university of groningen
faculty of mathematics and natural sciences
kapteyn astronomical institute

Klein onderzoek

Power spectrum analysis of galaxy potentials using strong lensing events

Report

Author:
Sander Bus

Supervisor:
Prof. dr. L.V.E. Koopmans

February 2, 2012

Abstract

The theory of gravitational lensing tells us how light travels in curved space time. Strong gravitational lensing can be used to quantify (dark) mass-substructure in galaxies. To find the scales on which these structures appear, we make a power spectrum analysis of strongly lensed images. The steps taken are: The non-linear lens equation is linearized, which confines its application to slow varying, single images of lensing events. The main equation derived is the power spectrum of how the image changes, the residuals, due to a change in the lens potential $\delta\psi$, which physically is mass substructure in the lens. If a best model for the lens potential is known, we take a correction on the lens potential $\delta\psi$ to second order. Observational effects are implemented, using a point spread function for smearing of the image and a window function for the limited field of view. We wish to have a signal-to-noise of at least 1, due to unknown systematic effects. In the S/N>1 regime the errors are sample dominated rather than noise dominated. Our results are the power spectrum of the residuals, as a function of mass substructure, its uncertainty and observational effects on this power spectrum. Future research must implement these results into a numerical method to be applied on data.

Contents

1 Introduction **1**
1.1 Short History 1
1.2 Physical importances 2

2 Problem setting **4**

3 Basics of gravitational lensing **5**

4 Power spectrum of image residuals **13**
4.1 Important formulas and definitions 13
4.2 The derivation of the residual's power spectrum 15

5 Error analysis **23**
5.1 How to model these processes? 23
5.2 Observed power spectrum 27
5.3 Error on measurement 29
5.4 A toy model 31

6 Conclusion **36**

7 Future research **38**

8 Acknowledgements **39**

9 References **40**

A Derivation by Hu **42**

B Getting to know lensing events **48**
B.1 Visual 48
B.2 Analytic 55

C Deriving the RMS **59**
C.1 Source: Cosine, lens potential: Cosine 62
C.2 Source: Gauss, lens potential: Cosine 63
C.3 Source: Cosine, lens potential: Gauss 66
C.4 Source: Gauss, lens potential: Gauss 67

D Plotting code in mathematica **75**
D.1 The effect of the PSF in Fourier space: Figure 10 75
D.2 Window function - figure 11(b) 75
D.3 Effect of the window function - figure 12 75
D.4 Original power spectrum and with PSF - figure 14 75
D.5 Window and PSF effect on power spectrum - figure D.5 76
D.6 Cosine source and a cosine lens potential - figure B.1.1 76
D.7 Gaussian source and a cosine lens potential - figure 17 76
D.8 Cosine source and a Gaussian lens potential - figure 18 77
D.9 Gaussian source and a Gaussian lens potential - figure 19 78

1 Introduction

This is a report on the research project or "Klein Onderzoek" done as the final part of the Bachelor phase at the Kapteyn Astronomical Institute in Groningen, The Netherlands. This project is theoretical in nature and deals with gravitational lensing. Especially with the influence a correction on a lens potential has on an image. Physically such a correction is mass substructure in the lens potential. Mass substructure can both be visible and dark matter. This first section introduces gravitational lensing by giving a brief historical overview and by showing why gravitational lensing is a useful tool for astronomers. In the second section the problems dealt with in this paper are sketched. The third section deals with the theory of gravitational lensing, starting from how much a lightray appears to deviate from a straight line due to a point mass, via the lens equation to the magnification matrix. With these introductions to the theory, the reader should have a firm foundation to understand the rest of the report. In the fourth section the true work starts, by linearization of the lens equation and by deriving the power spectrum of the first order difference between the real lens potential and the best model for the lens potential. The fifth section is on the physical limitations in observing lensing events, consisting of an estimate for errors present in the power spectrum and on how the power spectrum changes due to atmosphere and telescope. The sixth section summarizes the results found in this project. The final section gives an overview of steps which can be taken to expand and improve this research project in the future.

1.1 Short History

Gravitational lensing is the deviation of light from traveling in a straight line due to gravity. Many minds already thought about this idea, for example Johann Soldner [20], but it was Einstein who established it, with the theory of general relativity he published in 1916 [14]. His theory predicts that a light ray is deviated by an angle

$$\hat{\alpha} = \frac{4Gm}{c^2 b}$$

due to the gravitational field, induced by a point mass m. The impact parameter b is the closest, the lightray comes to the pointmass [1].

Not long after Einstein's prediction, Eddington showed that light is indeed deflected by the sun's gravitational field. He did this discovery during the solar eclipse of 1919 [19]. In 1936 Einstein published some notes in Science on gravitational lensing of the light of star A by a star B [18]. He suggested that the image of star A is magnified and that multiple images of star A can form. Furthermore, if the aligment of both stars with the observer is perfect, a continuous ring of images will form around star B. Even infinite magnification would be possible, according to Einstein. He conluded that all these effects would not be observed due to small probabilities and angles too small to be resolved by telescopes. A year later Fritz Zwicky predicted that these effects would certainly be visible, not by stars but by what are now known as galaxies [16],[17]. Zwicky predicted that mass estimates could be made, images of sources would be magnified and that finding a lensed image was a certainty. Unfortunately none were found for many decades. Not until the 1960s new interest was shown in gravitational lensing, with the most important event being the discovery of an

Figure 1 – The doubly imaged quasar Q0957+561 A,B, indicated by bars, together with the galaxy NGC 3079 [29]. This was not the image used to identify the two quasars as different images of the same source. The image was made by amateur astronomer Brian Peterson, reproduced with permission.

actual multiple imaged source in 1979 [15]. See figure 1 for this doubly imaged quasar Q0957+561 A,B, indicated by bars, together with the galaxy NGC 3079.

1.2 Physical importances

Gravitational lensing has multiple applications in astronomy. An important application is measuring mass and mass distribution of objects which act as lenses. These are most prominently stars, galaxies and clusters of galaxies. Most matter in the universe is believed to be dark matter [5], which is hard to detect directly. The bending of light due to gravitational lensing however, is independent of the nature of the matter. The dark matter distribution in a galaxy can thus be found by looking at how it deflects the light of a background source. These mass estimates can have an accuracy of about a percent, which is unprecedented in astronomy.

Lensing can be used to study multiple interesting subjects, which all effect the way the light reaches the observer. The three major subjects are summerized below.

Studying the lenses
Looking at lensed images can reveal the mass and the mass distribution of an extended object, as discussed above. Furthermore there might be unseen objects in our Milky Way, so called MACHOs (MAssive Compact Halo Objects). These objects might be a source of dark matter. They trigger so called micro lensing events, which is strong lensing by light, nearby objects. Micro lensing

is done statistically. When more microlensing events are observerd, than objects are visible, then MACHOs are likely to be present [1],[5]. These micro lensing events check if there is any mass, one can also look at how the mass is distributed in the lens. In this report we will use this feature to get information about substructure in lens potentials.

Studying the sources
The magnification property of lensing allows observers to see objects, which would have been to faint to see under normal circumstances. These natural telescopes allow the observation of high redshift objects. The furthest known objects are lensed objects [1].

Studying the cosmology
Light which has traveled a long time through space time will be influenced by the cosmology. Space is expanding by the Hubble parameter H_0 [5]. Light which has traveled through space is redshifted through this expansion. If one is able to compare two rays from a single source, but which have followed a different route through space time, one can derive the Hubble constant H_0. The measurements of cosmological parameters using lensing has turned out to be troublesome, due to degeneracies between the time delay and H_0 [1], [24].

In short gravitational lensing has many applications and therefore it is and will be a very useful tool for astronomers.

2 Problem setting

This report deals with the problem how strong gravitionally lensed images are affected, by changes in the lens' mass distribution. We are interested in this, because ΛCDM cosmology predicts mass substructure to be present in galaxies, in the form of dark matter [5]. We can not see this dark matter directly. However, it does affect gravitationally lensed images. Using extended images of lensed background sources, which were lensed by a foreground galaxy, we want to get information about the mass features of this foreground galaxy.

In this project we are interested in the scales present is these mass features. Scales can be represented by power spectra. We want to find a relation between the different power spectra present in the process. These power spectra re of the surface brightness profile of the source, of the features in the mass distribution of the galaxy, of the features in the surface brightness profile of the lensed image and of the total mass distribution of the galaxy. Applying the relation we want to derive will mean extracting information from extended lensed images, which is the only power spectrum, we will be able to observe.

Extended images of lensed objects contain a lot of information. Each resolution element is a piece of information, on the location on the sky (x, y) and on the luminosity. The problem is how to extract information on the lens' mass distribution, encoded in the lensed images. First of all one needs to construct the original image from the data, this will reduce the amount of information available to find the mass features in the lens, which we are interested in.

The features in the lensed image are extracted by subtracting a modeled image from the real image. This process is described in figure 2. An example of a lensed image is given in the upper right image of figure 2. To first order lenses can be described well as a smooth lens model. Smooth meaning that the lens potential can be described by a few parameters. Letting a smooth source be lensed by this lens model will result in a smooth modeled image. The modeled source is shown in the upper left image of 2, the resulting modeled image is given in the lower left. The differences between the true image and the modeled image are rather small in most cases. These residuals, lower right of figure 2, can not be described by a simple function of a few parameters. The remaining features in the surface brightness can be related to features in the surface density of the lens.

In this report we try to relate the surface brightness features to the surface density features through power spectrum analysis. To be able to make meaningful interpretations of these features, we need to know what effects are induced by observing. The main effects are noise, smearing of images due to equipment and atmosphere and the limited field of view. These considerations are useful for making observational strategies.

In section 3 we introduce the basics concerning gravitational lensing. In section 4 we derive the relation between the surface brightness in the residuals and the surface density in the lens. This is done by linearizing the lens equation, to be able to do the calculations analytically. Section 5 concerns the error analysis aspects noted above, especially its effects on the power spectrum of the images. This is done by deriving an equation to combine the power spectrum derived in section 4 with observing effects. We end with some conclusions in section 6 and suggestions to future work in section 7.

3 Basics of gravitational lensing

Gravitational lensing is geometric optics in curved space time. In this section we explain the basics of the theory of gravitational lensing. Geometric optics in curved space time means that light rays appear to deviate from a straight line due to the curvature of space time around a massive object. The deflection angle of a light ray passing a point mass m at a distance b, is given by

$$\hat{\alpha} = \frac{4Gm}{c^2 b} \tag{1}$$

Extended objects can be seen as an infinite sum of point masses. The deflection angle can thus be found by integrating over the combined effect of all the point

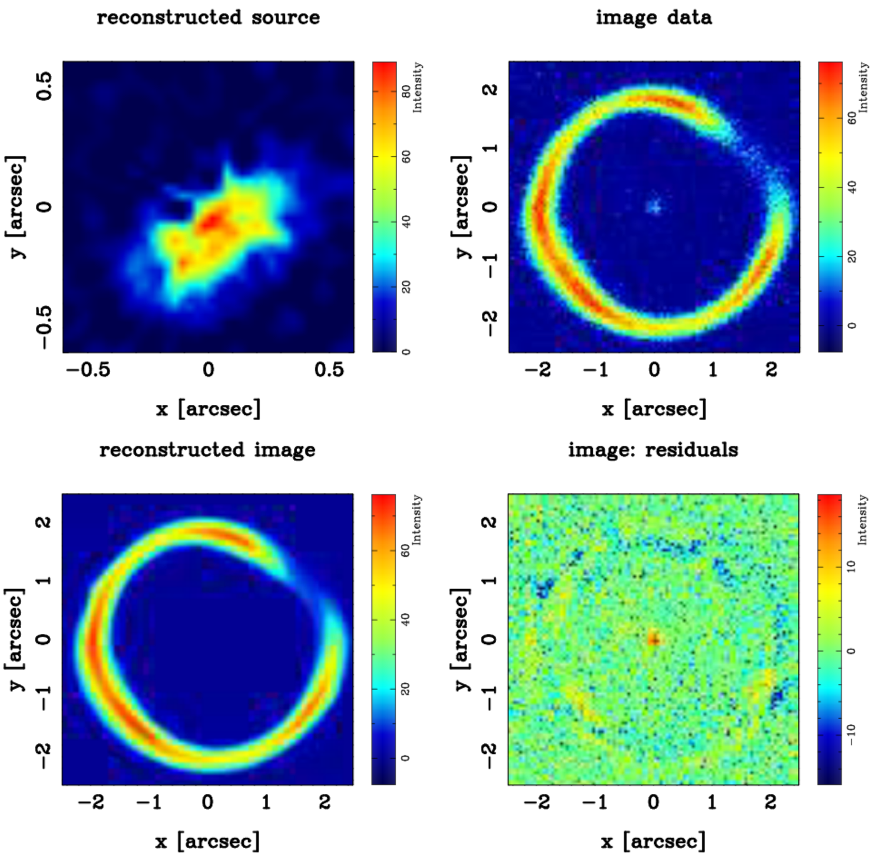


Figure 2 – The upper right image is a general example of an observed lensed image. The lower left image is a best model for the observed image. The difference between the true observation and the model is given by the residuals, the lower right image. The best guess for the source using a model for the lens and the observed image is given in the upper left image. These data were simulated by M. Barnabè et al. [21]

masses

$$\hat{\alpha}(\xi) = \frac{4G}{c^2} \int d^2\xi' \Sigma(\xi') \frac{\xi - \xi'}{|\xi - \xi'|^2} \tag{2}$$

where ξ is the vector describing the position where the light ray interacts with the surface mass density of the lens $\Sigma(\xi')$ and $|\xi - \xi'|$ is the impact parameter. The surface mass density is the density integrated along the line of sight.

There are some approximations in the description made here. The first is that the gravitational field should be weak. The light paths close to a black hole do not obey this approximation and need the full general relativistic treatment to be solved. Due to the weak gravitational field, light rays will not deviate much from a straight line. This is called the small angle approximation. The lensing object is often assumed to be much smaller than the distances involved. This is called the thin lens approximation and is the second assumption. In other words a light ray is assumed to bend at one point in the plane of the lens. In most cases this is a valid approximation. This means that the density of the lens can be collapsed into a surface density by integrating along the line of sight.

The best way to find where images form due to lensing is by studying the geometry between the source, the lens and the observer. Figure 3 shows a schematic picture of a gravitational lens. The derivation of the so called lens equation will be done with two dimensionsal vectors, working in planes, since all the action in gravitational lensing takes place in planes, due to the thin lens approximation.

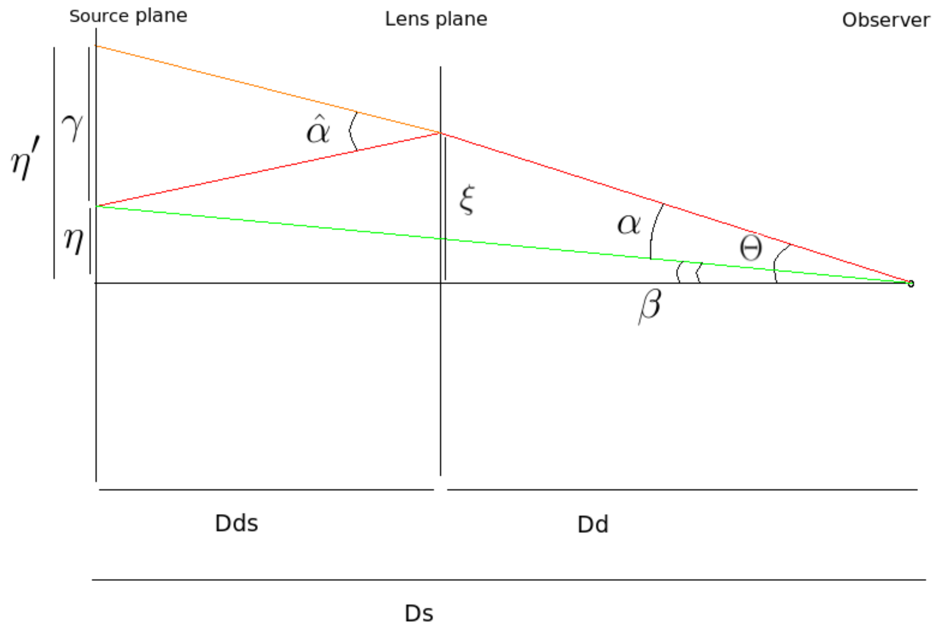


Figure 3 – A 1 dimensional schematic view of a gravitational lens

In figure 3 the lens, for example a galaxy, is in the lens plane at the origin. The source is in the source plane at position η. Light from the source would go in a straight line to the observer under an angle β, if lensing would not occur.

However, if there is lensing a light ray will be bent by an angle $\hat{\alpha}$. Resulting in redirection of light rays, one of these light rays will reach the observer under an angle θ. The distances between the observer and the lens D_d, between the observer and the source D_s and between the lens and the source D_{ds} are angular diameter distances, known from cosmology. To be able to use these distances, the universe should be homogeneous and isotropic, which are the third and fourth assumptions needed for the lens theory.

To derive the lens equation, some equalities are needed

$$\vec{\Theta} \simeq \tan(\vec{\Theta}) = \frac{\vec{\eta'}}{D_s} \tag{3}$$

$$\vec{\beta} \simeq \tan\vec{\beta} = \frac{\vec{\eta}}{D_s} \tag{4}$$

$$\vec{\hat{\alpha}} \simeq \tan\vec{\hat{\alpha}} \simeq \frac{\vec{\gamma}}{D_{ds}} \tag{5}$$

These equalities are allowed due to the small angle approximation.

With identities 3-5, it is easy to see that

$$\vec{\eta} = \vec{\eta'} - \vec{\gamma} \Rightarrow \vec{\beta}D_s = \vec{\Theta}D_s - \vec{\hat{\alpha}}D_{ds} \tag{6}$$

$$\Rightarrow \vec{\beta} = \vec{\Theta} - \vec{\hat{\alpha}}\frac{D_{ds}}{D_s} \tag{7}$$

If we define the reduced deflection angle α:

$$\vec{\alpha}(\vec{\theta}) \equiv \vec{\hat{\alpha}}\frac{D_{ds}}{D_s} \tag{8}$$

We get the lens equation

$$\vec{\beta} = \vec{\Theta} - \vec{\alpha}(\vec{\Theta}) \tag{9}$$

The lens equation is the most important equation in gravitational lensing and all processes we will describe will depend on it. One must remember that the lens equation is only applicable for weak gravitional fields.

In general the lens equation is non-linear. There can be multiple $\vec{\Theta}$'s which satisfy the lens equation for a given $\vec{\beta}$. This means that geometries can occur in which multiple images of the same source form. This results in the defining difference between strong gravitational lensing and weak gravitational lensing. The formation of a single image by lensing of a single source is called weak lensing and the formation of multiple images is called strong lensing.

To make the lens equation easier to handle, it is useful to write it in dimensionless units. To make the lens equation dimensionless one must look at a disk of constant surface mass density Σ. Since it is symmetric and constant we can see the deflection of this mass distribution as if the mass was concentrated in a point at the center. This is called Gauss' law for gravitation [8].

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\frac{4G}{c^2\vec{\xi}}M = \frac{D_{ds}}{D_s}\frac{4G}{c^2\vec{\xi}}\Sigma\pi\vec{\xi}^2$$

Using $\vec{\xi} = \vec{\Theta} D_d$

$$\vec{\alpha}(\vec{\theta}) = \frac{\Sigma}{\left(\frac{c^2 D_s}{4\pi G D_d D_{ds}}\right)} \vec{\Theta}$$

one defines the critical surface density as:

$$\Sigma_{crit} \equiv \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} \tag{10}$$

The critical surface density is the minimum density needed to get multiple images. The galaxies in the vicinity of the Milky Way have surface densities close to this critical value. The lens equation becomes

$$\vec{\beta} = \vec{\Theta} \left[1 - \frac{\Sigma}{\Sigma_{crit}} \right] \tag{11}$$

There are three regimes:

- $\Sigma < \Sigma_{crit}$: the density is lower than the critical density, which is too low to form multiple images, so there will only be one image visible.
- $\Sigma = \Sigma_{crit}$: the density is exactly the critical density, this is essentially in a perfectly focussing lens with the observer in the focal point. Due to the focussing there will be great magnification of the source.
- $\Sigma > \Sigma_{crit}$: the density is higher than the critical density, this results in an over focusing lens. Which causes multiple images to form from a single source.

The convergence, a dimensionless parameter, is defined by

$$k \equiv \frac{\Sigma}{\Sigma_{crit}} \tag{12}$$

The critical surface density only depends on the chosen cosmology, i.e. the positions of the source, the lens and the observer. The evolution of a source getting closer and closer to alignment with the lens and the observer is shown in figure 4. In this figure a point-mass like lens was used, which always results in two images.

The convergence can be used to make the deflection angle dimensionless :

$$\vec{\alpha}(\vec{\Theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} d^2\vec{\Theta}' k(\vec{\Theta}') \frac{\vec{\Theta} - \vec{\Theta}'}{|\vec{\Theta} - \vec{\Theta}'|^2} \tag{13}$$

Going one step further with defining the deflection potential:

$$\psi(\vec{\Theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} d^2\vec{\Theta}' k(\vec{\Theta}') \ln(|\vec{\Theta} - \vec{\Theta}'|) \tag{14}$$

using

$$\vec{\nabla} \ln(|\vec{\Theta}|) = \frac{\vec{\Theta}}{|\vec{\Theta}|^2}$$

one finally obtaines

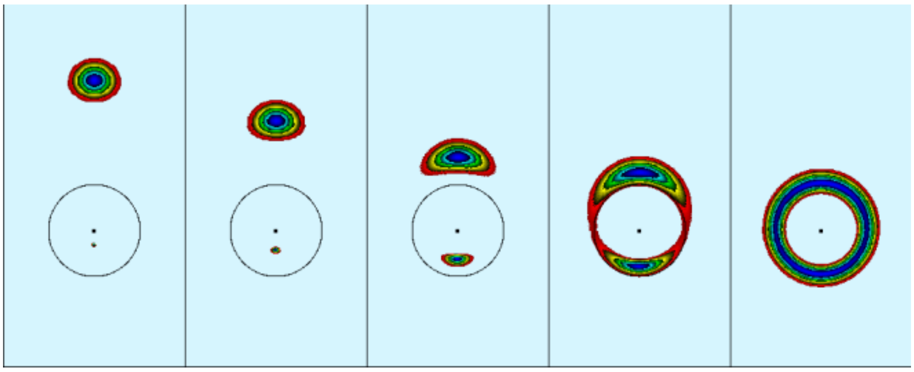


Figure 4 – Evolution of a source getting closer to the observer-lens line. From left to right: the source gets closer and closer to the lens-observer line, until in the uttermost right figure source, lens and observer are perfectly aligned. The circle visible in the left three images is drawn at the Einstein radius from the center of the lens, given by the dot. The ring seen in that figure is called an Einstein ring. The figure was taken from a "Living review in relativity" on gravitational lensing by Joachim Wambsganss [30]

$$\vec{\alpha}(\vec{\Theta}) = \vec{\nabla}\psi(\vec{\Theta}) \tag{15}$$

The deflection angle is now given by the gradient of a dimensionless deflection potential.
One can make the lens equation even more dimensionless using the so called Einstein radius Θ_E. The Einstein radius Θ_E is defined as the radius for which $\beta = 0$ in the lens equation

$$\vec{\alpha}(\vec{\theta}) = \vec{\theta} \quad \Rightarrow \quad \frac{4GM}{c^2\Theta D_d}\frac{D_{ds}}{D_s} = \Theta$$

so

$$\Theta_E = \left(\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}\right)^{1/2} \tag{16}$$

where the normalization is done as follows $\vec{y} = \frac{\vec{\beta}}{\Theta_E}$ and $\vec{x} = \frac{\vec{\Theta}}{\Theta_E}$.

Physically the Einstein radius is the typical distance of gravitationally lensed images from the center of the lensing mass. The Einstein radius can be totally covered with images, called an Einstein ring, when the source, lens and observer are perfectly aligned, see the utter most right image of figure 4. In figure 5 one can see a real lensed image, displaying an almost closed Einstein ring, which means the source, the lens and the observer were almost perfectly aligned.
In this notation $\vec{\alpha}$ can be written as:

$$\vec{\alpha}(\vec{x}) = \vec{\nabla}\psi(\vec{x}) \tag{17}$$

were ψ is the potential causing the lensing effect.
The lens equation then becomes:

$$\vec{y}(\vec{x}) = \vec{x} - \vec{\nabla}\psi(\vec{x}) \tag{18}$$

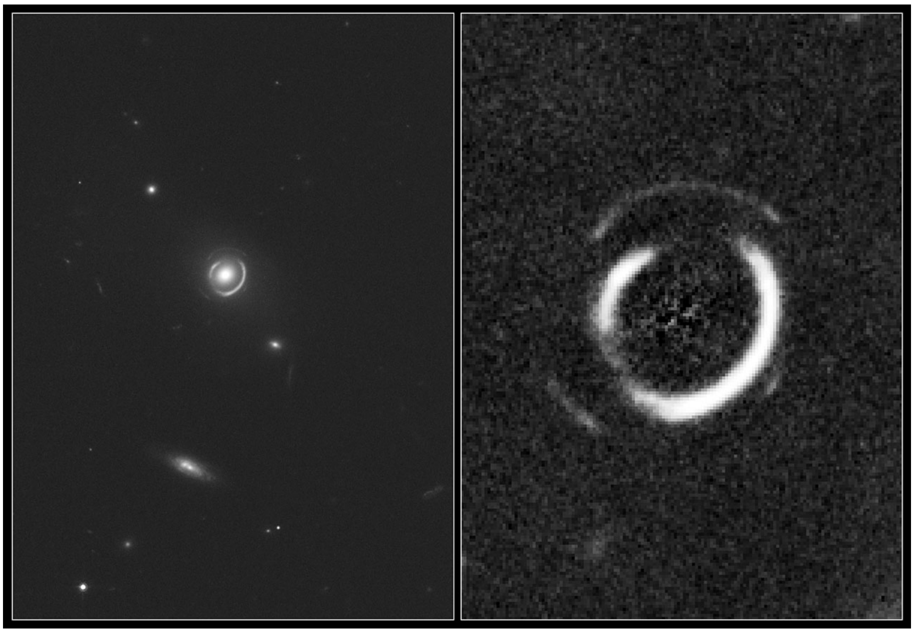


Figure 5 – A double Einstein ring discovered by the Sloan Lens ACS Survey [22]. The rings are not perfectly closed, because the alignment was not perfect between the source(s), the lens and the observer[27].

An important property of gravitational lensing, which is also true for lenses made out of glass, is that the surface brightness is conserved [1]. This is due to Liouville's theorem on phase space conservation [7].

Lenses do not only change the path of light rays, lensing also causes distortions of images. Because each individual point in an extended image has a different deflection angle in the lens equation, the lens equation is a non-linear equation and causes images to be distorted. Figure 5 shows an example. To first order the distortions of the images due to gravitational lensing can be described by the Jacobi matrix **A**. This is under the assumption that the deforming is locally linear. This is true if the source is much smaller than the angular scale on which the lens changes properties.

$$A_{ij} = \frac{\partial y_i}{\partial x_j} = \delta_{ij} - \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} \tag{19}$$

In matrix notation this becomes

$$\mathbf{A} = \begin{pmatrix} 1 - \psi_{,11} & -\psi_{,12} \\ -\psi_{,21} & 1 - \psi_{,22} \end{pmatrix} \tag{20}$$

Where $\psi_{,i}$ denotes $\frac{\partial \psi}{\partial x_i}$.

The inverse of this Jacobian is the magnification tensor. The magnification tensor gives the coordinate transformation from the lens plane to the source plane.

$$\mathbf{M} = \mathbf{A}^{-1} = \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \mu \begin{pmatrix} 1 - \psi_{,22} & \psi_{,21} \\ \psi_{,12} & 1 - \psi_{,11} \end{pmatrix} \tag{21}$$

where μ is defined as $1/\det(A)$.
If $\mathbf{A}$ is not invertable, i.e. $\det(A) = 0$, μ will go to infinity. This means there will be infinite magnification. In reality this will not happen, due to the fact that sources are never perfect points. The magnification will become extremely large at these so called critical points. The critical points form a closed line on the source plane, called the critical curve. When a source moves over such a critical curve two images will either be formed or destroyed. More information on critical curves and their relation to formation and destruction of images can be found in [1].

The goal of this paper is to linearize the highly non-linear lens equation. The behaviour near the critical curve is too non-linear to be linearized. This means the regime of application of the linearization will not include the region close to crititcal curves. This can be solved by looking at individual images of a source. For instance if four images are visible of one source, one can only consider 1 image at a time in this approximation. This is visualized in figure 6. To summerize, taking the magnification μ constant is only allowed for images smaller than the Einstein radius, for individual images and the image may not come too close to a critical curve. Considering these limitations we will try to find the which scales occur in gravitational lenses in the next section.

