Galaxy Dynamics

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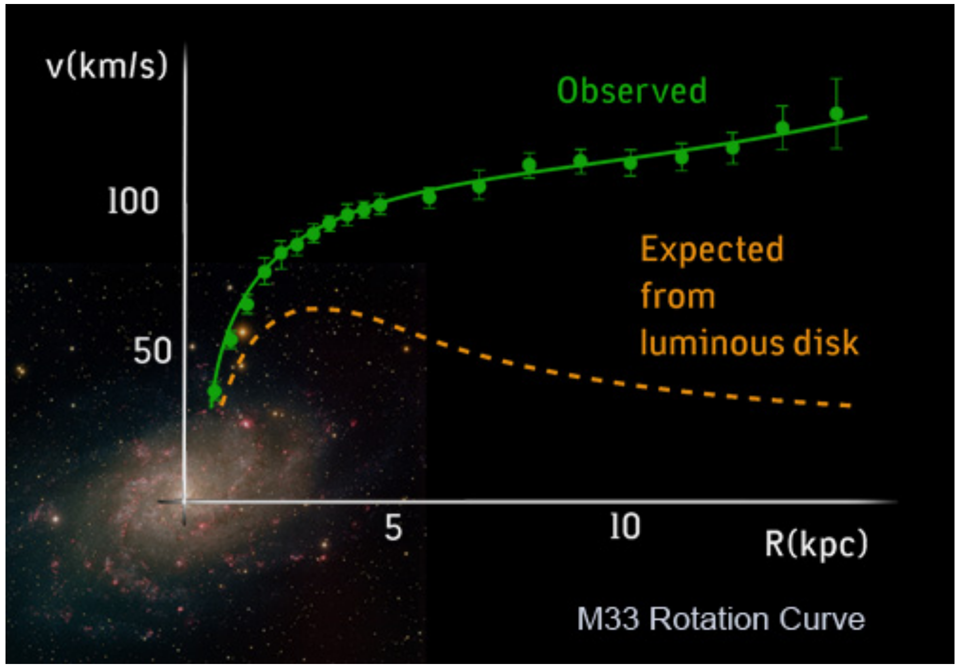
Stellar motions and galaxy mass distribution

A star moving on a circular orbit follows an Equation that relates to mass present inside orbit:

$$V_c^2 = G\,M(r)/r$$



M33 Rotation Curve

When comparing the mass in stars to the mass derived "dynamically", we find there is more mass than can be accounted for by the stars and gas

→ dark matter

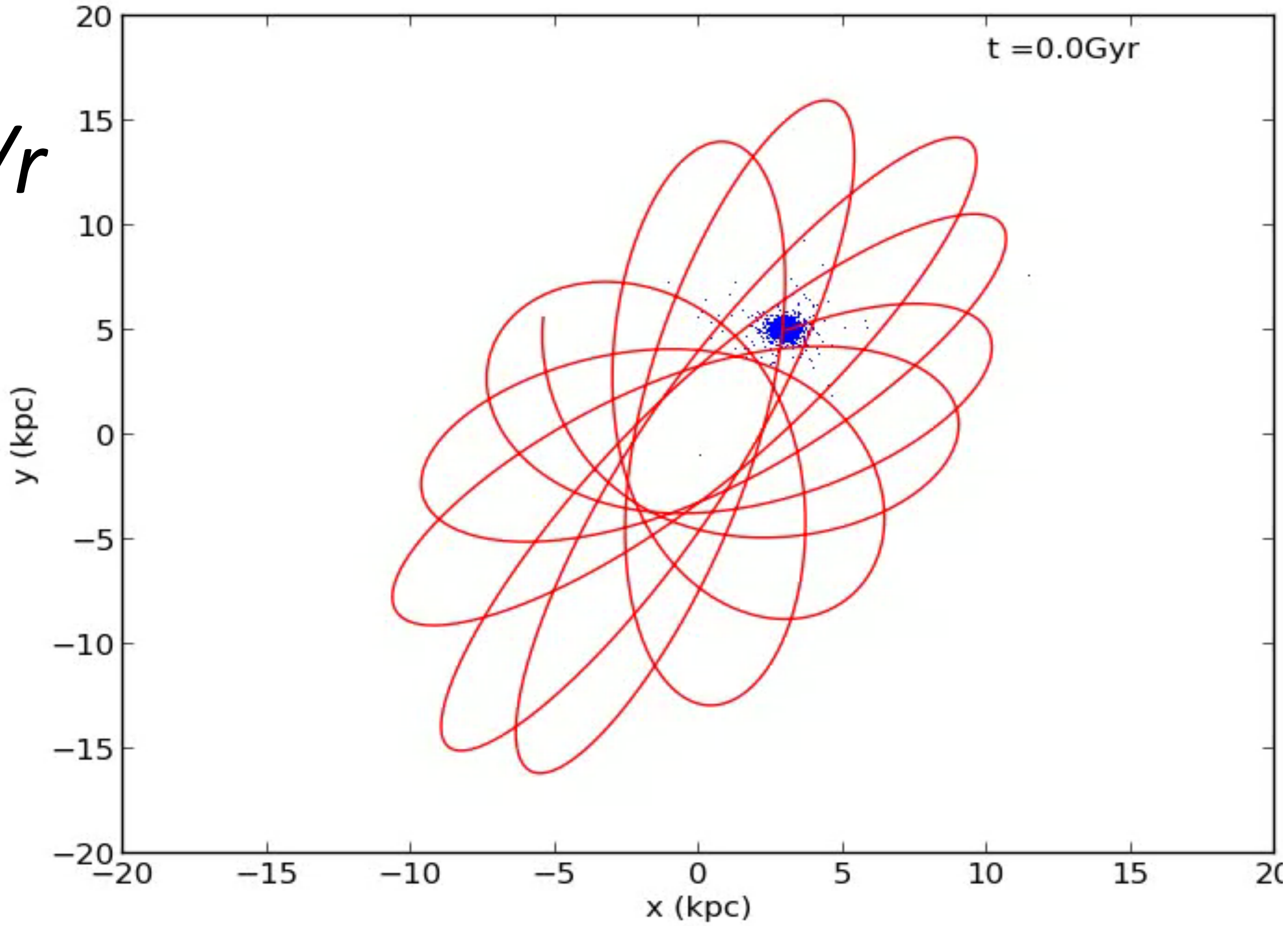
- Observed motions do not only tell us about mass distribution at present position of a star but along their orbit
  - – e.g. measurement of the escape velocity
  - – this is the velocity that a star must have to be able to escape the system:

$E = 1/2\ v^2 + \Phi(r) = 0 \rightarrow$ $v_{esc} = (2\,|\Phi(r)|)^{1/2}$ → the full potential matters

The main role of gravity

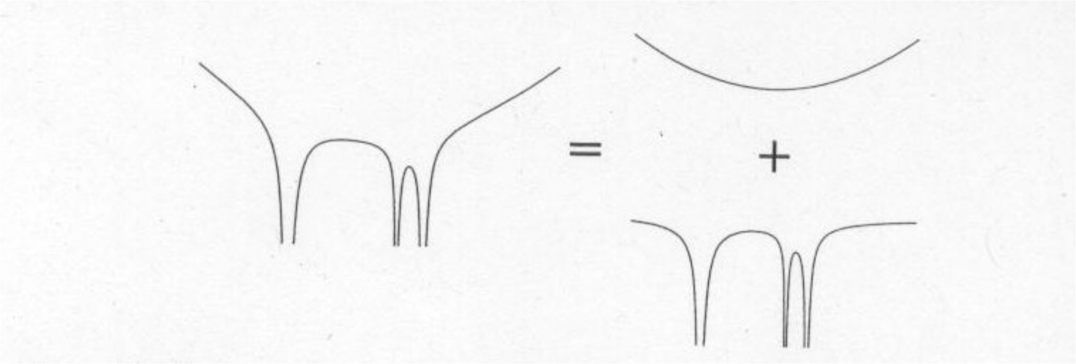
- Knowledge of the mass distribution of a galaxy allows to predict how the positions and velocities of stars will change over time.

