Galactic dynamics

- Stars in galaxies subject (only) the force of gravity.

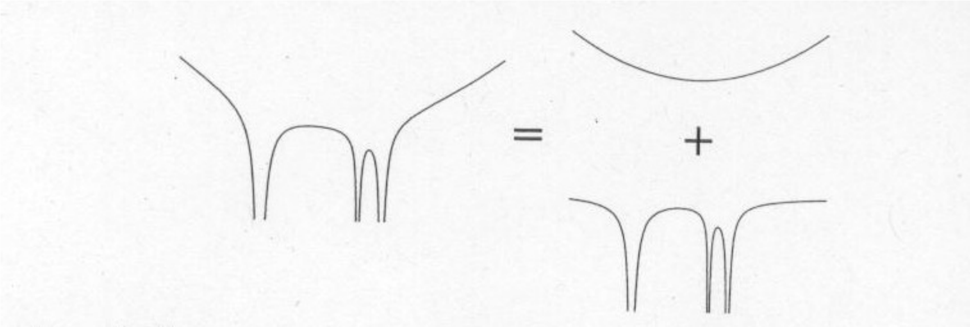
- Knowledge of the mass distribution of a galaxy allows to predict how the positions and velocities of stars will change over time.

- Conversely, we can also use the stellar motions to derive where the mass distribution.
  - Discovery of dark-matter

- Observed motions do not only tell us about mass distribution at present position of a star but along their orbit
  - e.g. measurement of the escape velocity

- We can consider the stars as point masses
  - their sizes are small compared to their separations

- The gravitational potential of a galaxy is the sum of:
  - a smooth component (the average over a region containing many stars)
  - the very deep potential well around each individual star



- The motion of stars within a galaxy is determined almost entirely by the smooth part of the force.
  - Two-body encounters are only important within dense star clusters

Motion under gravity

- Newton's law of gravity: $d(m_1\mathbf{v})/dt = -G\, m_1\, m_2\, \mathbf{r}_{12}/r_{12}^3$

- In a cluster of N stars with masses m_a, at positions $\mathbf{x}_a$

  $$d(m_b\, \mathbf{v}_b)/dt = -\Sigma_a\, G\, m_a\, m_b\, (\mathbf{x}_b - \mathbf{x}_a)\, / |\mathbf{x}_b - \mathbf{x}_a|^3$$

  (Note heavy and light stars suffer the same acceleration)

- In terms of the gradient of the gravitational potential $\Phi(\mathbf{x})$:

  $$d(m\mathbf{v})/dt = -m\, \nabla\Phi(\mathbf{x}), \text{ with } \Phi(\mathbf{x}) = -\Sigma_a\, G\, m_a/|\mathbf{x} - \mathbf{x}_a|$$

- The potential at point **x** produced by a continuous mass distribution represented by density $\rho(\mathbf{x})$

$$\Phi(\mathbf{x}) = - G \int \rho(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'|\, d^3x'$$

Essentially replaced the discrete summation by an integral, and the masses by $\rho(x)\, d^3x$

- If the potential is known, rather than the density, we obtain Poisson's equation:

$$\nabla^2\Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x})$$

  - Not all $\Phi(\mathbf{x})$ are physically meaningful: only those for which $\rho(\mathbf{x}) > 0$ everywhere (mass is always positive).
  - Note similarity to the electromagnetism and electric field: ($\nabla\Phi_e = - \mathbf{E}$) and the charge distribution ρ_e: $\nabla^2\Phi_e = - 4\pi k \rho_e$, where k is Coulomb's constant.
    ρ_e may be positive or negative: electric force can be repulsive or attractive.

Spherical systems: Newton's theorems

- Theorem 1: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

$\delta m_1 = \rho\, r_1^2\, dr_1\, d\Omega_1$ and $\delta m_2 = \rho\, r_2^2\, dr_2\, d\Omega_2$

But $dr_1 = dr_2 = dr$ and $d\Omega_1 = d\Omega_2 = d\Omega$.

Then $\delta\, m_1/r_1^2 = \delta\, m_2/r_2^2$.