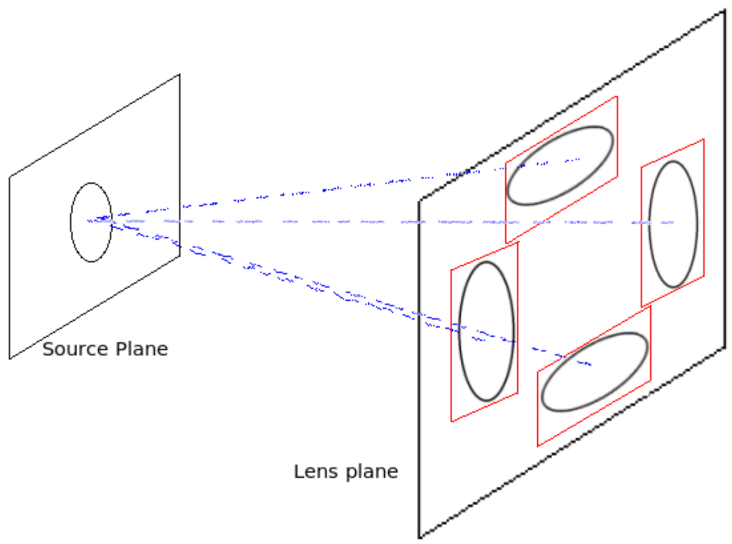


Figure 6 – The mapping from the source plane to the lens plane. In the approximation where for the description of an image μ is taken constant, one can only look at individual images. So each red square has to be evaluated as a single image.

For more in depth information, please see Schneider, Kochanek and Wambsganss [1], which is also the source of most information in this section. For readers interested in the behaviour of gravitational lenses we added appendix B. In this appendix cosine and Gaussian distributions are used to physically and analytically probe lensing events. This introduction should give the reader a firm basis to understand the physics this paper is based on.

4 Power spectrum of image residuals

In this section we relate the surface brightness fluctuations of the lensed image residual to surface density fluctuations. This is a two step proces, which is represented in figure 2. First we fit a general smooth model to the lens system, which describes the system reasonably well. Using this model one can find the difference between the observed surface brightness (upper right in figure 2) and the model surface brightness (the lower left in figure 2). These residuals can be described in a statistical manner with power spectra. Our goal is to find the power spectrum of the surface density residuals, if the power spectra of the source, the best lens potential and the correction to the best lens potential are known. This is of course the wrong way around, we detect the power spectrum of the residuals and want to obtain the power spectrum of the correction on the lens surface density, which is in fact the correction on the gravitational potential of the lens. It will probably not be possible to invert the equation analytically. This inversion is not part of this project, we will say a few words about it in the section on future research (section 7).

The lens equation (18) is non-linear. To enable analytic calculations the lens equation should first be linearized. We use the linearization method used by Hu (2000) [12] with the difference that we extend it to the strong lensing case. Hu's derivation is repeated in detail in appendix A. In weak lensing an object is only given new orientation and shape. In strong lensing, however, one has to deal with the fact that images will be distorted. This means one should incorporate the magnification tensor M into the equations. The derivation to be done includes some Fourier analysis and some notational issues. In the next subsection these are discussed.

4.1 Important formulas and definitions

This section introduces the mathematical objects and definitions needed in the rest of this report. Examples of these objects are the Fourier transform, the power spectrum and the convolution theorem.

Fourier transform
The Fourier transformation decomposes a function into a sum of wave functions. The amplitude of the wave as function of the wavelength indicates the power of that scale in the original function. The definition of the Fourier transform is

$$\mathcal{F}(f(\mathbf{x}))(\mathbf{k}) \equiv \tilde{f}(\mathbf{k}) \equiv \int d^2\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} \tag{22}$$

The Hessian of functions act under Fourier transformation as follows

$$\begin{aligned}\mathcal{F}(\nabla f(\mathbf{x}))(\mathbf{k}) &= \int d^2\mathbf{x}\nabla f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int d^2\mathbf{x}\nabla\left[f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] - \int d^2\mathbf{x}f(\mathbf{x})\nabla e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \left[f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]_{\text{boundaries}} + i\mathbf{k}\int d^2\mathbf{x}\nabla f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= i\mathbf{k}\tilde{f}(\mathbf{k}) \qquad (23)\end{aligned}$$

The last step assumes that a physical field, for example the density or a potential, goes to zero as the distances go to infinity. Similary the Fourier transforms of second derivatives can be calculated. The results are

$$\mathcal{F}(H(f(\mathbf{x})))(\mathbf{k}) = \mathbf{K}\tilde{f}(\mathbf{k}) \qquad (24)$$

Where

$$\mathbf{K}(\mathbf{k}) \equiv \begin{pmatrix} (k^{(1)})^2 & k^{(1)}k^{(2)} \\ k^{(1)}k^{(2)} & (k^{(2)})^2 \end{pmatrix}$$

where the vector $\mathbf{k}$ is given by

$$\mathbf{k} = \begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix}$$

Convolution theorem

The convolution theorem states that the Fourier transform of the product of two functions is the same as the the convolution of the Fourier transforms of the two terms seperatly, see for example the book on Fourier analysis by Stein and Shakarchi [4]. A convolution is defined as:

$$f(t) \otimes g(t) \equiv \int d\tau f(\tau)g(t-\tau) \qquad (25)$$

The $\otimes$ denotes the convolution operator. In this notation the convolution theorem states:

$$\mathcal{F}(f(t)g(t)) = \mathcal{F}(f(t)) \otimes \mathcal{F}(g(t)) \qquad (26)$$

The convolution theorem also works the other way around:

$$\mathcal{F}(f(t) \otimes g(t)) = \mathcal{F}(f(t)) \cdot \mathcal{F}(g(t)) \qquad (27)$$

Power spectrum

The power spectrum of a function f is defined as the Fourier transform of the function times its complex conjugate. The exact definition is:

$$C_k^{ff}(\mathbf{k}) \equiv \frac{\mathcal{F}^*(f)(\mathbf{k})\mathcal{F}(f)(\mathbf{k})}{(2\pi)^2} \qquad (28)$$

A power spectrum indicates which scales occur in a function and with which amplitude they occur. The phase information is lost.

Taylor expansion in two dimensions
A Taylor expansion in two dimensions is given by

$$p_3(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a}) \cdot (x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \cdot (x_i - a_i)(x_j - a_j) + O(x^3)$$

which can be found in the book on vector analysis by Colley [3].
Since it is customary in the field of strong lensing to work in the notation introduced in Schneider (1992) [2], this equation will be put in vector notation. An example of a book in which this notation is prominently used is the book by Schneider, Kochanek and Wambsganss, [1]. The Taylor expansion in vector notation is

$$p_3(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} - \mathbf{a})^T \nabla_x f(\mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T H_x f(\mathbf{a})(\mathbf{x} - \mathbf{a}) + O(\mathbf{x}^3) \quad (29)$$

Where H_x denotes the Hessian, in this two dimensional case this becomes

$$H_x f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

Notation
Derivatives will sometimes be denoted as follows:

$$\frac{df}{dx} = f_{,x}$$

Indices denoting the i^{th} element of vector v are denoted as

$$v^{(i)}$$

If in the convolutions multiple functions are used, it is convenient to give them indices like f_1 and f_2, in this context it means that an index denotes a different function, variable or vector.

4.2 The derivation of the residual's power spectrum

In this section we derive the power spectrum of the residuals given a best model for the lens potential, a model for the source and a model for the correction on the best model. We are particulary interested in the relation between the power spectrum of the surface brightness residuals and that of the surface mass density.

The derivation of the residual power spectrum starts from the priniciple of conservation of surface brightness. This is also known as Liouville's theorem, in which the surface brightness of the source S is the same as the surface brightness of the lensed image I [1],[7],

$$S(\mathbf{y}) = I(\mathbf{x}) \quad (30)$$

where $\mathbf{y}$ are the coordinates in the source plane and $\mathbf{x}$ are the coordinates in the lens plane.
The two coordinates are connected via the lens equation.

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} - \nabla\psi_t(\mathbf{x}) \quad (31)$$

Where $\psi_t(\mathbf{x})$ is the lens potential. Images form at all $\mathbf{x}$ where the lens equation is satisfied for a given $\mathbf{y}$.

The true lens potential is never known exactly. This is caused partially by limitations in observations and partially by its complexity. By complexity we mean that the true lens potential ψ_t can not be described by a few parameters. The best known model for the potential ψ_0, which is a function which describes the potential resonably well, without becoming too complex. Correcting the lens potential will change the image with respect to the smooth image. The difference between the two is called the residual. Figure 7 gives a graphic representation of the effect of differences between a model potential and a true potential. In a model potential the light is deflected by an angle α, but due to difference in potentials between the true and the modeled potential, there is also a difference in angle $\delta\alpha$. We first look at the lowest order correction to approach the true potential. This first order correction will be called $\delta\psi$

$$\psi_t = \psi_0 + \delta\psi$$

when inserting equation (31) into equation (30) we obtain

$$S(\mathbf{x} - \nabla\psi_{\mathbf{t}}) = I(\mathbf{x}) \tag{32}$$

However if the surface densities are calculated with the best guess for the potential ψ_0, S will not be exactly equal to I. To counter this difference, a correction term δI should be added to I,

$$S(\mathbf{x} - \nabla\psi_{\mathbf{0}}) = I(\mathbf{x}) + \delta I \tag{33}$$

Now the preliminaries are known, a derivation will be started like the derivation made by Hu (2000).

A Taylor expansion to second order in $\delta\psi$ of the surface brightness of the source S in the source plane $\mathbf{y}$ can be made around $\mathbf{y}_0$, where $\mathbf{y}_0$ is defined as

$$\mathbf{y}_0(\mathbf{x}) \equiv \mathbf{x} - \nabla\psi_0(\mathbf{x}) \tag{34}$$

The Taylor expansion of the surface brightness distribution of the source is given by

$$S(\mathbf{y}) = S(\mathbf{y}_0) + (\nabla_x\delta\psi)^T\nabla_y S(\mathbf{y}_0) + \frac{1}{2}(\nabla_x\delta\psi)^T H_y(S(\mathbf{y}_0))\nabla_x\delta\psi + \cdots \tag{35}$$

Because $\mathbf{y}$ is a function of $\mathbf{x}$ (see the lens equation (18)) one can also work in the lens plane. This means doing a coordinate transformation, using partial differentiation

$$\nabla_y S(\mathbf{y}) = \frac{\partial S}{\partial \mathbf{y}} = \mathbf{M}\nabla_x S(\mathbf{x})$$

Where

$$\mathbf{M} = \frac{\partial \mathbf{x}}{\partial \mathbf{y}},$$

is the magnification matrix mentioned earlier in equation (21). The magnification matrix is the Jacobian to go from the source plane to the lens plane.

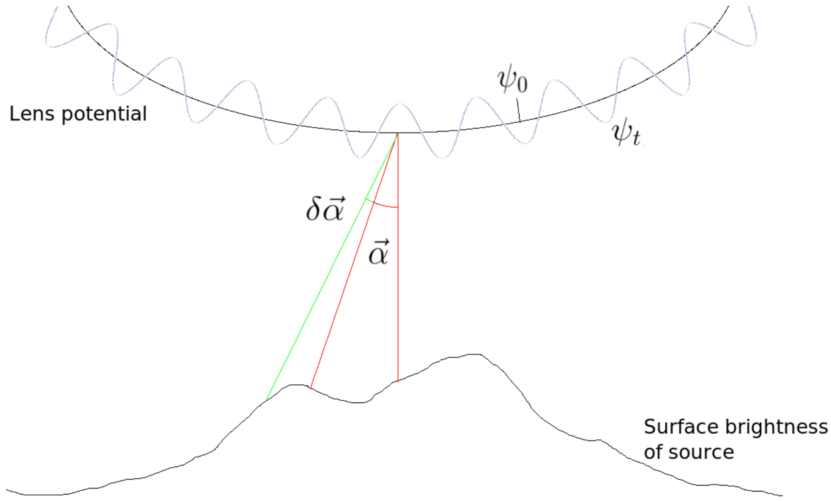


Figure 7 – A visual interpretation of the lens potential. The black line is the best lens model ψ_0 and the blue line is the true potential ψ_t. The black line below is the surface brightness distribution of the source. When the best model was the only potential light would deviate by an angle α, but the true potential is deviated extra by an angle $\delta\alpha$.

Bringing the Taylor expansion of the surface brightness distribution of the source to the lens plane

$$S(\mathbf{x}) = S(\mathbf{y}_0(\mathbf{x}))+(\nabla_x\delta\psi)^T\mathbf{M}\nabla_x S(\mathbf{y}_0(\mathbf{x}))+\frac{1}{2}(\nabla_x\delta\psi)^T\mathbf{M}^T H_x S(\mathbf{y}_0(\mathbf{x}))\mathbf{M}\nabla_x\delta\psi+\cdots \tag{36}$$

Now equations (32) and (33) will be inserted into equation (36).

$$I(\mathbf{x}) = I(\mathbf{x})+\delta I(\mathbf{x})+(\nabla_x\delta\psi)^T\mathbf{M}\nabla_x S(\mathbf{y}_0)+\frac{1}{2}(\nabla_x\delta\psi)^T\mathbf{M}^T H_x(S(\mathbf{y}_0))\mathbf{M}\nabla_x\delta\psi+\cdots \tag{37}$$

This becomes

$$\boxed{\delta I(\mathbf{x}) \quad = \quad -(\nabla_x\delta\psi)^T\mathbf{M}\nabla_x S(\mathbf{x}) - \tfrac{1}{2}(\nabla_x\delta\psi)^T\mathbf{M}^T H_x(S(\mathbf{x}))\mathbf{M}\nabla_x\delta\psi} \tag{38}$$

plus higher order terms, which can be neglected in the further analysis. Equation (38) is the second order correction on the best potential to approach the true potential, i.e. it gives an approximation of the residuals by varying the potential by a small amount.

The linearization is needed to be able to do Fourier transforms analytically. The goal is to find the scales on which the changes in the image occur. This can

be found via the power spectrum of δI, which is closely related to the Fourier transform of δI. The linearized formula for δI is a second order Taylor expansion and the power spectrum will also be taken to second order in $\delta\psi$. To prevent extra work, lets see to what order we need to take δI.
To do this a number of things are needed:

- The power spectrum, equation (28):

$$C_k^{ff} = \frac{\mathcal{F}^*(f)(\mathbf{k})\mathcal{F}(f)(\mathbf{k})}{(2\pi)^2} = \frac{|\mathcal{F}(f)(k)|^2}{(2\pi)^2}$$

  So the power spectrum consists of two Fourier transforms of the same function.

- The order of a function is the same in normal space as in Fourier space.

- Multiplying terms of orders m and n with each other, gives a new term of order $m+n$, for example a first and zeroth order term multiplied give a first order term.

- The power spectrum is required to second order, in other words $m+n \leq 2$ thus the Fourier transforms may only combine as: zeroth - zeroth, first - first, zeroth - first, second - zeroth and zeroth - second.

Conclusion: Since δI does not contain zeroth order terms, only the combination of a first order term times another first order term is allowed, i.e. it is sufficient to take δI to first order in $\delta\psi$ to get the power spectrum of δI to second order in $\delta\psi$.

Some simplifications
Before deriving the power spectra we need to define what regime we are going to consider. It is easiest to have only small changes in the lens potential and hence small changes in the deflection angle. This is generally true when looking at images which are much smaller than the Einstein radius. Furthermore we only look at individual images, which are seperated by the critical curves (where the magnification μ is infinite). With these assumptions it is allowed to assume the magnification to be constant over the image. With these assumptions it is however not possible to describe the entire system of images.
Keeping this in mind we can start with Fourier transforming the residuals

$$\delta I(\mathbf{x}) = -(\nabla_x \delta\psi)^T \mathbf{M} \nabla_x S(\mathbf{y}_0(\mathbf{x})) \tag{39}$$

where $\mathbf{M}$ is given by equation (21)

$$\mathbf{M} = \mu \begin{pmatrix} 1-\psi_{,22} & \psi_{,21} \\ \psi_{,12} & 1-\psi_{,11} \end{pmatrix} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mu \begin{pmatrix} \psi_{,22} & -\psi_{,21} \\ -\psi_{,12} & \psi_{,11} \end{pmatrix} \tag{40}$$

Where ψ will be taken as the best model ψ_0 to prevent excess indices. μ is also a function of ψ, but is considered constant for individual images in our approximation.

The last matrix looks like a Hessian of the total potential, but then in a different order. Lets define a new Hessian $\bar{H}$

$$\bar{H}(\psi) \equiv \begin{pmatrix} \psi_{,22} & -\psi_{,21} \\ -\psi_{,12} & \psi_{,11} \end{pmatrix} \tag{41}$$

So the magnification matrix can be written as

$$\mathbf{M} = \mu\mathcal{I} - \mu\bar{H}(\psi)$$

where $\mathcal{I}$ is the 2×2 identity matrix. We simplify our notation by dropping the subscript x in the gradient and by dropping the explicit $\mathbf{x}$ dependence of $\mathbf{y}_0$. Implicitly this is still true.
In this notation δI becomes

$$\delta I(\mathbf{x}) = -\mu(\nabla\delta\psi)^T\nabla S(\mathbf{y}_0) - \mu(\nabla\delta\psi)^T\bar{H}(\psi)\nabla S(\mathbf{y}_0) \tag{42}$$

To be able to Fourier transform the previous equation, the convolution theorem is needed.

$$\begin{aligned}\mathcal{F}(\delta I) &= \mathcal{F}(-\mu(\nabla\delta\psi)^T\nabla S(\mathbf{y}_0) - \mu(\nabla\delta\psi)^T\bar{H}(\psi)\nabla S(\mathbf{y}_0)) \\ &= -\mu\mathcal{F}((\nabla\delta\psi)^T\nabla S(\mathbf{y}_0)) - \mu\mathcal{F}((\nabla\delta\psi)^T\bar{H}(\psi)\nabla S(\mathbf{y}_0)) \\ &= -\mu\mathcal{F}((\nabla\delta\psi)^T) \otimes \mathcal{F}(\nabla S(\mathbf{y}_0)) - \mu\mathcal{F}(\nabla\delta\psi)^T) \otimes \mathcal{F}(\bar{H}(\psi)) \otimes \mathcal{F}(\nabla S(\mathbf{y}_0))\end{aligned}$$

using equations (23) and (24):

$$\mathcal{F}((\nabla\delta\psi(\mathbf{x})^T)(\mathbf{k}) = i\mathbf{k}^T\tilde{\delta\psi}(\mathbf{k}) \tag{43}$$

$$\mathcal{F}(\nabla S(\mathbf{x})(\mathbf{k}) = i\mathbf{k}\tilde{S}(\mathbf{k}) \tag{44}$$

$$\mathcal{F}(\bar{H}(\psi(\mathbf{x}))(\mathbf{k}) = \bar{\mathbf{K}}(\mathbf{k})\tilde{\psi}(\mathbf{k}) \tag{45}$$

where

$$\bar{\mathbf{K}}(\mathbf{k}) = \begin{pmatrix} -(k^{(2)})^2 & k^{(1)}k^{(2)} \\ k^{(1)}k^{(2)} & -(k^{(1)})^2 \end{pmatrix}$$

and the following two notations of the Fourier transformed are used

$$\mathcal{F}(\psi) = \tilde{\psi}$$

Thus the Fourier transformation of the correction on the image is

$$\begin{aligned}\mathcal{F}(\delta I) =& -\mu\mathcal{F}((\nabla\delta\psi)^T) \otimes \mathcal{F}(\nabla S(\mathbf{y}_0)) - \mu\mathcal{F}(\nabla\delta\psi)^T) \otimes \mathcal{F}(\bar{H}(\psi)) \otimes \mathcal{F}(\nabla S(\mathbf{y}_0)) \\ =& -\mu\left[i\mathbf{k}^T\mathcal{F}(\delta\psi)\right] \otimes \left[i\mathbf{k}\mathcal{F}(S)\right] \\ & -\mu\left[i\mathbf{k}^T\mathcal{F}(\delta\psi)\right] \otimes \left[\bar{\mathbf{K}}(\mathbf{k})\mathcal{F}(\psi)\right] \otimes \left[i\mathbf{k}\mathcal{F}(S)\right] \\ =&\mu\left[\mathbf{k}^T\mathcal{F}(\delta\psi)\right] \otimes \left[\mathbf{k}\mathcal{F}(S)\right] \\ & +\mu\left[\mathbf{k}^T\mathcal{F}(\delta\psi)\right] \otimes \left[\bar{\mathbf{K}}(\mathbf{k})\mathcal{F}(\psi)\right] \otimes \left[\mathbf{k}\mathcal{F}(S)\right]\end{aligned} \tag{46}$$

Working out the convolution of the first half of equation (46) gives

$$\left[\mathbf{k}^T\mathcal{F}(\delta\psi)(\mathbf{k})\right]\otimes[\mathbf{k}\mathcal{F}(S)(\mathbf{k})]=\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1 \qquad (47)$$

And the second half

$$\begin{aligned}&\left[\mathbf{k}^T\mathcal{F}(\delta\psi)\right]\otimes\left[\bar{\mathbf{K}}(\mathbf{k})\mathcal{F}(\psi)\right]\otimes[\mathbf{k}\mathcal{F}(S)]\\&=\left[\int d\mathbf{k}_2\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\right]\otimes\left[\mathbf{k}\tilde{S}\right]\\&=\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\int d\mathbf{k}_2\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1\\&=\iint d\mathbf{k}_1d\mathbf{k}_2\tilde{S}(\mathbf{k}_1)\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \qquad (48)\end{aligned}$$

Combining the two halves

$$\begin{aligned}\mathcal{F}(\delta I)=&\mu\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1\\&+\mu\iint d\mathbf{k}_1d\mathbf{k}_2\tilde{S}(\mathbf{k}_1)\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1\\=&\mu\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\Big(\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1\\&+\mu\int d\mathbf{k}_2\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1\Big)\\=&\mu\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\left(A(\mathbf{k},\mathbf{k}_1)+B(\mathbf{k},\mathbf{k}_1)\right) \qquad (49)\end{aligned}$$

where

$$\begin{aligned}A(\mathbf{k},\mathbf{k}_1)&\equiv\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1\\B(\mathbf{k},\mathbf{k}_1)&\equiv\textstyle\int d\mathbf{k}_2\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1\end{aligned}$$

With the Fourier transform of δI at hand, the power spectrum can be taken. The starting point is the expectation value of the power spectrum of δI

$$\begin{aligned}\left\langle\tilde{\delta I}^*(\mathbf{k})\tilde{\delta I}(\mathbf{k}')\right\rangle&=\int d^2\mathbf{k}\tilde{\delta I}^*(\mathbf{k})\tilde{\delta I}(\mathbf{k}')\mathrm{Pdf}(\mathbf{k})\\&=\begin{cases}\int d^2\mathbf{k}d^2\mathbf{k}'\tilde{\delta I}^*(\mathbf{k})\tilde{\delta I}(\mathbf{k}')\mathrm{Pdf}(\mathbf{k})=0 & \mathrm{if}\,\mathbf{k}\neq\mathbf{k}'\\\int d^2\mathbf{k}\tilde{\delta I}^*(\mathbf{k})\tilde{\delta I}(\mathbf{k})\mathrm{Pdf}(\mathbf{k})=(2\pi)^2\delta(\mathbf{k}-\mathbf{k}')C^{\delta I\delta I}_{\mathbf{k}} & \mathrm{if}\,\mathbf{k}=\mathbf{k}'\end{cases}\end{aligned} \qquad (50)$$

Continuing with the definition of the power spectrum, equation (28), the power spectrum of δI becomes

$$\begin{aligned}C^{\delta I\delta I}_k&=\left\langle\frac{\mathcal{F}^*(\delta I)(\mathbf{k})\mathcal{F}(\delta I)(\mathbf{k})}{(2\pi)^2}\right\rangle\\&=\left\langle\frac{\mu^2}{(2\pi)^2}\left(\int d\mathbf{k}_1\tilde{S}(\mathbf{k}_1)\left(A+B\right)\right)^*\int d\mathbf{k}_2\tilde{S}(\mathbf{k}_2)\left(A+B\right)\right\rangle\end{aligned}$$

The $(2\pi)^2$ comes from the definition in equation (50).
Which simplifies [1] to

$$\begin{aligned} C_k^{\delta I \delta I} &= \frac{\mu^2}{(2\pi)^2} \int d\mathbf{k}_1 \left(\tilde{S}(\mathbf{k}_1)\,(A+B)\right)^* \tilde{S}(\mathbf{k}_2)\,(A+B) \\ &= \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS}\,(A^*A + A^*B + B^*A + B^*B) \end{aligned} \quad (52)$$

The $(2\pi)^2$ drops out due to the definition of equation (50).
Thus the A^*A, A^*B, B^*A and B^*B terms must be evaluated.

$$\begin{aligned} A^*A &= (\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^* \tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1 \\ &= \tilde{\delta\psi}^*(\mathbf{k}-\mathbf{k}_1)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\ &= (2\pi)^2 C_{k1}^{\delta\psi\delta\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \end{aligned} \quad (53)$$

$$\begin{aligned} B^*B = &\left(\int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1\right)^* \\ &\cdot \int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \\ = &\int d\mathbf{k}_2 \tilde{\psi}^*(\mathbf{k}_2)\tilde{\delta\psi}^*(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)((\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1)^2 \\ = &(2\pi)^4 \int d\mathbf{k}_2 C_{k_2}^{\psi\psi} C_{k-k1-k2}^{\delta\psi\delta\psi}((\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1)^2 \end{aligned} \quad (54)$$

$$\begin{aligned} A^*B = &\left(\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1\right)^* \\ &\cdot \int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \\ = &\int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2)\tilde{\delta\psi}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\tilde{\delta\psi}^*(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \\ = &(2\pi)^2 \int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2) C_{k-k1}^{\delta\psi\delta\psi}\delta(\mathbf{k}-\mathbf{k}_1-(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2))(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \\ = &(2\pi)^2 \int d\mathbf{k}_2 \tilde{\psi}(\mathbf{k}_2) C_{k-k1}^{\delta\psi\delta\psi}\delta(\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T\bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1 \\ = &0 \end{aligned} \quad (55)$$

In the third step the definition of the expectation value is used, like in equation (50). The delta-function turns out to give the requirement that $\mathbf{k}_2 = 0$. The $\bar{\mathbf{K}}$ matrix only contains terms of the $\mathbf{k}_2$ vector, so $\bar{\mathbf{K}}$ will be entirely zero, so $A^*B = 0$. The same happens for B^*A, for symmetry reasons.

[1] When considering the integrals as sums of infinitisimal elements one gets

$$\begin{aligned} \left\langle \sum_i^{L/dx} x_i dx \sum_j^{L/dy} x_j dy \right\rangle &= \left\langle \sum_i \sum_j x_i x_j dx dx y \right\rangle \\ &= \sum_i \sum_j \langle x_i x_j \rangle dx dy \end{aligned} \quad (51)$$

which is only non-zero for $x_i = x_j$

Substituting equations (53), (54) and (55) into the equation for the power spectrum of δI, equation (52) gives,

$$\begin{aligned} C_k^{\delta I \delta I} &= \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS} \left(A^*A + A^*B + B^*A + B^*B\right) \\ &= \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS} \left(A^*A + B^*B\right) \end{aligned}$$

Using the above results of A^*A and B^*B finally gives

$$\boxed{\begin{aligned} C_k^{\delta I \delta I} = \quad & \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS} \Big[(2\pi)^2 C_{k-k1}^{\delta\psi\delta\psi}((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2 \\ & + (2\pi)^4 \int d\mathbf{k}_2 C_{k_2}^{\psi\psi} C_{k-k1-k2}^{\delta\psi\delta\psi}((\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T \bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1)^2\Big] \end{aligned}} \tag{56}$$

which is the power spectrum of the change of the image δI, due to the correction $\delta\psi$ on the best model ψ, .

Equation (56) is true under the assumptions that the size of the image is much smaller than the Einstein radius and if one only looks at one image at a time and where the magnification μ is approximatly constant. Since we assumed ψ to be slowly varying, which is approximatly constant, one can see the Fourier transformed of ψ as a delta function. So $C_{k_2}^{\psi\psi} \propto \delta(\mathbf{k}_2)$. This yields that in the second term of equation (56)

$$\bar{\mathbf{K}}(\mathbf{k}_2) = \bar{\mathbf{K}}(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} 0$$

in words the second term of equation (56) is zero, so (56) simplifies to

$$C_k^{\delta I \delta I} = \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS} (2\pi)^2 C_{k-k1}^{\delta\psi\delta\psi}((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2 \tag{57}$$

Equation (57) gives the power spectrum of a perfect image, observed under a perfect tranquille sky, with a perfect telescope, which even has an infinitly large field-of-view. In a real world, this is something which cannot be observed. We therefore need to know what the effects are of non-perfect telescopes and seeing. These effects will be dealt with in the next section.