$$d^2\mathbf{r}/dt^2 = - GM(r)/r^2\ \mathbf{r}/r$$



Two components in the gravitational potential

- We can consider the stars as point masses
  - their sizes are small compared to their separations
- The gravitational potential of a galaxy is the sum of:
  - a smooth component (the average over a region containing many stars)
  - the very deep potential well around each individual star



- The motion of stars within a galaxy is determined almost entirely by the smooth part of the force.
  - Two-body encounters are only important within dense star clusters

Stellar collisions

- The gravitational force is long-range
- The average (smoother) mass distribution determines the motion of stars

- Close encounters are not important in galaxies
- This is different from other physical systems such as molecules of air or dust particles. These reach equilibrium through collisions, where they exchange energy and momentum. The forces between molecules only strong when they are very close to each other; they experience violent and short-lived accelerations, in between long periods when they move at nearly constant speeds.

- The timescales/frequency of encounters tell how important they are in galaxies

- We can consider two types of encounters:
  - Strong (near): the trajectory of the star changes significantly
  - Weak (distant): perturbation on the initial trajectory

Strong encounters

- Take place when a star comes so close another that the collision completely changes its speed and direction of motion

- A strong encounter has happened when the change in potential energy is at least as large as the initial kinetic energy.

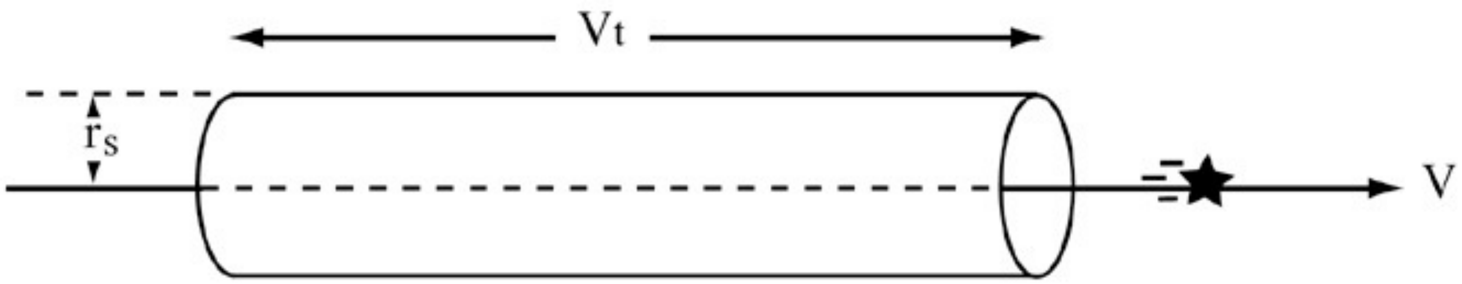
$$\frac{Gm^2}{r} \gtrsim \frac{mV^2}{2}, \quad \text{which means } r \lesssim r_\mathrm{s} \equiv \frac{2Gm}{V^2}$$

r_s is the strong encounter radius.

- Near the Sun, stars have random speeds v ~ 30 km/s, and for m=0.5 M– r_s ~ 1 AU

- The nearest star is at 4.2 light years away ~ 271,000 AU; the average separation between stars in galaxies is generally too low for a strong encounter …

Average time between collisions



- Any star within a distance r_s from another star will have a strong encounter
- As the star moves, it defines a cylinder of radius r_s centered on its path. The volume of this cylinder is $\pi r_s^2 V t$

- If there are n stars per unit volume, this star will on average have one close encounter in a time t_s such that $n \pi r_s^2 v t_s = 1$

- The characteristic time between collisions is $t_s \sim 1/(n\pi r_s^2 v)$, or $t_s = v^3/(4\pi G^2m^2n)$

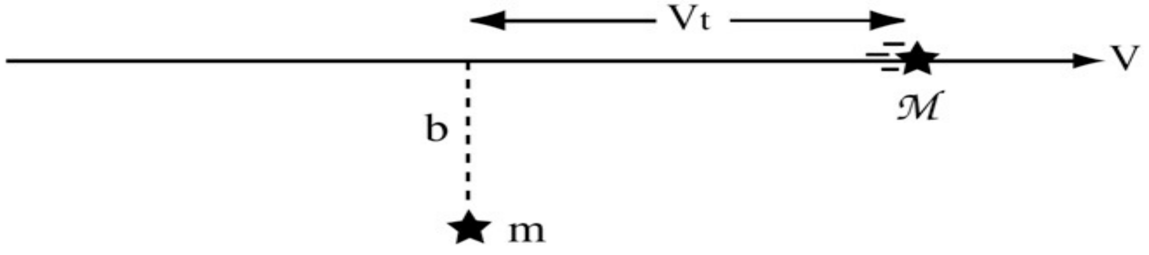
$$t_s = \frac{V^3}{4\pi G^2 m^2 n} \approx 4 \times 10^{12}\,\mathrm{yr} \left(\frac{V}{10\,\mathrm{km\,s^{-1}}}\right)^3 \left(\frac{m}{\mathcal{M}_\odot}\right)^{-2} \left(\frac{n}{1\,\mathrm{pc}^{-3}}\right)^{-1}$$

- Since n ~ 0.1 pc^{-3} near the Sun, $t_s \sim 10^{15}$ years (>> the age of the Universe)
- Strong encounters are only important in the dense cores of globular clusters.

Distant weak encounters

- In a distant encounter, the force of one star on another is so weak that the stars hardly deviate from their original paths after the encounter.

