In this task we will investigate both the Zeldovich Approximation (ZA) and Adhesion. Most of the content in this task is based on two articles,

- Sergei F. Shandarin & Yacob B. Zeldovich - The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium - Reviews of Modern Physics, Vol. 61, No. 2, April 1989
- David H. Weinberg & James E. Gunn - Large-scale structure and the adhesion approximation - Mon. Not. R. astr. Soc. (1990) 246, 260-286

We have previously seen how to generate a Gaussian random field. We will now need these to setup the initial conditions of the simulations. Also you have calculated the two-point correlation function from a point-set. This we will use to analyse some of the results. We first cover some of the theory behind ZA and Adhesion.

Zeldovich approximation.

The Zeldovich approximation is given by

$$\boldsymbol{x} = \boldsymbol{q} + D_+ \boldsymbol{\nabla}_q \Phi_0. \tag{1}$$

See the derivation in the appendix. There is another more general approach to the Zel'dovich approximation. From now on we will treat the growing mode solution as a de facto time coordinate. We introduce a new comoving velocity $\boldsymbol{u} \equiv (d\boldsymbol{x}/dD_+)$ and rewrite the equations of motion in terms of $\boldsymbol{u}$

$$\boldsymbol{v} = a\dot{\boldsymbol{x}} = a\dot{D}_+\boldsymbol{u}$$

a. Rewrite the non-linear Euler equation ($\dot{v} + Hv + (v\boldsymbol{\nabla})v = -\boldsymbol{\nabla}\phi/a$) for matter perturbations in terms of $\boldsymbol{u}$ to

$$\frac{\partial \boldsymbol{u}}{\partial D_+} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_x)\boldsymbol{u} = \frac{1}{(a\dot{D}_+)^2}(B\boldsymbol{u} - \boldsymbol{\nabla}_x \phi) \tag{2}$$

where $B = -(2a\dot{a}\dot{D}_+ + a^2\ddot{D}_+) = -\frac{\partial}{\partial t}\left(a^2\dot{D}_+\right)$.

b. Take equation (1), and the definition

$$-\boldsymbol{\nabla}_x\phi \equiv 2a\dot{a}\dot{\boldsymbol{x}} + a^2\ddot{\boldsymbol{x}},$$

and insert them into equation (2). Show that you get

$$\boxed{\frac{\partial \boldsymbol{u}}{\partial D_+} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_x)\boldsymbol{u} = 0} \tag{3}$$

c. express the parameter B in terms of H, and Ω. How do you interpret this result?

As a little extra, if we take a step back, we can rewrite equation (2) (which is still the full Euler equation) to the case of potential motion using $\boldsymbol{u} = -\boldsymbol{\nabla}_x \Phi$ and integrating

$$\frac{\partial \Phi}{\partial D_+} + \frac{1}{2}(\boldsymbol{\nabla}_x \Phi)^2 = -\frac{1}{a\dot{D}_+^2}(\Phi + A\phi) \tag{4}$$

This is often referred to in the literature as the Bernoulli equation. where D_+ is the growing-mode solution, $\boldsymbol{q}$ the original location of a test particle and Φ_0 the initial velocity potential.

$$\hat{\Phi}_0(\boldsymbol{k}) = \frac{1}{k^2}\hat{\delta}(\boldsymbol{k})$$

$$\frac{\partial \boldsymbol{u}}{\partial D_+} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_x)\boldsymbol{u} = 0. \tag{5}$$

For plane-symmetric initial conditions ZA is an exact solution of the full non-linear equations of structure formation, upto the formation of *multi-stream regions*. It is therefore illustrative to compute the ZA analytically for a simple example.

d. Take an initial density perturbation of

$$\delta_0 = \pi^2 \epsilon \cos(\pi \boldsymbol{q})$$

in a one-dimensional box of $\boldsymbol{q} \in [-1, 1)$. Compute the velocity u_0 and the density

$$1 + \delta = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1}$$

At what time t_0 is the first singularity? Sketch graphs of $\boldsymbol{x}(\boldsymbol{q})$, $\boldsymbol{u}(\boldsymbol{x})$ and $\delta(\boldsymbol{x})$ for three times $t\{<, =, >\}t_0$ (or plot them on a computer; make sure locations of critical points are well motivated).

e. show particle displacements for your gaussian random field realisations in 2D, for several times D_+. If the results look ugly, try filtering with a small gaussian kernel. What happens when D_+ gets large?

Anisotropic collapse.

Along with the equation comes a beautiful interpretation of early structure formation, namely that of collapsing ellipsoids. To see what a local perturbation will do; collapse as a wall, filament, cluster or expand to become void, we can study the eigenvalues of the deformation tensor.