5 Error analysis

Working on observations means dealing with distortions in the data and with error analysis. Due to physical limitations observations can never be perfect, the only thing an astronomer can do, is be aware of the faults in his or her data. Light traveling from a source, through the lens potential, to the observer can be compromised by various phenomena. The most important disturbances are inflicted in the final stages of its journey. The earth's atmosphere contains turbulent layers, which cause seeing. Mirrors and filters in the telescope will blur the image even more. Leakage of electrons in the CCD chip will also make objects appear larger than they really are. These three effects can be combined in a point spread function (PSF). The PSF describes a point source is smeared, when it is read from the CCD chip. The PSF limits the smallest scales one can see. In addition the field of view (FoV) of the telescope is not infinitely large. Not being able to see the entire sky limits the largest scales one can observe. The finite FoV can be simulated using a window function. The more modes fit inside the FoV the better the statistics become. This is called sample variance. A limited sample size gives uncertainties which have to be taken into account. The FoV is in general not the sky visible on the CCD chip. We can only look at single images of a lensed object. This means the the FoV we are interested in is just the size of one image. So the scale of the image above the noise level is the largest scale one can see. We want the amplitude of the mode to exceed or to be equal to the level of the noise, as will be explained in section 5.3. We are therefore in the sample dominated regime and errors due to noise can be neglected. We give an overview of error analysis in one dimension, using the same methods described below it is possible to go to two dimensions.

5.1 How to model these processes?

We would like to implement these atmospheric and instrumental effects in our equation for the power spectrum of the residuals, equation (56) or in approximation equation (57). This equation gives the power spectrum of the image

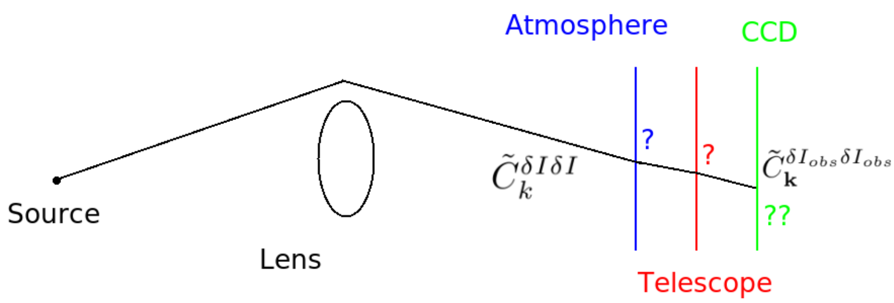


Figure 8 – In section 4 the power spectrum for the distortions on an image was derived for light which was still moving through space. However we have to deal with effects in the atmosphere, the telescope and the CCD-chip, which will alter the power spectrum we will observe.

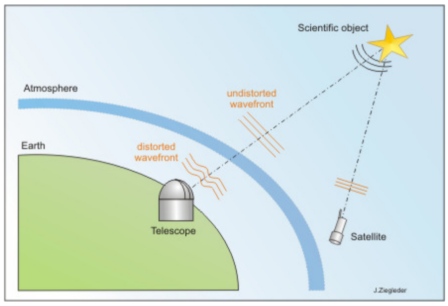


Figure 9 – The main effect of smearing for a ground-based telescope is seeing. Space-based telescopes are unaffected by this effect, but do have smearing due to electron leakage and non-perfect mirrors. These effects can be summarized in a point spread function [28]

before it enters the atmosphere and/or the telescope (see figure 8). As stated in the introduction of this section both the atmosphere and the telescope have distorting effects due to smearing, incorporated in the point spread function. Furthermore the finite field of view of the telescope has also an effect. We can use a window function to model this. We would like to determine, what the power spectrum looks like as it is observed, if $C_{\mathbf{k}}^{\delta I \delta I}$ is known.

5.1.1 Point spread function

The point spread function quantifies how the light of a point source is scattered during its passage through the atmosphere and the telescope (see figure 9). The point spread function convolved with the surface brightness distribution entering the atmosphere gives the surface brightness distribution registered by the telescope [23].

$$I_{obs}(\mathbf{x}) = PSF(\mathbf{x}) \otimes I(\mathbf{x})$$

The point spread function can often be approximated by a Gaussian profile, although the PSF can be made as complicated as needed. We will assume the PSF to be spatially invariant. In this derivation we assume the PSF to be larger than the pixel size on the CCD chip. If this is not the case additional smearing occurs. The spatial invariance allows us to go from two dimensions to one dimension. Another assumption to be made is that all light entering the atmosphere will reach the telescope, in other words the point spread function is normalized to unity.

We use the following definition for the PSF:

$$PSF(x) = \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left(-\frac{x^2}{2\sigma_p^2}\right) \tag{58}$$

and have it work on an image with a Gaussian power spectrum, defined by:

$$I(x) = A \exp\left(-\frac{x^2}{2\sigma_I^2}\right)$$

Ignoring the error analysis for the time being, we just want to see the effect of a PSF.

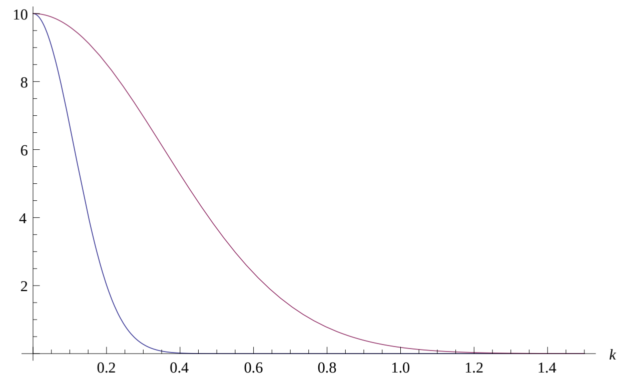


Figure 10 – The effect of a PSF in Fourier space. The red line represents the original power spectrum before applying the PSF. The blue line represents the power spectrum after the PSF has been applied. It is clear small scales are no longer distinghuisable

In Fourier space the observed image becomes

$$\mathcal{F}(I_{obs})(k) = \mathcal{F}(PSF)(k)\mathcal{F}(I)(k)$$

due to the convolution theorem: a convolution is real space, is a multiplication in Fourier space, where

$$\mathcal{F}(PSF)(k) = \phi \cdot e^{-2\sigma_p^2 k^2} \tag{59}$$

$$\mathcal{F}(I)(k) = \phi' \cdot e^{-2\sigma_I^2 k^2} \tag{60}$$

where ϕ and ϕ' are arbitrary phases. The derivation of these equations is analog to the derivation of equation (136) in appendix C.
Using the definition of the power spectrum (28) we get the power spectrum:

$$C_k^{IobsIobs} = A^2 \exp\left(-4(\sigma_I^2 + \sigma_P^2)k^2\right)$$

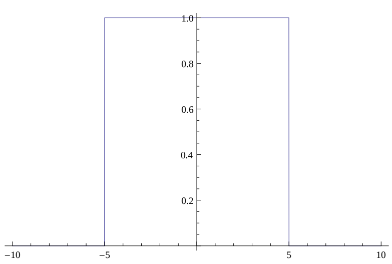
The effect of the PSF can clearly be seen in figure 10. The larger **k** disappear. The reason is that large **k** correspond to small scales, which are smeared by the effect of the PSF.

5.1.2 Window function

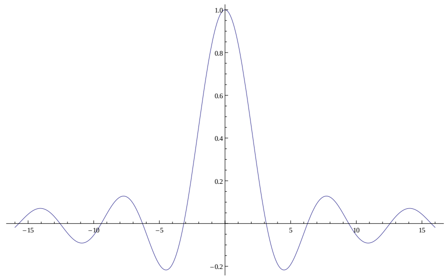
The window function deals with the fact that the field of view (FoV) is not infinitly large. The simplest window function is a top-hat function

$$w(\mathbf{x}) = \begin{cases} w = 1 & \Rightarrow \quad if\ x_1 < x < x_2\ and\ y_1 < y < y_2 \\ w = 0 & \Rightarrow \quad otherwise \end{cases}$$

where x_1 and x_2 are the boundaries of the window in the x-direction and y_1 and y_2 are the bounderies in the y-direction. Where x and y are the coordinates



(a) The step function



(b) The Fourier transformed of the step function: the sinc function

Figure 11 – Window function

of the CCD chip. See figure 11(a) for one dimensional plot of the step function. One should multiply the profile which falls onto the chip with the window function.

$$I_{obs}(\mathbf{x}) = w(\mathbf{x}) \cdot I(\mathbf{x})$$

This step function has a Fourier transform called a sinc function. See figure 11(b) for the plot of a sinc function. The sinc function has two main problems, the first it is hard to integrate over in combination with other functions. This property is needed due to the convolution theorem: a multiplication in real space is a convolution in Fourier space. Since power spectra are in Fourier space and convolutions are essentially integrations, we have a problem with the sinc function. The second problem has to do with the shape of the sinc function. It has side lobes that might cause additional peaks at high frequencies in the power spectrum. These higher frequencies are not caused by physical processes in space, but by our equipment, so we would like to suppress of them.

First we would like to see what a window function does. To maintain some simplicity let us use a Gaussian window function and consider it in one dimension. The Gaussian window function maintains general properties of a step function, but without the mathematical nuisance. The true step function is given in figure 11(a) and the Gaussian window function is given in figure 12(b).

The window fuction is mathematically given by

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_w}} e^{-\frac{x^2}{2\sigma_w^2}}$$

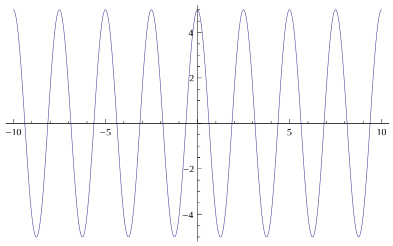
The Fourier transform of the window is in this simplified case the Fourier transformed of a Gaussian, which was already done in equation (136).

$$\begin{aligned} \mathcal{F}\left(w(x)\right) &= \mathcal{F}\left(\frac{1}{\sqrt{2\pi\sigma_w}} e^{-\frac{x^2}{2\sigma_w^2}}\right) \\ &= e^{-2\sigma_w^2 k^2} \end{aligned} \tag{61}$$

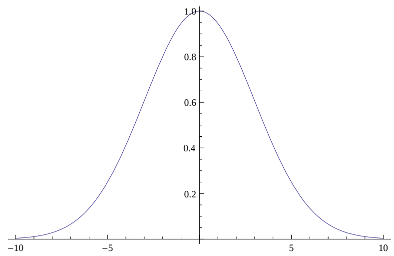
The power spectrum of the window function becomes

$$C_k^{w,w} = \frac{\mathcal{F}^*(w)\mathcal{F}(w)}{(2\pi)^2} = \frac{1}{(2\pi)^2} e^{-4\sigma_w^2 k^2} \tag{62}$$

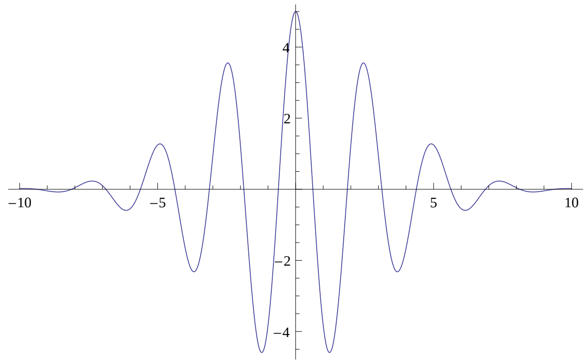
The effect of the window function can best be seen in real space, see figure 12. The further on goes from the center of the FoV the lower the amplitude of the cosine becomes. In Fourier space this would mean that the largest scales, the smallest modes, would get a lower amplitude.



(a) Original cosine source



(b) The window function, at least a first order approximation



(c) The cosine source multiplied by the window function

Figure 12 – The effects of a window function in real space

Thus the PSF cuts of the power spectrum at large **k** and the window function smears the power spectrum on all, but especially the largest, modes (i.e. the smallest k).

5.2 Observed power spectrum

Combining the original δI with the PSF and the window function in an appropriate way, gives us

$$\delta I_{observed}(\mathbf{x}) = [\delta I(\mathbf{x}) \otimes PSF(\mathbf{x})] \cdot w(\mathbf{x}) \tag{63}$$

We would like to know the power spectrum of $\delta I_{observed}$.

$$C_{\mathbf{k}}^{\delta I_{obs} \delta I_{obs}} = \frac{\mathcal{F}^*(\delta I_{obs}) \mathcal{F}(\delta I_{obs})}{(2\pi)^2} \tag{64}$$

The Fourier transformed of $\delta I_{observed}$, equation (63) is

$$\begin{aligned}\mathcal{F}(\delta I_{obs}) &= \mathcal{F}\left([\delta I(\mathbf{x}) \otimes PSF(\mathbf{x})] \cdot w(\mathbf{x})\right) \\ &= [\mathcal{F}\left(\delta I(\mathbf{x}) \otimes PSF(\mathbf{x})\right)] \otimes \mathcal{F}(w(\mathbf{x})) \\ &= [\mathcal{F}(\delta I(\mathbf{x})) \cdot \mathcal{F}(PSF(\mathbf{x}))] \otimes \mathcal{F}(w(\mathbf{x}))\end{aligned} \tag{65}$$

Substituting this into the equation for the power spectrum

$$\begin{aligned}C_{\mathbf{k}}^{\delta I_{obs}\delta I_{obs}} &= \frac{\mathcal{F}^*(\delta I_{obs})\mathcal{F}(\delta I_{obs})}{(2\pi)^2} \\ &= \left[[\mathcal{F}(\delta I(\mathbf{x})) \cdot \mathcal{F}(PSF(\mathbf{x}))] \otimes \mathcal{F}(w(\mathbf{x}))\right]^* \left[[\mathcal{F}(\delta I(\mathbf{x})) \cdot \mathcal{F}(PSF(\mathbf{x}))] \otimes \mathcal{F}(w(\mathbf{x}))\right] \\ &= [\mathcal{F}^*(\delta I(\mathbf{x})) \cdot \mathcal{F}^*(PSF(\mathbf{x}))] \otimes \mathcal{F}^*(w(\mathbf{x}))\,[\mathcal{F}(\delta I(\mathbf{x})) \cdot \mathcal{F}(PSF(\mathbf{x}))] \otimes \mathcal{F}(w(\mathbf{x}))\end{aligned} \tag{66}$$

We are dealing with the multiplication of two integrals, since convolutions are defined using integrals, see equation (25). For simplicity let us call f the Fourier transform of δI, g the transform of the PSF and h the transform of the window function, from equation (66) we then find

$$\int f^*(k')g^*(k')h^*(k-k')dk' \cdot \int f(l')g(l'h(k-l')dl' \tag{67}$$

when we regard an integral as an infinite sum we get

$$\left\langle \sum_i f_i^* g_i^* h_{i,j}^* \cdot \sum_{i'} f_{i'} g_{i'} h_{i',j} \right\rangle \tag{68}$$

The image δI entering earth's atmosphere is assumed to be a Gaussian random variable, in other words, no two points are correlated. This means when one is multiplying the terms of the sums, the only non-zero component will be for $i = i'$. This means we can write the two sums of equation (68) as one sum

$$\left\langle \sum_i f_i^* f_i g_i^* g_i h_{i,j}^* \right\rangle \tag{69}$$

going back to integral form and filling in our definitions of f, g and h

$$\int |f(k')|^2|g(k')|^2|h(k-k')|^2dk' \Rightarrow \int |\mathcal{F}(\delta I)(k')|^2|\mathcal{F}(PSF)(k')|^2|\mathcal{F}(w)(k-k')|^2dk' \tag{70}$$

which is the convolution of the power spectrum of δI multiplied by the power spectrum of the PSF with the power spectrum of the window function

$$C_{\mathbf{k}}^{\delta I_{obs}\delta I_{obs}} = (2\pi)^2 \int d\mathbf{k}_1 C_{\mathbf{k1}}^{\delta I\delta I} C_{\mathbf{k1}}^{PSFPSF} C_{\mathbf{k\text{-}k1}}^{ww} \tag{71}$$

or in compact notation

$$\boxed{C_{\mathbf{k}}^{\delta I_{obs}\delta I_{obs}} = 4\pi^2 \left[C_{\mathbf{k}}^{\delta I\delta I} \cdot C_{\mathbf{k}}^{PSFPSF}\right] \otimes C_{\mathbf{k}}^{ww}} \tag{72}$$

5.3 Error on measurement

We now know which scales to expect on observing strong lensing events. However to do a proper analysis one needs to know what errors are present in the observations. The most important way in lowering errors is making more observations. So we would like to measure each scale as often as we can. This "often as we can" is limited by the point spread function and the window function. Measurements closer together than the effective size of the PSF are blurred, making them indistinghuisable. Scales comparable to the window size can only be measured a few times. Say that n point spread functions fit effectively next to each other in the observed image. This means the error due to noise will go down with $1/\sqrt{n}$.
If we go to Fourier space, we are also limited by the same effects as in normal space, the window function and the point spread function.

Sample variance is the variance associated with a limited sample size due to the finite field of view. At the largest scale k_{min} we can only have 1 measure-

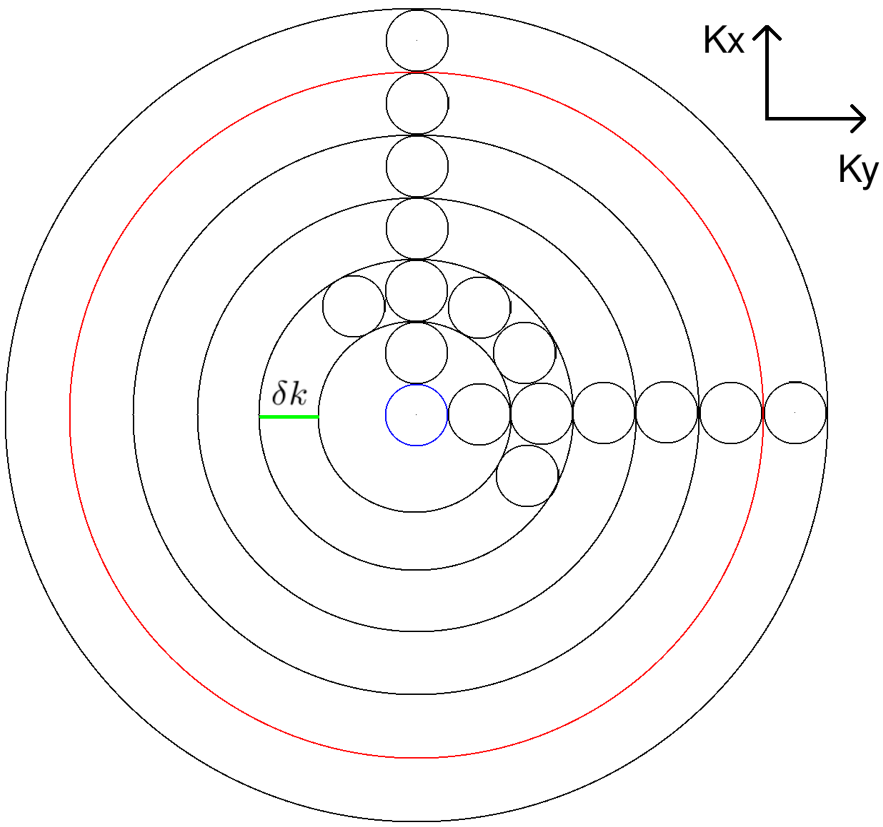


Figure 13 – The two dimensional Fourier space. In this figure the important scales are shown. The red line assigns the highest $|\mathbf{k}|$ one can possibly measure, this is the scale limited by the point spread function. The blue line assigns the lowest $|\mathbf{k}|$ one can measure, this is the scale limited by the field of view.

ment. This is a really bad statistic, since the measurment is now its own error. Moving to smaller scales, and thus larger k values in Fourier space, we will have more measurement possibilities. The limit will be at k_{max}, the point beyond which all information is destroyed by the blurring of the PSF.

We want to have an amplitude to noise ratio of at least 1. This means that we want data that is visible above the noise. Data buried under noise can be extracted statistically, but systematic effects can affect the data. To ward systematic effects, the amplitude to noise ratio is chosen this strict. In this regime the sample variance is dominant over the noise.

We would like to know which errors belong to a power spectrum. The variance decreases linearly with the number of measurement possibilities M. To determine the number of measurment possibilities we need to go the two dimensional Fourier space (figure 13). In this figure the largest scale possible, determined by the FoV, is given by the blue line. The smallest scale possible, determined by the PSF, is given by the red line.

The size of the blue circle is also the size of the resolution element. This means that the lowest $|\mathbf{k}|$ can only be measured once. The number of times a resolution element fits into a certain annulus is equal to the number of measurements one can take of the scale determined by that same annulus. An annulus has a surface area of

$$A_{annulus} = 2\pi\delta k\ k$$

and the surface area of a resolution element is

$$A_{resolution} \approx \delta k^2$$

So the number of measurements one can take is of a scale $\mathbf{k}$ is given by

$$M(\mathbf{k}) = \frac{2\pi k}{\delta k}$$

Measuring a certain scale once gives an uncertainty equal to the measurement itself due to sample variance. Doing M extra measurements of a scale lowers the uncertainty by a factor $M^{1/2}$. The variance on the power spectrum at a certain scale is given by:

$$\sigma_C^2(k) = \frac{C_k^{\delta I\delta I}}{\sqrt{M}} = \frac{C_k^{\delta I\delta I}\ \sqrt{\delta k}}{\sqrt{2\pi k}} \tag{73}$$

5.4 A toy model

Let us try to do a simple example using equation (57):

$$C_k^{\delta I \delta I} = \mu^2 \int d\mathbf{k}_1 (2\pi)^2 C_{k_1}^{SS} C_{k-k1}^{\delta\psi\delta\psi} ((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2$$

We will take a simple configuration:

- μ is constant
- We assume the lens potential pertubation to be a Gaussian random field.
- We take the average over many lens systems and assume the source a Gaussian distribution as well.
- For symplicity we also go to one dimension.

$$C_k^{\delta I \delta I} = B \int d\mathbf{k}_1 C_{k_1}^{SS} ((k-k_1)k_1)^2$$

The power spectrum of a Gaussian function was calculated in (137)

$$C_k^{SS} = Ce^{-\sigma^2 k^2}$$

So

$$C_k^{\delta I \delta I} = D \int d\mathbf{k}_1 e^{-\sigma^2 k_1^2} (k-k_1)^2 k_1^2$$

$$= D \int d\mathbf{k}_1 e^{-k_1^2 \sigma^2} (k^2 k_1^2 - 2k_1^3 k + k_1^4)$$

The Gaussian is a symmetric function, so integrating over all space while it is multiplied by an odd k_1^3 gives zero. The other two terms will just give numbers, so what we can conclude from this is that the power spectrum of the image is an quadratic function:

$$C_k^{\delta I \delta I} = E \cdot k^2 + F \qquad (74)$$

where E and F are constants, see the green line in figure 14.
Adding the point spread function cuts of the smallest scales, see figure 14. The function plotted in this figure is given by

$$C_k^{\delta I \delta I\ \prime} = C_k^{\delta I \delta I} \cdot C_k^{PSFPSF} = (E \cdot k^2 + F) \cdot \exp\left(-4\sigma_P^2 k^2\right) \qquad (75)$$

where the power spectrum of the image is given by $E \cdot k^2 + F$ and the power spectrum of the PSF by equation (59).
The sample variance, equation (73), is also plotted, in red, according to the following equations

$$\sigma_C^2(k) = \frac{C_k^{\delta I \delta I}\ \delta k^{1/2}}{(2\pi k)^{1/2}}$$

$$\sigma_C = \frac{C_k^{\delta I \delta I\ 1/2} \delta k^{1/4}}{(2\pi k)^{1/4}} = \frac{(D \cdot k^2 + E)^{1/2} \cdot \exp\left(-2\sigma_P^2 k^2\right)\ \delta k^{1/4}}{(2\pi k)^{1/4}} \qquad (76)$$

It is clear what the point spread function does in figure 14. As the original power spectrum begins to rise significantly after k=1, it has a cut-off at $k =$

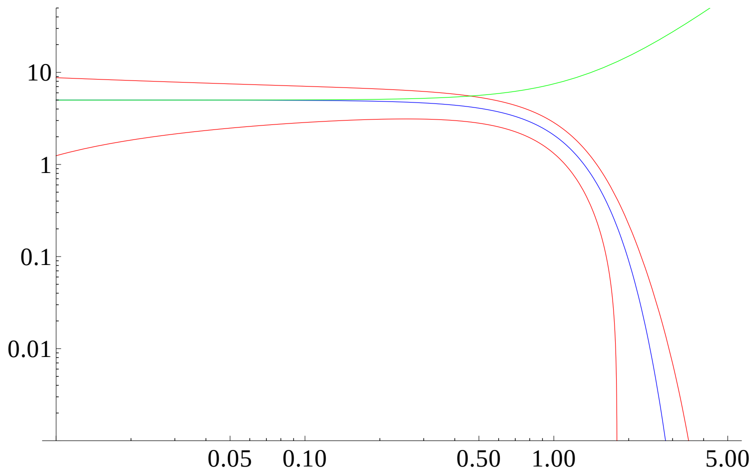


Figure 14 – The original power spectrum (equation (74) is given by the green line, the observational effect of the point spread function (equation (75) is added in blue. This supresses scales smaller than the PSF itself. The red lines are the error bars (equation (76) belonging to the observed power spectrum when the PSF is taken into account.