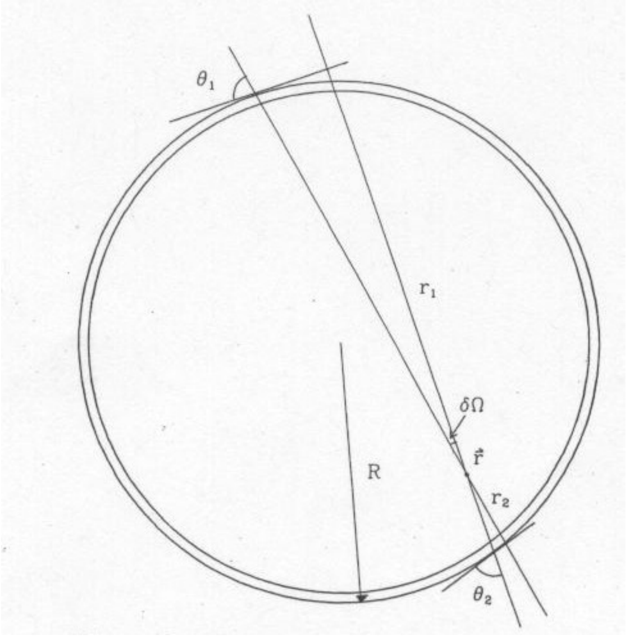
A particle M located at **r** experiences a force

$\mathbf{F} = \mathbf{f_1} + \mathbf{f_2}$ where

$\mathbf{f_1} = -\,GM\,\delta m_1/r_1^2\, \boldsymbol{\varepsilon_1}$ and $\mathbf{f_2} = -GM\,\delta m_2/r_2^2\, \boldsymbol{\varepsilon_2}$

Since $\boldsymbol{\varepsilon_1} = -\boldsymbol{\varepsilon_2}$,

then $F = -GM(\delta\, m_1/r_1^2 - \delta\, m_2/r_2^2) = 0$



Theorem 2: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shells' mass was concentrated in a point at its centre.

_The gravitational force within a spherical system that a particle feels at a radius R is only due to the mass inside that radius.

- Therefore if a star moves on a circular orbit, its acceleration is given by

$$v_c^2/r = GM(<r)/r^2$$

- For a point mass, the circular velocity $v_c^2 = GM/r$, and so $v_c / r^{-1/2}$

- Since M generally increases with radius this implies that for a spherical galaxy, the circular velocity never falls off more rapidly than the Kepler case $r^{-1/2}$.

Are collisions important in gravitational systems?

- The gravitational force is **long-range**
- In galaxies, close encounters are not important.
- The average (smoother) mass distribution determines the motion of stars

- Molecules of air or dust particles reach equilibrium through collisions, where they exchange energy and momentum.
- The forces between molecules only strong when they are very close to each other; they experience violent and short-lived accelerations, in between long periods when they move at nearly constant speeds.

- The net gravitational force acting on a star in a galaxy is determined by the gross structure of the galaxy, and not by its nearest stars.

- Example:

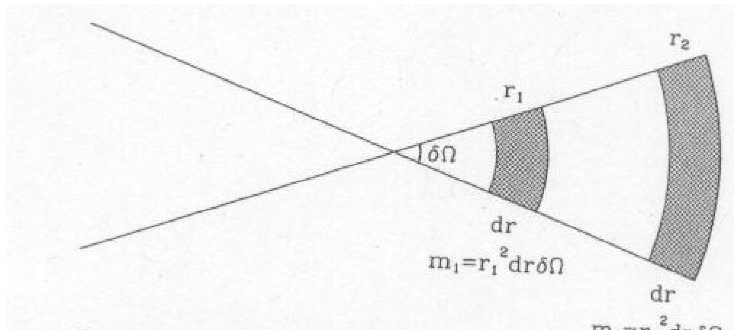


Figure 4-1. If the density of ...