If we look at the density evolution in the Zel'dovich approximation, as before

$$\frac{\rho}{\bar{\rho}} = \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right)^{-1}$$

and using expression (1)

$$\frac{\rho}{\bar{\rho}} = \det\left(\delta_{ij} - D_+ d_{ij}\right)^{-1}$$

where we defined the *deformation tensor*

$$d_{ij} \equiv \frac{\partial^2 \Phi_0}{\partial q_i \partial q_j}$$

We know this determinant to be equal to that of the diagonalised Jacobian matrix.

$$\boxed{\rho(\boldsymbol{q}, t) = \frac{\bar{\rho}}{(1 - D_+\lambda_1(\boldsymbol{q}))(1 - D_+\lambda_2(\boldsymbol{q}))(1 - D_+\lambda_3(\boldsymbol{q}))}} \tag{6}$$

where $\lambda_1 > \lambda_2 > \lambda_3$ are the eigenvalues of the deformation tensor.

The signatures of the eigenvalues indicate the shape of an ellipsoidal overdensity (spherical, oblong, oblate, or void) The corresponding eigenvectors give the direction of the principal axes of the ellipsoid.

Equation (6) contains singularities where eigenvalues are positive. Locally the density becomes infinite when $D_+ = 1/\lambda_1$. At this moment the structure is collapsed in one direction but still free-streaming in two other directions. In the Zel'dovich formalism this signals the formation of a *pancake*. If $\lambda_1 \approx \lambda_2$, the ellipsoid will collapse in two directions at the same time, and a filament forms. At a density peak all eigenvalues will be positive: a cluster appears. If all of the eigenvalues are negative, the particle will never collapse and can be said to live in a void.

The distribution of eigenvalues in Gaussian random fields was derived by Doroshkevich (1970),

$$\begin{aligned}P(\lambda_1, \lambda_2, \lambda_3) \sim\ & (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \\ & \times \exp\left\{-\frac{15}{2\sigma^2}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \frac{1}{2}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\right]\right\}.\end{aligned}$$

a. Calculate the mass-fraction of particles residing in a void by the criterium of all eigenvalues being negative. What does this tell you about the fractions for the other eigenvalue signatures?

extra. Check these percentages numerically, using gaussian random field realisations in 3D.

b. Show that $\lambda_1 + \lambda_2 + \lambda_3 \sim \delta_0$. What does this mean for the collapse time of a spherical peak ($\lambda_1 = \lambda_2 = \lambda_3$) versus that of a more ellipsoidal structure? What structures do you expect to collapse first?

Adhesion.

We have seen that ZA breaks down after particles cross orbits. To prevent orbit crossing from happening we add an ad-hoc viscosity ν to the equation, emulating gravitational adhesion at smaller scales. This gives us Burgers' equation

$$\frac{\partial \boldsymbol{u}}{\partial D_+} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_x)\boldsymbol{u} = \nu \nabla_x^2 \boldsymbol{u}. \tag{7}$$

Fortunately this equation has an exact solution hopf

$$\boldsymbol{u}(\boldsymbol{x}, D_+) = -2\nu \boldsymbol{\nabla}_x \ln U(\boldsymbol{x}, D_+) \tag{8}$$

$$U(\boldsymbol{x}, D_+) = \int \exp\left[\frac{-\Phi_0}{2\nu}\right] \exp\left[\frac{-(\boldsymbol{x}-\boldsymbol{q})^2}{4D_+\nu}\right] d\boldsymbol{q} \tag{9}$$

Applying the Hopf-Cole transformation ($u \to U$) to Burgers' equation results in the diffusion equation

$$\frac{\partial U}{\partial D_+} = \nu \boldsymbol{\nabla}^2 U$$

This equation is solved with a convolution with a Gaussian with variance $t = 2D_+\nu$.

$$U_0 = \exp\left[\frac{-\Phi_0}{2\nu}\right]$$

$$U(\boldsymbol{x}, D_+) = U_0(\boldsymbol{q}) * G(\boldsymbol{x} - \boldsymbol{q}, 2D_+\nu)$$

We know how to perform these operations in Fourier space, so an implementation of this solution should be relatively straight forward. However there are a few issues to attend that have to do with the discretisation of the equation.

Task.

1. Calculate random fields with power spectrum $P(k) = k^n$ for $n = 1, 0, -1$. Calculate the eigenvalues of the deformation tensor. When will the structures on a scale of a tenth of the boxsize first collapse in the Zeldovich approximation? How can we make this happen at $D_+ = 1$?

2. Displace a set of particles using the Zeldovich approximation to $D_+ = 1$. Calculate the 2-point corellation function of the result.