Statistical Signal Processing
Assignment 4

Submission deadline: 16th, December, 2015

Note: **C**: Computer Assignment, **W**: Workout Problem. For computer assignments, please submit both the code (properly indented) and the results in form of plots (properly labelled).

1. (**W**) We wish to estimate the amplitudes of exponentials in noise. The observed data are

$$x[n] = \left(\sum_{i=1}^{i=p} A_i r_i^n \right) + w[n], \quad i = [0, 1, ..., N-1]$$

   $w[n]$ is White Gaussian Noise(WGN) with distribution $\mathcal{N}(0, \sigma^2)$. Find the Minimum Variance Unbiased (MVU) Estimator of the vector of amplitudes $\mathbf{A} = [A_1, A_2, ...A_p]^T$ and also find the covariance of $\hat{\mathbf{A}}$. Evaluate your results for $p = 2$ with $r_1 = 1$ and $r_2 = -1$, and N is even.

2. (**W**) Suppose you make a set of measurements $x[n]$, $n = [0, 1, .., N-1]$ where,

$$x[n] = a\cos\left(2\pi \frac{k}{N} n\right) + b\sin\left(2\pi \frac{k}{N} n\right) + w[n]$$

   $w[n]$ is WGN with $\mathcal{N}(0, \sigma^2)$ and a & b are the parameters to be estimated.

   (a) Find a MVU Estimator for $\theta = [a\ b]^T$.

   (b) We would like to estimate the power (P) of the signal portion of $x[n]$. The estimator for P is given by

$$\hat{P} = \frac{\hat{a}^2 + \hat{b}^2}{2}$$

   Find the variance of P as $N \to \infty$ (Use vector transformation to find the covariance).

3. (**W**) A biased coin is tossed N times and the outcomes are recorded in a vector $x[n]$, $n = [1, 2, ...N]$. This is a typical example of a Bernoulli experiment.

$$P\{x[n]\} = \begin{cases} p, & \text{if } x[n] = 1 \text{ (Head)} \\ 1-p, & \text{if } x[n] = 0 \text{ (Tail)}. \end{cases}$$

   Derive the expression for the Maximum Likelihood Estimator (MLE) of p.

4. (**C**) Plot the function

$$f(x) = \exp\left[-\frac{x^2}{2}\right] + 0.1\exp\left[-\frac{(x-10)^2}{2}\right]$$

over the domain $-3 \leq x \leq 13$. Use Newton-Raphson iteration method to find the maximum of the function. Use the initial guesses $x_0 = 0.5,\ 3.5,\ 9.5$. What can you say about the importance of intial guess?

5. (**C+W**) Consider a parameter model given by

$$x_i = r^i + w_i, \quad i = [1, 2, ...N]$$

where r is the parameter to be estimated and w_i is WGN with distribution $\mathcal{N}(0, \sigma^2)$.

   (a) Write down the expression for the MLE of parameter r. Can you solve this equation analytically?

   (b) Write down an iterative solution for the MLE equation using the Newton-Raphson method.

   (c) Write a program to iteratively solve for the parameter r. Run the program with three different initial guesses: $r_0 = (0.8, 0.2, 1.2)$, using $N = 10$ and $\sigma^2 = 0.01$ Plot the estimated parameter value as a function of iteration number. Interpret your results.

6. (**C+W**) Consider the model

$$x_i = A\cos(2\pi f_0 i + \phi) + w_i,\ i = [1, 2..., N]$$

where $w_i \sim \mathcal{N}(0, \sigma^2)$ (IID) and $A,\ f_0,\ \sigma^2$ are known.

   (a) Show that the MLE for ϕ is

$$\hat{\phi} = -\tan^{-1}\left(\frac{\sum_{i=1}^{N} x_i \sin(2\pi f_0 i)}{\sum_{i=1}^{N} x_i \cos(2\pi f_0 i)}\right)$$

   (b) Assume $A = 1,\ f_0 = 0.125,\ \phi = \pi/4,\ \sigma^2 = 0.04$ and $N = 100$. Run Monte Carlo simulations for 1000 and 10000 different realizations and generate the PDF of $\hat{\phi}$.

Important formulae:

$$\sum_{n=0}^{N-1} \cos\left(\frac{2\pi i n}{N}\right) \cos\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$$

$$\sum_{n=0}^{N-1} \sin\left(\frac{2\pi i n}{N}\right) \sin\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$$

$$\sum_{n=0}^{N-1} \cos\left(\frac{2\pi i n}{N}\right) \sin\left(\frac{2\pi j n}{N}\right) = 0, \text{ for all } i, j.$$