$1/\sigma_p$, which was chosen around 1, in the case where the PSF is taken into account. This is exactly what we expected the PSF to do: supress small scales. We want to implement all effects from equation (72)

$$C_{\mathbf{k}}^{\delta I_{obs}\delta I_{obs}} = 4\pi^2 \left[C_{\mathbf{k}}^{\delta I\delta I} \cdot C_{\mathbf{k}}^{PSFPSF}\right] \otimes C_{\mathbf{k}}^{ww}$$

this means the window function should be added. We already have $C_{\mathbf{k}}^{\delta I\delta I} \cdot C_{\mathbf{k}}^{PSFPSF}$ from equation (75). The convolution still has to be done, which we do in one dimension. We assume the window function to be

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{x^2}{2\sigma_w^2}}$$

the power spectrum is given in (62)

$$C_k^{w,w} = \frac{1}{(2\pi)^2} e^{-2\sigma_w^2 k^2}$$

$$\begin{aligned}
C_k^{\delta Iobs\delta Iobs} &= C_k^{\delta I\delta I\ \prime} \otimes C_k^{ww} \\
&= C_k^{\delta I\delta I} \cdot C_k^{PSFPSF} \otimes C_k^{ww} \\
&= \left[(2\pi)^{-2}(E \cdot k^2 + F) \cdot \exp\left(-2\sigma_P^2 k^2\right)\right] \otimes \exp\left(-2\sigma_w^2 k^2\right)
\end{aligned}$$

Doing the convolution and absorbing the 2π's into the constants

$$
\begin{aligned}
C_k^{\delta Iobs\delta Iobs} &= \left[(Ek^2+F)\exp(-2\sigma_p^2k^2)\right]\otimes\exp(-2\sigma_w^2k^2)\\
&= \int_{-\infty}^{\infty} dk_1 \exp(-2\sigma_w^2k_1^2)\exp(-2\sigma_p^2(k-k_1)^2)(E(k-k_1)^2+F)\\
&= \int_{-\infty}^{\infty} dk_1 \exp(-2\sigma_w^2k_1^2-2\sigma_p^2k_1^2+4\sigma_p^2kk_1-2\sigma_p^2k^2)(E(k_1^2-2kk_1+k^2)+F)\\
&= \exp(-2\sigma_p^2k^2)\int_{-\infty}^{\infty} dk_1 \exp(-2(\sigma_w^2+\sigma_p^2)k_1^2+4\sigma_p^2kk_1)(E(k_1^2-2kk_1+k^2)+F)
\end{aligned}
$$

Now we use completing the squares to get a more standard Gaussian integral, in other words we want an exponent of the form $(ak_1+b)^2$.

$$
\begin{aligned}
a^2 = \sigma_w^2+\sigma_p^2 &\Rightarrow a = (\sigma_w^2+\sigma_p^2)^{1/2}\\
2ab = 2\sigma_p^2k &\Rightarrow b = \frac{2\sigma_p^2k}{2a} = \frac{\sigma_p^2k}{(\sigma_w^2+\sigma_p^2)^{1/2}}\\
&\Rightarrow b^2 = \frac{\sigma_p^4k^2}{\sigma_w^2+\sigma_p^2}
\end{aligned}
$$

putting this in our convolution integral

$$
\begin{aligned}
C_k^{\delta Iobs\delta Iobs} &= \exp(-\sigma_p^2k^2)\int_{-\infty}^{\infty} dk_1 \exp(-[(\sigma_w^2+\sigma_p^2)k_1^2-2\sigma_p^2kk_1+\frac{\sigma_p^4k^2}{\sigma_w^2+\sigma_p^2}]+\frac{\sigma_p^4k^2}{\sigma_w^2+\sigma_p^2})\\
&\quad\cdot(E(k_1^2-2kk_1+k^2)+F)\\
&= \exp(-\sigma_p^2k^2+\frac{\sigma_p^4k^2}{\sigma_w^2+\sigma_p^2})\int_{-\infty}^{\infty} dk_1 \exp(-[\sqrt{\sigma_w^2+\sigma_p^2}k_1-\frac{\sigma_p^2k}{\sqrt{\sigma_w^2+\sigma_p^2}}]^2)\\
&\quad\cdot(E(k_1^2-2kk_1+k^2)+F)\\
&= \exp(-\frac{\sigma_p^2\sigma_w^2k^2}{\sigma_w^2+\sigma_p^2})\int_{-\infty}^{\infty} dk_1 \exp(-(\sigma_w^2+\sigma_p^2)[k_1-\frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2}]^2)\\
&\quad\cdot(E(k_1^2-2kk_1+k^2)+F) \qquad (77)
\end{aligned}
$$

since we integrate over all frequency space the constant in the exponent only causes a shift of

$$k_1 \rightarrow k_1 + \frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2}$$

This causes $k_1^2-2kk_1+k^2$ to shift to

$$
\begin{aligned}
&\left(k_1+\frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2}\right)^2 - 2k\left(k_1+\frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2}\right)+k^2\\
=&k_1^2+2\frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2}k_1+(\frac{\sigma_p^2k}{\sigma_w^2+\sigma_p^2})^2-2kk_1-\frac{2\sigma_p^2k^2}{\sigma_w^2+\sigma_p^2}+k^2\\
=&k_1^2+\left(2\frac{\sigma_p^2}{\sigma_w^2+\sigma_p^2}-2\right)kk_1+\left(1+\frac{\sigma_p^4}{(\sigma_w^2+\sigma_p^2)^2}-\frac{2\sigma_p^2}{\sigma_w^2+\sigma_p^2}\right)k^2 \qquad (78)
\end{aligned}
$$

Insert equation (78) into (77), while using the fact that the Gaussian is symmetric and $2kk_1$ is antisymmetric, so they will give zero in an integral over all space

$$C_k^{\delta Iobs\delta Iobs} = \exp(-\frac{\sigma_p^2\sigma_w^2k^2}{\sigma_w^2+\sigma_p^2})\int_{-\infty}^{\infty} dk_1 \exp(-(\sigma_w^2+\sigma_p^2)k_1^2)$$
$$\cdot (E(k_1^2 + \left(1+\frac{\sigma_p^4}{(\sigma_w^2+\sigma_p^2)^2} - \frac{2\sigma_p^2}{\sigma_w^2+\sigma_p^2}\right)k^2) + F) \qquad (79)$$

Let us define

$$\alpha \equiv \sigma_w^2 + \sigma_p^2$$

and

$$\beta \equiv \left(1+\frac{\sigma_p^4}{(\sigma_w^2+\sigma_p^2)^2} - \frac{2\sigma_p^2}{\sigma_w^2+\sigma_p^2}\right)Ek^2 + F$$

We will evaluate the integral of equation (79) using the definitions of α and β given above

$$\int_{-\infty}^{\infty} dk_1 (Ek_1^2+\beta)\exp(-\alpha k_1^2) \qquad (80)$$
$$= \int_{-\infty}^{\infty} dk_1 \left(\beta - E\frac{\partial}{\partial\alpha}\right)\exp(-\alpha k_1^2) \qquad (81)$$
$$= \left(\beta - E\frac{\partial}{\partial\alpha}\right)\int_{-\infty}^{\infty} dk_1 \exp(-\alpha k_1^2) \qquad (82)$$
$$= \left(\beta - E\frac{\partial}{\partial\alpha}\right)\sqrt{\frac{\pi}{\alpha}} \qquad (83)$$
$$= \left(\frac{E}{2\alpha}+\beta\right)\sqrt{\frac{\pi}{\alpha}} \qquad (84)$$

Going from equation (80) to (81) we used the fact that taking a derivative of an exponential doesn't change the exponential. From (81) to (82) we used that E, α and β are independent of k_1. The evaluation of the Gaussian integral in equation (83) was done for example in appendix C in equation (145). The last step is combining (84) with with the definitions of α and β in equation (79)

$$C_k^{\delta Iobs\delta Iobs} = \exp(-\frac{\sigma_p^2\sigma_w^2k^2}{\sigma_w^2+\sigma_p^2})\sqrt{\frac{\pi}{\sigma_w^2+\sigma_p^2}}$$
$$\left(\frac{E}{\sigma_w^2+\sigma_p^2} + \left(1+\frac{\sigma_p^4}{(\sigma_w^2+\sigma_p^2)^2} - \frac{2\sigma_p^2}{\sigma_w^2+\sigma_p^2}\right)Ek^2 + F\right) \qquad (85)$$

To check which terms are of importance in equation (85) we take the limit in which the scale of the field of view is much larger than the scale of the point spread function, i.e. $\sigma_w \gg \sigma_p$. In this regime equation (85) becomes

$$C_k^{\delta Iobs\delta Iobs} \approx \sqrt{\frac{\pi}{\sigma_w^2}}\exp\left(-\sigma_p k^2\right)\left[\frac{E}{\sigma_w^2} + Ek^2 + F\right] \qquad (86)$$

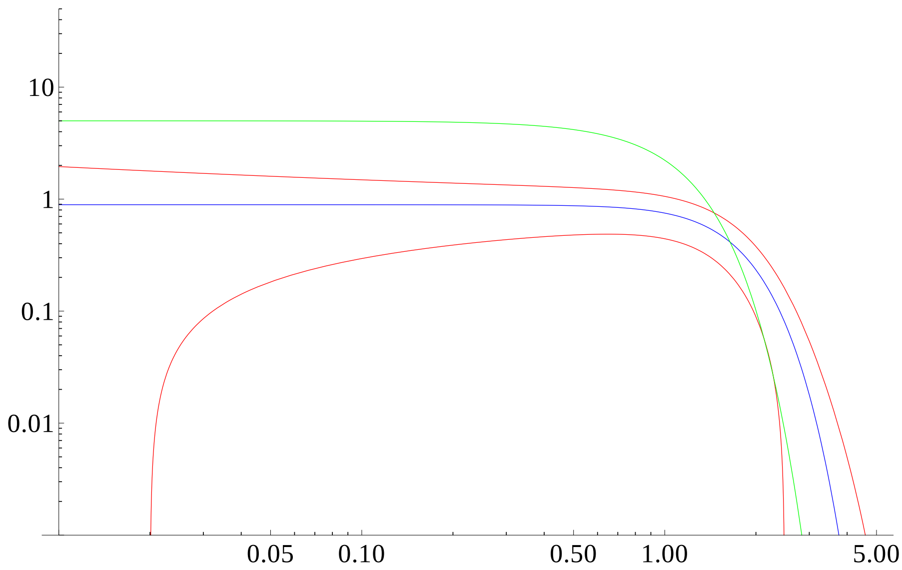


Figure 15 – The effect of the window function, the green line is if only the PSF is taken into account, with other words, we have an infinitly large window function. The blue line is when the window function is taken finite. It is clear that flux at all large scales drop. The uncertainties are given by the red lines, the uncertainties are dominated by sample variance is the S/N>1 regime. The window function causes less flux to be present at large scales.

To show the effect of the window function on the power spectrum we plot in figure 15 the power spectrum which is only affected by the point spread function (equation (76)), the power spectrum in which the PSF and the FoV are implemented (equation (86)) and the uncertainties belonging to the latter. The finite field of view causes the power spectrum to be lowered over all k. The largest scales are redistributed to smaller scales.

This toy model is a simplified version of reality, the most important simplification is that we have only considered one dimension. What we can tell from this model is that the point spread function causes the smallest scales to be supressed and that the finite field of view causes all scales to supressed. This behavoir was expected, so this tells us that equation (72) behaves as it should.

6 Conclusion

The goal of this report was to find the power spectrum of the features in the surface brightness distribution of stronlgy gravitational lensed images. The lens is often a foreground galaxy. These features can be associated with mass substructure in the lens. To get the equation for the power spectrum we needed to make some assumptions. We have listed them below, with the reason why we assumed it and why it is a good assumption. The general assumptions are:

- General relativity is correct: We used general relativity to find the angle a lightray deviates from a straigth line due to a point mass. General relativity seems to hold, so this is a good assumption.
- Small angle approximation: We used that the gravitational fields are weak, this allows us to use the lens equation. As long as we do not try to describe lightrays too close to black holes this assumption holds.
- Thin lens approximation: The lens is much smaller than the distances between observer, lens and source. Since lenses are most of the time galaxies of kpc scale and the distances involved are of Mpc scale this assumption holds.
- Cosmological principle holds: the universe is isotropic and homogeneous, we used this assumption to be able to use angular diameter distances. The cosmological principle seems to hold in the visible universe.
- Slow varying lens potential: This assumption allowed us to take the magnification μ constant over the image. This is good assumption if we look at one image at a time, not too close to critical curves and have a source which is much smaller than the Einstein radius.
- S/N larger than 1: allows us to ignore noise, this convines us to the sample dominated regime. This assumption is true if multiple point spread functions fit into one image.

The first five assumptions have allowed us to derive an equation for the power spectrum of the residuals of an image $C_k^{\delta I \delta I}$, when the power spectra of the source C_k^{SS}, the best model for the lens potential C_k^{II} and of the correction on the lens potential $C_k^{\delta\psi\delta\psi}$ are known. The power spectrum of the residuals is given by:

$$\begin{aligned} C_k^{\delta I \delta I} = \quad & \mu^2 \int d\mathbf{k}_1 C_{k_1}^{SS} \Big[(2\pi)^2 C_{k-k1}^{\delta\psi\delta\psi} ((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2 \\ & + (2\pi)^4 \int d\mathbf{k}_2 C_{k_2}^{\psi\psi} C_{k-k1-k2}^{\delta\psi\delta\psi} ((\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)^T \bar{\mathbf{K}}(\mathbf{k}_2)\mathbf{k}_1)^2 \Big] \end{aligned} \tag{87}$$

The power spectrum of the difference between the true lens potential and the best model is of course not known. The residual can be measured. With numerical techniques one can find the most likely power spectrum of the correction on the lens potential. This still has to be done, in the next section a more elaborate list of subjects which have to be explored more thoroughly. This power spectrum gives the scales at which substructure occurs in lens potentials.

One of the assumptions we made to get (87) was that the lens potential ψ is slow varying. Slow varying means almost constant. A constant function in Fourier space is a dirac delta function. By assuming a constant ψ one can simplify (87) to

$$C_k^{\delta I \delta I} = \mu^2 \int d\mathbf{k}_1 (2\pi)^2 C_{k_1}^{SS} C_{k-k1}^{\delta\psi\delta\psi} ((\mathbf{k} - \mathbf{k}_1)^T \mathbf{k}_1)^2 \tag{88}$$

We can not observe this, since one would need a telescope of infinite size and infinite resolution. Observational effects alter the power spectrum. First of all the shape of the power spectrum at high scales is compromised by smearing of light, which can be modeled by a point spread function. The power spectrum is surpressed by the limited field of view, this can be modeled by a window function. The observed power spectrum is not significantly affected by noise, due to our last assumption. The observed power spectrum is given by

$$C_{\mathbf{k}}^{\delta I_{obs} \delta I_{obs}} = 4\pi^2 \left[C_{\mathbf{k}}^{\delta I \delta I} C_{\mathbf{k}}^{PSFPSF} \right] \otimes C_{\mathbf{k}}^{ww} \tag{89}$$

where $C_{\mathbf{k}}^{PSFPSF}$ is the power spectrum of the point spread function and $C_{\mathbf{k}}^{ww}$ is the power spectrum of the window function.

The uncertainties in the power spectrum of the residuals $C_k^{\delta I \delta I}$ are dominated by sample variance, due to the last assumption. Doing the calculations gives a variance on the observed power spectrum of

$$\sigma_C^2(k) = \frac{C_k^{\delta I \delta I}}{\sqrt{M}} = \frac{C_k^{\delta I \delta I} \sqrt{\delta k}}{\sqrt{2\pi k}} \tag{90}$$

It is clear that at larger k, the relative error becomes smaller.

In the next section on future research we talk about how these equations can be used and how these equations can be improved. In short we found the relations between the different power spectra involved in strong gravitional lensing (equations (87) and (88)). We found which effects observing has (equation (89)) and we found the variance (equation (90)) belonging to the power spectrum of the observed surface brightness distribution.

7 Future research

In this final section we give a short overview of improvements and implementations which still can be made on this research project. We start with the improvements one could make. They are due to some of the assumptions we made along the way. The equations we have derived are subject to a number of limitations, some of which can be lifted or made less severe when they are examined closer. I will try to point out to which topics this applies. First of all we go from the source plane to the lens plane without much of a fuss (equation (32)). However, it might not be so obvious we can do this, this should still be proven rigorously. Second we have chosen to vary the second derivative of the potential ψ and to keep the magnification μ constant (equation (21)). This means our equations are only applicable on slowly varying images. It is interesting to see what difference it makes if we take μ not to be constant. We did the examples and the toy model (section 5.4) in one dimension, it would be interesting to see what happens if two dimensional or more complicated behavoir is modeled.

We have derived some equations to calculate the power spectrum of residuals surface brightness distribution. To be able to derive this you need the power spectrum of the source's surface brightness distribution, the power spectrum of the non-smooth components of the galaxy potential and also the power spectrum of the smooth potential. The problem is that we do not have these power spectra. What we do have is a population of strongly lensed images and the knowledge that most galaxies can be approximatly considered to have spherical mass profiles. If one would take an assemble of about a thousand lensed images and average over them, you will get that on average the images and sources becomes circular. This would mean you now have power spectra of the smooth lens potential and the source's light profile. The real power spectrum of the lensed images we can observe. By extracting the average power spectrum you get the residuals, the light profile's substructure. Now one has an equation with just one unknown, the mass substructure term. Numerical methods will probably be needed to extract the mass substructure term. This process of taking real observations, numerical methods and the equations derived in this report to get mass substructure is subject to future research. I can already announce that at the moment of writing Robin Kooistra has started his own Bachelor research project. He will try to implement the equations developed here to a numerical code.

Another approach to determine the power spectra of the sources could be taking a piece of the sky which is not close to a foreground galaxy and examining the properties of the galaxies there. Using they cosmological principle we can say that these galaxies have probably the same properties as the galaxies which act as sources in the gravitationally lensed images.

To summarize: this results of this report need to be used in studying observations. Furthermore, some of the assumptions made can be lifted by a more close examination of the equations and the physics involved.

8 Acknowledgements

This project would not have been a succes without a number of people, I would like to thank them here. I am most grateful to my supervisor, Leon Koopmans. He pointed me towards this subject and supported me with literature and advice. I would also like to thank Tom Hafkenscheid for the discussions we had on this and many other topics. Trying to explain what I was doing helped me a lot in figuring out how to proceed. Furthermore I would like to thank Daniël Siepman, Robin Kooistra, Boudewijn Hut, Anja Huizing and Mirjam Soeten for reading and giving useful hints to improve this report.

9 References

[1] *P. Schneider, C. Kochanek, J. Wambsganss*, `Gravitational lensing: Strong, Weak and Micro`, Springer-Verlag Berlin Heidelberg, 2006.

[2] *P. Schneider, J. Ehlers, E. Falco*, `Gravitational lensing`, Springer, New York, 1992

[3] *S.J. Colley*, `Vector analysis`, Pearson, 2005

[4] *E. Stein, R. Shakarchi*, `Fourier analysis, an introduction`, princeton university press, 2003, Princeton, NJ, US

[5] *B.S. Ryden*, `Introduction to Cosmology`, Addison Wesley, 2003, San Fransisco, CA, US

[6] *I.S. Gradshteyn, I.M. Ryzhik, ed. A. Jeffrey*, `Table of Integrals, Series, and Products`, Acedemic Press Inc., 1980, San Diego

[7] *R.C. Tolman*, `The principles of statistical mechanics`, Oxford university press, 1959, London, UK

[8] *G.R. Fowles & G.L. Cassidy*, `Analytic mechanics`, Thomson Brooks/Cole, 2005, Belmont, CA, USA

[9] *R.D. Blandford, C.S. Kochanek, Israel Kovner, Ramesh Narayan*, `Gravitational lens optics`, Science, New Series, Vol. 245, No. 4920 (Aug. 25, 1989), pp. 924-830.

[10] *S. Refsdal, J. Surdej*, `Gravitational lenses`, Rep. Prog. Phys. **56** (1994) 117-185.

[11] *R. Narayan, M. Bartelmann*, `Lectures on gravitational lensing`, arXiv:astro-ph/9606001v2.

[12] *W. Hu*, `Weak lensing of the CMB: A harmonic approach`, PHYSICAL REVIEW D **62** 043007

[13] *S. Mao, P. Schneider*, `Evidence for substructure in lens galaxies`, 1998 RAS, MNRAS **295**, 587-594

[14] *A. Einstein*, `Die Grundlage der allgemeinen Relativittstheorie`, Ann. Phys. (Leipzig), **49**, 769, (1916)

[15] *D. Walsh, R.F. Carswell, R.J. Weymann*, `0957 + 561 A, B - Twin quasistellar objects or gravitational lens`, Nature, vol. 279, May 31, 1979, p. 381-384

[16] *F. Zwicky*, `Nebulae as gravitational lenses`, Phys. Rev. **51**, 290290 (1937)

[17] *F. Zwicky*, On the Probability of Detecting Nebulae Which Act as Gravitational Lenses, Phys. Rev. **51**, 679679 (1937)

[18] *A. Einstein*, Lens-like action of a star by the deviation of light in the gravitational field, Science, **84**, 506, (1936).

[19] *F.W. Dyson, A.S. Eddington, C.R. Davidson*, A determination of the deflection of light by the sun's gravitational field from observations made at total eclipse of May 29, 1919, Mem. R. Astron. Soc., **62**, 291 (1920)

[20] *J. Soldner*, ber die Ablenkung eines Lichtstrahls von seiner geradlinigen Bewegung durch die Attraktion eines Weltkrpers, an welchem er nahe vorbeigeht, Berliner Astronomisches Jahrbuch, 1804, 161172, (1801).

[21] *M. Barnabè, C. Nipoti, L.V.E. Koopmans, S. Vegetti, L. Ciotti*, Crash testing the CAULDRON code for joint lensing and dynamics analysis of early type galaxies, arXiv:0808.3916v2 [astro-ph]

[22] *R. Gavazzi, T. Treu, L.V.E. Koopmans, A.S. Bolton, L.A. Moustakas, S. Burles, P.J. Marshall*, THE SLOAN LENS ACS SURVEY. VI: DISCOVERY AND ANALYSIS OF A DOUBLE EINSTEIN RING, arXiv:0801.1555 [astro-ph]

[23] personal correspondance with Prof. dr. L.V.E. Koopmans

[24] lectures given by Prof. dr. L.V.E. Koopmans for the Capita Selecta course "Gravitational lensing" in the spring of 2011.

[25] information on cosmic variance, http://en.wikipedia.org/wiki/Cosmic_variance, July 7th 2011

[26] *wikipedia*, Convolution theorem, http://en.wikipedia.org/wiki/Convolution_theorem, 23-2-2011

[27] Figure of the Einstein rings, courtesy: http://www.spacetelescope.org/news/heic0803/, found on July 6th 2011.

[28] The effect of the atmosphere, http://www.mpe.mpg.de/ir/projects/LBTLaser/images/atmosphere.jpg, November 7th 2011

[29] Figure of the first observed doubly imaged quasar, http://www.pbase.com/bkpeterson/image/123479749, July 12th 2011

[30] Figure of source moving wrt to lens observer geometry, from the living reviews in astronomy on Gravitational lensing, by Joachim Wambsganss, last revised on August 31th, http://relativity.livingreviews.org/Articles/lrr-1998-12/, July 12th 2011

A Derivation by Hu

In Hu (2000) [12] a power spectrum is derived for how the light distribution of the CMB is changed due to weak lensing. A derivation of the same sort is made in section 4 for strong lensing. In this appendix we rdo the derivation made by Hu, to understand which steps were taken in the paper.
The paper starts from the fact that gravitational lensing obeys conservation of surface brightness. This means that the surface brightness of the source $S(\mathbf{y})$ is equal to the surface brightness of the image $I(\mathbf{x})$.

$$I(\mathbf{x}) = S(\mathbf{y}) \tag{91}$$

Now we fill in the lens equation, equation (18).

$$I(\mathbf{x}) = S(\mathbf{x} - \nabla\phi(\mathbf{x})) \tag{92}$$

To compare with the notation used by Hu, we give here a conversion table.

$$I \leftrightarrow \tilde{\Theta}$$

$$S \leftrightarrow \Theta$$

$$\mathbf{x} \leftrightarrow \hat{n}$$

$$\phi \leftrightarrow -\phi$$

So equation (92) looks in Hu's article like this:

$$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + \nabla\phi(\hat{n})) \tag{93}$$

As is done in the main part of the paper, the vector notation introduced by Peter Schneider (1992) [2] will be used. So equation (93) will become

$$I(\mathbf{x}) = S(\mathbf{x} + \nabla\psi) \tag{94}$$

We will use this notation. To be able to use equation (93) in analytic and numeric models one needs to get rid of its highly nonlinear terms. This can be done by taking a Taylor expansion of $I(\mathbf{x})$ untill second order in $\nabla\psi$. This is a standard problem in vector analysis [3]. This standard Taylor expansion for $f(\mathbf{x})$ around $\mathbf{a}$ in vector notation,

$$p_3(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} - \mathbf{a})^T \nabla_x f(\mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T H_x(f(\mathbf{a}))(\mathbf{x} - \mathbf{a}) + O(x^3)$$

In the case of the lens equation one works in the lens plane and the image plane, i.e. in two dimensions.

$$I(\mathbf{x}) = S(\mathbf{y}) + (\nabla_x\psi(\mathbf{x}))^T \nabla_y S(\mathbf{y}) + \frac{1}{2}(\nabla_x\psi(\mathbf{x}))^T H_y(S(\mathbf{y}))\nabla_x\psi(\mathbf{x}) + O(\Psi^3) \tag{95}$$

Where H_y is the Hessian.
In the case of weak lensing of the CMB there is no difference between a derivative in the lens or the source plane, since the images are not distorted. They are

only magnified, this is the weak lensing limit. So the Taylor expansion of $I(\mathbf{x})$ becomes

$$I(\mathbf{x}) = S(\mathbf{y}) + (\nabla_x \psi(\mathbf{x}))^T \nabla_x S(\mathbf{y}) + \frac{1}{2}(\nabla_x \psi(\mathbf{x}))^T H_x(S(\mathbf{y}))\nabla_x \psi(\mathbf{x})$$

omitting the x index

$$I(\mathbf{x}) = S(\mathbf{y}) + (\nabla \psi(\mathbf{x}))^T \nabla S(\mathbf{y}) + \frac{1}{2}(\nabla \psi(\mathbf{x}))^T H(S(\mathbf{y}))\nabla \psi(\mathbf{x}) \qquad (96)$$

To be able to get to the power spectrum, equation (96) needs to be Fourier transformed. Equation (96) will be split in three parts, respectively **(a)**, **(b)** and **(c)**.

(a):
This is the easiest case, as this is the way Fourier transforms are defined, see equation (22):

$$\mathcal{F}(S(\mathbf{y})(\mathbf{k}) = \tilde{S}(\mathbf{k}) \equiv \int d^2\mathbf{y} S(\mathbf{y}) e^{-i\mathbf{x}\cdot\mathbf{k}}$$

(b):
The second case contains two multiplied divergences. This means a Fourier transformation has to be made of a product of two terms. The convolution theorem [26] states that the fourier transform of a product, the same is as the convolution fourier transformed terms. The definition of a convolution is given in equation (26)
This applied to the second case gives:

$$\mathcal{F}((\nabla \psi(\mathbf{x}))^T \nabla S(\mathbf{y})) = \mathcal{F}((\nabla \psi(\mathbf{x}))^T) \otimes \mathcal{F}(\nabla S(\mathbf{y}))$$

The Fourier transform of the derivative of a function is given by equation (23). So

$$\mathcal{F}(\nabla \psi(\mathbf{x}))(\mathbf{k}) = i\mathbf{k}\mathcal{F}(\psi)(\mathbf{k})$$

and

$$\mathcal{F}(\nabla S(\mathbf{y})) = i\mathbf{k}\mathcal{F}(S)(\mathbf{k})$$

Now these have to be combined via the convolution:

$$\begin{aligned}
&(i\mathbf{k}\mathcal{F}(\psi)(\mathbf{k}))^T \otimes (i\mathbf{k}\mathcal{F}(S)(\mathbf{k})) \\
&= -\left(\mathbf{k}^T\mathcal{F}(\psi)(\mathbf{k})\right) \otimes (\mathbf{k}\mathcal{F}(S)(\mathbf{k})) \\
&= -\int_{-\infty}^{\infty} d^2\mathbf{k}_1 \mathcal{F}(\psi)(\mathbf{k}-\mathbf{k}_1)\mathcal{F}(S)(\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k} \\
&= -\int_{-\infty}^{\infty} d^2\mathbf{k}_1 \tilde{\psi}(\mathbf{k}-\mathbf{k}_1)\tilde{S}(\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}
\end{aligned}$$

(c)
The third case consists of three multiplied divergences. Extending the methods used in case 2, one needs to do two convolutions:

$$\mathcal{F}((\nabla \psi(\mathbf{x}))^T H(S(\mathbf{y}))\nabla \psi(\mathbf{x})) = \mathcal{F}((\nabla \psi(\mathbf{x}))^T) \otimes \mathcal{F}(H(S(\mathbf{y}))) \otimes \mathcal{F}(\nabla \psi(\mathbf{x}))$$

The Fourier transforms of the first order derivatives are just like **(b)**. The second order derivative is can be found in equation (24).

$$\mathcal{F}((\nabla\psi(\mathbf{x}))) = i\mathbf{k}\mathcal{F}(\psi)(\mathbf{k})$$

and

$$\mathcal{F}(H(S(\mathbf{y})))(\mathbf{k}) = -\mathbf{K}(\mathbf{k})\mathcal{F}(S)(\mathbf{k})$$

where

$$\mathbf{K}(\mathbf{k}) = \begin{pmatrix} (\mathbf{k}^{(1)})^2 & \mathbf{k}^{(1)}\mathbf{k}^{(2)} \\ \mathbf{k}^{(1)}\mathbf{k}^{(2)} & (\mathbf{k}^{(2)})^2 \end{pmatrix}$$

Now the Fourier transforms for the seperate terms are known, one can begin combining them via convolutions. The first convolution to be done will be between the Hessian and the vertical vector.

$$\begin{aligned}\mathcal{F}((\nabla\psi(\mathbf{x}))^T) \otimes \mathcal{F}(H(S(\mathbf{y}))) &= \left(i\mathbf{k}^T\mathcal{F}(\psi)(\mathbf{k})\right) \otimes (-\mathbf{K}(\mathbf{k})\mathcal{F}(S)(\mathbf{k})) \\ &= -i\int d^2\mathbf{k}_1\tilde{S}(\mathbf{k_1})\tilde{\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{K}(\mathbf{k_1})\end{aligned}$$

the second convolution

$$\begin{aligned}&(i\mathbf{k}\mathcal{F}(\psi)(\mathbf{k}))^T \otimes (-\mathbf{K}(\mathbf{k})\mathcal{F}(S)(\mathbf{k})) \otimes \left(i\mathbf{k}\tilde{\psi}(\mathbf{k})\right) \\ &= \left(-i\int d^2\mathbf{k}_1\tilde{S}(\mathbf{k_1})\tilde{\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{K}(\mathbf{k_1})\right) \otimes \left(i\mathbf{k}\tilde{\psi}(\mathbf{k})\right) \\ &= \iint d^2\mathbf{k}_1 d^2\mathbf{k}_2\tilde{S}(\mathbf{k_1})\tilde{\psi}(\mathbf{k}-\mathbf{k}_2-\mathbf{k}_1)\tilde{\psi}(\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_2-\mathbf{k}_1)^T\mathbf{K}(\mathbf{k}_1)\mathbf{k}_2\end{aligned}$$

Combining parts **(a)**,**(b)** and **(c)** gives the following Fourier coefficients:

$$\tilde{I}(\mathbf{k}) = \tilde{S}(\mathbf{k}) - \int d^2\mathbf{k}_1\tilde{S}(\mathbf{k}_1)L(\mathbf{k},\mathbf{k}_1) \tag{97}$$

where

$$L(\mathbf{l},\mathbf{l}_1) = \tilde{\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k} - \frac{1}{2}\int d^2\mathbf{k}_2\tilde{\psi}(\mathbf{k}-\mathbf{k}_2-\mathbf{k}_1)\tilde{\psi}^*(\mathbf{k}_2)(\mathbf{k}-\mathbf{k}_2-\mathbf{k}_1)^T\mathbf{K}(\mathbf{k}_1)\mathbf{k}_2 \tag{98}$$

Now the Fourier transform is known, information about the potentials can be taken from the this. This will be done via power spectra and expectation values, which of course are related. We start from the expectation value of $\tilde{I}(k)$.

