Exercises of Dynamics of Galaxies 22/05/2014

Exercise 1 (Problem 3.15 from Binney & Tremaine)

Using Poisson's equation for an axisymmetric system

$$\frac{\partial^2 \Phi}{\partial z^2}(R, z) = 4\pi G \rho(R, z) - \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial \Phi}{\partial R}(R, z)\right), \tag{1}$$

prove that at any point in an axisymmetric system at which the local density is negligible, the epicycle, vertical and circular frequencies are related by $\kappa^2 + \nu^2 = 2\Omega^2$.

Exercise 2 (Problem 3.18 from Binney & Tremaine)

Let $\Phi(R, z)$ be the Galactic potential. At the solar location, $(R, z) = (R_0, 0)$, prove that

$$\frac{\partial^2 \Phi}{\partial z^2}(R_0, 0) = 4\pi G \rho_0 + 2\left(A^2(R_0) - B^2(R_0)\right),$$

where $\rho_0 = \rho(R_0, 0)$ and Oort's constants $A(R)$ and $B(R)$ are defined

$$A(R) = \frac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right), \quad B(R) = -\frac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right).$$

Hint: use equation 1.

Exercise 3

For a distribution function of the form

$$f(\mathcal{E}) = \begin{cases} F\mathcal{E}^{n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0), \end{cases}$$

the density depends on the potential as

$$\rho = c_n \Psi^n \quad (\Psi > 0),$$

with c_n a constant, and is 0 otherwise. Imposing that the problem is self-consistent, Poisson's equation becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) + 4\pi G c_n \Psi^n = 0.$$

Show that

- $\rho \propto r^{-\alpha}$ is a solution of this Poisson's equation, with $\alpha = 2n/(n-1)$,
- that the correspondent potential goes with the radius as $\Psi \propto r^{-\alpha/n}$,
- that the mass contained within radius r goes as $M(r) \propto r^{1-\alpha/n}$,
- that the circular velocity goes as $v_c(r) \propto r^{-\alpha/n}$.

Exercise 4 (Problem 4.11 from Binney & Tremaine)

Prove that the following DF generates a stellar system in which the density distribution is that of a homogeneous sphere of density ρ and radius a:

$$f(E,L) = \frac{9}{16\pi^4 G\rho a^5} \frac{1}{\sqrt{L^2/a^2 + \frac{4}{3}\pi G\rho a^2 - 2E}} \qquad \left(L^2 < \tfrac{4}{3}\pi G\rho a^4\right) \ .$$

Here it is understood that $f = 0$ when the argument of the square root is not positive, the DF is normalized so that $\int d^3x d^3v f = 1$, the potential $\Phi = 0$ at $r = 0$, and the system is isolated (Polyachenko & Shukhman 1973).

Hint: Let v_r and $\vec{v}_t$ (for tangential) be the components of $\vec{v}$ parallel and perpendicular to the radial direction, so $v_t^2 = v_\theta^2 + v_\phi^2$ in spherical coordinates (r, θ, ϕ). Then $L = rv_t$ and $E = \frac{1}{2}(v_r^2 + v_t^2) + \Phi(r)$. An integral over all possible velocities then becomes:

$$\int d^3v = \int_{v_{r,min}}^{v_{r,max}} dv_r \int_0^{v_{t,max}} 2\pi v_t \ ,$$

where the integration bounds can be found from the aforemention conditions where $f = 0$.