Radiative Processes in Astrophysics

2. Thermal radiation

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Thermal radiation
emitted by matter in thermal equilibrium

Black-Body radiation comes from a body that is in local thermodynamic equilibrium (LTE) with its surroundings. Thus all populations (atomic, ionic, molecular) are described entirely by the local temperature, and given by Saha-Boltzmann statistics:

\[
\frac{N_i}{N} \propto \exp\left(-\frac{E_i}{kT}\right)
\]

In this case \( I_\nu \) depends only on \( T \) (and \( \nu \)), and we define the Planck function:

\[
B_\nu(T) = I_\nu
\]
Kirchoff’s Law

for thermal emission

matter & radiation in thermal equilibrium:

\[ \frac{dI_\nu}{d\tau} = S_\nu - I_\nu \]

in equilibrium the intensity must be spatially constant:

\[ \frac{dI}{d\tau} = 0 \]

and hence:

\[ S_\nu = I_\nu = B_\nu(T) \]

i.e., source function & intensity are equal, and since

\[ S_\nu \equiv \frac{j_\nu}{\alpha_\nu} \]

i.e., if a material absorbs well at some wavelength it will also radiate well at the same wavelength.

Kirchoff’s law holds for all thermal radiation, but not all thermal radiation is blackbody radiation. Thermal radiation only becomes blackbody radiation for optically thick media. When this is not the case then \( I_\nu \neq B_\nu \) and \( dI_\nu/ds \neq 0 \). However since \( S_\nu = B_\nu(T) \) still holds, \( j_\nu = \alpha_\nu B_\nu(T) \)

\[ \frac{dI_\nu}{ds} = -I_\nu + B_\nu(T) \]

only zero for BB-radiation.
BB Intensity

An important property of $I_\nu$ is that it is independent of the properties of the enclosure and depends only on TEMPERATURE. $I_\nu \equiv B_\nu(T)$

$B_\nu(T)$ is called the PLANCK FUNCTION

Any object with a temperature above absolute zero emits light of all wavelengths with varying degrees of efficiency; an ideal emitter is an object that absorbs all of the light energy incident upon it and reradiates this energy with a characteristic spectrum. Because it reflects no light it is called a blackbody, and the radiation is called Blackbody radiation.

Thermodynamics of BB radiation

A BB enclosure with a piston, so that work may be done on or extracted from the radiation.

First Law of Thermodynamics: $dQ = dU + PdV$

Second Law of Thermodynamics:

$U = uV$ \hspace{1cm} $P = u/3$ \hspace{1cm} $dU = d(uV) = udV + Vdu$

$dS = \left( \frac{V}{T} \frac{du}{dT} \right) dT + \left( \frac{4u}{3T} \right) dV$

Since total entropy is a function only of temperature and volume:

$dS = \frac{\partial S}{\partial T}dT + \frac{\partial S}{\partial V}dV$
The Stefan-Boltzmann law

Since temperature and volume are independent quantities for BB:

\[ \left( \frac{\partial S}{\partial T} \right)_V = \frac{V}{T} \frac{du}{dT} \]

\[ \frac{\partial^2 S}{\partial T \partial V} = \frac{1}{T} \frac{du}{dT} = \frac{\partial^2 S}{\partial V \partial T} = \frac{4}{3} \left( \frac{-u}{T^2} + \frac{1}{T} \frac{du}{dT} \right) \]

\[ \frac{du}{u} = 4 \frac{dT}{T} \]

the soln of which:

\[ u(T) = aT^4 \]

Integrated Planck fn. is defined by:

\[ u = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T) \]

\[ B(T) = \int B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 \]

The emergent flux from an isotropically emitting surface (BB) is \( \pi \times \) brightness

\[ F = \sigma T^4 \]

\[ \sigma \equiv \frac{ac}{4} = 5.67 \times 10^{-5} \text{erg cm}^{-2} \text{K}^{-4} \text{s}^{-1} \]

Stefan-Boltzmann constant

The Planck Spectrum

To derive the form of a blackbody spectrum we need to employ some elementary quantum mechanics and to know:

1. The number density of photon states for a given energy level
2. The average energy of each state, using Boltzmann formula

Figure 5.2: Left panel shows an idealised critical box of side L. Also indicated is a standing wave (actually the fundamental mode). On the right is the corresponding lattice of states in wave-number space. As the box becomes very large, the spacing of the states becomes very fine. The number of states in some volume \( \Delta k^3 \) of \( k \)-space is \( 2 \times L^3 \Delta k^3/(2\pi)^3 \).
**Number density of photon states**

Consider a photon of frequency $\nu$ propagating in direction $\mathbf{n}$ inside a box. The wave vector of the photon is $\mathbf{k} = (2\pi/\lambda)\mathbf{n} = (2\pi\nu/c)\mathbf{n}$. If each dimension of the box $L_x, L_y, L_z$ is much longer than the wavelength of the photon, then it can be represented by a standing wave in the box. The number of nodes in the wave in each direction $x, y, z$ is thus, $n_x = k_x L_x / 2\pi$, since there is one node for each integral number of wavelengths in given orthogonal directions. Now the wave can be said to have changed states in a distinguishable manner when the number of nodes in a given direction changes by more than one.