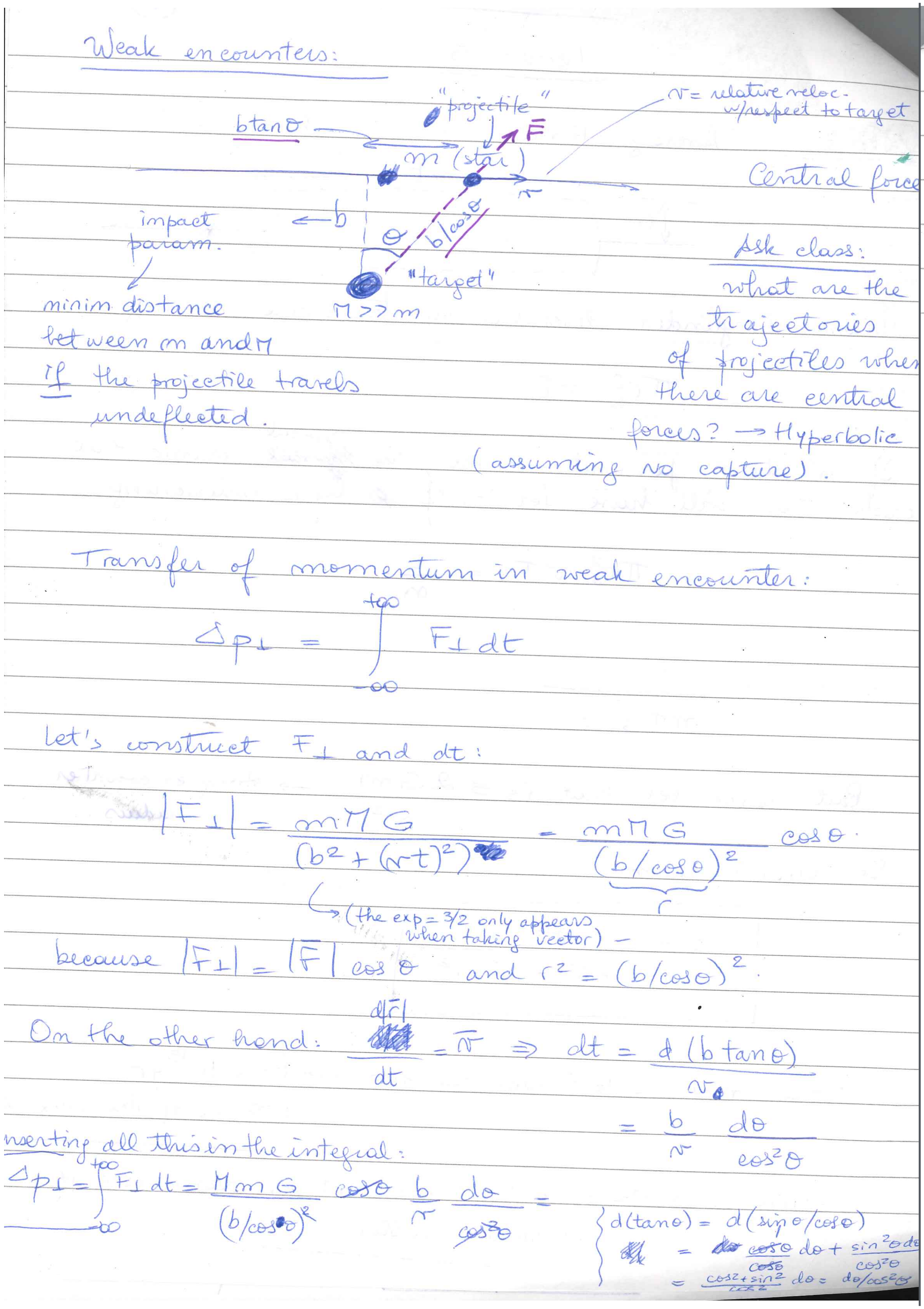
- We will consider the case of a star moving through a system of N identical stars of mass m.



- We assume that
  - the change in velocity is very small: $\delta v/v << 1$,
  - the perturbing star is stationary

  - This is known as the impulse approximation.

Momentum transfer in weak encounters



Weak encounters:

"projectile"

$\vec{F}$

$b\tan\theta$

m (star)

v

v = relative veloc. w/respect to target

impact param. $\leftarrow b$

minim distance between m and M if the projectile travels undeflected.

θ

$b/\cos\theta$

"target"

$M >> m$

Central force

Ask class: what are the trajectories of projectiles when there are central forces? → Hyperbolic (assuming no capture).

Transfer of momentum in weak encounter:

$$\Delta p_\perp = \int_{-\infty}^{+\infty} F_\perp \, dt$$

let's construct $F_\perp$ and dt:

$$|F_\perp| = \frac{mMG}{(b^2+(vt)^2)} = \frac{mMG}{\underbrace{(b/\cos\theta)}_{r}{}^2} \cos\theta .$$

(the exp = 3/2 only appears when taking vector) –

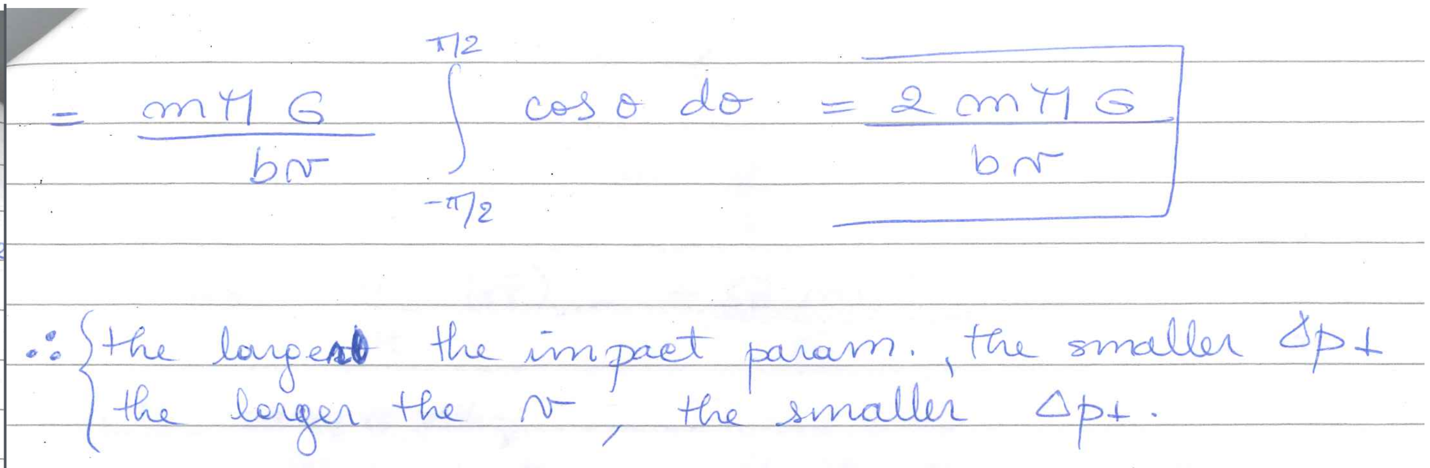
because $|F_\perp| = |\vec{F}| \cos\theta$ and $r^2 = (b/\cos\theta)^2$.

On the other hand: $\frac{d|\bar{r}|}{dt} = \bar{v} \Rightarrow dt = \frac{d(b\tan\theta)}{v} = \frac{b}{v}\frac{d\theta}{\cos^2\theta}$

nserting all this in the integral:

$$\Delta p_\perp = \int_{-\infty}^{+\infty} F_\perp dt = \frac{MmG}{(b/\cos\theta)^2} \cancel{\cos\theta} \frac{b}{v} \frac{d\theta}{\cancel{\cos^2\theta}} =$$

$$\left\{ \begin{aligned} d(\tan\theta) &= d(\sin\theta/\cos\theta) \\ &= \frac{\cos\theta}{\cos\theta} d\theta + \frac{\sin^2\theta \, d\theta}{\cos^2\theta} \\ &= \frac{\cos^2+\sin^2}{\cos^2} d\theta = d\theta/\cos^2\theta \end{aligned} \right.$$