- The force on a star located at the apex of the cone of constant density:

$$dF_1 = -G\, m_* \, dm_1/r_1^2 = -G\, m_* \, r_1^2 \rho \, dr \, d\Omega \, /r_1^2 = -G\, m_* \, \rho \, dr \, d\Omega,$$

while the force from the more distant shell is

$$dF_2 = -Gm_* \, dm_2/r_2^2 = -G\, m_* \, \rho \, dr \, d\Omega.$$

The force produced by shells at different distances is the same

- makes explicit the long-range nature of gravity

Encounters

Two types of encounters:

- Strong (near): the trajectory of the star changes significantly

- Weak (distant): perturbation on the initial trajectory

The timescales/frequency of these encounters tell how important they are in the dynamics of galaxies

Strong encounters

- Strong encounter: one star comes so close another that the collision completely changes its speed and direction of motion

- A strong encounter has happened when the change in potential energy is at least as large as the initial kinetic energy.
  - (equivalent to saying that the final kinetic energy has doubled)

$$Gm^2/r > m\, v^2/2, \text{ or } r < r_s = 2\, G\, m/v^2$$

- The distance r_s is the strong encounter radius.

- Near the Sun, stars have random speeds v ~ 30 km/s, and for m=0.5 $M_\odot$ r_s ~ 1 AU

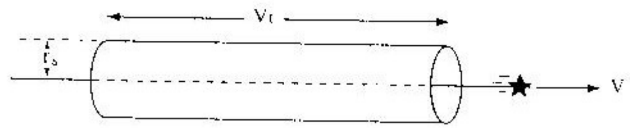


Figure 3.4 During time t, this star will have a strong encounter with any other star lying within the cylinder of radius r_s.

- Any star within a distance r_s from another star will have a strong encounter
- As the star moves, it defines a cylinder of radius r_s centered on its path. The volume of this cylinder is $\pi r_s^2 v t$
- If there are n stars per unit volume, this star will on average have one close encounter in a time t_{coll} such that $n \pi r_s^2 v t_{coll} = 1$
- The characteristic time between collisions is

$$t_{coll} \sim 1/(n\pi r_s^2 v), \text{ or } t_{coll} = v^3/(4\pi G^2 m^2 n)$$

- Normalizing to some characteristic values

$$t_{coll} \sim 4\times10^{12}\ \text{yr}\ (v/10\ \text{km/s})^3\ (m/M_\odot)^{-2}\ (n/1\ \text{pc}^{-3})^{-1}$$

- Since $n \sim 0.1\ \text{pc}^{-3}$ near the Sun, $t_{coll} \sim 10^{15}$ years (>> the age of the Universe)

- Strong encounters are only important in the dense cores of globular clusters.

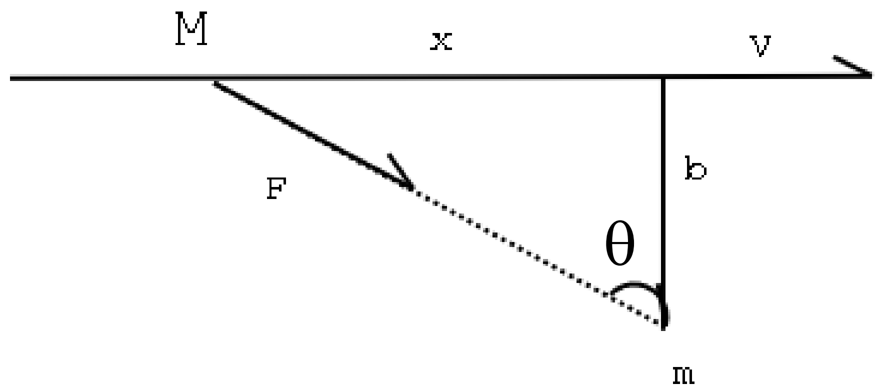
Distant weak encounters

- In a distant encounter, the force of one star on another is so weak that the stars hardly deviate from their original paths after the encounter.

- We will consider the case of a star moving through a system of N identical stars of mass m.

- We assume that
  - the change in velocity is very small: $\delta v/v \ll 1$,
  - the perturbing star is stationary

  - This is known as the impulse approximation.