With the Fourier transform of S at hand, the power spectrum can be taken. The starting point is the expectation value of the Fourier transform of S, $\tilde{S}$.

$$\begin{aligned}\left\langle \tilde{I}^*(\mathbf{k})\tilde{I}(\mathbf{k}')\right\rangle &= \int d^2\mathbf{k}\tilde{I}^*(\mathbf{k})\tilde{I}(\mathbf{k}')\mathrm{Pdf}(\mathbf{k}) \\ &= \begin{cases} \tilde{I}(\mathbf{k}')\int d^2\mathbf{k}\tilde{I}^*(\mathbf{k})\mathrm{Pdf}(\mathbf{k}) = 0 & \text{if}\mathbf{k} \neq \mathbf{k}' \\ \int d^2\mathbf{k}\tilde{I}^*(\mathbf{k})\tilde{I}(\mathbf{k})\mathrm{Pdf}(\mathbf{k}) = (2\pi)^2\delta(\mathbf{k}-\mathbf{k}')C_{\mathbf{k}}^{II} & \text{if}\mathbf{k} = \mathbf{k}' \end{cases}\end{aligned} \tag{99}$$

Continuing with the definition over the power spectrum, equation (28), the power spectrum of S becomes

$$\begin{aligned}
C_k^{II} &= \frac{\left\langle \tilde{I}^*(\mathbf{k})\tilde{I}(\mathbf{k})\right\rangle}{(2\pi)^2} \\
&= \frac{1}{(2\pi)^2}\left\langle \left(\tilde{S}(\mathbf{k}) - A\right)^* \left(\tilde{S}(\mathbf{k}) - A\right)\right\rangle \\
&= \frac{1}{(2\pi)^2}\left\langle \left(\tilde{S}^*(\mathbf{k})\tilde{S}(\mathbf{k}) + A^*A - \tilde{S}^*(\mathbf{k})A - A^*\tilde{S}^(\mathbf{k})\right)\right\rangle \\
&= C_k^{SS} + \frac{1}{(2\pi)^2}\left\langle A^*A - \tilde{S}^*(\mathbf{k})A - A^*\tilde{S}(\mathbf{k})\right\rangle
\end{aligned} \tag{100}$$

Where

$$A = \int d^2\mathbf{k}_1 \tilde{S}(\mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1)$$

Some terms need to be worked out.

$$\begin{aligned}
A^*A &= \left(\int d^2\mathbf{k}_1 \tilde{S}(\mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1)\right)^* \left(\int d^2\mathbf{k}_1' \tilde{S}(\mathbf{k}_1') L(\mathbf{k}, \mathbf{k}_1')\right) \\
&= \iint d^2\mathbf{k}_1 d^2\mathbf{k}_1' \tilde{S}^*(\mathbf{k}_1)\tilde{S}(\mathbf{k}_1') L^*(\mathbf{k}, \mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1') \\
&= (2\pi)^2 \iint d^2\mathbf{k}_1 d^2\mathbf{k}_1' C_{k1}^{SS} L^*(\mathbf{k}, \mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1') \delta^2(\mathbf{k}_1 - \mathbf{k}_1') \\
&= (2\pi)^2 \int d^2\mathbf{k}_1 C_{k1}^{SS} L^*(\mathbf{k}, \mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1)
\end{aligned} \tag{101}$$

To be able to continue L^*L should be found. In the derivation it is used that work is done to second order in ψ. So higher order terms can be left out.

$$\begin{aligned}
L^*(\mathbf{k}, \mathbf{k}_1) L(\mathbf{k}, \mathbf{k}_1) &= \left(\tilde{\psi}(\mathbf{k} - \mathbf{k}_1)(\mathbf{k} - \mathbf{k}_1)^T \mathbf{k} - ...\right)^* \\
&\quad \times \left(\tilde{\psi}(\mathbf{k} - \mathbf{k}_1)(\mathbf{k} - \mathbf{k}_1)^T \mathbf{k} - ...\right) \\
&= \tilde{\psi}^*(\mathbf{k} - \mathbf{k}_1)\tilde{\psi}(\mathbf{k} - \mathbf{k}_1)((\mathbf{k} - \mathbf{k}_1)^T \mathbf{k})^2 \\
&= (2\pi)^2 C_{k-k1}^{\psi\psi}((\mathbf{k} - \mathbf{k}_1)^T \mathbf{k})^2
\end{aligned} \tag{102}$$

Putting this back into A^*A

$$A^*A = (2\pi)^4 \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} C_{k1}^{SS} C_{k-k1}^{\psi\psi}((\mathbf{k} - \mathbf{k}_1)^T \mathbf{k})^2 \tag{103}$$

The cross terms still have to be done

$$
\begin{aligned}
\tilde{S}^*(\mathbf{k})A + A^*\tilde{S}(\mathbf{k}) &= \tilde{S}^*(\mathbf{k})\int d^2\mathbf{k}_1\tilde{S}(\mathbf{k}_1)L(\mathbf{k},\mathbf{k}_1) + \tilde{S}(\mathbf{k})\int d^2\mathbf{k}_1\tilde{S}^*(\mathbf{k}_1)L^*(\mathbf{k},\mathbf{k}_1) \\
&= \int d^2\mathbf{k}_1\tilde{S}^*(\mathbf{k})\tilde{S}(\mathbf{k}_1)L(\mathbf{k},\mathbf{k}_1) + \int d^2\mathbf{k}_1\tilde{S}(\mathbf{k})\tilde{S}^*(\mathbf{k}_1)L^*(\mathbf{k},\mathbf{k}_1) \\
&= (2\pi)^2\int d^2\mathbf{k}_1 C_k^{SS}\delta(\mathbf{k}-\mathbf{k}_1)L(\mathbf{k},\mathbf{k}_1) + (2\pi)^2\int d^2\mathbf{k}_1 C_k^{SS}\delta(\mathbf{k}-\mathbf{k}_1)L^*(\mathbf{k},\mathbf{k}_1) \\
&= (2\pi)^2 C_k^{SS}(L(\mathbf{k},\mathbf{k}) + L^*(\mathbf{k},\mathbf{k})) \\
&= (2\pi)^2 C_k^{SS}\left(\left(\tilde{\psi}(0)(0)^T\mathbf{k} - \frac{1}{2}\int d^2\mathbf{k}_2\tilde{\psi}^*(-\mathbf{k}_2)\tilde{\psi}(\mathbf{k}_2)(-\mathbf{k}_2)^T\mathbf{K}(\mathbf{k})\mathbf{k}_2\right)^*\right. \\
&\quad \left. + \tilde{\psi}(0)(0)^T\mathbf{k} - \frac{1}{2}\int d^2\mathbf{k}_2\tilde{\psi}(-\mathbf{k}_2)\tilde{\psi}^*(\mathbf{k}_2)(-\mathbf{k}_2)^T\mathbf{K}(\mathbf{k})\mathbf{k}_2\right) \\
&= (2\pi)^4 C_k^{SS}\int d^2\mathbf{k}_2 C_{k2}^{\psi\psi}\left((\mathbf{k}_2)^T\mathbf{K}(\mathbf{k})\mathbf{k}_2\right) \qquad (104)
\end{aligned}
$$

Combining all terms for the power spectrum of I

$$
\begin{aligned}
C_k^{II} &= C_k^{SS} + \frac{1}{(2\pi)^2}\left\langle A^*A - \tilde{S}^*(\mathbf{k})A - A^*\tilde{S}(\mathbf{k})\right\rangle \\
&= C_k^{SS} + (2\pi)^2\int\frac{d^2\mathbf{k}_1}{(2\pi)^4}C_{k1}^{SS}C_{k-k1}^{\psi\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k})^2 - (2\pi)^2 C_k^{SS}\int d^2\mathbf{k}_2 C_{k2}^{\psi\psi}\left((\mathbf{k}_2)^T\mathbf{K}(\mathbf{k})\mathbf{k}_2\right) \qquad (105)
\end{aligned}
$$

Lets compare this result with the result Hu got in equation 38 of his article [12].

$$
C_k^{II} = (1 - l^2R)C_k^{SS} + \int\frac{d^2\mathbf{k}_1}{(2\pi)^2}C_{|\mathbf{k}-\mathbf{k}_1|}^{SS}C_{\mathbf{k}_1}^{\psi\psi}[(\mathbf{k}-\mathbf{k}_1)\cdot\mathbf{k}_1]^2
$$

where

$$
R = \frac{1}{4\pi}\int\frac{dk}{k}k^4C_k^{\psi\psi}
$$

These two look kind of alike, the only thing to do is to show that R and the last part of equation (105) are the same.

There are two main differences between the two. The first thing is the notation and the second has to do with the isotropy of the CMB. Lets first change some notation. Lets call $\mathbf{k}_2 = \mathbf{l}$ for the moment and rewrite

$$
\begin{aligned}
\mathbf{l}^T\mathbf{K}(\mathbf{k})\mathbf{l} &= \begin{pmatrix} l^{(1)} & l^{(2)} \end{pmatrix}\begin{pmatrix} (k^{(1)})^2 & k^{(1)}k^{(2)} \\ k^{(1)}k^{(2)} & (k^{(2)})^2 \end{pmatrix}\begin{pmatrix} l^{(1)} \\ l^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} l^{(1)}(k^{(1)})^2 + l^{(2)}k^{(1)}k^{(2)} & l^{(1)}k^{(1)}k^{(2)} + l^{(2)}(k^{(2)})^2 \end{pmatrix}\begin{pmatrix} l^{(1)} \\ l^{(2)} \end{pmatrix} \\
&= (l^{(1)})^2(k^{(1)})^2 + 2l^{(1)}l^{(2)}k^{(1)}k^{(2)} + (l^{(2)})^2(k^{(2)})^2 \\
&= (\mathbf{l}\cdot\mathbf{k})^2 \qquad (106)
\end{aligned}
$$

Since the CMB is isotropic, directions have no meaning, only the length of a vector matters. This allows for a simplification of the integral

$$
d^2\mathbf{l} \rightarrow 2ldl
$$

and

$$\mathbf{l} \to l$$

So

$$\mathbf{l}^T\mathbf{K}(\mathbf{k})\mathbf{l} = l^2k^2$$

So

$$\begin{aligned} C_k^{II} &= C_k^{SS} + (2\pi)^2 \int d^2\mathbf{k}_1 C_{k1}^{SS} C_{k-k1}^{\psi\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k})^2 - (2\pi)^2 C_k^{SS} k^2 \int dk_2 2k_2 C_{k2}^{\psi\psi} k_2^2 \\ &= C_k^{SS}\left(1 - (2\pi)^2 k^2 \int dk_2 2k_2 C_{k2}^{\psi\psi} k_2^2\right) + (2\pi)^2 \int d^2\mathbf{k}_1 C_{k1}^{SS} C_{k-k1}^{\psi\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k})^2 \end{aligned} \tag{107}$$

The terms are the same as derived by Hu, the factors of 2π are different due to a different definition of the Fourier transformed used.

The last thing to show is that the equations conserves power, i.e. the last integral in equation (107) should be equal to $C_k^{SS}k^2 \int dk_2 2k_2 C_{k2}^{\psi\psi} k_2^2$. The integral will be evaluated in the limit in which C_k^{SS} is slowly varying and at $\mathbf{k} - \mathbf{k}_1 = \mathbf{k}$. This means that C_k^{SS} is independent of $\mathbf{l}_1$ and can be taken out of the integral.

$$\int d^2\mathbf{k}_1 C_{k1}^{SS} C_{k-k1}^{\psi\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k})^2 = C_k^{\psi\psi} \int d^2\mathbf{k}_1 C_{k1}^{SS}((\mathbf{k}_1)^T\mathbf{k})^2 \tag{108}$$

$$= C_k^{SS} k^2 (2\pi)^4 \int dk_1 2k_1 k_1^2 C_{k1}^{\psi\psi} \tag{109}$$

This is indeed the same, so the power is conserved in

$$\boxed{C_k^{II} = C_k^{SS}\left(1 - k^2 \int dk_2 2k_2 C_{k2}^{\psi\psi} k_2^2\right) + \int \frac{d^2\mathbf{k}_1}{(2\pi)^4} C_{k1}^{SS} C_{k-k1}^{\psi\psi}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k})^2} \tag{110}$$

B Getting to know lensing events

The goal of this appendix is to get more familiar with lensing events. We will start with some visualisations of lensing events using cosine and gaussian distributions. The next thing is comparing the root mean squares of the same distributions. We will use two general distributions to model the lens and the source: the cosine distribution and the Gaussian distribution. This means we assume to know shape of the source, which in reality is not true. Let us define the distributions.
A cosine in two dimensions is defined as

$$F_{cosine} \equiv C \cos(\mathbf{m}^T(\mathbf{x} - \mathbf{x}_0)) \tag{111}$$

where C is the amplitude, $\mathbf{m}$ is the wavenumber in two dimensions and $\mathbf{x}_0$ is the displacement of the the wave.
A Gaussian distribution in two dimensions is given by

$$F_{Gauss} \equiv De^{-(\mathbf{x}-\mathbf{x}_0)^T\mathbf{M}(\mathbf{x}-\mathbf{x}_0)} \tag{112}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2\sigma_1^2} & 0 \\ 0 & \frac{1}{2\sigma_2^2} \end{pmatrix}$$

Which causes the two dimensional Gaussian distribution to get an ellipsoidal shape. D is the amplitude of the Gaussian.

B.1 Visual

We will visualize the effects of lensing by plugging the lens equation (18) into the equation for the conservation of surface brightness. These visualisations will be done in mathematica. The equation for conservation of surface brightness

$$I(\mathbf{x}) = S(\mathbf{y}) \tag{113}$$

becomes

$$\delta I(\mathbf{x})) = S(\mathbf{x} - \nabla\psi(\mathbf{x})) \tag{114}$$

when the lens equation was put into it.

Section 5.1.1 deals with the cosine distribution both as source as as lens potential. Section 5.1.2 deals with a Gaussian source distribution and a cosine lens potential. This is reversed in section 5.1.3. The final visualisation will be done in section 5.1.4, where the lens potential and the source distribution are both given by Gaussian distributions.

B.1.1 Source: cosine, lens potential: cosine

Cosine distributions are simple in mathematical manipulations and when distorted they are easy to distinghuis from their original shape. That is why we start our visual exploration of lensing events with cosine distributions.
In this section the following lens potential ψ and source distribution S are examined

$$\psi(\mathbf{x}) = d\cos(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0)) \tag{115}$$

$$S(\mathbf{y}) = e\cos(\mathbf{n}^T(\mathbf{y}-\mathbf{y}_0)) \tag{116}$$

Since it is needed to fill the lens potential ψ into the lens equation, it is needed to calculate $\nabla\psi$

$$\begin{aligned}\nabla\psi &= \nabla d\cos(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0)) \\ &= -\mathbf{m}d\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))\end{aligned}$$

filling this into equation (114)

$$\begin{aligned}\delta I(\mathbf{x}) &= S(\mathbf{y}(\mathbf{x})) \\ &= e\cos(\mathbf{n}^T(\mathbf{y}-\mathbf{y}_0)) \\ &= e\cos(\mathbf{n}^T(\mathbf{x}-\nabla\psi(\mathbf{x})-\mathbf{y}_0)) \\ &= e\cos(\mathbf{n}^T(\mathbf{x}+\mathbf{m}d\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))-\mathbf{y}_0))\end{aligned} \tag{117}$$

The code in mathematica to plot this function is in appendix D.6 and the plot itself is in figure 8.

In figure 8 one sees how the straight lines of the cosine source are distorted more and more as the lens becomes more and more massive, where in figure 16(b) the lens potential is quite low and in 16(d) the lens potential is high. Figure 16(a) is the unlensed cosine source for comparison. The lens potential is also a cosine, which is under an angle of $-45°$ with the source. It is clear that a small potential in 16(b) leaves the source resonably intact, whereas the structure of the source is deformed a lot in 16(d).

B.1.2 Source: Gauss, lens potential: cosine

Gaussian distributions for sources are very common in astronomy. Since they are easy to manipulate and resemble light curves of stars, quasars and Galaxies quite a lot. So it is useful to see what the effect of a simple lens is on such a source distribution. This simple lens will have a cosine lens potential.
Again we start by defining the distributions

$$\psi(x) = C\cos(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0)) \tag{118}$$

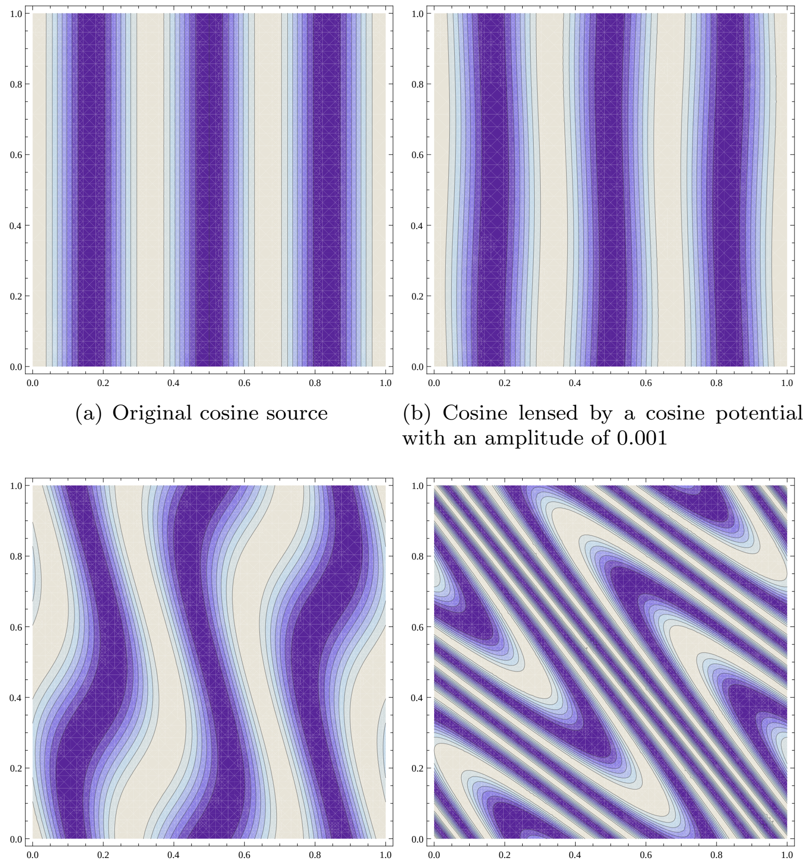
$$S(y) = De^{-(\mathbf{y}-\mathbf{y}_0)^T\mathbf{M}(\mathbf{y}-\mathbf{y}_0)} \tag{119}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2\sigma_1^2} & 0 \\ 0 & \frac{1}{2\sigma_1^2} \end{pmatrix}$$

Since it is needed to fill the lens potential ψ into the lens equation, it is needed to calculate $\nabla\psi$

$$\begin{aligned}\nabla\psi &= \nabla C\cos(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0)) \\ &= -\mathbf{m}C\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))\end{aligned} \tag{120}$$



(a) Original cosine source

(b) Cosine lensed by a cosine potential with an amplitude of 0.001

(c) Cosine source lensed by a cosine potential with an amplitude of 0.01

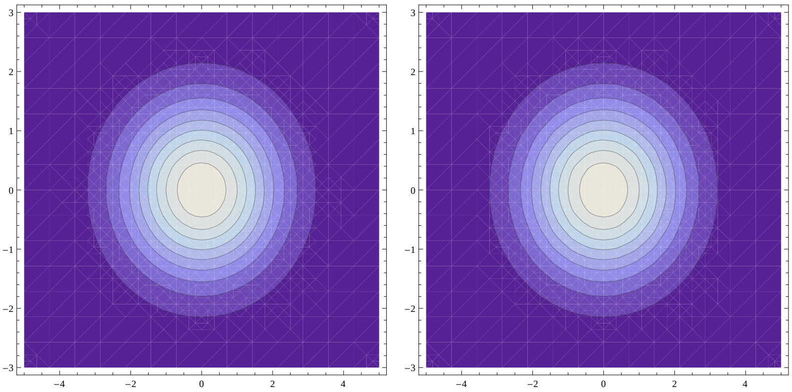
(d) Cosine source lensed by a cosine potential with an amplitude of 0.1

Figure 16 – A cosine source lensed by a cosine lens potential

filling this into equation (114)

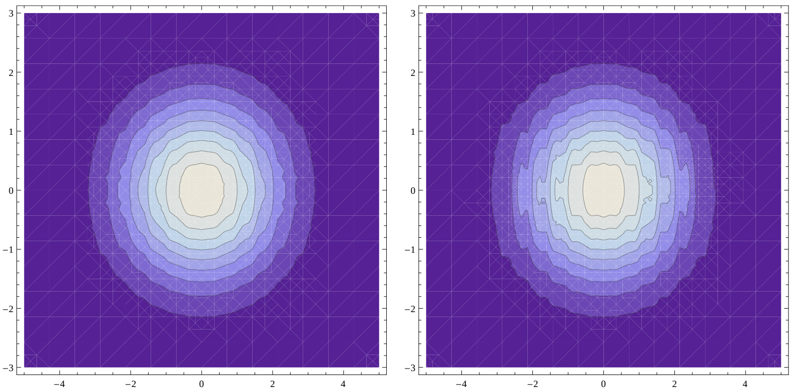
$$
\begin{aligned}
\delta I(\mathbf{x}) =& S(\mathbf{y}(\mathbf{x})) \\
=& De^{-(\mathbf{y}-\mathbf{y}_0)^T \mathbf{M}(\mathbf{y}-\mathbf{y}_0)} \\
=& De^{-(\mathbf{x}-\nabla\psi(\mathbf{x})-\mathbf{y}_0)^T \mathbf{M}(\mathbf{x}-\nabla\psi(\mathbf{x})-\mathbf{y}_0)} \\
=& De^{-(\mathbf{x}+\mathbf{m}C\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))-\mathbf{y}_0)^T \mathbf{M}(\mathbf{x}+\mathbf{m}C\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))-\mathbf{y}_0)} \\
=& D\exp\left(-\frac{1}{2}\left(\frac{[\mathbf{x}+\mathbf{m}C\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))-\mathbf{y}_0]^{(1)}}{\sigma_1}\right)^2\right. \\
&\left.-\frac{1}{2}\left(\frac{[\mathbf{x}+\mathbf{m}C\sin(\mathbf{m}^T(\mathbf{x}-\mathbf{x}_0))-\mathbf{y}_0]^{(2)}}{\sigma_2}\right)^2\right)
\end{aligned}
\tag{121}
$$

The code in mathematica to plot this function is in appendix D.7 and the plots themselves are in figure 17.



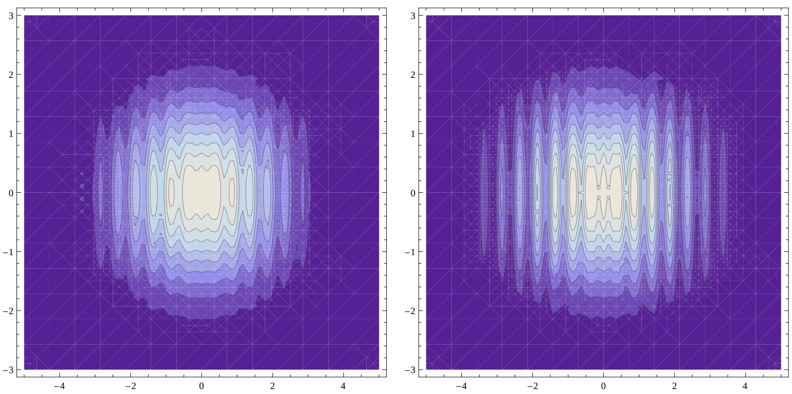
(a) Original source

(b) Gaussian source lensed by a cosine potential with an amplitude of 0.001



(c) Gaussian source lensed by a cosine potential with an amplitude of 0.005

(d) Gaussian source lensed by a cosine potential with an amplitude of 0.01



(e) Gaussian source lensed by a cosine potential with an amplitude of 0.025

(f) Gaussian source lensed by a cosine potential with an amplitude of 0.05

Figure 17 – A Gaussian source lensed by a cosine lens potential

Figure 17 one can see the effect of an increaslingly massive lens on a Gaussian source. Figure 17(a) is the unlensed source. The extrema of the cosine run vertically in the figure.

B.1.3 Source: cosine, lens potential: Gauss

Gaussian models for lens potentials are very common due to their well known and relatively simple mathematical properties and their resemblence to a lot off lensing candidates, such as elliptical galaxies and stars. In this section we will look at the effect of a Gaussian lens on a simple source.
Again we start by defining the distributions

$$\psi(x) = Ce^{-(\mathbf{x}-\mathbf{x}_0)^T\mathbf{M}(\mathbf{x}-\mathbf{x}_0)} \tag{122}$$

$$S(y) = D\cos(\mathbf{m}^T(\mathbf{y}-\mathbf{y}_0)) \tag{123}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2\sigma_1^2} & 0 \\ 0 & \frac{1}{2\sigma_1^2} \end{pmatrix}$$

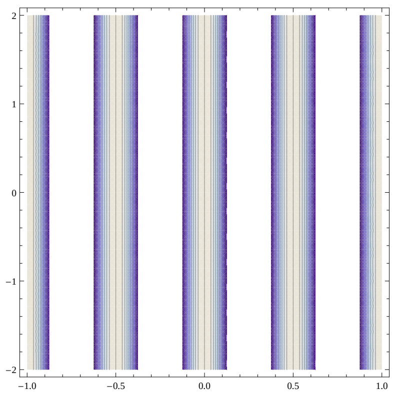
Since it is needed to fill the lens potential ψ into the lens equation, it is needed to calculate $\nabla\psi$

$$\begin{aligned}
\nabla\psi &= \nabla\left(Ce^{-(\mathbf{x}-\mathbf{x}_0)^T\mathbf{M}(\mathbf{x}-\mathbf{x}_0)}\right) \\
&= C\nabla\left(e^{-\frac{(x-x_0)^2}{2\sigma_1^2}-\frac{(y-y_0)^2}{2\sigma_2^2}}\right) \\
&= -C\left(\frac{x-x_0}{\sigma_1^2}\begin{pmatrix}1\\0\end{pmatrix}+\frac{y-y_0}{\sigma_2^2}\begin{pmatrix}0\\1\end{pmatrix}\right)e^{-\frac{(x-x_0)^2}{\sigma_1^2}-\frac{(y-y_0)^2}{\sigma_2^2}} \\
&= -C\begin{pmatrix}\frac{(\mathbf{x}-\mathbf{x}_0)^{(1)}}{\sigma_1^2}\\ \frac{(\mathbf{x}-\mathbf{x}_0)^{(2)}}{\sigma_2^2}\end{pmatrix}e^{-\frac{((\mathbf{x}-\mathbf{x}_0)^{(1)})^2}{\sigma_1^2}-\frac{((\mathbf{x}-\mathbf{x}_0)^{(2)})^2}{\sigma_2^2}}
\end{aligned}$$

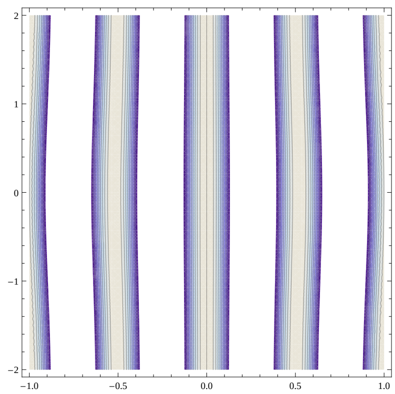
filling this into equation (114)

$$\begin{aligned}
\delta I(\mathbf{x}) &= S(\mathbf{y}(\mathbf{x})) \\
&= D\cos(\mathbf{m}^T(\mathbf{y}-\mathbf{y}_0)) \\
&= D\cos(\mathbf{m}^T(\mathbf{x}-\nabla\psi(\mathbf{x})-\mathbf{y}_0)) \\
&= D\cos((\mathbf{m}^T\mathbf{x}-\frac{d}{\sigma_1\sigma_2}(m^{(1)}\frac{(\mathbf{x}-\mathbf{x}_0)^{(1)}}{\sigma_1^2}+m^{(2)}\frac{(\mathbf{x}-\mathbf{x}_0)^{(2)}}{\sigma_2^2}) \\
&\exp\left(-\frac{((\mathbf{x}-\mathbf{x}_0)^{(1)})^2}{\sigma_1^2}-\frac{((\mathbf{x}-\mathbf{x}_0)^{(2)})^2}{\sigma_2^2}\right)-\mathbf{m}^T\mathbf{y}_0))
\end{aligned} \tag{124}$$

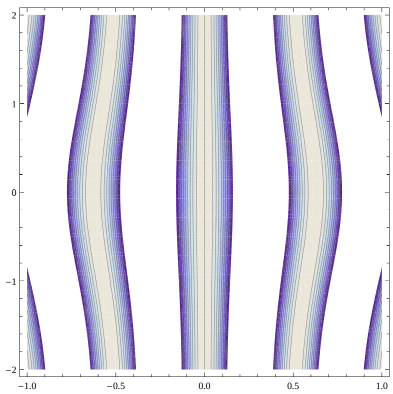
The code in mathematica to plot this function is in appendix D.8 and the plots themselves are in figure 18.