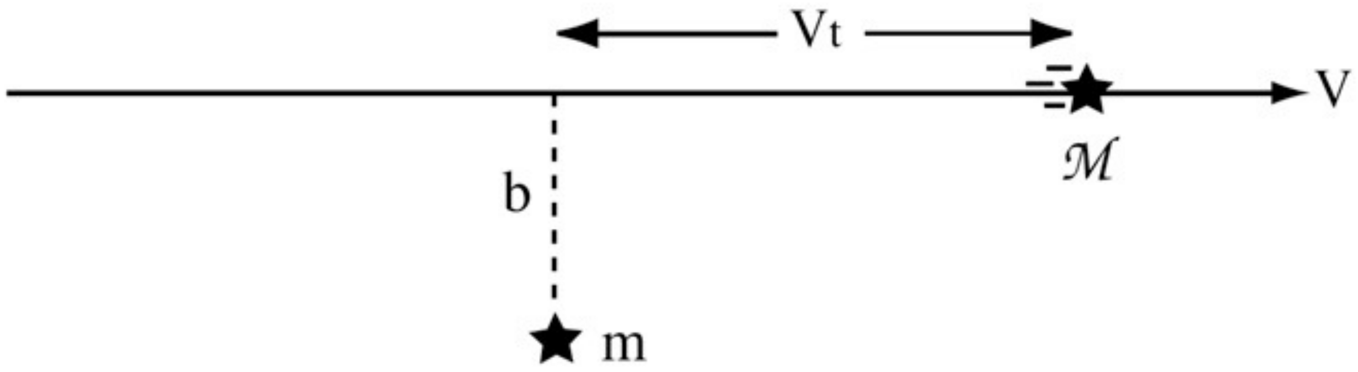


$$= \frac{mMG}{bv} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \boxed{\frac{2mMG}{bv}}$$

∴ { the largest the impact param., the smaller $\Delta p_\perp$

the larger the v, the smaller $\Delta p_\perp$.

Star deflection due to weak encounter



- The pull by m induces a motion dv perpendicular to the original trajectory. The full force is $\mathbf{F} = - Gm\mathcal{M}/r^2\ \varepsilon$, and that in the perpendicular direction is

$$\mathbf{F}_\perp = \frac{Gm\mathcal{M}}{(b^2 + V^2t^2)^{3/2}} = \mathcal{M}\frac{\mathrm{d}V_\perp}{\mathrm{d}t}$$

- The change in velocity of star M

$$\Delta V_\perp = \frac{1}{\mathcal{M}}\int_{-\infty}^{\infty}\mathbf{F}_\perp(t)\mathrm{d}t = \frac{2Gm}{bV}$$

→ the faster the relative velocity, the smaller the perturbation is.

Define impact parameter: minimum distance between m and the trajectory of M in the case that the latter is undeflected. This would correspond to the linear trajectory with v=v(inf).

Cumulative effect of individual encounters

- If the surface density of stars in system is n, the number of (weak) encounters dn_e with impact parameter b that a star suffers when crossing the system over time t

$$dn_e = n\ V\ t\ \ 2\ \pi\ b\ db$$

- Each of these encounters will produce a change in d**v**, but because the perturbations are randomly oriented, the mean **vector** change is zero

(there is no change in direction, but the road is "bumpy")

- The accumulation of weak encounters can be measured by Sum of $(\Delta V_\perp)^2$

$$\langle \Delta V_\perp^2 \rangle = \int_{b_{\min}}^{b_{\max}} n V t \left(\frac{2Gm}{bV} \right)^2 2\pi b \, \mathrm{d}b = \frac{8\pi G^2 m^2 n t}{V} \ln\left(\frac{b_{\max}}{b_{\min}} \right)$$

- After a time t_{relax}, such that $\langle \Delta V_\perp{}^2 \rangle = V^2$ the memory of the initial path is lost.

Relaxation time

Relaxation is the cumulative effect of individual encounters

In a relaxation time, the stellar velocity distribution randomizes

- This is called the relaxation timescale:

$$t_{\text{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda} = \frac{t_{\text{s}}}{2 \ln \Lambda}$$

$$\approx \frac{2 \times 10^9 \text{ yr}}{\ln \Lambda} \left(\frac{V}{10 \text{ km s}^{-1}}\right)^3 \left(\frac{m}{\mathcal{M}_\odot}\right)^{-2} \left(\frac{n}{10^3 \text{ pc}^{-3}}\right)^{-1}$$

$$\Lambda = (\text{b}_{\text{max}}/\text{b}_{\text{min}})$$

- It is the timescale required for a star to change its velocity by the same order, due to weak encounters with a "sea" of stars. Compared to the strong collisions timescale, $t_{\text{relax}} < t_s$

- Some characteristic values:
  - Typically $\ln \Lambda \sim 20$.
  - Exact values of b_{min} and b_{max} are not very important (logarithmic dependence): b_{max} = system size, and $b_{\text{min}} = r_s$, for example for 300 pc $< b_{\text{max}} <$ 30 kpc, and r_s = 1 AU (near the Sun), $\ln \Lambda \sim 18 - 22$.
- For an elliptical galaxy, N ~ 10^{11} stars, R ~ 10 kpc, and the average relative velocity of stars is v ~ 200 km/s, then $t_{\text{relax}} \sim 10^4$ Gyr!
- For stars in a globular cluster like Ω Cen, $t_{\text{relax}} \sim 0.4$ Gyr, so relaxation will be important over a Hubble time.

- *This implies that when calculating the motions of stars like the Sun, we can ignore pulls of individual stars, and consider them to move in the smooth potential of the entire Galaxy.*

Motion under gravity

- Newton's law of gravity: $$\boxed{\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r}}$$

- In a cluster of N stars with masses m_α, at positions $\mathbf{x}_\alpha$

$$\boxed{\frac{d}{dt}(m_\alpha \mathbf{v}_\alpha) = -\sum_{\beta,\beta\neq\alpha} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta)}$$

(Note heavy and light stars suffer the same acceleration)

- In terms of the gradient of the gravitational potential $\Phi(\mathbf{x})$:

$$\boxed{\frac{d}{dt}(m\mathbf{v}) = -m\nabla\Phi(\mathbf{x})} \quad \text{with} \quad \boxed{\Phi(\mathbf{x}) = -\sum_\alpha \frac{Gm_\alpha}{|\mathbf{x} - \mathbf{x}_\alpha|} \text{ for } \mathbf{x} \neq \mathbf{x}_\alpha}$$

Poisson's equation

- The potential at point **x** produced by a continuous mass distribution represented by density $\rho(\mathbf{x})$

$$\boxed{\Phi(\mathbf{x}) \equiv -G \int \mathrm{d}^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}}$$

Essentially replaced the discrete summation by an integral, and the masses by $\rho(x)\ d^3x$

- If the potential is known, rather than the density, we obtain Poisson's equation:

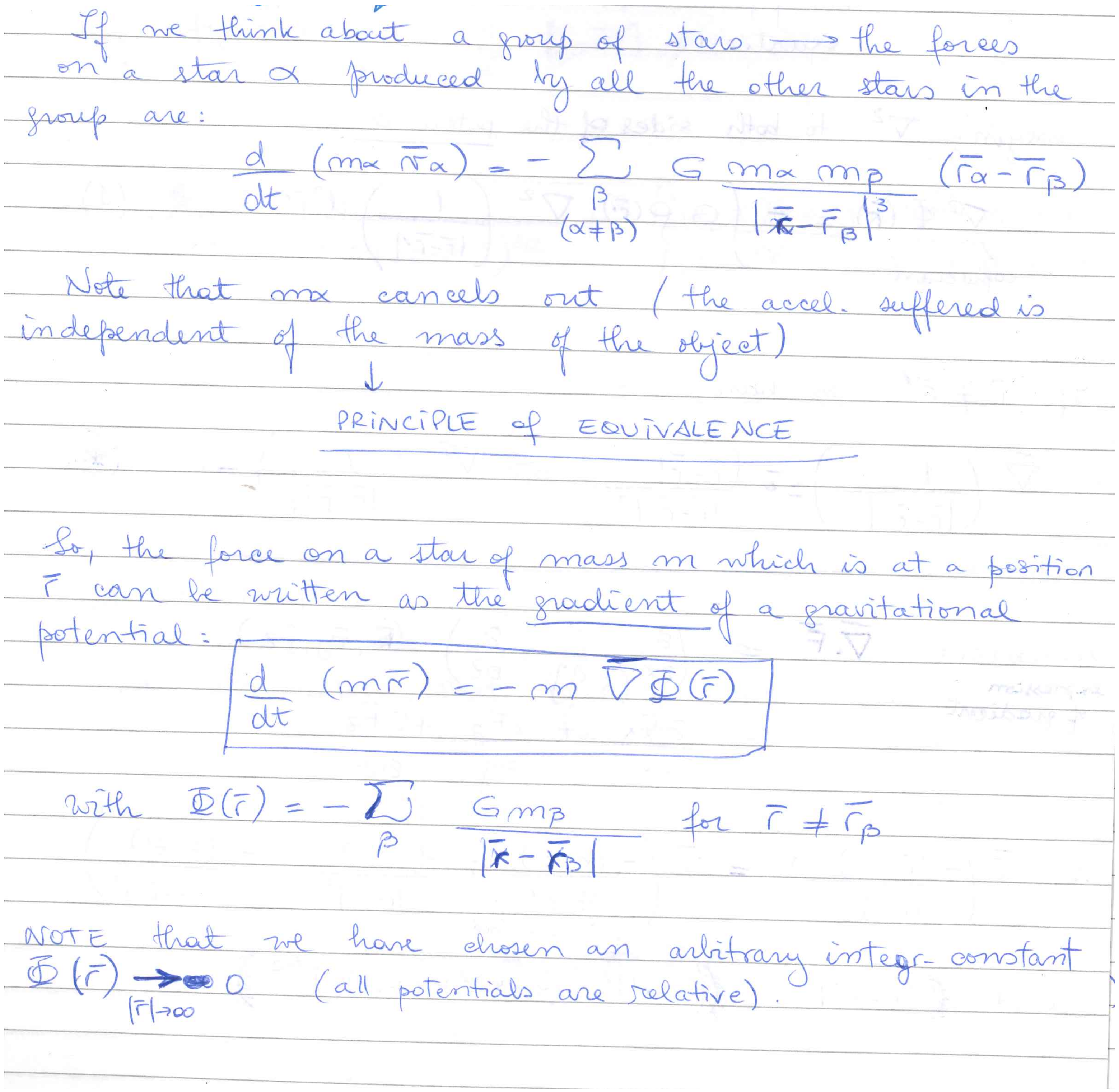
$$\boxed{\nabla^2\Phi = 4\pi G\rho.}$$

- Not all $\Phi(\mathbf{x})$ are physically meaningful: only those for which $\rho(\mathbf{x}) > 0$ everywhere (mass is always positive).

- Note similarity to the electromagnetism and electric field: ($\nabla\Phi_e = -\mathbf{E}$) and the charge distribution ρ_e: $\nabla^2\Phi_e = -4\pi\ k\ \rho_e$, where k is Coulomb's constant.

ρ_e may be positive or negative: electric force can be repulsive or attractive.

Derivation of Poisson’s equation



If we think about a group of stars → the forces on a star α produced by all the other stars in the group are:

$$\frac{d}{dt}(m_\alpha \bar{v}_\alpha) = -\sum_{\substack{\beta \\ (\alpha\neq\beta)}} G \frac{m_\alpha m_\beta}{|\bar{r}_\alpha-\bar{r}_\beta|^3}(\bar{r}_\alpha-\bar{r}_\beta)$$

Note that m_α cancels out (the accel. suffered is independent of the mass of the object)

↓

PRINCIPLE of EQUIVALENCE

So, the force on a star of mass m which is at a position $\bar{r}$ can be written as the gradient of a gravitational potential:

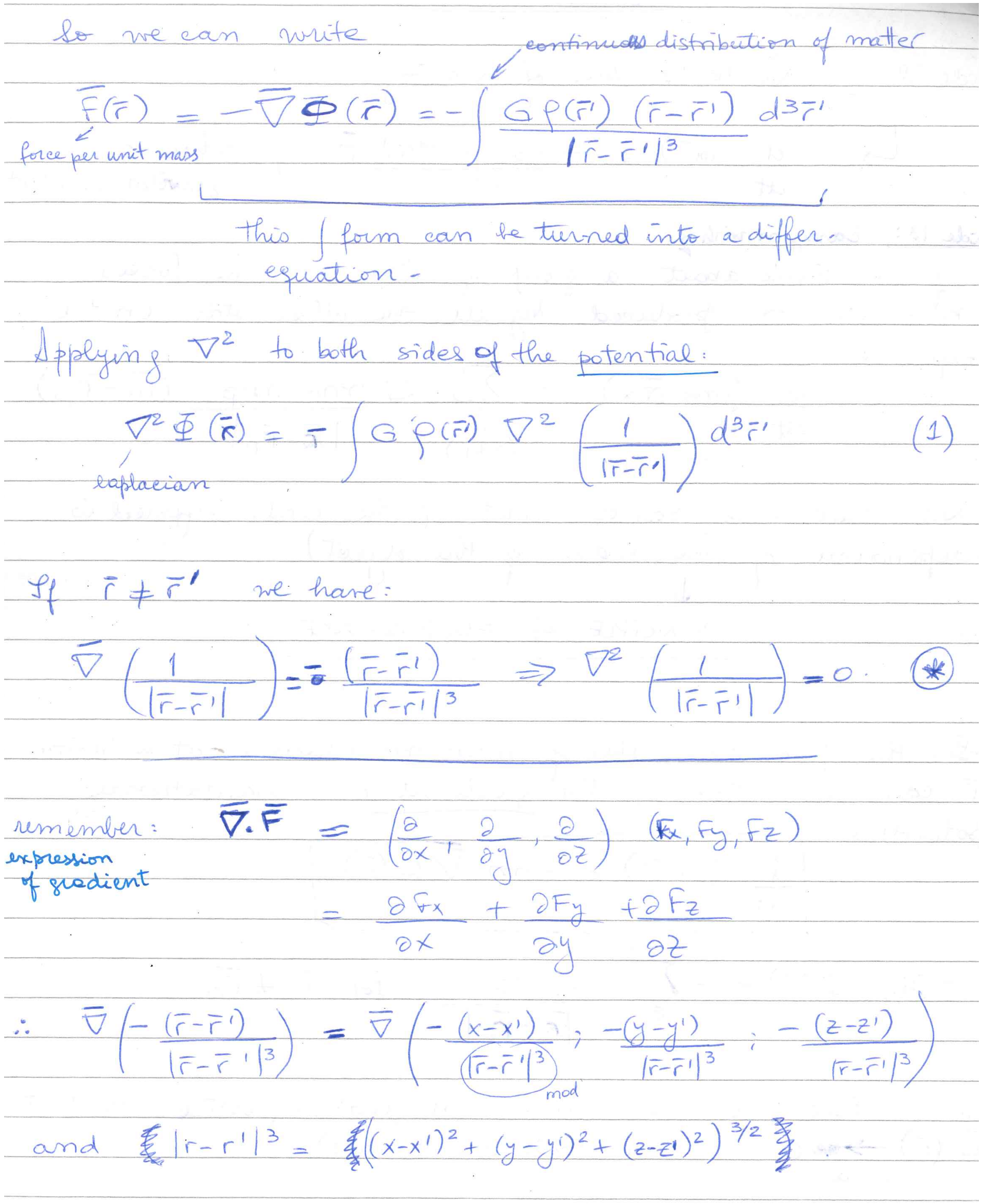
$$\boxed{\frac{d}{dt}(m\bar{v}) = -m\bar{\nabla}\Phi(\bar{r})}$$

with $\Phi(\bar{r}) = -\sum_{\beta} \frac{G m_\beta}{|\bar{r}-\bar{r}_\beta|}$ for $\bar{r} \neq \bar{r}_\beta$

NOTE that we have chosen an arbitrary integr. constant

$\Phi(\bar{r}) \xrightarrow[|\bar{r}|\to\infty]{} 0$ (all potentials are relative).