- The pull by m induces a motion $\delta v_\perp$ perpendicular to the original trajectory. The force is $\mathbf{F} = - GmM/r^2\, \boldsymbol{\varepsilon}_\mathbf{r}$, and
  $F_\perp = GmM/r^2 \cos\theta$, where $r^2 = x^2 + b^2$, $\cos\theta = b/r$ and $x = vt$.

  Therefore $F_\perp = GmM/b^2\, (1 + (vt/b)^2)^{-3/2}$

- Since $M\, dv_\perp/dt = F_\perp$, the change in velocity is obtained by integrating over time.

- Finally $\delta v_\perp = 2\, Gm/(bv)$.

- Therefore the faster the star M, the smaller the perturbation is.

- Now we compute the cumulative effect of the individual encounters.

- If the surface density of stars in the system is $N/(\pi R^2)$, where R is some characteristic radius, the number of encounters dn_e with impact parameter b that a star suffers when crossing the system is

  $dn_e = N/(\pi R^2)\, 2\pi b\, db = 2 N/R^2\, b\, db$.

- Each of these encounters will produce a change in $\delta v_\perp$, but because the perturbations are randomly oriented, the mean **vector** change is zero: :

  - $\mathbf{dv}_\perp = \Sigma_i\, \delta v_\perp\, \varepsilon_\iota = 0$

- But there is a change in modulus

$$dv_\perp^2 = \Sigma_i\, (\delta v_\perp)^2\, \varepsilon_i .\, \varepsilon_i = \Sigma_i\, (\delta v_\perp)^2 = dn_e\, (\delta v_\perp)^2$$

$$dv_\perp^2 = (2\,Gm/bv)^2\, 2\,N/R^2\, b\, db$$

- Integrating this equation:

$\Delta v_\perp^2 = 8\,N\,(Gm/vR)^2 \ln\Lambda$, where $\Lambda = b_{max}/b_{min}$.

- Define n_{relax} as the number of weak encounters that a star has to experience to change its velocity by the same order as its incoming velocity

$n_{relax}\,\Delta v_\perp^2 = v^2$, or $n_{relax} = v^4 R^2/(8\,G^2 m^2\,N \ln\Lambda)$

- We can define a timescale

$$t_{relax} = n_{relax}\, R/v = v^3 R^3/ (8\, G^2 m^2\, N \ln\Lambda)$$

- This is the relaxation timescale: it estimates the timescale required for a star to change its velocity by the same order, due to weak encounters with a "sea" of stars.

- We can compare the relaxation timescale to the collision timescale derived previously: $t_{coll} = v^3/(4\pi G^2 m^2 n)$. If we use that $n \sim N/(\pi R^3)$, then

$$t_{relax} = t_{coll}/(2 \ln \Lambda)$$

The relaxation timescale is always shorter than the timescale for 2-body encounters.

- Typically $\ln \Lambda \sim 20$.
- The exact values of b_{min} and b_{max} are not very important (logarithmic dependence)
- b_{max} = system size, and $b_{min} = r_s$,
  - for example for 300 pc < b_{max} < 30 kpc, and r_s = 1 AU (near the Sun), $\ln \Lambda \sim 18 – 22$.

•For example, for an elliptical galaxy, $N \sim 10^{11}$ stars, $R \sim 10$ kpc, and the average relative velocity of stars is $v \sim 200$ km/s, then $t_{relax} \sim 10^8$ Gyr!

•This implies that when calculating the motions of stars like the Sun, we can ignore the pulls of the individual stars, and consider them to move in the smoothed-out potential of the entire Galaxy.

•For stars in a globular cluster like Ω Cen, $t_{relax} \sim 0.4$ Gyr, so relaxation will be important over a Hubble time.

The orbits of stars in spherical systems

- In a time-independent gravitational potential: energy is conserved

- In a spherical potential: angular momentum is conserved.
  - The motion of a star is restricted to the *orbital* plane

  - Only two coordinates are needed to describe the location of a star. Typically: polar coordinates in the plane (r, ϕ) to describe the motion.