If $n_i \gg 1$

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

The number of states in the 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3k$ is

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z d^3 k}{(2\pi)^3}$$

$$d^3 k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 dv d\Omega}{c^3}$$

$\nu^2$ is the density of states per solid angle per volume per frequency.

**Mean Energy per state**

The next step is to compute the mean energy for an oscillator of a given frequency, $\nu$ (and polarisation). According to elementary QM, each photon of frequency $\nu$ has energy $h\nu$, so each state may contain $n$ photons of energy $h\nu$, and according to statistical mechanics, the probability of a state of energy $E_\nu$ is proportional to $e^{-\beta E_\nu}$ where $\beta = (kT)^{-1}$ and $k$ is the Boltzmann constant.

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$

But the sum here is a simple geometric series:

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-n\hbar \nu} = (1 - e^{-\beta \hbar \nu})^{-1}$$

$$\bar{E} = \frac{\hbar \nu e^{-\beta \hbar \nu}}{1 - e^{-\beta \hbar \nu}} = \frac{\hbar \nu}{e^{\beta \hbar \nu} - 1} = \frac{\hbar \nu}{e^{\hbar \nu/kT} - 1}$$

Bose-Einstein statistics with a limitless number of particles
Occupation number in Planck spectrum

If we divide the mean energy by the energy per photon we get the mean occupation number:

$$\bar{n} = \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

If

$$h\nu \ll kT$$

$$h\nu \approx kT$$

$$h\nu \gg kT$$

Specific energy density & brightness

Multiplying the number of states, \(dN = \frac{2\nu^2}{c^3} \ d\nu d\nu d\Omega\) by the mean energy for a state:

$$dE = u_\nu(\Omega) d\nu d\nu d\Omega = \frac{2\nu^2}{c^3} \frac{h\nu}{e^{\beta h\nu} - 1} \ d\nu d\nu d\Omega$$

So the specific energy density is:

$$u_\nu(\Omega) = \frac{2\nu^3/c^3}{e^{\beta h\nu} - 1}$$

And the brightness \(B_\nu = I_\nu = cu_\nu\) is:

$$B_\nu = \frac{2\nu^3/c^2}{e^{\beta h\nu} - 1}$$ Planck Law

The Planck function grows as:

$$B \propto \nu^2 \quad (h\nu \ll kT)$$

and then falls rather abruptly for

$$h\nu \geq kT$$

Consistent with \(u \propto T^4\) behaviour derived from thermodynamics, total energy density:

$$u = \frac{4\pi}{c} \int B_\nu d\nu$$

Dominated by modes around characteristic frequency \((kT/h)\) and the value of the integrand at the peak is of order \(B_{max} \sim h^3/c^2 \sim (kT)^3/(h^2c^2)\)

$$u \sim v^* B_{max}/c \sim (kT)^4/(h^3c^3) \propto T^4$$
The Planck Spectrum

The energy density at frequency $\nu$, $u_\nu$, is a product of the number of photons with energy, $E$ with their density, and since $u_\nu c = B_\nu$.

$$B_\nu(T) = \frac{2\hbar \nu^3 / c^2}{\exp(h\nu/kT) - 1}$$
$$B_\lambda(T) = \frac{2hc^2 / \lambda^5}{\exp(hc/\lambda kT) - 1}$$

The hotter the blackbody, the bluer the peak of the radiation and the higher the value of $B_\nu$ at any given $\nu$. 
Properties of the Planck Law

The form of $B_\nu (T)$ is one of the most important results for radiation processes. There are a number of properties and consequences of this law:

Rayleigh-Jeans Law: $h \nu \ll k T$

Argument of the exponential is small and so expanding gives a mean energy per state, $E \sim kT$ so we can say these modes are `in equipartition', and the intensity is: $B_\nu^{RJ} \approx 2\nu^2 kT/c^2$

applies at low frequencies (radio); otherwise `ultra-violet catastrophe'

Wien Law: $h \nu \gg k T$

$B_\nu^{\text{Wien}} \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$

which falls sharply with rising frequency

Monotonicity with temperature

From Planck law we find, for all $\nu$: $\frac{\partial B_\nu (T)}{\partial T} > 0$

thus $B_\nu (T' > T)$ lies everywhere above $B_\nu (T)$

Properties of the Planck Law (contd)