Derivation of Poisson's equation (cont.)



So we can write

continuous distribution of matter

$$\bar{F}(\bar{r}) = -\bar{\nabla}\Phi(\bar{r}) = -\int \frac{G\rho(\bar{r}')\,(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\, d^3\bar{r}'$$

force per unit mass

this ∫ form can be turned into a differ. equation.

Applying ∇^2 to both sides of the potential:

$$\nabla^2 \Phi(\bar{r}) = -\int G\rho(\bar{r}')\,\nabla^2\left(\frac{1}{|\bar{r}-\bar{r}'|}\right) d^3\bar{r}' \qquad (1)$$

laplacian

If $\bar{r} \neq \bar{r}'$ we have:

$$\bar{\nabla}\left(\frac{1}{|\bar{r}-\bar{r}'|}\right) = -\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} \Rightarrow \nabla^2\left(\frac{1}{|\bar{r}-\bar{r}'|}\right) = 0. \qquad \circledast$$

---

remember:

expression of gradient

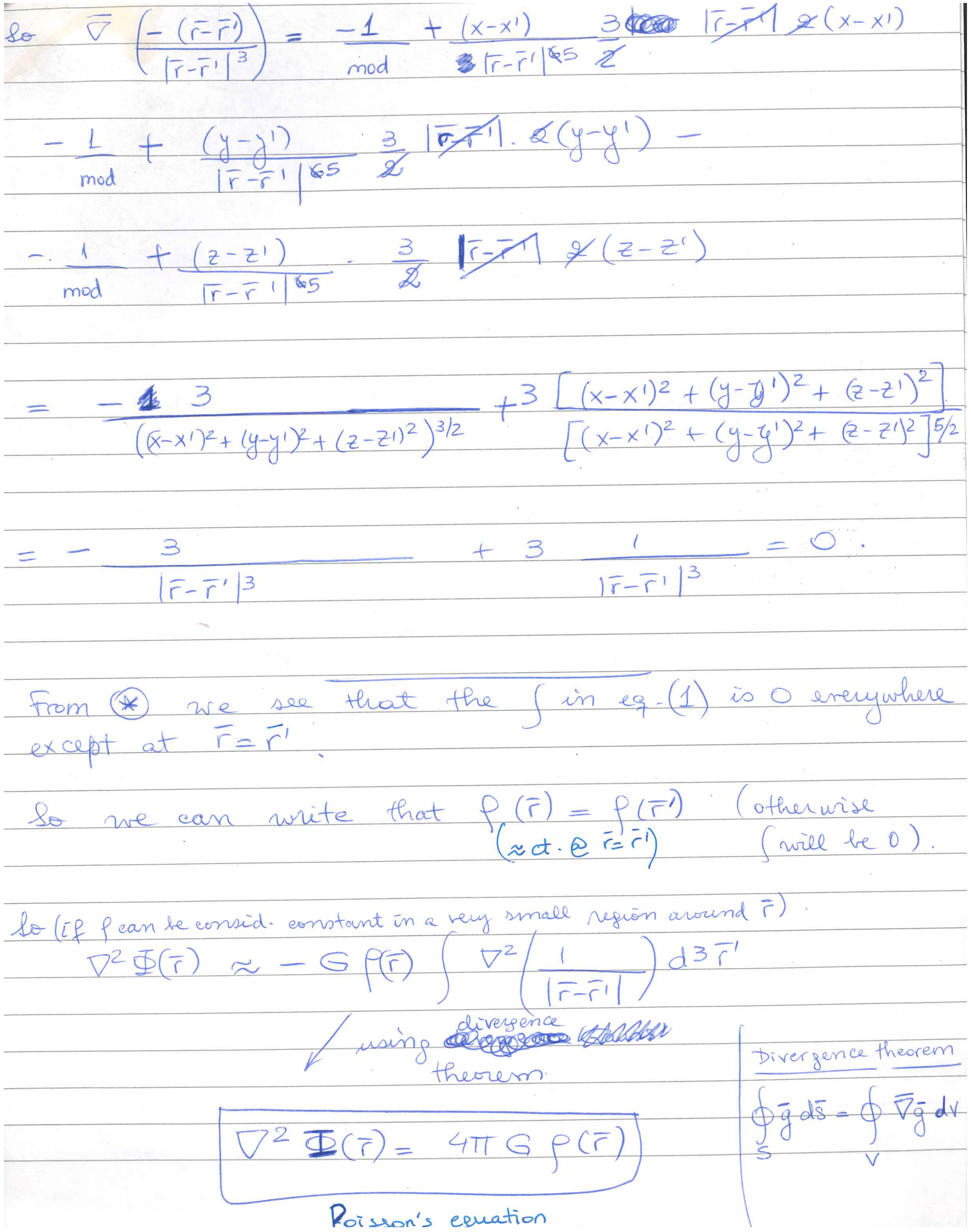
$$\bar{\nabla}\cdot\bar{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(F_x, F_y, F_z)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\therefore \quad \bar{\nabla}\left(-\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\right) = \bar{\nabla}\left(-\frac{(x-x')}{\underset{\text{mod}}{\boxed{|\bar{r}-\bar{r}'|^3}}}; -\frac{(y-y')}{|\bar{r}-\bar{r}'|^3}; -\frac{(z-z')}{|\bar{r}-\bar{r}'|^3}\right)$$

and $|r-r'|^3 = \left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}$

Derivation of Poisson's equation (cont.)



So $$\bar{\nabla}\left(\frac{-(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\right) = \frac{-1}{\text{mod}} + \frac{(x-x')}{\cancel{3}|\bar{r}-\bar{r}'|^{\cancel{6}5}}\frac{3}{\cancel{2}}\cancel{|\bar{r}-\bar{r}'|}\,\cancel{2}(x-x')$$

$$-\frac{1}{\text{mod}} + \frac{(y-y')}{|\bar{r}-\bar{r}'|^{\cancel{6}5}}\frac{3}{\cancel{2}}\cancel{|\bar{r}-\bar{r}'|}.\cancel{2}(y-y') -$$

$$-\frac{1}{\text{mod}} + \frac{(z-z')}{|\bar{r}-\bar{r}'|^{\cancel{6}5}}\cdot\frac{3}{\cancel{2}}\cancel{|\bar{r}-\bar{r}'|}\,\cancel{2}(z-z')$$

$$= \frac{-\cancel{1}\,3}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} + 3\frac{[(x-x')^2+(y-y')^2+(z-z')^2]}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}}$$

$$= -\frac{3}{|\bar{r}-\bar{r}'|^3} + 3\frac{1}{|\bar{r}-\bar{r}'|^3} = 0.$$

From ⊛ we see that the $\int$ in eq.(1) is 0 everywhere except at $\bar{r}=\bar{r}'$.

