Exercises for Dynamics of Galaxies, 14/05/2014

Exercise 1

In a spherical potential

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = \frac{1}{L^2 u^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r}(1/u),$$

where $u \equiv 1/r$, Φ is the potential, L the magnitude of the angular momentum and (r, ψ) the polar coordinates in the plane of the orbit.
Consider the Kepler potential

$$\Phi = -\frac{GM}{r}.$$

Show that, for this potential, the form of the orbit is

$$r(\psi) = \frac{a(1-e^2)}{1 + e\cos(\psi - \psi_0)},$$

where the eccentricity e is defined:

$$e \equiv \frac{CL^2}{GM},$$

a is the semi-major axis

$$a \equiv \frac{L^2}{GM(1-e^2)},$$

and C and ψ_0 are constants of motion. Show also that, in the case $e < 1$, the radii of pericenter and apocenter, r_1 and r_2, are

$$r_1 = a(1-e) \quad \text{and} \quad r_2 = a(1+e),$$

and that the periods of the orbit are

$$T_r = T_\psi.$$

What does it mean that the radial and azimuthal periods are always equal?

Given that

$$2\frac{\mathrm{d}A}{\mathrm{d}t} = L,$$

(where $\mathrm{d}A/\mathrm{d}t$ is the “areolar velocity”), that the semi-minor axis is

$$b = a\sqrt{1-e^2},$$

and that the area of an ellipse of axis a and b is

$$A = \pi ab,$$

demonstrate that the periods of the orbit are

$$T_r = T_\psi = 2\pi\frac{a^2}{L}\sqrt{1-e^2} = 2\pi\sqrt{\frac{a^3}{GM}}.$$

Exercise 2 (Problem 3.6 from Binney & Tremaine)

A star orbiting in a spherical potential suffers an arbitrary instantaneous velocity change while it is at pericenter. Show that the pericenter distance of the ensuing orbit cannot be larger than the initial pericenter distance.