Orbits of stars in an axisymmetric galaxy

- We use a cylindrical coordinate system (R,ϕ,z), where $z = 0$ corresponds to the symmetry plane (in the case of a disk: it is its mid-plane)
  - Preferred because of the symmetries of the mass distribution.
  - The disk is axisymmetric: it is independent of the angular coordinate ϕ.
    - We neglect non-axisymmetric features such as the bar, the spiral arms...
- For an axisymmetric system, the gravitational potential Φ is independent of ϕ. Therefore, $\partial\Phi/\partial\phi = 0$, and the force in the ϕ- direction is zero.
- Stars in a disk conserve angular momentum about the z-axis

- The equations of motion for a star in the disk are

$$d^2\mathbf{r}/dt^2 = -\nabla\Phi,$$

or, in each direction, and using that $\mathbf{r} = R\varepsilon_{\mathbf{R}} + z\,\varepsilon_{\mathbf{z}}$,

$$d^2R/dt^2 - R(d\phi/dt)^2 = -\partial\Phi/\partial R \qquad (1)$$

$$d^2z/dt^2 = -\partial\Phi/\partial z \qquad (2)$$

$$d(R^2\, d\phi/dt)/dt = -\,\partial\Phi/\partial\phi = 0 \qquad (3)$$

- Eq. (3) $\boxed{L_z = R^2\, d\phi/dt = \text{cst.}}$

reflects the conservation of angular momentum about z-axis

- Eq.(1) can also be written as $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$ (4)

where $\Phi_{eff} = \Phi(R,z) + L_z^2/(2\,R^2).$

- If we multiply Eq. (4) by dR/dt, and integrate wrt t, then

$$\frac{1}{2}(dR/dt)^2 + \Phi_{eff}(R,z;L_z) = \text{cst.}$$

which is like an energy-conservation law.

- The effective potential Φ_{eff} ($= \Phi(R,z) + L_z^2/(2R^2)$) behaves like a potential energy for the star's motion in R and z.

- The effective potential is constant if:

* $\partial\Phi_{eff}/\partial R = 0$ thus $\partial\Phi/\partial R - L_z^2/R^3 = 0$, and
* $\partial\Phi_{eff}/\partial z = \partial\Phi/\partial z = 0$

- The second eq. is satisfied for z=0 (since the disk is symmetric with respect to its mid-plane $\Phi(R,z) = \Phi(R,-z)$).

- In combination with $dR/dt = 0$, this implies a circular orbit in the disk-plane

- The radius of this circular orbit is R_g .where:

$$\partial\Phi/\partial R|_{R_g} = L_z^2/R_g^3 = R_g\,(d\phi/dt)^2$$

- This circular orbit is the orbit with least energy for a given angular momentum L_z.

Epicycles

- We will now derive approximate solutions to the eq. of motion for stars on nearly circular orbits in the symmetry plane (e.g. the disk).

- Define: $x = R - R_g$, and expand the effective potential around the point $(R_g,0)$:

$$\Phi_{eff}(R,z) \sim \Phi_{eff}(R_g,0) + \frac{1}{2}\, \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\, x^2 + \frac{1}{2}\, \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\, z^2 + \ldots$$

(the linear terms disappear because this expansion is performed around a stationary point of the potential).

- Let us define

$\kappa^2 = \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}$

and

$\nu^2 = \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}$

The eq. of motion become

- $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$, or $d^2x/dt^2 = - \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\ x$

$$d^2x/dt^2 = -\kappa^2 x$$

- $d^2z/dt^2 = -\partial\Phi_{eff}/\partial z$, or $d^2z/dt^2 = - \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\ z$

$$d^2z/dt^2 = -\nu^2 z$$

- These are the equations of motion of two decoupled harmonic oscillators with frequencies κ and ν.