So we can write that $\rho(\bar{r}) = \rho(\bar{r}')$ (otherwise $\int$ will be 0).
($\approx$ ct. @ $\bar{r}=\bar{r}'$)

So (if ρ can be consid. constant in a very small region around $\bar{r}$).

$$\nabla^2\Phi(\bar{r}) \approx -G\rho(\bar{r})\int\nabla^2\left(\frac{1}{|\bar{r}-\bar{r}'|}\right)d^3\bar{r}'$$

↙ using divergence theorem

$$\boxed{\nabla^2\Phi(\bar{r}) = 4\pi G\rho(\bar{r})}$$

Poisson's equation

Divergence theorem

$$\oint_S \bar{g}\,d\bar{s} = \oint_V \bar{\nabla}\bar{g}\,dV$$

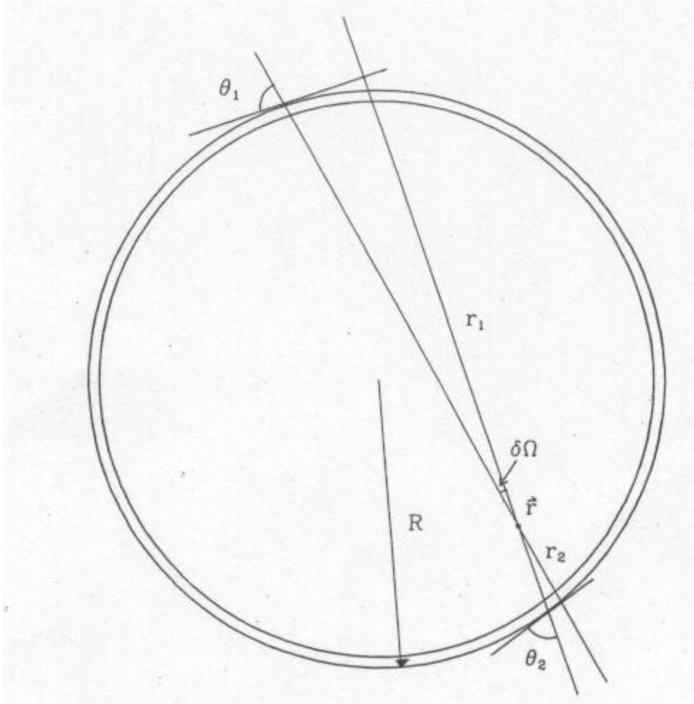
Spherical Systems: Newton’s theorem I

- Theorem 1: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

$\delta m_1 = \rho\, r_1^2\, dr_1\, d\Omega_1$ and $\delta m_2 = \rho\, r_2^2\, dr_2\, d\Omega_2$
But $dr_1 = dr_2 = dr$ and $d\Omega_1 = d\Omega_2 = d\Omega$.
Then $\delta\, m_1/r_1^2 = \delta\, m_2/r_2^2$.

A particle M located at $\mathbf{r}$ experiences a force
$\mathbf{F} = \mathbf{f_1} + \mathbf{f_2}$ where
$\mathbf{f_1} = -\,GM\, \delta m_1/r_1^2\, \boldsymbol{\varepsilon}_1$ and $\mathbf{f_2} = -GM\, \delta m_2/r_2^2\, \boldsymbol{\varepsilon}_2$

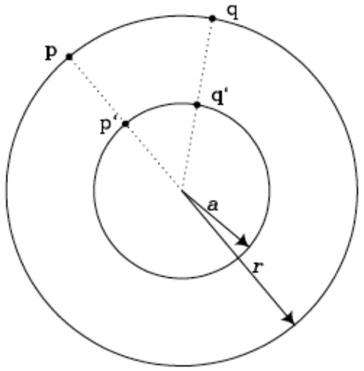
Since $\boldsymbol{\varepsilon}_1 = -\boldsymbol{\varepsilon}_2$,
then $F = -GM(\delta\, m_1/r_1^2 - \delta\, m_2/r_2^2) = 0$



Spherical Systems: Newton’s theorem II

Theorem 2: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shells’ mass was concentrated in a point at its centre.

The gravitational force within a spherical system that a particle feels at a radius R is only due to the mass inside that radius.



- Therefore if a star moves on a circular orbit, its acceleration is given by

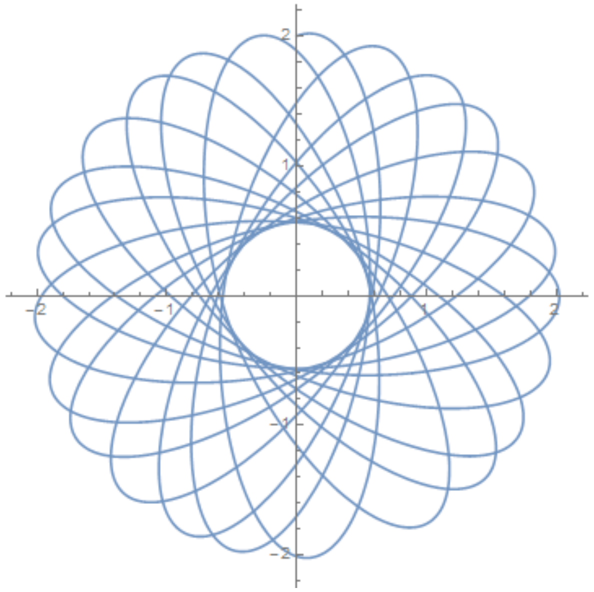
$$v_c^2/r = GM(r)/r^2$$

- For a point mass, the circular velocity $v_c^2 = GM/r$, and so $v_c \propto 1/r^{1/2}$
- Since M generally increases with radius this implies that for a spherical galaxy, the circular velocity never falls off more rapidly than the Kepler case $1/r^{1/2}$, except beyond its edge…

The orbits of stars on a plane

- In a time-independent gravitational potential: energy is conserved
  - try to prove analytically this is the case using $dE/dt = d(1/2\ v^2 + \phi(\mathbf{r}))$, recalling what $d\mathbf{r}/dt$ and $d\mathbf{v}/dt$ are

- In a spherical potential: angular momentum is conserved.
  - The motion of a star is restricted to the *orbital* plane
    - try to prove this using $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ and taking $d\mathbf{L}/dt$, using derivatives and definitions
  - Only two coordinates are needed to describe the location of a star. Typically: polar coordinates in the plane (r, ϕ) to describe the motion.