Wien Displacement Law: $x \equiv h\nu_{\text{max}} / kT$

$\left. \frac{\partial B_\nu}{\partial \nu} \right|_{\nu=\nu_{\text{max}}} = 0$

$x = 3(1 - e^{-x})$ which has approx. root $x = 2.82$

The brightness $B_\nu$ peaks at $h\nu_{\text{max}} = 2.82kT$ so $\nu_{\text{max}} = (5.88 \times 10^{10} \text{ Hz}) kT/h$

thus the peak frequency of blackbody law shifts linearly with temperature

in wavelength, $B_\lambda$ peaks at $\lambda_{\text{max}}T = 0.29 \text{ cm} K$

$\nu_{\text{max}}$ gives a convenient way of characterising the frequency range for which the Rayleigh-Jeans is valid ($\nu << \nu_{\text{max}}$) and similarly for the Wien law ($\nu >> \nu_{\text{max}}$).
**Radiation constants**

The constants occurring in the various versions of the Stefan-Boltzmann law can be computed in terms of the ‘fundamental’ constants: $h, k, c$

$$B = \int B_\nu(T) d\nu = \left(\frac{2h}{c^3}\right) \left(\frac{kT}{h}\right)^4 \int \frac{x^3}{e^x - 1} dx$$

Where the value of the dimensionless integral is $\pi^4/15$, so

$$B = \frac{2\pi^4 k^4}{15c^2 h^3} T^4$$

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$$

$$a = \frac{8\pi^5 k^4}{15c^3 h^3}$$

**Characteristic Temperature**

**Brightness Temperature**

If we observe a brightness $I_\nu$ at some frequency $\nu$ then we can define a brightness temperature, $T_b$, such that $I_\nu = B_\nu(T_b)$

This has the advantage of being closely connected to physical properties of the emitter if it is resolved, and has simple units ($K$). This is especially used in radio astronomy, where Rayleigh-Jeans applies:

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu \quad h\nu \ll kT$$

The transfer eqn for thermal emission takes a particularly simple form in terms of brightness temperature in the R-J limit:

$$\frac{dT_b}{d\nu} = -T_b + T$$

when $T$ is constant,

$$T_b = T_b(0) e^{-\tau_\nu} + T(1 - e^{-\tau_\nu})$$

$h\nu \ll kT$

thus is the optical depth is large the brightness temperature of the radiation approaches the temperature of the material.

In the Wien region of the Planck law the concept of brightness temperature is not so useful because of the rapid decrease in $B_\nu$ with $\nu$, and because it is not possible to formulate a transfer equation linear in brightness temperature.
**Characteristic Temperature (contd.)**

**Colour Temperature**

We can deduce the characteristic temperature from the shape of the spectrum - measuring the flux at a range of frequencies and determining which of the possible Planck curves a source lies on, i.e., estimating the peak of the spectrum and applying Wien's displacement law:

![Graph showing the shape of the spectrum and Wien's displacement law]

**Effective Temperature**

A bolometer provides the total flux density $F$ integrated over all frequencies but without any detailed frequency distribution information. Can deduce $T_{\text{eff}}$, if the size of the source, $d\Omega$, is known by equating the actual flux $F$ to the flux of a BB at temperature $T_{\text{eff}}$:

$$F = \int \cos \theta \, I_\nu \, d\nu \, d\Omega \equiv \sigma T_{\text{eff}}^4$$

Both $T_{\text{eff}}$ and $T_b$ depend upon the magnitude of the source intensity, but $T_c$ depends only on the shape of the observed spectrum.
Bose-Einstein distribution

The Planck spectrum is fully equilibriated distribution function for photons, and in the derivation we placed no restrictions on the occupation numbers, implicitly assuming that reactions which create or destroy photons are efficient.

Under certain circumstances this may not be true (e.g., electron scattering). In this case the photon energy distribution will deviate from the Planck spectrum and one has the more general Bose-Einstein distribution, with a mean occupation number:

$$\bar{n} = \frac{1}{e^{(h\nu + \mu)/kT} - 1}$$

$\mu$ is the chemical potential. If it is positive then this leads to finite occupation number at zero energy. This arises if there are too few photons for a given energy.

Stars are Black-Bodies... almost...

Stellar photospheres share some properties with this container:

The basic condition for the black body as an emitting source is that a negligible small fraction of the radiation escapes.

At the bottom of the stellar photosphere the optical depth to the surface is high enough to prevent escape of most photons. They are reabsorbed close to where they were emitted - thermodynamic equilibrium - and radiation laws of BB apply.