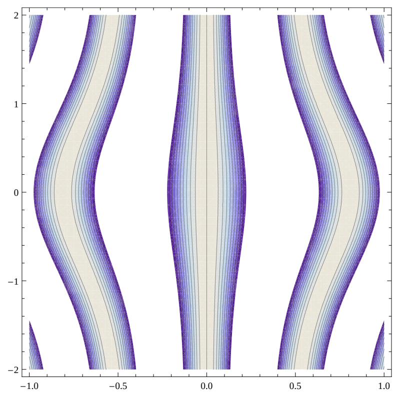
(a) Original source



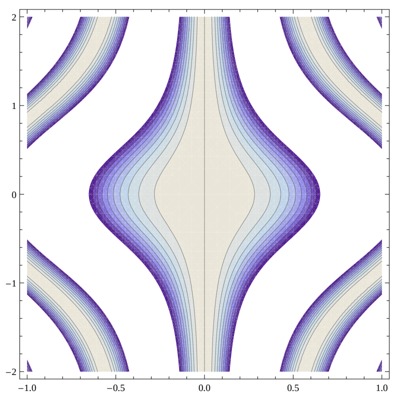
(b) Cosine source lensed by a Gaussian potential with an amplitude of 0.1



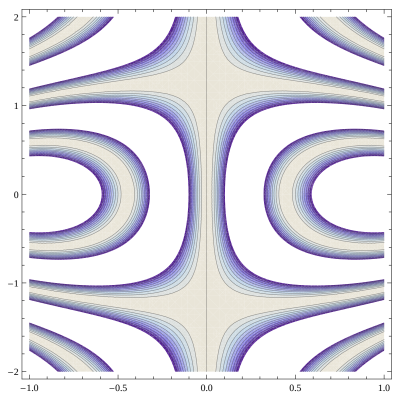
(c) Cosine source lensed by a Gaussian potential with an amplitude of 0.5



(d) Cosine source lensed by a Gaussian potential with an amplitude of 1.0



(e) Cosine source lensed by a Gaussian potential with an amplitude of 2.0



(f) Cosine source lensed by a Gaussian potential with an amplitude of 5.0

Figure 18 – A cosine source lensed by a Gaussian lens potential

The amplitudes of the lens potential must be much bigger here, due to the normalization of the gaussian $(1/(\sigma_1\sigma_2))$. The effects of an increasingly strong

Gaussian lens potential can be seen in figure 18. Figure 18(a) is the non-lensed source. Especially in figures (d)&(e) the magnifying effect of lensing comes out. The equibrightness lines, the lines which indicate points of equal brightness, still give the same amount of intensity, energy per surface area, but become broader, so the total energy will become higher.Figure 18(f) is a strong lensing event, with very clear distortions and makes the cosine source almost irrecognizable.

B.1.4 Source: Gauss, lens potential: Gauss

As noted in the previous sections the Gaussian distribution is a very important and natural distribution to work with both as source and as lens potential. To get a reasonably natural lensing event, we had best combine them. This is what we will do in this section.
Again we start by defining the distributions

$$\begin{aligned} \psi(x) &= Ce^{-(\mathbf{x}-\mathbf{x}_0)^T\mathbf{M}(\mathbf{x}-\mathbf{x}_0)} \\ &= \frac{d}{\sigma_1\sigma_2} e^{-\frac{(\mathbf{x}-\mathbf{x}_0)_1^2}{2\sigma_1^2}+\frac{(\mathbf{x}-\mathbf{x}_0)_2^2}{2\sigma_2^2}} \end{aligned} \tag{125}$$

$$\begin{aligned} S(y) &= De^{-(\mathbf{y}-\mathbf{y}_0)^T\mathbf{M}'(\mathbf{y}-\mathbf{y}_0)} \\ &= \frac{e}{\sigma_3\sigma_4} e^{-\frac{(\mathbf{y}-\mathbf{y}_0)_1^2}{2\sigma_3^2}+\frac{(\mathbf{y}-\mathbf{y}_0)_2^2}{2\sigma_4^2}} \end{aligned} \tag{126}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_1^2} \end{pmatrix}$$

and

$$\mathbf{M}' = \begin{pmatrix} \frac{1}{2\sigma_3^2} & 0 \\ 0 & \frac{1}{2\sigma_4^2} \end{pmatrix}$$

The lens equation is $\mathbf{y} = \mathbf{x} - \nabla\psi$. So we need to know what $\nabla\psi$ is.

$$\nabla\psi = -C \begin{pmatrix} \frac{(\mathbf{x}-\mathbf{x}_0)^{(1)}}{\sigma_1^2} \\ \frac{(\mathbf{x}-\mathbf{x}_0)^{(2)}}{\sigma_2^2} \end{pmatrix} e^{-\frac{((\mathbf{x}-\mathbf{x}_0)^{(1)})^2}{\sigma_1^2}-\frac{((\mathbf{x}-\mathbf{x}_0)^{(2)})^2}{\sigma_2^2}} \tag{127}$$

filling the lens equation into the $S(\mathbf{y})$ gives

$$\begin{aligned} S(\mathbf{y}) &= S(\mathbf{x}-\nabla\psi) \\ &= De^{-\frac{((\mathbf{y}-\mathbf{y}_0)^{(1)})^2}{2\sigma_3^2}+\frac{((\mathbf{y}-\mathbf{y}_0)^{(2)})^2}{2\sigma_4^2}} \\ &= D\exp\left(-\frac{((\mathbf{x}-\nabla\psi-\mathbf{y}_0)^{(1)})^2}{2\sigma_3^2}+\frac{((\mathbf{x}-\nabla\psi-\mathbf{y}_0)^{(2)})^2}{2\sigma_4^2}\right) \\ &= D\exp\left(-\frac{(\mathbf{x}+C\frac{(\mathbf{x}-\mathbf{x}_0)^{(1)}}{\sigma_1^2}\exp{(-\frac{((\mathbf{x}-\mathbf{x}_0)^{(1)})^2}{\sigma_1^2}-\frac{((\mathbf{x}-\mathbf{x}_0)^{(2)})^2}{\sigma_2^2})}-(\mathbf{y}_0)^{(1)})^2}{2\sigma_3^2}\right) \\ &\cdot\exp\left(\frac{(\mathbf{x}+C\frac{(\mathbf{x}-\mathbf{x}_0)^{(2)}}{\sigma_2^2}exp(-\frac{((\mathbf{x}-\mathbf{x}_0)^{(1)})^2}{\sigma_1^2}-\frac{((\mathbf{x}-\mathbf{x}_0)^{(2)})^2}{\sigma_2^2})-(\mathbf{y}_0)^{(2)})^2}{2\sigma_4^2}\right) \end{aligned} \tag{128}$$

The code in mathematica to plot this function is in appendix D.9 and the plots themselces are in figure 19.

The first figure, 19(a), is again the unperturbed source. In the other figures in 19 the source has been distorted due to lensing effects. The source is elliptical with its orientated horizontally and centered at (1,1). It is clear that strong distortions appear quite quickly. The final three images are already in the strong lensing regime, showing multiple images.

We hope these four examples of lensing will help to make gravitational lensing more intuitive. In appendices D.6-D.9 the code used to make these plots in mathematica is present. Using these simple lines of code and the method used to make usable functions of both lens potentials and sources, you will be able to make plots of your favourite distributions and see how they change due to lensing events.

B.2 Analytic

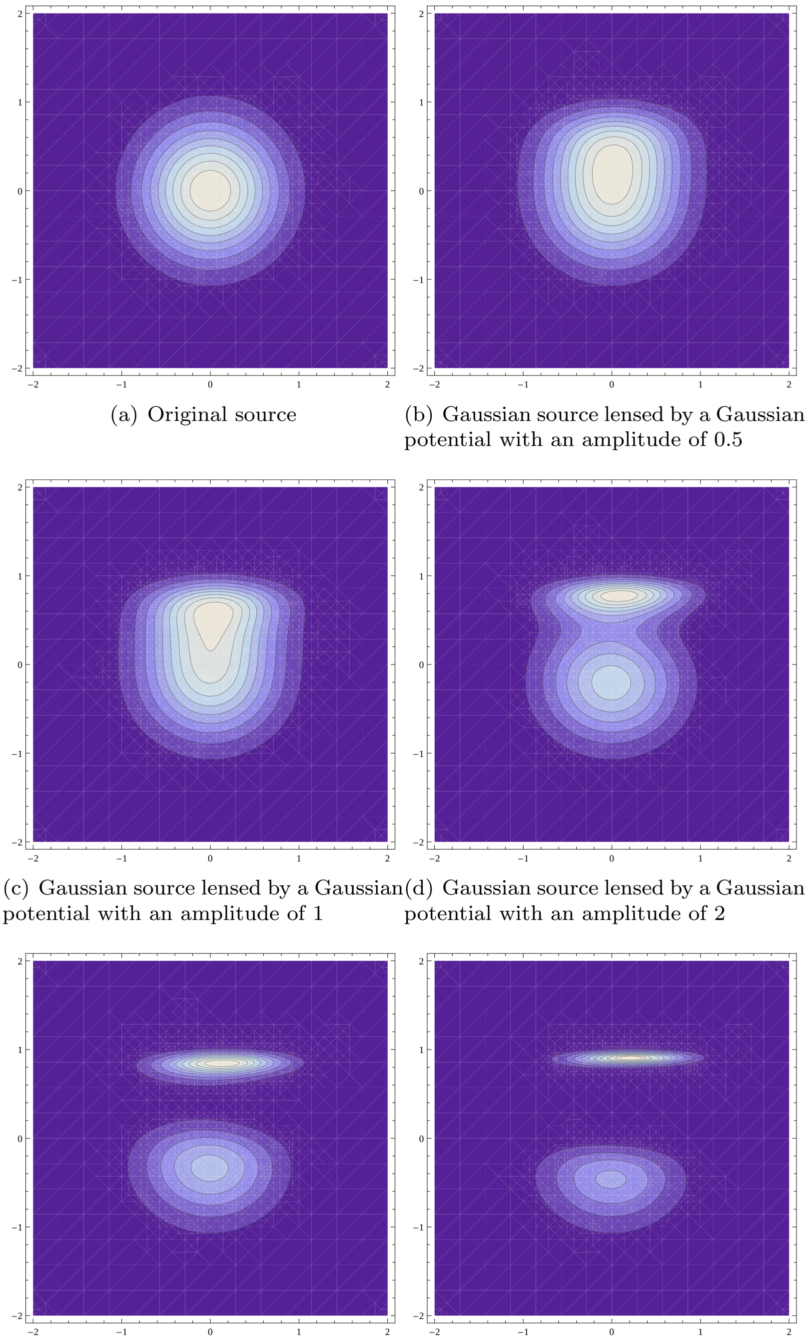
In the previous section we tried to make lensing clear visually, but one can also tell a lot about how distributions behave under lensing by looking at their analytic functions.

A power spectrum tells us which scales occur and how often they occur. The power spectrum of the image clearly depends on the source and on the lens potential. In appendix C it is explained how they depend on each other, yielding that the power spectrum of the image depends on the convolution between the power spectrum of the lens potential times the scale convolved with the power spectrum of the source times the scale:

$$C^{II}_{\mathbf{k}} = \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} C^{\tilde{S}\tilde{S}}_{\mathbf{k}_1} C^{\tilde{\phi}\tilde{\phi}}_{\mathbf{k}-\mathbf{k}_1} ((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2$$

We will again use the general Gaussian and cosine distributions in our treatise. To be able to make some sensible arguments about them we will take the root mean square of the functions as last step, in other words we will integrate the power spectrum of the image over all Fourier space. The step by step evaluation of the mathematics is done in appendix C. Here we will only give the results and we will try to give a meaning to them. The itemization of 4 combinations of the distributions begins with a definition of the functions, for a cosine the amplitude and the wavenumber are given as $(A, \mathbf{m})$ and for a Gaussian distribution the amplitude and the standard deviations in both spatial directions are given as (B, σ_1, σ_2). The titles of the items refer to the appendix where the derivation is presented.

The root mean square (RMS) of a function is the magnitude of a varying quantity. To see why this is useful for us we need two things. The first is that we work in Fourier space, in Fourier space everything is transformed into an infinite sum of sines. The second is that sines are varying quantities. In other words if one takes the integral over all Fourier space one looks at how all sines change, or how the field behaves. This means that integrating over the entier Fourier space of a power spectrum is taking the root mean square of the system.



(a) Original source

(b) Gaussian source lensed by a Gaussian potential with an amplitude of 0.5

(c) Gaussian source lensed by a Gaussian potential with an amplitude of 1

(d) Gaussian source lensed by a Gaussian potential with an amplitude of 2

(e) Gaussian source lensed by a Gaussian potential with an amplitude of 3

(f) Gaussian source lensed by a Gaussian potential with an amplitude of 5

Figure 19 – A Gaussian source lensed by a Gaussian lens potential

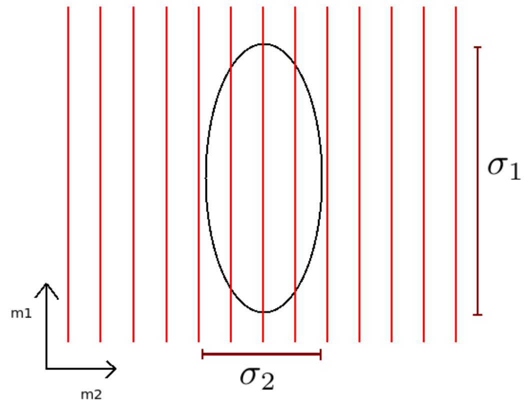


Figure 20 – The Gaussian distribution (the ellipsiod) and the cosine distribution (the lines) drawn in one figure. The orientation of the Gaussian is the direction of the elongation, in this case pointing up. The representation of the cosine distribution is given by equipotential lines.

Appendix C.1
The cosine source $(D, \mathbf{m})$ and the cosine lens potential $(C, \mathbf{n})$:

$$RMS = \frac{C^2 D^2}{4(2\pi)^2} m^2 n^2 \cos^2(\Theta) \tag{129}$$

If the lens potential and the source are aligned, the distortion to the image will be largest. If they are perpendicular the rms of the total image will be zero, i.e. the magnitude of change is zero.

Appendix C.2
The gaussian source (C, σ_1, σ_2) and the cosine lens potential $(D, \mathbf{m})$:

$$RMS(\mathbf{m}, \sigma) = \frac{C^2 D^2}{2} \left(m_1^2 \frac{\sigma_2}{\sigma_1} + m_2^2 \frac{\sigma_1}{\sigma_2} \right) \tag{130}$$

So if the orientation of the Gaussian is along a equipotential line of the lens potential, the root mean square will be maximum. If the orientation of the Gaussian is perpendicular to a equipotential line of the lens potential, the RMS will be minimized. See figure 20, for a graphical description of the situation.

Appendix C.3
The cosine source $(C, \mathbf{m})$ and the gaussian lens potential (D, σ_1, σ_2):

$$RMS(\mathbf{m}, \sigma) = \frac{C^2 D^2}{2} \left(m_1^2 \frac{\sigma_2}{\sigma_1} + m_2^2 \frac{\sigma_1}{\sigma_2} \right) \tag{131}$$

So if the equi-intensity lines of the source have the same orientation as the Gaussian lens potential, the power will be maximum. if the equi-intensity lines of the source are perpendicular to the orientation of the Gaussian, the RMS will be minimized. It is again useful to look at figure 20.

Appendix C.4
The gaussian source (C, σ_1, σ_2) and the gaussian lens potential (D, σ_3, σ_4):

$$
\begin{aligned}
RMS = 4\sigma_1^2\sigma_2^2\sigma_3^2 C^2 \sigma_4^2 D^2 \pi^2 \Bigg(& -\frac{\left(3\sigma_1^8 + 24\sigma_1^2\sigma_3^2 + 6\sigma_1^6\sigma_3^2 + 8\sigma_3^4 + 4\sigma_1^4\left(-10+\sigma_3^4\right)\right)\pi}{32\sigma_1^2\sigma_2\left(3\sigma_1^2-\sigma_3^2\right)\left(\sigma_1^2+\sigma_3^2\right)^2\sqrt{-3\sigma_1^4+\sigma_1^2\sigma_3^2}\,\sigma_4} \\
& - \frac{\left(24\sigma_1^4\left(5\sigma_2^4 - 3\sigma_2^2\sigma_4^2 - \sigma_4^4\right) + 8\sigma_1^2\sigma_3^2\left(-5\sigma_2^4 + 3\sigma_2^2\sigma_4^2 + \sigma_4^4\right) - \sigma_2^6\left(9\sigma_2^6 + 15\sigma_2^4\sigma_4^2 + 6\sigma_2^2\sigma_4^4 - 4\sigma_4^6\right)\right)\pi}{32\sigma_1^3\sigma_2^2\sigma_3\left(-3\sigma_1^2+\sigma_3^2\right)\left(3\sigma_2^2-\sigma_4^2\right)\left(\sigma_2^2+\sigma_4^2\right)^2\sqrt{-3\sigma_2^4+\sigma_2^2\sigma_4^2}} \\
& + \frac{\left(\sigma_1^4 + 5\sigma_1^2\sigma_3^2 + 2\sigma_3^4\right)\left(\sigma_2^4 + 5\sigma_2^2\sigma_4^2 + 2\sigma_4^4\right)\pi^2}{8\sigma_3^2\sqrt{\sigma_1^2\left(-\sigma_1^2+\sigma_3^2\right)}\sqrt{\sigma_1^2+\sigma_3^2}\left(2\sigma_1^2+\sigma_3^2\right)\sigma_4^2\sqrt{\sigma_2^2\left(-\sigma_2^2+\sigma_4^2\right)}\sqrt{\sigma_2^2+\sigma_4^2}\left(2\sigma_2^2+\sigma_4^2\right)}\Bigg)
\end{aligned}
\tag{132}
$$

This analytic solution is not useful for comparison, too many and too complicated terms.

C Deriving the RMS

In section B.2 the root mean squares of a few general functions are presented. In this appendix we will show the steps taken to get to them. We will use the linearized model of equation 56, but now untill first order in $\delta\psi$. To make things simpler we will take the correction on the lens potential as the total lens potential, so $\delta\psi$ becomes ψ and we assume the shape of the source to be known. This last assumption allows us the differentiate in the source plane, avoiding the magnification matrix. This simplifications will give us the following image

$$I = -(\nabla_x \psi)^T \nabla_y S_y \tag{133}$$

We will use this equation to become more familiar with the behaviour of the images under lensing. This behaviour will be examined via the power spectrum of I. The power spectrum of I is defined as

$$C_{\mathbf{k}}^{II} = \frac{\mathcal{F}^*(I(\mathbf{k}))\mathcal{F}(I(\mathbf{k}))}{(2\pi)^2}$$

So the Fourier transform of I is needed

$$\begin{aligned}
\mathcal{F}(I(\mathbf{k})) &= -\mathcal{F}((\nabla_x\psi)^T \nabla_y S_y) \\
&= -\mathcal{F}((\nabla_x\psi)^T) \otimes \mathcal{F}(\nabla_y S_y) \\
&= (\mathbf{k}^T \mathcal{F}(\psi)) \otimes (\mathbf{k}\mathcal{F}(S_y)) \\
&= \int d^2\mathbf{k}_1 \tilde{S}(\mathbf{k}_1)\tilde{\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1
\end{aligned}$$

The third equality was taken from equation (23). The convolution in the fourth equality was taken from equation (25). The next step is to take the power spectrum of I

$$\begin{aligned}
C_{\mathbf{k}}^{II} &= \frac{\mathcal{F}^*(I(\mathbf{k}))\mathcal{F}(I(\mathbf{k}))}{(2\pi)^2} \\
&= \frac{1}{(2\pi)^2} \int d^2\mathbf{k}_1 \tilde{S}(\mathbf{k}_1)\tilde{\psi}(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1 \int d^2\mathbf{k}'_1 \tilde{S}(\mathbf{k}'_1)\tilde{\psi}(\mathbf{k}-\mathbf{k}'_1)(\mathbf{k}-\mathbf{k}'_1)^T \mathbf{k}'_1 \\
&= (2\pi)^2 \iint d^2\mathbf{k}_1\mathbf{k}'_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}} \delta^2(\mathbf{k}'_1-\mathbf{k}_1)(\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1(\mathbf{k}-\mathbf{k}'_1)^T\mathbf{k}'_1 \\
&= (2\pi)^2 \int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}} ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2
\end{aligned} \tag{134}$$

The only unknowns in equation (134) are the Fourier transforms of the source S and of the lens potential ψ. Since earlier on it was explained that the functions used in both are the cosine and the Gaussian distributions. Thus the logical thing to do is compute the Fourier transforms of the cosine function and the Gaussian function and use these later on. The first equation to be taken on is the Gaussian distribution.

The Fourier transforms of standard functions in 2D is given by equation (22).

$$\begin{aligned}\mathcal{F}(F_{Gauss}) &= \int_{-\infty}^{\infty} d^2\mathbf{x} F_{Gauss}(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} \\ &= D\int_{-\infty}^{\infty} d^2\mathbf{x} e^{-(\mathbf{x}-\mathbf{x}_0)^T\mathbf{M}(\mathbf{x}-\mathbf{x}_0)} e^{-i\mathbf{x}\cdot\mathbf{k}}\end{aligned}$$

A coordinate transformation is done to get an integral which can be found in literature. $(\mathbf{x}-\mathbf{x}_0) \to \mathbf{x}$ and thus also $\mathbf{x} \to (\mathbf{x}+\mathbf{x}_0)$.

$$\begin{aligned}\mathcal{F}(F_{Gauss}) &= D\int_{-\infty}^{\infty} d^2\mathbf{x} e^{-\mathbf{x}^T\mathbf{M}\mathbf{x}} e^{-i(\mathbf{x}+\mathbf{x}_0)\cdot\mathbf{k}} \\ &= D\int_{-\infty}^{\infty} d^2\mathbf{x} e^{-\mathbf{x}^T\mathbf{M}\mathbf{x}-i(\mathbf{x}+\mathbf{x}_0)\cdot\mathbf{k}} \\ &= De^{-i\mathbf{x}_0\cdot\mathbf{k}}\int_{-\infty}^{\infty} d^2\mathbf{x} e^{-\mathbf{x}^T\mathbf{M}\mathbf{x}-i\mathbf{x}\cdot\mathbf{k}} \\ &= De^{-i\mathbf{x}_0\cdot\mathbf{k}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2 e^{-\frac{x_1^2}{2\sigma_1^2}-\frac{x_2^2}{2\sigma_2^2}-i(x_1k_1+x_2k_2)} \\ &= De^{-i\mathbf{x}_0\cdot\mathbf{k}}\int_{-\infty}^{\infty} dx_1 e^{-\frac{x_1^2}{2\sigma_1^2}-ix_1k_1}\int_{-\infty}^{\infty} dx_2 e^{-\frac{x_2^2}{2\sigma_2^2}-ix_2k_2}\end{aligned}$$

From the integral book of Gradshteyn [6], equation 3.323.2, the following integral was found:

$$\int_{-\infty}^{\infty} e^{-p^2x^2\pm qx} = e^{\frac{q^2}{4p^2}}\frac{\sqrt{\pi}}{p} \qquad \text{where p} > 0 \tag{135}$$

This can easily be repeated by completing the squares, by which this integral becomes the standard Gaussian integral of equation (145).
Applying this to the Fourier transformation gives

$$\begin{aligned}\mathcal{F}(F_{Gauss}) &= De^{-i\mathbf{x}_0\cdot\mathbf{k}}\left(e^{\frac{(ik_1)^2}{4\frac{1}{2\sigma_1^2}}}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2\sigma_1^2}}}\right)\left(e^{\frac{(ik_2)^2}{4\frac{1}{2\sigma_2^2}}}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2\sigma_2^2}}}\right) \\ &= De^{-i\mathbf{x}_0\cdot\mathbf{k}}\left(e^{\frac{2(\sigma_1 ik_1)^2}{4}}\sqrt{2\pi}\sigma_1\right)\left(e^{\frac{2(\sigma_2 ik_2)^2}{4}}\sqrt{2\pi}\sigma_2\right) \\ &= 2\pi D\sigma_1\sigma_2 e^{-i\mathbf{x}_0\cdot\mathbf{k}} e^{-\frac{1}{2}(\sigma_1^2k_1^2+\sigma_2^2k_2^2)} \\ &= 2\pi D\sigma_1\sigma_2 e^{-i\mathbf{x}_0\cdot\mathbf{k}} e^{-\frac{1}{2}\mathbf{k}^T P\mathbf{k}}\end{aligned} \tag{136}$$

Where

$$P = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

The Fourier transformation of F_{Gauss} was taken in the first place because it is needed in the power spectrum.

$$C_{\mathbf{k}}^{\tilde{F_{Gauss}}\tilde{F_{Gauss}}} = \frac{\mathcal{F}^*(F_{Gauss})\mathcal{F}(F_{Gauss})}{(2\pi)^2}$$
$$= D^2\sigma_1^2\sigma_2^2 e^{-\mathbf{k}^T P\mathbf{k}} \quad (137)$$

The power spectrum of the Gaussian distribution is now known. The next thing to do is find the power spectrum of the cosine. To do this one needs the Fourier transformation of F_{cosine}.

$$\begin{aligned}
\mathcal{F}(F_{cosine}) &= \int_{-\infty}^{\infty} d^2\mathbf{x} C\cos(\mathbf{m}\cdot(\mathbf{x}-\mathbf{x}_0))e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= C\int_{-\infty}^{\infty} d^2\mathbf{x}\frac{e^{i\mathbf{m}\cdot(\mathbf{x}-\mathbf{x}_0)}+e^{-i\mathbf{m}\cdot\mathbf{x}}}{2}e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= Ce^{-i\mathbf{m}\cdot\mathbf{x}_0}\int_{-\infty}^{\infty} d^2\mathbf{x}\left(e^{i(\mathbf{m}-\mathbf{k})\cdot\mathbf{x}}+e^{-i(\mathbf{m}+\mathbf{k})\cdot\mathbf{x}}\right) \\
&= Ce^{-i\mathbf{m}\cdot\mathbf{x}_0}\frac{\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k})}{2}
\end{aligned}$$

Giving a power spectrum of.

$$\begin{aligned}
C_{\mathbf{k}}^{\tilde{F_{cosine}}\tilde{F_{cosine}}} &= \frac{\mathcal{F}^*(F_{cosine})\mathcal{F}(F_{cosine})}{(2\pi)^2} \\
&= C^2\frac{(\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k}))^2}{(4\pi)^2} \\
&= C^2\frac{\delta^2(\mathbf{m}-\mathbf{k})+2\delta(\mathbf{m}-\mathbf{k})\delta(\mathbf{m}+\mathbf{k})+\delta^2(\mathbf{m}+\mathbf{k})}{(4\pi)^2} \\
&= C^2\frac{\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k})}{(4\pi)^2} \quad (138)
\end{aligned}$$

In the last step the fact was used that the different Dirac delta functions don't have anything to do with one another, so when they are multiplied they are zero. When multiplying two Dirac delta functions which are the same one gets the same delta function.
Now all terms are known to fill in the power spectrum of I from equation (134). The three equations needed are

I

$$C_{\mathbf{k}}^{II} = (2\pi)^2\int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2$$

II

$$C_{\mathbf{k}}^{\tilde{F_{Gauss}}\tilde{F_{Gauss}}} = D^2\sigma_1^2\sigma_2^2 e^{-\mathbf{k}^T P\mathbf{k}}$$

III

$$C_{\mathbf{k}}^{\tilde{F_{cosine}}\tilde{F_{cosine}}} = C^2\frac{\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k})}{(4\pi)^2}$$

