edge of sphere

$R$: maximum radius of sphere ($R \equiv r_{\text{max}}$)

We use *Birkhoff's theorem*, which is the GR analogue of Gauss's law. Basically, we can ignore the force from any matter outside of our sphere. Then we choose a radius $r \ll ct$ (the horizon radius), so we can use Netwonian dynamics for our calculations.

Conservation of energy for the outermost spherical *shell* $dM$ gives us

$$\frac{v^2}{2}dM - \frac{GM\,dM}{r} = -\frac{GM\,dM}{R}$$

Note that $v = dr/dt$, so

$$t = \int_0^r dr' \left(\frac{2GM}{r'} - \frac{2GM}{R}\right)^{-1/2}$$

At early times, $r' \ll R$, so we can show that $r \sim t^{2/3}$, which is what it should be for the underlying, unperturbed Universe. Doing the integral, we get

$$t_{\text{max}} \equiv t(R) = \frac{\pi}{2^{3/2}}\left(\frac{R^3}{GM}\right)^{1/2} \qquad (2)$$

which means $t_{\text{max}} \propto \rho^{-1/2}$, which is good, since all dynamical times scale as $\rho^{-1/2}$.

Next, let's calculate $R$ as a function of the initial overdensity and radius. Assume we begin in the linear regime, $\delta \ll 1$. Define two spheres, one in the average universe ($\delta = 0$), the other containing a lump ($\delta > 0$), both with the same mass $M$. At early times, we can write

$$\rho_1 = (1+\delta)\rho_0$$

By mass conservation,

$$\frac{\delta\rho}{\rho} = -3\frac{\delta r}{r},$$

so

$$r_1 = \left(1 - \frac{\delta}{3}\right) r_0 \tag{3}$$

and thus the radial growth is retarded by $\delta$. Differentiating $r_1$,

$$\begin{aligned} v_1 &= \dot{r}_1 = \dot{r}_0 \left(1 - \frac{\delta}{3}\right) - r_0 \frac{\dot{\delta}}{3} \\ &= \dot{r}_0 \left(1 - \frac{\delta}{3} - \frac{\dot{\delta}}{3}\frac{r_0}{\dot{r}_0}\right) \end{aligned}$$

Therefore

$$\begin{aligned} v_1^2 &= \dot{r}_0^2 \left(1 - \frac{2\delta}{3} - \frac{2\dot{\delta}}{3}\frac{r_0}{\dot{r}_0}\right) \\ &= \dot{r}_0^2 \left(1 - \frac{2\delta}{3} - \dot{\delta} t\right), \qquad (4) \end{aligned}$$

where we have assumed $\Omega = 1$ and therefore $r_0 = At^{2/3}$ and $r_0/\dot{r}_0 = 3t/2$.

Energy conservation for a shell at the edge of the lump gives

$$\frac{v_1^2}{2} - \frac{GM}{r_1} = -\frac{GM}{R}$$

(remember that $R$ is the maximum radius of the lump). Then use Eq. 4 to write

$$\frac{\dot{r}_0^2}{2}\left(1 - \frac{2\delta}{3} - \dot{\delta} t\right) - \frac{GM}{r_0}\left(1 + \frac{\delta}{3}\right) = -\frac{GM}{R}$$

Note that $\dot{r}_0^2/2 = GM/r_0$, because the unperturbed Universe has zero net energy and infinite $R$.

So

$$-\frac{2\delta}{3} - \dot{\delta}t - \frac{\delta}{3} = -\frac{r_0}{R}$$

and we get the equation of motion for $\delta(t)$:

$$\delta + \dot{\delta}t = \frac{r_0}{R} \tag{5}$$

We can solve this by assuming $\delta = Bt^n$. To get the dimensions correct, we choose $n = 2/3$ because $r_0 \propto t^{2/3}$ and we find

$$\delta = \frac{3}{5}\frac{r_0}{R} = \frac{3}{5}\frac{r_1}{R} \tag{6}$$

since at early times $r_1 \approx r_0$, and

$$R = \frac{3}{5}\frac{r_0}{\delta} = \frac{3}{5}\frac{r_1}{\delta}, \tag{7}$$

which gives us the maximum radius of the lump as a function of the the *initial* radius $r_1$ and overdensity $\delta$.