Higher layers deviate increasingly from Black Body case as this leakage becomes more significant.

There is a continuous transition from near perfect local thermodynamic equilibrium (LTE) deep in the photosphere to complete non-equilibrium (non-LTE) high in the atmosphere.
The Einstein A, B Coefficients

**spontaneous emission:**
System in level 2 and drops to level 1 by emitting a photon, and this can occur even in the absence of a radiation field.

**Absorption:**

\[
\int \phi(\nu) d\nu = 1
\]

\[
\bar{J} = \int_{0}^{\infty} J_\nu \phi(\nu) d\nu
\]

**Stimulated emission**
External photons with correct energy triggering emission (lasers)

Energy density sometimes \(\nu\) instead of \(J\), then definitions will differ by \(c/4\pi\).

**Relations between Einstein Coefficients**

\[
n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J} \quad \bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}
\]

in thermodynamic equilibrium, the ratio of \(n_1\) to \(n_2\):

\[
\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp(-(E + h\nu_0)/kT)} = \frac{g_1}{g_2} \exp(h\nu_0/kT)
\]

\[
\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}
\]

For this to equal the Planck fn for all temperatures we must have the following Einstein relations:

\[
g_1 B_{12} = g_2 B_{21}
\]

\[
A_{21} = \frac{2h\nu^3}{c^2} B_{21}
\]

Extensions to Kirchoff's law to include non-thermal emission that occurs when matter is not in thermodynamic equilibrium.

these connect atomic properties & have no reference to \(T\), unlike Kirchoff's law
Absorption & Emission

defining absorption and emission in terms of these coefficients
the amount of energy emitted in a volume dV, solid angle dΩ, frequency range dν, and time dt is, by definition:

\[ j_ν = \frac{hν_0}{4π} n_2 A_{21}(ν) \]

emission coefficient is:

To find the absorption coefficient, first note that the total energy absorbed in time dt in volume dV is:

\[ dV dt \ ν_0 n_1 B_{12} (4π)^{-1} \int dΩ \int dν θ(ν) I_ν \]

Therefore the energy absorbed out of the beam in frequency range dν, solid angle dΩ, time dt, and volume dV is

\[ dV dtdΩdν \frac{ν_0}{4π} n_1 B_{12} θ(ν) I_ν \]

assuming volume dν = dAdσ (cylinder)
absorption coefficient is:

\[ α_ν = \frac{hν}{4π} θ(ν) (n_1 B_{12} - n_2 B_{21}) \]

Transfer Equation, in Einstein coeffs

\[ \frac{dI_ν}{ds} = -\frac{hν}{4π} (n_1 B_{12} - n_2 B_{21}) θ(ν) I_ν + \frac{hν}{4π} n_2 A_{21} θ(ν) \]

Source fn: \[ S_ν = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \]

Using the Einstein relations:

\[ α_ν = \frac{hν}{4π} n_1 B_{12} (1 - g_1 n_2 / g_2 n_1) θ(ν) \]

\[ S_ν = \frac{2hν^3}{c^2} \left( \frac{g_2 n_1}{g_1 n_2} - 1 \right)^{-1} \]

generalised Kirchoff's law

There are 3 interesting cases: thermal emission (LTE); non-thermal emission; inverted populations (masers)
**Thermal Emission (LTE):**

If matter in thermal equilibrium with itself (not necessarily the radiation):

\[
\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu}{kT}\right)
\]

**LOCAL THERMODYNAMIC EQUILIBRIUM (LTE)**

\[
\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[ 1 - \exp\left(\frac{-h\nu}{kT}\right) \right] \phi(\nu)
\]

\[
S_\nu = B_\nu(T)
\]

Thermal value for source function is just a statement of Kirchoff's law. The addition is the correction factor, \(1 - \exp(-hn/kT)\) in the absorption coefficient which is due to stimulated emission.

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**Non-thermal Emission:**

This term covers all other cases in which

\[
\frac{n_1}{n_2} \neq \frac{g_1}{g_2} \exp\left(\frac{h\nu}{kT}\right)
\]

This occurs if the radiating particles do not have a Maxwellian velocity distribution, or if the atomic populations don’t obey the Maxwell-Boltzmann distribution law. This can also be applied when scattering is present.

For normal populations in thermal equilibrium

\[
\frac{n_2 g_1}{n_1 g_2} = \exp\left(\frac{-h\nu}{kT}\right) < 1
\]

**Inverted Populations**

\[
\frac{n_1}{g_1} < \frac{n_2}{g_2}
\]

In this case \(\alpha < 0\) and rather than decrease along the ray the intensity increases - maser - the amplification involved can be very large. \(\tau = -100\) means amplification of \(10^{8}\).