  κ is the *epicyclic frequency* and

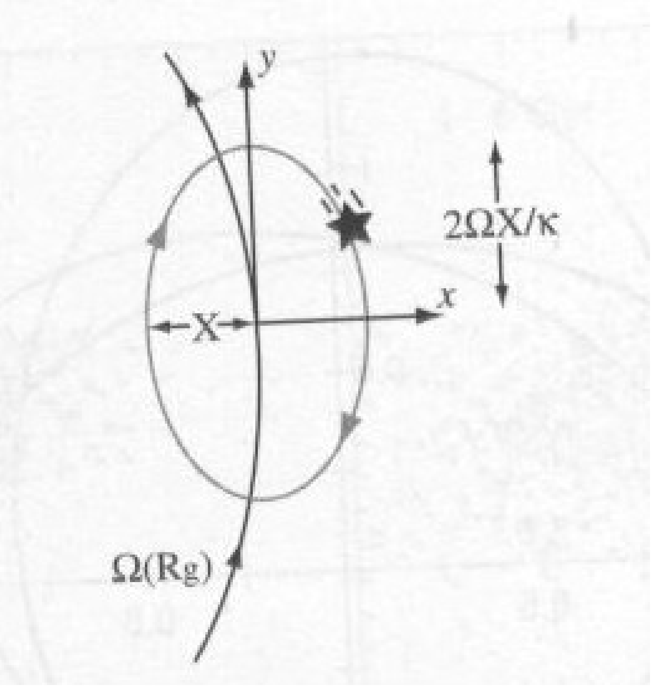
  ν as the *vertical frequency*:

$$\kappa^2(R_g) = \partial^2\Phi/\partial R^2|_{R_g,0} + 3\ L_z^2/R_g^4$$
$$\nu^2(R_g) = \partial^2\Phi/\partial z^2|_{R_g,0}$$

- The solution to the eq. of motion is

$$x = X_o \cos(\kappa t + \Psi) \quad \text{and} \quad z = Z_o \cos(\nu t + \theta) \qquad \text{for } \kappa^2 > 0.$$

The motion of a star in the disk can be described as an oscillation about a guiding center that is moving on a circular orbit.



- Note as well that

$$d\phi/dt = L_z/R^2 = \Omega(R_g)\, R_g^2/(R_g + x)^2 \sim \Omega_g(1 - 2x/R_g)$$

which can be integrated to obtain

$$\boxed{\phi(t) = \phi_0 + \Omega_g\, t - 2\, \Omega_g/\kappa\, X_o/R_g \sin(\kappa\, t + \Psi)}$$

- The first two terms give the guiding center motion.
- The third represents harmonic motion with the same frequency as the x-oscillation, but 90 deg out of phase, and with a different amplitude.

- This motion is known as the **epicyclic motion**. It is retrograde because it is in the opposite sense of the guiding centre.

- The approximation to 2nd order in z in the effective potential ($\Phi_{eff} \propto z^2$) is only valid if $\rho(z) \sim$ cst (since $\nabla^2\Phi \sim \rho$). However, the disk density decreases exponentially. This means that the approximation can at most be valid for 1 scale-height (z < 300 pc). Since a good fraction of the disk stars move to higher heights, the motion in the z-direction is not well-described as an harmonic oscillation.

- There is a relation between the epicyclic frequency κ and the angular frequency Ω:

$$\kappa^2 = [R\, d\Omega^2/dR + 4\,\Omega^2]_{R_g}.$$

This relation derives from

- $R\,\Omega^2 = d\Phi/dR$ (centrifugal force = gravitational pull)
- and $\Omega^2 = L_z^2/R^4$
- These equations can be replaced in the definition of κ

- In general $\Omega \leq \kappa \leq 2\,\Omega$. For example:
  - for a sphere of uniform density $\Omega(R)$ = cst, and $\kappa = 2\Omega$
  - for the Kepler problem (point mass), $\Omega \propto r^{-3/2}$, and $\kappa = \Omega$

- The epicyclic frequency is related to the Oort constants:

- Recall that
  - $A = - 1/2\, R\, d\Omega/dR|_{R_o}$ and $B = - (½\, R\, d\Omega/dR + \Omega)_R$, where R_o is the location of the Sun, and Ω the angular frequency of the LSR motion.

- Therefore, at the Sun $\kappa_o^2 = - 4\, B(A – B) = - 4\, B\, \Omega_o$

- Using the measured value of B, we find that

$$\kappa_o/\Omega_o \sim 1.3 \pm 0.2$$

Therefore the Sun makes 1.3 radial oscillations in the time it takes to complete one revolution around the Galactic centre.