C.1 Source: Cosine, lens potential: Cosine

In this section the following lens potential ψ and source distribution S are examined

$$\psi(x) = C\cos(\mathbf{m}\cdot(\mathbf{x}-\mathbf{x}_0)) \tag{139}$$

$$S(y) = D\cos(\mathbf{n}\cdot(\mathbf{y}-\mathbf{y}_0)) \tag{140}$$

When considering the three main equations from this section, equation II is needed twice:

$$C_{\mathbf{k}}^{\tilde{\psi}\tilde{\psi}} = C^2\frac{\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k})}{(4\pi)^2}$$

$$C_{\mathbf{k}}^{\tilde{S}\tilde{S}} = D^2\frac{\delta(\mathbf{n}-\mathbf{k})+\delta(\mathbf{n}+\mathbf{k})}{(4\pi)^2}$$

filling this into the power spectrum for the change of surface brightness, equation I

$$\begin{aligned}
C_{\mathbf{k}}^{II} &= (2\pi)^2\int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= (2\pi)^2\int d^2\mathbf{k}_1 D^2\frac{\delta(\mathbf{n}-\mathbf{k}_1)+\delta(\mathbf{n}+\mathbf{k}_1)}{(4\pi)^2}C^2\frac{\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)}{(4\pi)^2}((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= \frac{C^2D^2}{16}\int\frac{d^2\mathbf{k}_1}{(2\pi)^2}\left(\delta(\mathbf{n}-\mathbf{k}_1)+\delta(\mathbf{n}+\mathbf{k}_1)\right)\left(\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)\right)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= \frac{C^2D^2}{16}\int\frac{d^2\mathbf{k}_1}{(2\pi)^2}\left(\delta(\mathbf{n}-\mathbf{k}_1)\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{n}-\mathbf{k}_1)\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)\right. \\
&\quad \left.+\,\delta(\mathbf{n}+\mathbf{k}_1)\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{n}+\mathbf{k}_1)\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)\right)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= \frac{C^2D^2}{16}\int\frac{d^2\mathbf{k}_1}{(2\pi)^2}(\delta(\mathbf{n}-\mathbf{k}_1)\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&\quad +\delta(\mathbf{n}-\mathbf{k}_1)\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&\quad +\delta(\mathbf{n}+\mathbf{k}_1)\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&\quad +\delta(\mathbf{n}+\mathbf{k}_1)\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2) \\
&= \frac{C^2D^2}{16}\frac{1}{(2\pi)^2}(\delta(\mathbf{m}-\mathbf{k}+\mathbf{n})((\mathbf{k}-\mathbf{n})^T\mathbf{n})^2 \\
&\quad +\delta(\mathbf{m}+\mathbf{k}-\mathbf{n})((\mathbf{k}-\mathbf{n})^T\mathbf{n})^2 \\
&\quad +\delta(\mathbf{m}-\mathbf{k}-\mathbf{n})(-(\mathbf{k}+\mathbf{n})^T\mathbf{n})^2 \\
&\quad +\delta(\mathbf{m}+\mathbf{k}+\mathbf{n})(-(\mathbf{k}+\mathbf{n})^T\mathbf{n})^2) \\
&= \frac{C^2D^2}{16(2\pi)^2}((\mathbf{k}-\mathbf{n})^T\mathbf{n})^2(\delta(\mathbf{m}-\mathbf{k}+\mathbf{n})+\delta(\mathbf{m}+\mathbf{k}-\mathbf{n})) \\
&\quad +((\mathbf{k}+\mathbf{n})^T\mathbf{n})^2(\delta(\mathbf{m}-\mathbf{k}-\mathbf{n})+\delta(\mathbf{m}+\mathbf{k}+\mathbf{n}))
\end{aligned}$$

When integrating over all momentum space (i.e. one calculates the root mean square (RMS) of the distribution)

$$\begin{aligned} RMS(m,n) &= \frac{C^2D^2}{16}\frac{1}{(2\pi)^2}\int d^2\mathbf{k}(\delta(\mathbf{m}-\mathbf{k}+\mathbf{n})((\mathbf{k}-\mathbf{n})^T\mathbf{n})^2 \\ &+ \delta(\mathbf{m}+\mathbf{k}-\mathbf{n})((\mathbf{k}-\mathbf{n})^T\mathbf{n})^2 \\ &+ \delta(\mathbf{m}-\mathbf{k}-\mathbf{n})((\mathbf{k}+\mathbf{n})^T\mathbf{n})^2 \\ &+ \delta(\mathbf{m}+\mathbf{k}+\mathbf{n})((\mathbf{k}+\mathbf{n})^T\mathbf{n})^2) \\ &= \frac{C^2D^2}{16}\frac{1}{(2\pi)^2}4(\mathbf{m}^T\mathbf{n})^2 \\ &= \frac{C^2D^2}{4(2\pi)^2}(\mathbf{m}^T\mathbf{n})^2 \\ &= \frac{C^2D^2}{4(2\pi)^2}m^2n^2\cos^2(\Theta) \end{aligned} \tag{141}$$

Where m is the length of vector $\mathbf{m}$ and n is the length of the vector $\mathbf{n}$. Θ is the angle between the two vectors. If the lens potential and the source are aligned ($\Theta = 0$) the total power is maximized, if they are perpendicular ($\Theta = 0$) the power will be zero.

What does this mean? The total change on the image is dependent on the amplitudes of both the source and the lens potential. This means that when $\mathbf{m}^T\mathbf{n}$ gets bigger, i.e. the wavenumbers of the waves which constitute the source and the potential get bigger, the total power of the images gets bigger. Also if the amplitudes CD gets bigger the power will increase. This is caused by the fact that more images are being formed.

C.2 Source: Gauss, lens potential: Cosine

In this section the following lens potential ψ and source distribution S are examined

$$\psi(x) = C\cos(\mathbf{m}\cdot(\mathbf{x}-\mathbf{x}_0)) \tag{142}$$

$$S(y) = De^{-(\mathbf{y}-\mathbf{y}_0)^T\mathbf{M}(\mathbf{y}-\mathbf{y}_0)} \tag{143}$$

Thus equations II and III are needed

$$C_{\mathbf{k}}^{\tilde{S}\tilde{S}} = D^2\sigma_1^2\sigma_2^2e^{-\mathbf{k}^TP\mathbf{k}}$$

$$C_{\mathbf{k}}^{\tilde{\psi}\tilde{\psi}} = C^2\frac{\delta(\mathbf{m}-\mathbf{k})+\delta(\mathbf{m}+\mathbf{k})}{(4\pi)^2}$$

Filling this into equation I

$$C_{\mathbf{k}}^{II} = (2\pi)^2 \int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}} ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2$$
$$= (2\pi)^2 \int d^2\mathbf{k}_1 D^2\sigma_1^2\sigma_2^2 e^{-\mathbf{k}_1^T P \mathbf{k}_1} C^2 \frac{\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)}{(4\pi)^2} ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2$$
$$= C^2D^2\sigma_1^2\sigma_2^2 \int \frac{d^2\mathbf{k}_1}{4} e^{-\mathbf{k}_1^T P \mathbf{k}_1} \left(\delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)+\delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)\right) ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2$$
$$= C^2D^2\sigma_1^2\sigma_2^2 \int \frac{d^2\mathbf{k}_1}{4} (e^{-\mathbf{k}_1^T P \mathbf{k}_1} \delta(\mathbf{m}-\mathbf{k}+\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2$$
$$+ e^{-\mathbf{k}_1^T P \mathbf{k}_1} \delta(\mathbf{m}+\mathbf{k}-\mathbf{k}_1)((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2)$$
$$= \frac{C^2D^2\sigma_1^2\sigma_2^2}{4} (e^{-(\mathbf{k}-\mathbf{m})^T P (\mathbf{k}-\mathbf{m}}) (\mathbf{m}^T(\mathbf{k}-\mathbf{m}))^2$$
$$+ e^{-(\mathbf{k}+\mathbf{m})^T P (\mathbf{k}+\mathbf{m})} ((-\mathbf{m})^T(\mathbf{k}+\mathbf{m}))^2)$$

When integrating over all momentum space (i.e. one calculates the total power in the power spectrum)

$$RMS(m,\sigma) = \frac{C^2D^2\sigma_1^2\sigma_2^2}{4} (\int_{-\infty}^{\infty} d^2\mathbf{k} e^{-(\mathbf{k}-\mathbf{m})^T P (\mathbf{k}-\mathbf{m}}) (\mathbf{m}^T(\mathbf{k}-\mathbf{m}))^2$$
$$+ \int_{-\infty}^{\infty} d^2\mathbf{k} e^{-(\mathbf{k}+\mathbf{m})^T P (\mathbf{k}+\mathbf{m})} ((-\mathbf{m})^T(\mathbf{k}+\mathbf{m}))^2)$$

The boundaries are shown to make clear that a coordinate transformation in the first line of $\mathbf{k}-\mathbf{m} \to \mathbf{k}$ and in the second line of $\mathbf{k}+\mathbf{m} \to \mathbf{k}$ is allowed. This yields two indentical integrals. Furthermore in the second step $k^{(1)}$ will be defined as k_1 and $k^{(2)}$ will be defined as k_2.

$$RMS(m,\sigma) = \frac{C^2D^2\sigma_1^2\sigma_2^2}{4} \cdot 2 \cdot \int d^2\mathbf{k} e^{-\mathbf{k}^T P \mathbf{k}} (\mathbf{m}^T\mathbf{k})^2$$
$$= \frac{C^2D^2\sigma_1^2\sigma_2^2}{2} \int dk_1 dk_2 e^{-(\sigma_1^2 k_1^2 + \sigma_2^2 k_2^2)} (k_1 m_1 + k_2 m_2)^2$$
$$= \frac{C^2D^2\sigma_1^2\sigma_2^2}{2} \int dk_1 dk_2 e^{-\sigma_1^2 k_1^2} e^{-\sigma_2^2 k_2^2} (k_1^2 m_1^2 + 2k_1 m_1 k_2 m_2 + k_2^2 m_2^2)$$
$$= \frac{C^2D^2\sigma_1^2\sigma_2^2}{2} \left(m_1^2 \int dk_1 dk_2 e^{-\sigma_1^2 k_1^2} e^{-\sigma_2^2 k_2^2} k_1^2 \right.$$
$$+ 2m_1 m_2 \int dk_1 dk_2 e^{-\sigma_1^2 k_1^2} e^{-\sigma_2^2 k_2^2} k_1 k_2$$
$$\left. + m_2^2 \int dk_1 dk_2 e^{-\sigma_1^2 k_1^2} e^{-\sigma_2^2 k_2^2} k_2^2 \right)$$
$$= \frac{C^2D^2\sigma_1^2\sigma_2^2}{2} \left(m_1^2 \int dk_1 e^{-\sigma_1^2 k_1^2} k_1^2 \int dk_2 e^{-\sigma_2^2 k_2^2} \right.$$
$$+ 2m_1 m_2 \int dk_1 e^{-\sigma_1^2 k_1^2} k_1 \int dk_2 e^{-\sigma_2^2 k_2^2} k_2$$
$$\left. + m_2^2 \int dk_1 e^{-\sigma_1^2 k_1^2} \int dk_2 e^{-\sigma_2^2 k_2^2} k_2^2 \right)$$

To get the equation into a standard from, a transformation must be made of $\sigma_1 k_1 \rightarrow k_1$ and $\sigma_2 k_2 \rightarrow k_2$

$$\begin{aligned} RMS(m,\sigma) = \frac{C^2}{2} D^2 \sigma_1^2 \sigma_2^2 \bigg(& m_1^2 \sigma_1^{-3} \sigma_2^{-1} \int dk_1 e^{-k_1^2} k_1^2 \int dk_2 e^{-k_2^2} \\ & + 2 m_1 m_2 \sigma_1^{-2} \sigma_2^{-2} \int dk_1 e^{-k_1^2} k_1 \int dk_2 e^{-k_2^2} k_2 \\ & + m_2^2 \sigma_1^{-1} \sigma_2^{-3} \int dk_1 e^{-k_1^2} \int dk_2 e^{-k_2^2} k_2^2 \bigg) \end{aligned} \tag{144}$$

The standard equations needed are

$$\int dx e^{-\alpha x^2}$$

$$\int dx e^{-\alpha x^2} x$$

$$\int dx e^{-\alpha x^2} x^2$$

where α is a positive number.

$$\begin{aligned} \int dx e^{-\alpha x^2} &= (\alpha)^{-1/2} \int dx e^{-x^2} \\ &= (\alpha)^{-1/2} \left(\int dx e^{-x^2} \int dx e^{-x^2} \right)^{1/2} \\ &= (\alpha)^{-1/2} \left(\int dx e^{-x^2} \int dy e^{-y^2} \right)^{1/2} \\ &= (\alpha)^{-1/2} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-(x^2+y^2)} \right)^{1/2} \\ &= (\alpha)^{-1/2} \left(\int_0^{2\pi} \int_0^{\infty} r dr d\psi e^{-r^2} \right)^{1/2} \\ &= (\alpha)^{-1/2} \left(2\pi \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty} \right)^{1/2} \\ &= \sqrt{\frac{\pi}{\alpha}} \end{aligned} \tag{145}$$

In the fourth line a coordinate transformation from Cartesian coordinates to circular coordinates was made.
The second equation is quite trivial, it is an even equation (the e^{-x^2}) times an odd equation (x). This means when one integrates the function, it must yield zero

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2} x = 0 \tag{146}$$

The third equation is less trivial and actually needs parametrization.

$$\begin{aligned}\int dx e^{-\alpha x^2} x^2 &= -\int dx \frac{\partial}{\partial \alpha} e^{-\alpha x^2} \\ &= -\frac{\partial}{\partial \alpha} \int dx e^{-\alpha x^2} \\ &= -\frac{\partial}{\partial \alpha} \sqrt{\frac{\pi}{\alpha}} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \end{aligned} \tag{147}$$

Filling back in that $\alpha = 1$ gives the answer. All the answers summarized gives

$$\int dx e^{-x^2} = \sqrt{\pi}$$

$$\int dx e^{-x^2} x = 0$$

$$\int dx e^{-x^2} x^2 = \frac{\sqrt{\pi}}{2}$$

Now that this is known, equation (144) can be evaluated

$$\begin{aligned} RMS(m,\sigma) = \frac{C^2D^2}{2}\sigma_1^2\sigma_2^2 \Bigg(& m_1^2\sigma_1^{-3}\sigma_2^{-1}\frac{1}{4\pi^4}\int dk_1 e^{-k_1^2}k_1^2 \int dk_2 e^{-k_2^2} \\ & + 2m_1m_2\sigma_1^{-2}\sigma_2^{-2}\frac{1}{4\pi^4}\int dk_1 e^{-k_1^2}k_1 \int dk_2 e^{-k_2^2}k_2 \\ & + m_2^2\sigma_1^{-1}\sigma_2^{-3}\frac{1}{4\pi^4}\int dk_1 e^{-k_1^2} \int dk_2 e^{-k_2^2}k_2^2 \Bigg) \end{aligned} \tag{148}$$

$$= \frac{C^2D^2}{2}\left(m_1^2\frac{\sigma_2}{\sigma_1} + 0 + m_2^2\frac{\sigma_1}{\sigma_2}\right) \tag{149}$$

So

$$RMS(\mathbf{m},\sigma) = \frac{C^2D^2}{2}\left(m_1^2\frac{\sigma_2}{\sigma_1} + m_2^2\frac{\sigma_1}{\sigma_2}\right) \tag{150}$$

So the power increases with the amplitudes of both the source and the potential. Also a higher wave number in the perturbing cosine and a lower standard deviation of the Gaussian shaped source.

C.3 Source: Cosine, lens potential: Gauss

In this section the following potential ψ and source distribution S are examined

$$\psi(x) = Ce^{-(\mathbf{x}-\mathbf{x}_0)^T P(\mathbf{x}-\mathbf{x}_0)} \tag{151}$$

$$S(y) = D\cos(\mathbf{m}^T(\mathbf{y}-\mathbf{y}_0)) \tag{152}$$

Thus equations II and III are needed

$$C_{\mathbf{k}}^{\tilde{\psi}\tilde{\psi}} = D^2\sigma_1^2\sigma_2^2 e^{-\mathbf{k}^T P\mathbf{k}}$$

$$C_{\mathbf{k}}^{\tilde{S}\tilde{S}} = C^2 \frac{\delta(\mathbf{m}-\mathbf{k}) + \delta(\mathbf{m}+\mathbf{k})}{(4\pi)^2}$$

Looking at the power spectrum of the distortion of the image I, equation I is

$$C_{\mathbf{k}}^{II} = (2\pi)^2 \int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}} ((\mathbf{k}-\mathbf{k}_1)^T \mathbf{k}_1)^2$$

It is easy to see it doesn't make a difference if the source is a cosine and the lens potential a gaussian or vice versa. The only difference is that everything is shifted in the $\mathbf{k}_1$ space. But since integration goes over all of $\mathbf{k}_1$ space, this does not have any effect on the power spectrum. This means that the total power also stays the same, i.e.

$$RMS(\mathbf{m}, \sigma) = \frac{C^2 D^2}{2} \left(m_1^2 \frac{\sigma_2}{\sigma_1} + m_2^2 \frac{\sigma_1}{\sigma_2} \right) \tag{153}$$

C.4 Source: Gauss, lens potential: Gauss

In this section the following lens potential ψ and source distribution S are examined

$$\psi(x) = Ce^{-(\mathbf{x}-\mathbf{x}_0)^T \mathbf{M}(\mathbf{x}-\mathbf{x}_0)} \tag{154}$$

$$S(y) = De^{-(\mathbf{y}-\mathbf{y}_0)^T \mathbf{M}'(\mathbf{y}-\mathbf{y}_0)} \tag{155}$$

The magnification matrices $\mathbf{M}$ and $\mathbf{M}'$ belong respectively to the pertubation and the source functions. Where $\mathbf{M}$ consitst of the diagonal elements $\frac{1}{2\sigma_1^2}$ and $\frac{1}{2\sigma_2^2}$. $\mathbf{M}'$ consists of the diagonal elements $\frac{1}{2\sigma_3^2}$ and $\frac{1}{2\sigma_4^2}$.
Thus equation II is needed twice

$$C_{\mathbf{k}}^{\tilde{\psi}\tilde{\psi}} = C^2 \sigma_1^2 \sigma_2^2 e^{-\mathbf{k}^T P \mathbf{k}}$$

$$C_{\mathbf{k}}^{\tilde{S}\tilde{S}} = D^2 \sigma_3^2 \sigma_4^2 e^{-\mathbf{k}^T P' \mathbf{k}}$$

where P has σ_1^2 and σ_2^2 as its diagonal elements and P' σ_3^2 and σ_4^2

Looking at the power spectrum of the distortion of the image I, equation I is

$$\begin{aligned}
C_{\mathbf{k}}^{II} &= (2\pi)^2 \int d^2\mathbf{k}_1 C_{\mathbf{k}_1}^{\tilde{S}\tilde{S}} C_{\mathbf{k}-\mathbf{k}_1}^{\tilde{\psi}\tilde{\psi}} ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= (2\pi)^2 \int d^2\mathbf{k}_1 C^2\sigma_1^2\sigma_2^2 e^{-\mathbf{k}_1^T P'\mathbf{k}_1} D^2\sigma_3^2\sigma_4^2 e^{-(\mathbf{k}-\mathbf{k}_1)^T P(\mathbf{k}-\mathbf{k}_1)} ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= (2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \int d^2\mathbf{k}_1 \exp\left(-\left(\sigma_3^2(k_1^{(1)})^2 + \sigma_4^2(k_1^{(2)})^2\right)\right) \\
&\quad \exp\left(-\left(\sigma_1^2((k-k_1)^{(1)})^2 + \sigma_2^2((k-k_1)^{(2)})^2\right)\right) ((\mathbf{k}-\mathbf{k}_1)^T\mathbf{k}_1)^2 \\
&= (2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \int d^2\mathbf{k}_1 \exp\left(-\left(\sigma_3^2(k_1^{(1)})^2 + \sigma_4^2(k_1^{(2)})^2\right)\right) \\
&\quad \exp\left(-\sigma_1^2\left((k^{(1)})^2 - 2k^{(1)}k_1^{(1)} + (k_1^{(1)})^2\right)\right) \\
&\quad \exp\left(-\sigma_2^2\left((k^{(2)})^2 - 2k^{(2)}k_1^{(2)} + (k_1^{(2)})^2\right)\right) ((k^{(1)} - k_1^{(1)})k_1^{(1)} + (k^{(2)} - k_1^{(2)})k_1^{(2)})^2 \\
&= (2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \exp\left(-\left(\sigma_1^2(k^{(1)})^2 + \sigma_2^2(k^{(2)})^2\right)\right) \\
&\quad \int d^2\mathbf{k}_1 \exp\left(-\left((k_1^{(1)})^2(\sigma_1^2+\sigma_3^2) + (k_1^{(2)})^2(\sigma_2^2+\sigma_4^2)\right)\right) \\
&\quad \exp\left(2\left(\sigma_1^2 k^{(1)}k_1^{(1)} + \sigma_2^2 k^{(2)}k_1^{(2)}\right)\right) ((k^{(1)}k_1^{(1)})^2 + (k^{(2)}k_1^{(2)})^2 + (k_1^{(1)})^4 \\
&\quad + (k_1^{(2)})^4 + 2k^{(1)}k^{(2)}k_1^{(1)}k_1^{(2)} - 2k^{(1)}(k_1^{(1)})^3 - 2k^{(1)}(k_1^{(1)})^2k_1^{(2)} - 2k^{(2)}k_1^{(1)}(k_1^{(2)})^2 \\
&\quad - 2k^{(2)}(k_1^{(2)})^3 + (k_1^{(1)})^2(k_1^{(2)})^2)
\end{aligned}$$

Lets define some constants

$$p_1 \equiv \sigma_1^2 + \sigma_3^2 \qquad q_1 \equiv 2\sigma_1^2 k^{(1)}$$

and

$$p_2 \equiv \sigma_2^2 + \sigma_4^2 \qquad q_2 \equiv 2\sigma_2^2 k^{(2)}$$

Using this notation in the formula for the power spectrum and putting appropriate terms in the right integral gives

$$\begin{aligned}
C_{\mathbf{k}}^{II} =& (2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \exp\left(-\mathbf{k}^T P\mathbf{k}\right) \Big[\\
& \int dk_1^{(1)} \exp\left(-p_1(k_1^{(1)})^2 + q_1k_1^{(1)}\right) \left[(k^{(1)})^2(k_1^{(1)})^2 + (k_1^{(1)})^4 - 2k^{(1)}(k_1^{(1)})^3\right] \\
& \cdot \int dk_1^{(2)} \exp\left(-p_2(k_1^{(2)})^2 + q_2k_1^{(2)}\right) \\
& + \int dk_1^{(1)} \exp\left(-p_1(k_1^{(1)})^2 + q_1k_1^{(1)}\right) \\
& \cdot \int dk_1^{(2)} \exp\left(-p_2(k_1^{(2)})^2 + q_2k_1^{(2)}\right) \cdot \left[(k^{(2)})^2(k_1^{(2)})^2 + (k_1^{(2)})^4 - 2k^{(2)}(k_1^{(2)})^3\right] \\
& + 2k^{(1)}k^{(2)} \int dk_1^{(1)} \exp\left(-p_1(k_1^{(1)})^2 + q_1k_1^{(1)}\right) k_1^{(1)} \\
& \cdot \int dk_1^{(2)} \exp\left(-p_2(k_1^{(2)})^2 + q_2k_1^{(2)}\right) k_1^{(2)} \\
& + 2\int dk_1^{(1)} \exp\left(-p_1(k_1^{(1)})^2 + q_1k_1^{(1)}\right) (k_1^{(1)})^2
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int dk_1^{(2)} \exp\left(-p_2 (k_1^{(2)})^2 + q_2 k_1^{(2)}\right) (k_1^{(2)})^2 \\
&- 2k^{(1)} \int dk_1^{(1)} \exp\left(-p_1 (k_1^{(1)})^2 + q_1 k_1^{(1)}\right) k_1^{(1)} \\
&\cdot \int dk_1^{(2)} \exp\left(-p_2 (k_1^{(2)})^2 + q_2 k_1^{(2)}\right) (k_1^{(2)})^3 \\
&- 2k^{(2)} \int dk_1^{(1)} \exp\left(-p_1 (k_1^{(1)})^2 + q_1 k_1^{(1)}\right) (k_1^{(1)})^3 \\
&\cdot \int dk_1^{(2)} \exp\left(-p_2 (k_1^{(2)})^2 + q_2 k_1^{(2)}\right) k_1^{(2)} \quad \Big]
\end{aligned}
$$

Again a useful formula can be found in the book by Gradshteyn and Ryzhik[6], where equation 3.462.2 was taken:

$$\int_{-\infty}^{\infty} x^n e^{-px^2+2qx} dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\frac{d^{n-1}}{dq^{n-1}}\left(qe^{q^2/p}\right) \quad \text{where} \quad [p>0] \tag{156}$$

This equation can be solved using by completing the squares in the exponent first and after that the trick used to derive (147).
To be able to solve the power spectrum of I, it is necessary to know the functions for which $n = 0, 1, 2, 3, 4$.

n=0 Can't be calculate from (156), but it can with equation (135)

$$\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} = \frac{\sqrt{\pi}}{p} e^{\frac{q^2}{4p^2}} \tag{157}$$

n=1

$$\int_{-\infty}^{\infty} x e^{-px^2+2qx} dx = p^{-1}\sqrt{\frac{\pi}{p}} q e^{q^2/p} \tag{158}$$

n=2

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-px^2+2qx} dx &= \frac{1}{2p}\sqrt{\frac{\pi}{p}}\frac{d}{dq}\left(qe^{q^2/p}\right) \\
&= \frac{1}{2p}\sqrt{\frac{\pi}{p}}\left(1+\frac{2q^2}{p}\right)e^{q^2/p}
\end{aligned} \tag{159}
$$

n=3

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^3 e^{-px^2+2qx} dx &= \frac{1}{4p}\sqrt{\frac{\pi}{p}}\frac{d^2}{dq^2}\left(qe^{q^2/p}\right) \\
&= \frac{1}{4p}\sqrt{\frac{\pi}{p}}\frac{d}{dq}\left(1+\frac{2q^2}{p}\right)e^{q^2/p} \\
&= \frac{1}{4p}\sqrt{\frac{\pi}{p}}\left(\frac{4q}{p}+\frac{4q^3}{p^2}\right)e^{q^2/p}
\end{aligned} \tag{160}
$$

n=4

$$\int_{-\infty}^{\infty} x^4 e^{-px^2+2qx} dx = \frac{1}{8p}\sqrt{\frac{\pi}{p}}\frac{d^3}{dq^3}\left(qe^{q^2/p}\right)$$
$$= \frac{1}{8p}\sqrt{\frac{\pi}{p}}\frac{d}{dq}\left(\frac{4q}{p}+\frac{4q^3}{p^2}\right)e^{q^2/p}$$
$$= \frac{1}{8p}\sqrt{\frac{\pi}{p}}\left(\frac{4}{p}+\frac{16q^2}{p^2}+\frac{8q^4}{p^3}\right)e^{q^2/p} \tag{161}$$

The next step is to work out the formula for the power spectrum

$$\begin{aligned}
C_{\mathbf{k}}^{II} =&(2\pi)^2C^2D^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\exp\left(-\mathbf{k}^TP\mathbf{k}\right)\Bigg[\\
&\left((k^{(1)})^2\frac{1}{2p_1}\sqrt{\frac{\pi}{p_1}}\left(1+\frac{2q_1^2}{p_1}\right)e^{q_1^2/p_1}-2k^{(1)}\frac{1}{4p_1}\sqrt{\frac{\pi}{p_1}}\left(\frac{4q_1}{p_1}+\frac{4q_1^3}{p_1^2}\right)e^{q_1^2/p_1}\right.\\
&\left.+\frac{1}{8p_1}\sqrt{\frac{\pi}{p_1}}\left(\frac{4}{p_1}+\frac{16q_1^2}{p_1^2}+\frac{8q_1^4}{p_1^3}\right)e^{q_1^2/p_1}\right)\frac{\sqrt{\pi}}{p_2}e^{\frac{q_2^2}{4p_2^2}}\\
&+\frac{\sqrt{\pi}}{p_1}e^{\frac{q_1^2}{4p_1^2}}\left((k^{(2)})^2\frac{1}{2p_2}\sqrt{\frac{\pi}{p_2}}\left(1+\frac{2q_2^2}{p_2}\right)e^{q_2^2/p_2}-2k^{(2)}\frac{1}{4p_2}\sqrt{\frac{\pi}{p_2}}\left(\frac{4q_2}{p_2}+\frac{4q_2^3}{p_2^2}\right)e^{q_2^2/p_2}\right.\\
&\left.+\frac{1}{8p_2}\sqrt{\frac{\pi}{p_2}}\left(\frac{4}{p_2}+\frac{16q_2^2}{p_2^2}+\frac{8q_2^4}{p_2^3}\right)e^{q_2^2/p_2}\right)2k^{(1)}k^{(2)}\left(p_1^{-1}\sqrt{\frac{\pi}{p_1}}q_1e^{q_1^2/p_1}p_2^{-1}\sqrt{\frac{\pi}{p_2}}q_2e^{q_2^2/p_2}\right)\\
&+2\left(\frac{1}{2p_1}\sqrt{\frac{\pi}{p_1}}\left(1+\frac{2q_1^2}{p_1}\right)e^{q_1^2/p_1}\frac{1}{2p_2}\sqrt{\frac{\pi}{p_2}}\left(1+\frac{2q_2^2}{p_2}\right)e^{q_2^2/p_2}\right)\\
&-2k^{(1)}\left(p_1^{-1}\sqrt{\frac{\pi}{p_1}}q_1e^{q_1^2/p_1}\frac{1}{4p_2}\sqrt{\frac{\pi}{p_2}}\left(\frac{4q_2}{p_2}+\frac{4q_2^3}{p_2^2}\right)e^{q_2^2/p_2}\right)\\
&-2k^{(2)}\left(\frac{1}{4p_1}\sqrt{\frac{\pi}{p_1}}\left(\frac{4q_1}{p_1}+\frac{4q_1^3}{p_1^2}\right)e^{q_1^2/p_1}p_2^{-1}\sqrt{\frac{\pi}{p_2}}q_2e^{q_2^2/p_2}\right)\Bigg]
\end{aligned}$$

