Exercises for Dynamics of Galaxies, 30/4/2014

Exercise 1

The relaxation time for stellar systems may be defined as

$$t_{\text{relax}} = n_{\text{relax}} t_{\text{cross}} \ ,$$

where

$$n_{\text{relax}} \simeq \frac{N}{8 \ln \Lambda} \ ,$$

with N the number of particles of the system. The crossing time is

$$t_{\text{cross}} = R/v \ .$$

In the Coulomb logarithm, Λ is

$$\Lambda \approx \frac{Rv^2}{Gm},$$

where m represents the typical mass of a star of the system.

Question: The speed v represents a typical speed in the system. What can you use as a rough estimate of it?

Question: Taking $m = M_\odot$, calculate the relaxation time of a galaxy like the Milky Way, a typical globular cluster and a typical open cluster using:

1. for the galaxy: $N \approx 10^{11}$, $R \approx 10$ kpc,
2. for the globular cluster: $N \approx 10^5$, $R \approx 5$ pc,
3. for the open cluster: $N \approx 100$, $R \approx 5$ pc.

Given a lifetime for the galaxy, the globular cluster and open cluster of respectively $\approx$ 10Gyr, 10Gyr and 100Myr, which one of these systems is collisionless?

Exercise 2 (Problem 2.1 from Binney & Tremaine)

Question: Show that the gravitational potential energy of a spherical system of finite mass in which the density satisfies

$$\lim_{r \to 0} \rho(r)\, r^{5/2} = 0,$$

can be written

$$W = -\frac{G}{2} \int_0^\infty dr \frac{M^2(r)}{r^2},$$

where

$$M(r) \equiv \int_0^r dr' 4\pi \rho(r') r'^2,$$

is the mass interior to radius r.
Hint: start from the expression for the potential energy of a spherical body and integrate by parts.

$$W = -4\pi G \int_0^\infty dr r \rho(r) M(r).$$

Exercise 3

Many galaxies have luminosity profiles that approximate a power law over a large range in radius. Consider the structure of a system in which the mass density drops off as some power of the radius:

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{\alpha}.$$

Question: Show that:

1. The mass interior to r, $M(r)$ is

$$M(r) \equiv \int_0^r dr' 4\pi\rho(r')r'^2 = \frac{4\pi\rho_0 r_0^{\alpha}}{3-\alpha} r^{3-\alpha},$$

2. The circular speed is

$$v_c^2(r) = \frac{GM(r)}{r} = \frac{4\pi G\rho_0 r_0^{\alpha}}{3-\alpha} r^{2-\alpha},$$

3. The potential difference between radius r and the reference radius r_0 is

$$\Phi(r) - \Phi(r_0) = G\int_{r_0}^{r} dr' \frac{M(r')}{r'^2} = \begin{cases} \dfrac{v_c^2(r_0) - v_c^2(r)}{\alpha - 2} & \text{for } \alpha \neq 2 \\ v_c^2 \ln(r/r_0) & \text{for } \alpha = 2, \end{cases}$$

4. The escape speed $v_e(r)$ from radius r, for $\alpha > 2$, is given by

$$v_e^2(r) \equiv 2[\Phi(\infty) - \Phi(r)] = 2\frac{v_c^2(r)}{\alpha - 2}.$$

Question: What happens to these quantities changing the value of the parameter α? Which expressions still "make sense"?

Exercise 4

Logarithmic potentials provide a simple model for a flat rotation curve at large radii. Consider the axisymmetric Logarithmic potential defined by:

$$\Phi_L(R, z) = \frac{1}{2} v_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right) + \text{constant},$$

where R_c and v_0 are constants, and q_Φ is the axis ratio of the equipotential surfaces. The density distribution to which Φ_L corresponds is

$$\rho_L(R, z) = \frac{v_0^2}{4\pi G q_\Phi^2} \frac{(2q_\Phi^2 + 1)R_c^2 + R^2 + (2 - q_\Phi^{-2})z^2}{(R_c^2 + R^2 + q_\Phi^{-2} z^2)^2}.$$

The flattening of the potential is represented by $1 - q_\Phi$.

Question: If we define the axial ratio q_ρ of the isodensity surfaces by the ratio z_m/R_m of the distances down the z and R axes at which a given isodensity surface cuts the z axis and the $x - y$ plane respectively, show that, in the limit $r \gg R_c$ we find

$$q_\rho^2 \simeq q_\Phi^4 \left(2 - \frac{1}{q_\Phi^2}\right) \quad (r \gg R_c).$$

Question: What happens in these regions to the density for $q_\Phi < 1/\sqrt{2}$?
Hint: Express the density as a function of R/R_c and z/R_c.