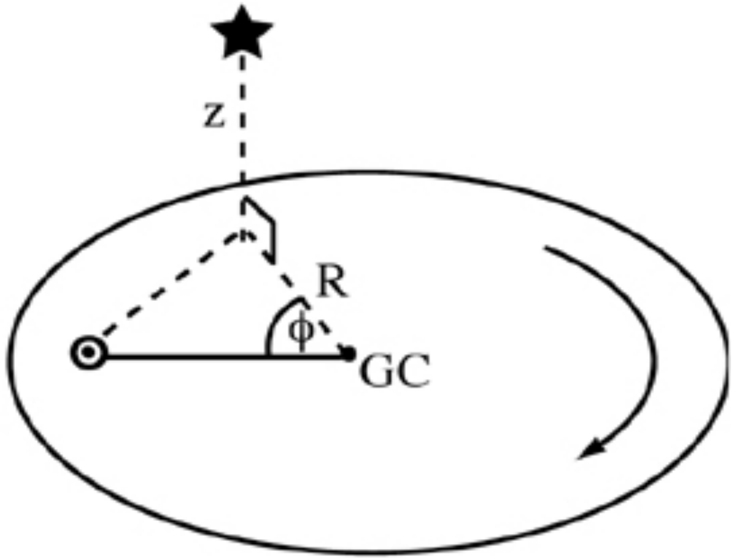


"rosetta" or precessing "ellipses"

Orbits of a star in a disk galaxy

- We use a cylindrical coordinate system (R,ϕ,z), where z = 0 corresponds to the symmetry plane/ mid-plane for the disk
  - Preferred because of the symmetries of the mass distribution.
  - The disk is axisymmetric: it is independent of the angular coordinate ϕ.
    - We neglect non-axisymmetric features such as the bar, the spiral arms…

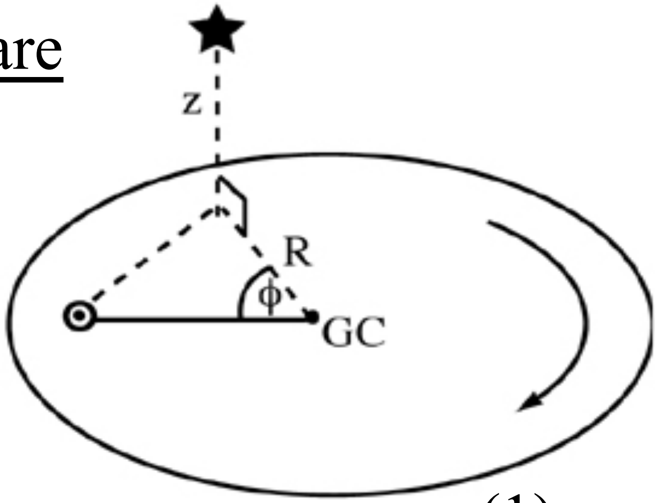
- For an axisymmetric system, the gravitational potential Φ is independent of ϕ → Φ = Φ(R,z)



Equations of motion

- The equations of motion for a star in the disk are

$$d^2\mathbf{r}/dt^2 = -\nabla\Phi,$$



or, in each direction, and using that $\mathbf{r} = R\boldsymbol{\varepsilon}_\mathbf{R} + z\,\boldsymbol{\varepsilon}_\mathbf{z}$,

$$d^2R/dt^2 - R(d\phi/dt)^2 = -\partial\Phi/\partial R \qquad (1)$$

$$d^2z/dt^2 = -\partial\Phi/\partial z \qquad (2)$$

$$d(R^2\, d\phi/dt)/dt = -\ \partial\Phi/\partial\phi = 0 \qquad (3)$$

- Eq. (3) $\boxed{L_z = R^2\, d\phi/dt = \text{cst.}}$

conservation of angular momentum about z-axis

- Eq.(1) can also be written as $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$ (4)

where $\Phi_{eff} = \Phi(R,z) + L_z^2/(2\,R^2)$.

Equations of motion (cont.)

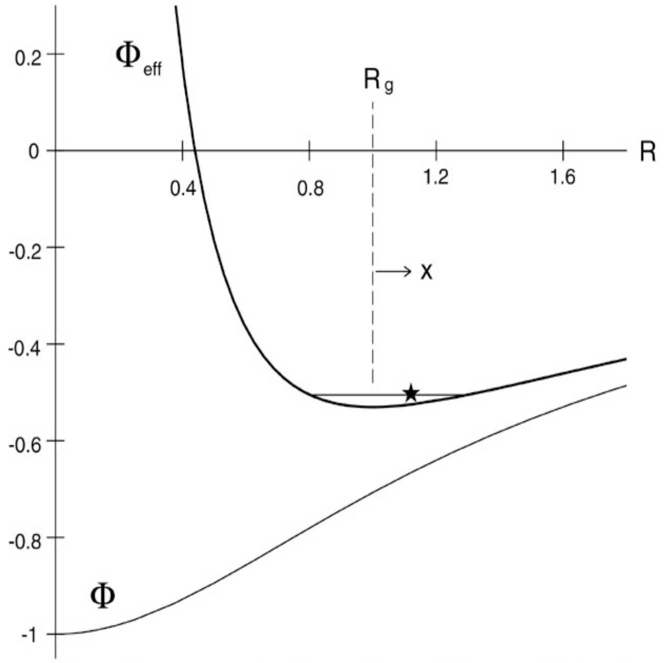
- If we multiply Eq. (4) by dR/dt, and integrate wrt t, then

$$\boxed{\tfrac{1}{2}\,(dR/dt)^2 + \Phi_{eff}(R,z;L_z) = \text{cst.}}$$

which is like an *energy-conservation* law.

- The effective potential Φ_{eff} ($= \Phi(R,z) + L_z^2/(2\,R^2)$) behaves like a potential energy for the star's motion in R and z.

- Note that the effective potential is minimum when:



$* \; \partial\Phi_{eff}/\partial R = 0 \rightarrow \partial\Phi/\partial R - L_z^2/R^3 = 0,$

at $R = R_g$ $\quad \partial\Phi/\partial R|_{R_g} = L_z^2/R_g^3 = R_g\,(d\phi/dt)^2$

this is the radius of a circular orbit

and

$* \; \partial\Phi_{eff}/\partial z = \partial\Phi/\partial z = 0 \rightarrow z = 0$ *on the plane*

Epicycles

- We will now derive approximate solutions to the eq. of motion for stars on nearly circular orbits in the symmetry plane (e.g. the disk).
  - Since gas moves on circular orbits (why?), the stars born will also move on very nearly circular orbits

- Define: $x = R - R_g$, and expand the effective potential around the point $(R_g,0)$:

$$\Phi_{eff}(R,z) \sim \Phi_{eff}(R_g,0) + ½\ \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\ x^2 + ½\ \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\ z^2 + \ldots$$

(the linear terms disappear because this expansion is performed around a stationary point of the potential).

- Let us define

$$\kappa^2 = \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}$$

and

$$\nu^2 = \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}$$

Epicycles (cont.)

The eq. of motion become

- $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$, or $d^2x/dt^2 = -\ \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\ x$

$$\boxed{d^2x/dt^2 = -\kappa^2\ x}$$

- $d^2z/dt^2 = -\partial\Phi_{eff}/\partial z$, or $d^2z/dt^2 = -\ \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\ z$

$$\boxed{d^2z/dt^2 = -\nu^2\ z}$$

- These are the equations of motion of two decoupled harmonic oscillators with frequencies κ and ν.

  κ is the *epicyclic frequency* and

  ν as the *vertical frequency*:

$$\boxed{\begin{array}{l}\kappa^2(R_g) = \partial^2\Phi/\partial R^2|_{R_g,0} + 3\ L_z^2/R_g^4 \\ \nu^2\ (R_g) = \partial^2\Phi/\partial z^2|_{R_g,0}\end{array}}$$

Solution to epicycle equation of motion

- The solution to the eq. of motion is

$x = X_o \cos(\kappa\, t + \Psi)$ and $z = Z_o \cos(\nu\, t + \theta)$ for $\kappa^2 > 0$.

The motion of a star in the disk can be described as an oscillation about a guiding center that is moving on a circular orbit.

The approximation to 2nd order in z in the effective potential is only valid if $\rho(z) \sim$ cst (from Poisson's eq). However, the disk density decreases exponentially. This means that the approximation can at most be valid for 1 scale-height ($z < 300$ pc).