Simplifying

$$\begin{aligned}
&C_{\mathbf{k}}^{II} = (2\pi)^2C^2D^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\exp\left(-\frac{1}{4}\mathbf{k}^TP\mathbf{k}\right)\Bigg[\\
&\frac{\pi}{8p_1^{3/2}p_2}\exp\left(\frac{q_1^2}{p_1}+\frac{q_2^2}{4p_2^2}\right)\left(4(k^{(1)})^2\left(1+\frac{2q_1^2}{p_1}\right)-2k^{(1)}\left(\frac{4q_1}{p_1}+\frac{4q_1^3}{p_1^2}\right)+\frac{4}{p_1}+\frac{16q_1^2}{p_1^2}+\frac{8q_1^4}{p_1^3}\right)\\
&+\frac{\pi}{8p_1p_2^{3/2}}\exp\left(\frac{q_1^2}{4p_1^2}+\frac{q_2^2}{p_2}\right)\left(4(k^{(2)})^2\left(1+\frac{2q_2^2}{p_2}\right)-2k^{(2)}\left(\frac{4q_2}{p_2}+\frac{4q_2^3}{p_2^2}\right)+\frac{4}{p_2}+\frac{16q_2^2}{p_2^2}+\frac{8q_2^4}{p_2^3}\right)\\
&+\frac{\pi}{2p_1^{3/2}p_2^{3/2}}\exp\left(\frac{q_1^2}{p_1}+\frac{q_2^2}{p_2}\right)\left(4k^{(1)}k^{(2)}+\left(1+\frac{2q_1^2}{p_1}\right)\left(1+\frac{2q_2^2}{p_2}\right)\right.\\
&\left.-2k^{(1)}q_1\left(\frac{4q_2}{p_2}+\frac{4q_2^3}{p_2^2}\right)-2k^{(2)}q_2\left(\frac{4q_1}{p_1}+\frac{4q_1^3}{p_1^2}\right)\right)
\end{aligned}$$

Remember that q_1 and q_2 are dependent on the wavenumber $\mathbf{k}$.
Integrating over momentum space gives the total power of the spectrum. Integrating over a symmetric term times an assymmetric term gives zero. The

terms which are zero are omitted in the next step.

$$\begin{aligned}
RMS =& \int d\mathbf{k} C^{II}_{\mathbf{k}} \\
=& \int dk^{(1)} dk^{(2)} C^{II}_{k^{(1)},k^{(2)}} \\
=&(2\pi)^2 C^2 D^2 \sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2 \, [\int dk^{(1)} \exp\left(-\sigma_1^2 (k^{(1)})^2\right) \frac{\pi}{8 p_1^{3/2} p_2} \exp\left(\frac{q_1^2}{p_1}\right) \\
&\cdot \left(4(k^{(1)})^2 \left(1 + \frac{2q_1^2}{p_1}\right) - 2k^{(1)} \left(\frac{4q_1}{p_1} + \frac{4q_1^3}{p_1^2}\right) + \frac{4}{p_1} + \frac{16 q_1^2}{p_1^2} + \frac{8q_1^4}{p_1^3}\right) \\
&\cdot \int dk^{(2)} \exp\left(-\sigma_2^2 (k^{(2)})^2\right) \exp\left(\frac{q_2^2}{4p_2^2}\right) \\
&+ \int dk^{(1)} \exp\left(-\sigma_1^2 (k^{(1)})^2\right) \exp\left(\frac{q_1^2}{4p_1^2}\right) \\
&\cdot \int dk^{(2)} \exp\left(-\sigma_2^2 (k^{(2)})^2\right) \frac{\pi}{8 p_1 p_2^{3/2}} \exp\left(\frac{q_2^2}{p_2}\right) \\
&\cdot \left(4(k^{(2)})^2 \left(1 + \frac{2q_2^2}{p_2}\right) - 2k^{(2)} \left(\frac{4q_2}{p_2} + \frac{4q_2^3}{p_2^2}\right) + \frac{4}{p_2} + \frac{16 q_2^2}{p_2^2} + \frac{8q_2^4}{p_2^3}\right) \\
&+ \frac{\pi}{2 p_1^{3/2} p_2^{3/2}} \int dk^{(1)} \exp\left(-\left(\sigma_1^2 (k^{(1)})^2 - \frac{q_1^2}{p_1}\right)\right) \left(1 + \frac{2q_1^2}{p_1}\right) \\
&\cdot \int dk^{(2)} \exp\left(\left(-\sigma_2^2 (k^{(2)})^2 - \frac{q_2^2}{p_2}\right)\right) \left(1 + \frac{2q_2^2}{p_2}\right) \;]
\end{aligned}$$

Simplifying and filling in the q's.

$$\begin{aligned}
RMS =&(2\pi)^2 C^2 D^2 \sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2 \left[\frac{\pi}{8 p_1^{3/2} p_2} \int dk^{(1)} \exp\left(-\left(\sigma_1^2 - \frac{4\sigma_1^4}{p_1}\right)(k^{(1)})^2\right) \right. \\
&\cdot \Big(\frac{4}{p_1} + \left(4 - \frac{16\sigma_1^2}{p_1} + \frac{64\sigma_1^4}{p_1^2}\right)(k^{(1)})^2 \\
&+ \left(\frac{32\sigma_1^4}{p_1} - \frac{64\sigma_1^6}{p_1^2} + \frac{128\sigma_1^8}{p_1^3}\right)(k^{(1)})^4 \Big) \\
&\cdot \int dk^{(2)} \exp\left(-\left(\sigma_2^2 - \frac{\sigma_2^4}{p_2^2}\right)(k^{(2)})^2\right) + \frac{\pi}{8 p_1 p_2^{3/2}} \\
&\cdot \int dk^{(1)} \exp\left(-\left(\sigma_1^2 - \frac{\sigma_1^4}{p_1^2}\right)(k^{(1)})^2\right) \int dk^{(2)} \exp\left(-\left(\sigma_2^2 - \frac{4\sigma_2^4}{p_2}\right)(k^{(2)})^2\right) \\
&\cdot \Big(\frac{4}{p_2} + \left(4 - \frac{16\sigma_2^2}{p_2} + \frac{64\sigma_2^4}{p_2^2}\right)(k^{(2)})^2 \\
&+ \left(\frac{32\sigma_2^4}{p_2} - \frac{64\sigma_2^3}{p_2^2} + \frac{128\sigma_2^8}{p_2^3}\right)(k^{(2)})^4 \Big) \\
&+ \frac{\pi}{2 p_1^{3/2} p_2^{3/2}} \int dk^{(1)} \exp\left(-\left(\sigma_1^2 - \frac{4\sigma_1^4}{p_1^2}\right)(k^{(1)})^2\right) \left(1 + \frac{8\sigma_1^4}{p_1}(k^{(1)})^2\right) \\
&\left. \cdot \int dk^{(2)} \exp\left(-\left(\sigma_2^2 - \frac{4\sigma_2^4}{p_2^2}\right)(k^{(2)})^2\right) \left(1 + \frac{8\sigma_2^4}{p_2}(k^{(2)})^2\right) \; \right]
\end{aligned}$$

This can again be solved using equation (156), which was taken from an integral book. Solutions were already derived up to fourth order in equations (157)-(161). In this case q=0 and only even n are needed, so

n=0

$$\int_{-\infty}^{\infty} e^{-px^2} = \sqrt{\frac{\pi}{p}}$$

n=2

$$\int_{-\infty}^{\infty} x^2 e^{-px^2} dx = \frac{1}{2p}\sqrt{\frac{\pi}{p}}$$

n=4

$$\int_{-\infty}^{\infty} x^4 e^{-px^2} dx = \frac{1}{2p^2}\sqrt{\frac{\pi}{p}}$$

Note that the definition of the equation for $n = 0$ has been changed a bit, from p^2 to p.

taking

$$\alpha_1 = \frac{p_1\sigma_1^2 - 4\sigma_1^4}{p_1} \qquad \beta_1 = \frac{p_2^2\sigma_2^2 - \sigma_2^4}{p_2^2}$$

$$\alpha_2 = \frac{p_1^2\sigma_1^2 - \sigma_1^4}{p_1^2} \qquad \beta_2 = \frac{p_2\sigma_2^2 - 4\sigma_2^4}{p_2}$$

$$\alpha_3 = \frac{p_1^2\sigma_1^2 - 4\sigma_1^4}{p_1^2} \qquad \beta_3 = \frac{p_2^2\sigma_2^2 - 4\sigma_2^4}{p_2^2}$$

this gives us

$$\begin{aligned}
RMS =&(2\pi)^2C^2D^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \Bigg[\frac{\pi}{8p_1^{3/2}p_2} \cdot \frac{\pi}{\sqrt{\alpha_1\beta_1}}\Bigg(\frac{4}{p_1} + \left(4 - \frac{4\sigma_1^2}{p_1} + \frac{4\sigma_1^4}{p_1^2}\right)\frac{1}{2\alpha_1} \\
&+ \left(\frac{2\sigma_1^4}{p_1} - \frac{\sigma_1^3}{p_1^2} + \frac{\sigma_1^8}{2p_1^3}\right)\frac{1}{2\alpha_1^2} \Bigg) \\
&+ \frac{\pi}{8p_1p_2^{3/2}}\frac{\pi}{\sqrt{\alpha_2\beta_2}}\Bigg(\frac{4}{p_2} + \left(4 - \frac{4\sigma_2^2}{p_2} + \frac{4\sigma_2^4}{p_2^2}\right)\frac{1}{2\beta_2} \\
&+ \left(\frac{2\sigma_2^4}{p_2} - \frac{\sigma_2^3}{p_2^2} + \frac{\sigma_2^8}{2p_2^3}\right)\frac{1}{2\beta_2^2} \Bigg) \\
&+ \frac{\pi}{2p_1^{3/2}p_2^{3/2}}\frac{\pi}{\sqrt{\alpha_3\beta_3}}\left(1 + \frac{\sigma_1^4}{2p_1}\frac{1}{2\alpha_3}\right) \\
&\cdot \left(1 + \frac{\sigma_2^4}{2p_2}\frac{1}{2\beta_3}\right) \Bigg]
\end{aligned}$$

fill in the α's and $\beta's$:

$$
\begin{aligned}
RMS =&(2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \Bigg[\frac{\pi}{8p_1^{3/2}p_2} \\
&\cdot \pi\sqrt{\frac{p_1p_2^2}{(p_1\sigma_1^2-4\sigma_1^4)(p_2^2\sigma_2^2-\sigma_2^4)}}\Bigg(\frac{4}{p_1}+\left(4-\frac{4\sigma_1^2}{p_1}+\frac{4\sigma_1^4}{p_1^2}\right)\frac{1}{2}\frac{p_1}{p_1\sigma_1^2-4\sigma_1^4} \\
&+\left(\frac{2\sigma_1^4}{p_1}-\frac{\sigma_1^3}{p_1^2}+\frac{\sigma_1^8}{2p_1^3}\right)\frac{1}{2}\frac{p_1^2}{(p_1\sigma_1^2-4\sigma_1^4)^2}\Bigg) \\
&+\frac{\pi}{8p_1p_2^{3/2}}\pi\sqrt{\frac{p_1^2p_2}{(p_2\sigma_2^2-4\sigma_2^4)(4p_1^2\sigma_1^2-\sigma_1^4)}}\Bigg(\frac{4}{p_2}+\left(4-\frac{4\sigma_2^2}{p_2}+\frac{4\sigma_2^4}{p_2^2}\right)\frac{1}{2}\frac{p_2}{p_2\sigma_2^2-4\sigma_2^4} \\
&+\left(\frac{2\sigma_2^4}{p_2}-\frac{\sigma_2^3}{p_2^2}+\frac{\sigma_2^8}{2p_2^3}\right)\frac{1}{2}\frac{p_2^2}{(p_2\sigma_2^2-4\sigma_2^4)^2}\Bigg) \\
&+\frac{\pi}{2p_1^{3/2}p_2^{3/2}}\pi\sqrt{\frac{p_1^2p_2^2}{(p_1^2\sigma_1^2-4\sigma_1^4)(p_2^2\sigma_2^2-4\sigma_2^4)}}\left(1+\frac{\sigma_1^4}{2p_1}\frac{p_1^2}{p_1^2\sigma_1^2-4\sigma_1^4}\right) \\
&\cdot\left(1+\frac{\sigma_2^4}{2p_2}\frac{p_2^2}{p_2^2\sigma_2^2-4\sigma_2^4}\right)\Bigg]
\end{aligned}
$$

simplify:

$$
\begin{aligned}
RMS =&(2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \Bigg[\frac{\pi^2}{8p_1} \\
&\cdot \sqrt{\frac{1}{(p_1\sigma_1^2-4\sigma_1^4)(p_2^2\sigma_2^2-\sigma_2^4)}}\Bigg(\frac{4}{p_1}+\left(4-\frac{4\sigma_1^2}{p_1}+\frac{4\sigma_1^4}{p_1^2}\right)\frac{1}{2}\frac{p_1}{p_1\sigma_1^2-4\sigma_1^4} \\
&+\left(\frac{2\sigma_1^4}{p_1}-\frac{\sigma_1^6}{p_1^2}+\frac{\sigma_1^8}{2p_1^3}\right)\frac{1}{2}\frac{p_1^2}{(p_1\sigma_1^2-4\sigma_1^4)^2}\Bigg) \\
&+\frac{\pi^2}{8p_2}\sqrt{\frac{1}{(p_2\sigma_2^2-4\sigma_2^4)(p_1^2\sigma_1^2-\sigma_1^4)}}\Bigg(\frac{4}{p_2}+\left(4-\frac{4\sigma_2^2}{p_2}+\frac{4\sigma_2^4}{p_2^2}\right)\frac{1}{2}\frac{p_2}{p_2\sigma_2^2-4\sigma_2^4} \\
&+\left(\frac{2\sigma_2^4}{p_2}-\frac{\sigma_2^6}{p_2^2}+\frac{\sigma_2^8}{2p_2^3}\right)\frac{1}{2}\frac{p_2^2}{(p_2\sigma_2^2-4\sigma_2^4)^2}\Bigg) \\
&+\frac{\pi^2}{2p_1^{1/2}p_2^{1/2}}\sqrt{\frac{1}{(p_1^2\sigma_1^2-4\sigma_1^4)(p_2^2\sigma_2^2-4\sigma_2^4)}}\left(1+\frac{1}{2}\frac{p_1\sigma_1^2}{p_1^2-4\sigma_1^2}\right) \\
&\cdot\left(1+\frac{1}{2}\frac{p_2\sigma_2^2}{p_2^2-4\sigma_2^2}\right)\Bigg]
\end{aligned}
$$

Fill in the p's

$$
\begin{aligned}
RMS = & (2\pi)^2 C^2 D^2 \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \Bigg[\ \frac{\pi}{8\sigma_2\sigma_4(\sigma_1^2+\sigma_3^2)} (\sigma_1^2\sigma_3^2 - 3\sigma_1^4)^{-1/2} \\
& \cdot \left(\frac{4}{\sigma_1^2+\sigma_3^2} + \left(4 - \frac{4\sigma_1^2}{\sigma_1^2+\sigma_3^2} + \frac{4\sigma_1^4}{(\sigma_1^2+\sigma_3^2)^2}\right) \frac{1}{2} \frac{\sigma_1^2+\sigma_3^2}{\sigma_1^2\sigma_3^2 - 3\sigma_1^4} \right. \\
& \left. + \left(\frac{2\sigma_1^4}{\sigma_1^2+\sigma_3^2} - \frac{\sigma_1^6}{(\sigma_1^2+\sigma_3^2)^2} + \frac{\sigma_1^8}{2(\sigma_1^2+\sigma_3^2)^3} \right) \frac{1}{2} \frac{(\sigma_1^2+\sigma_3^2)^2}{(\sigma_1^2\sigma_3^2 - 3\sigma_1^4)} \right) \\
& + \frac{\pi}{8\sigma_1\sigma_3(\sigma_2^2+\sigma_4^2)} (\sigma_2^2\sigma_4^2 - 3\sigma_2^4)^{-1/2} \\
& \cdot \left(\frac{4}{\sigma_2^2+\sigma_4^2} + \left(4 - \frac{4\sigma_2^2}{\sigma_2^2+\sigma_4^2} + \frac{4\sigma_2^4}{(\sigma_2^2+\sigma_4^2)^2}\right) \frac{1}{2} \frac{\sigma_2^2+\sigma_4^2}{\sigma_2^2\sigma_4^2 - 3\sigma_2^4} \right. \\
& \left. + \left(\frac{2\sigma_2^4}{\sigma_2^2+\sigma_4^2} - \frac{\sigma_2^6}{(\sigma_2^2+\sigma_4^2)^2} + \frac{\sigma_2^8}{2(\sigma_2^2+\sigma_4^2)^3} \right) \frac{1}{2} \frac{(\sigma_2^2+\sigma_4^2)^2}{(\sigma_1^2\sigma_3^2 - 3\sigma_1^4)} \right) \\
& + \frac{\pi^2}{2} (\sigma_1^2+\sigma_3^2)^{-1/2} (\sigma_2^2+\sigma_4^2)^{-1/2} (\sigma_1^2\sigma_3^2 - \sigma_1^4)^{-1/2} (\sigma_2^2\sigma_4^2 - \sigma_2^4)^{-1/2} \\
& \cdot \left(1 + \frac{1}{2} \frac{(\sigma_1^2+\sigma_3^2)\sigma_1^2}{(\sigma_1^2+\sigma_3^2)^2 - \sigma_1^4}\right) \left(1 + \frac{1}{2} \frac{(\sigma_2^2+\sigma_4^2)\sigma_2^2}{(\sigma_2^2+\sigma_4^2)^2 - \sigma_2^4}\right) \Bigg]
\end{aligned}
$$

Simplified by mathematica:

$$
\begin{aligned}
RMS = & \, 4\sigma_1^2\sigma_2^2\sigma_3^2 C^2 \sigma_4^2 D^2 \pi^2 \Bigg(-\frac{\left(3\sigma_1^8 + 24\sigma_1^2\sigma_3^2 + 6\sigma_1^6\sigma_3^2 + 8\sigma_3^4 + 4\sigma_1^4\left(-10+\sigma_3^4\right)\right)\pi}{32\sigma_1^2\sigma_2\left(3\sigma_1^2-\sigma_3^2\right)\left(\sigma_1^2+\sigma_3^2\right)^2\sqrt{-3\sigma_1^4+\sigma_1^2\sigma_3^2}\,\sigma_4} \\
& - \frac{\left(24\sigma_1^4\left(5\sigma_2^4 - 3\sigma_2^2\sigma_4^2 - \sigma_4^4\right) + 8\sigma_1^2\sigma_3^2\left(-5\sigma_2^4 + 3\sigma_2^2\sigma_4^2 + \sigma_4^4\right) - \sigma_2^6\left(9\sigma_2^6 + 15\sigma_2^4\sigma_4^2 + 6\sigma_2^2\sigma_4^4 - 4\sigma_4^6\right)\right)\pi}{32\sigma_1^3\sigma_2^2\sigma_3\left(-3\sigma_1^2+\sigma_3^2\right)\left(3\sigma_2^2-\sigma_4^2\right)\left(\sigma_2^2+\sigma_4^2\right)^2\sqrt{-3\sigma_2^4+\sigma_2^2\sigma_4^2}} \\
& + \frac{\left(\sigma_1^4 + 5\sigma_1^2\sigma_3^2 + 2\sigma_3^4\right)\left(\sigma_2^4 + 5\sigma_2^2\sigma_4^2 + 2\sigma_4^4\right)\pi^2}{8\sigma_3^2\sqrt{\sigma_1^2\left(-\sigma_1^2+\sigma_3^2\right)}\sqrt{\sigma_1^2+\sigma_3^2}\left(2\sigma_1^2+\sigma_3^2\right)\sigma_4^2\sqrt{\sigma_2^2\left(-\sigma_2^2+\sigma_4^2\right)}\sqrt{\sigma_2^2+\sigma_4^2}\left(2\sigma_2^2+\sigma_4^2\right)} \Bigg)
\end{aligned}
\tag{162}
$$

D Plotting code in mathematica

We have added the code used in the sections and appendices of this paper to allow the reader to play with them. This can enhance understanding and it is of course fun to find out what your favorite function does as a lens potential.

D.1 The effect of the PSF in Fourier space: Figure 10

```
sigmai = 1
sigmap = 3
CI[k_] = 10*Exp[-4*sigmai^2*k^2]
CIP[k_] = 10*Exp[-(4*sigmai^2 + 4*sigmap^2)*k^2]
plaatje =
 Plot[{CIP[k], CI[k]}, {k, 0, 1.5}, AxesLabel -> Automatic,
 PlotRange -> Full ]
```

D.2 Window function - figure 11(b)

```
Step = Plot[HeavisidePi[x/10], {x, -10, 10}, Exclusions -> None]
sinc=Plot[Sinc[x], {x, -17, 17}, PlotRange -> Full]
Export["step.pdf", Step]
Export["sinc.pdf",sinc]
```

D.3 Effect of the window function - figure 12

```
m = 0.4
sigmaw = 3
CCC[x_] = 5 Cos[2 Pi m x]
w[x_] = Exp[-x^2/(2 sigmaw^2)]
A = Plot[{CCC[x]}, {x, -10, 10}]
B = Plot[CCC[x]*w[x], {x, -10, 10}]
Cc = Plot[w[x], {x, -10, 10}]

Export["cosine.pdf", A]
Export["window.pdf", Cc]
Export["combcoswindow.pdf", B]
```

D.4 Original power spectrum and with PSF - figure 14

```
Dc = 2.5
Ec = 5
sigmap = 0.8
dk = 0.5
psf[k_] := (Dc*k^2 + Ec) Exp[-sigmap^2*k^2]
sigma[k_] := psf[k]^0.5*(dk/(2*Pi*k))^(0.25)
LogLogPlot[{psf[x], psf[x] + sigma[x],
  psf[x] - sigma[x], (Dc*x^2 + Ec)}, {x, 0.01, 5},
 PlotStyle -> {Blue, Red, Red, Green}, PlotRange -> {0.001, 50},
 AxesStyle -> Directive[Black, 20]]
```

D.5 Window and PSF effect on power spectrum - figure D.5

```
sw = 10
sp = 0.8
Ee = 3
F = 5
dk = 0.1
FoV[k_] := Sqrt[Pi/sw^2] Exp[-sp^2 k^2] (Ee/sw^2 + Ee k^2 + F)
psf[k_] := (Ee*k^2 + F) Exp[-2 sp^2*k^2]
sigma[x_] := FoV[x]^0.5*(dk/(2*Pi*x))^(0.25)
LogLogPlot[{FoV[x], FoV[x] + sigma[x], FoV[x] - sigma[x], psf[x]}, {x,
   0.01, 5}, PlotStyle -> {Blue, Red, Red, Green},
 PlotRange -> {0.001, 50}, AxesStyle -> Directive[Black, 20]]
```

D.6 Cosine source and a cosine lens potential - figure B.1.1

```
e = 2
d=0
n = {3, 0}
m = {1, 1}
sigma = 0.5
x0 = {6, 1}
y0 = {0, 0}
Funct[x1_, x2_] :=
 e*Cos[2*Pi*
    Dot[n, {x1, x2} + 2*Pi*m*d Sin[2 Pi Dot[m, {x1, x2} - x0]] - y0]]

P1 = ContourPlot[Funct[x, y], {x, 0, 1}, {y, 0, 1}]
d = 0.001
P3 = ContourPlot[Funct[x, y], {x, 0, 1}, {y, 0, 1}]
d = 0.01
P4 = ContourPlot[Funct[x, y], {x, 0, 1}, {y, 0, 1}]
d = 0.1
P5 = ContourPlot[Funct[x, y], {x, 0, 1}, {y, 0, 1}]

Export["A000.pdf", P1]
Export["A001.pdf", P3]
Export["A010.pdf", P4]
Export["A100.pdf", P5]
```

D.7 Gaussian source and a cosine lens potential - figure 17

```
sigma1 = 1.5
sigma2 = 1
d = 0.01
e = 1
m = {2, 0}
```

```
x0 = {0, 0}
y0 = {0, 0}
Func2[x_, y_] :=
 e* Exp[-1/
     2*((x + 2 Pi m[[1]] d Sin[2 Pi Dot[m, {x, y} - x0]] - y0[[1]])/
       sigma1)^2] *
  Exp[-1/2*((y + 2 Pi m[[2]] d Sin[2 Pi Dot[m, {x, y} - x0]] -
         y0[[2]])/sigma2)^2]
d = 0
P0 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]
d = 0.001
P1 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]
d = 0.005
P2 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]
d = 0.01
P3 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]
d = 0.025
P4 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]
d = 0.05
P5 = ContourPlot[Func2[x, y], {x, -5, 5}, {y, -3, 3},
  PlotRange -> {0, All}]

Export["B000.pdf", P0]
Export["B001.pdf", P1]
Export["B005.pdf", P2]
Export["B010.pdf", P3]
Export["B025.pdf", P4]
Export["B050.pdf", P5]
```

D.8 Cosine source and a Gaussian lens potential - figure 18

```
sigma1 = 1.5
sigma2 = 1
m = {2, 0}
d = 0.01
e = 1
x0 = {0, 0}
y0 = {0, 0}
Func2[x_, y_] :=
 e Cos[2 Pi (Dot[m, {x, y}] -
      d ((x - x0[[1]])/sigma1^2* m[[1]] + (y - x0[[2]])/sigma2^2 *
          m[[2]]) Exp[-(x - x0[[1]])^2/(2 sigma1^2) - (y -
             x0[[2]])^2/(2 sigma2^2)] - Dot[y0, m])]
```

```
d = 0
P0 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 0.1
P1 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 0.5
P2 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 1
P3 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 2
P4 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 5
P5 = ContourPlot[Func2[x, y], {x, -1, 1}, {y, -2, 2},
  PlotRange -> {0, All}]

Export["C00.pdf", P0]
Export["C01.pdf", P1]
Export["C05.pdf", P2]
Export["C10.pdf", P3]
Export["C20.pdf", P4]
Export["C50.pdf", P5]
```

D.9 Gaussian source and a Gaussian lens potential - figure 19

```
sigma1 = 3
sigma2 = 0.5
sigma3 = 0.5
sigma4 = 0.5
d = 5
e = 1
x0 = {1, 1}
y0 = {0, 0}
Func3[x_, y_] :=
 e*Exp[-(x +
        d (x - x0[[
             1]])/(2 sigma1^2) Exp[-(x -
                x0[[1]])^2/(2 sigma1^2) - (y -
               x0[[2]])^2/(2 sigma2^2)] - y0[[1]])^2/(2 sigma3^2)]*
  Exp[-(y +
        d (y - x0[[
             2]])/(2 sigma2^2) Exp[-(x -
                x0[[1]])^2/(2 sigma1^2) - (y -
               x0[[2]])^2/(2 sigma2^2)] - y0[[2]])^2/(2 sigma4^2)]
f = 0
Func4[x_, y_] :=
```

```
 e*Exp[-(x +
       f (x - x0[[
            1]])/(2 sigma1^2) Exp[-(x -
               x0[[1]])^2/(2 sigma1^2) - (y -
              x0[[2]])^2/(2 sigma2^2)] - y0[[1]])^2/(2 sigma3^2)]*
  Exp[-(y +
       f (y - x0[[
            2]])/(2 sigma2^2) Exp[-(x -
               x0[[1]])^2/(2 sigma1^2) - (y -
              x0[[2]])^2/(2 sigma2^2)] - y0[[2]])^2/(2 sigma4^2)]

d = 0
P0 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 0.5
P1 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 1
P2 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 2
P3 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 3
P4 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]
d = 5
P5 = ContourPlot[Func3[x, y], {x, -2, 2}, {y, -2, 2},
  PlotRange -> {0, All}]

Export["E000.pdf", P0]
Export["E005.pdf", P1]
Export["E010.pdf", P2]
Export["E020.pdf", P3]
Export["E030.pdf", P4]
Export["E050.pdf", P5]
```