Another useful relation comes from taking the square root of Eq. 4 (in the limit where $\delta \ll 1$):

$$v_1 = \dot{r}_0\left(1 - \frac{2\delta}{3}\right), \tag{8}$$

which gives the velocity perturbation as a function of $\delta$.

Now we substitute Eq. 7 into Eq. 2 to get

$$t_{\text{max}} = \frac{9}{20}\left(\frac{\pi}{10}G^{-1}\rho_0^{-1}\delta^{-3}\right)^{1/2} \qquad (9)$$

so $t_{\text{max}} \sim \delta^{-3/2}$. Therefore all perturbations with the same overdensity $\delta$ at a given epoch (defined by $\rho_0$, the mean density of the Universe) *collapse at the same time.* We can also see that bigger perturbations (higher $\delta$) collapse faster than small perturbations that begin collapsing at the same time.

So how does $\delta$ grow in the linear regime? Using Eq. 7 at two epochs,

$$\frac{\delta_1}{r_1} = \frac{\delta_2}{r_2} = \text{constant} = \frac{3}{5}R^{-1}$$

Hence

$$\delta \propto r \propto a \propto \frac{1}{1+z} \qquad (10)$$

Therefore, the density constrast grows linearly with the scale factor of the Universe.

*Final Equilibrium of Collapsed Lumps*

In the absence of random motions with the lumps, the collapse of a lump would be the mirror image of expansion and the sphere would collapse into a black hole. However, real perturbations contain small *subperturbations* that are collapsing within the sphere, and these generate entropy. The sphere settles down to a cluster of mass points with equilibrium radius $r_{eq}$.

In the absence of dissipation (e.g., dark matter halos), we define $E$, the total energy of the system (constant), $T$, the kinetic energy, $V$, the potential energy, $V_{\max}$, the potential energy at maximum expansion (when $T_{\max} = 0$), and $T_{eq}$ and $V_{eq}$, the kinetic and potential energies in equilibrium. Then the virial theorem gives

$$T_{eq} = -V_{\max}$$
$$V_{eq} = 2V_{\max}$$

Define

$$
\begin{aligned}
V_{eq} &\equiv -\frac{GM^2}{r_{eq}}, \\
V_{\max} &= \frac{3}{5}\frac{GM^2}{R}, \\
T_{eq} &\equiv \frac{1}{2}Mv_{eq}^2, \\
t_{dyn} &\equiv \frac{r_{eq}}{v_{eq}},
\end{aligned}
$$

where $r_{eq}$ is the harmonic radius in equilibrium, $R$ is the maximum radius (as before), $v_{eq}$ is the RMS internal velocity in equilibrium, and $t_{dyn}$ is the crossing time in equilibrium.

Let $\bar{\rho}$ and $\delta$ be the initial mean density of the Universe and the initial overdensity of the perturbation *at the same epoch*. Note that

$$
\begin{aligned}
R &= \frac{3}{5}\delta^{-1}r_{initial} \\
M &= \frac{4\pi}{3}\bar{\rho}r_{initial}^3
\end{aligned}
$$

We can combine all of those equations together to find

$$r_{eq} = \frac{5}{6}R = \frac{1}{2}\delta^{-1}\left(\frac{3}{4\pi}\frac{M}{\bar{\rho}}\right)^{\frac{1}{3}}, \quad (11)$$

$$v_{eq} = \left(\frac{32\pi}{3}\right)^{\frac{1}{6}}\delta^{\frac{1}{2}}G^{\frac{1}{2}}M^{\frac{1}{3}}\bar{\rho}^{\frac{1}{6}}, \quad (12)$$

$$t_{dyn} = \left(\frac{3}{32\pi}\right)^{\frac{1}{2}}\delta^{-\frac{3}{2}}G^{-\frac{1}{2}}\bar{\rho}^{-\frac{1}{2}}. \quad (13)$$

We can therefore conclude that the initial epoch ($\bar{\rho}$), initial overdensity ($\delta$), and mass ($M$) suffice to determine the radius, density, and other properties of the final collapsed object.