

COSMIC STRUCTURE FORMATION

Computer Assignment

Saikat Chatterjee: room 192, saikat@astro.rug.nl, phone: 8689

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In this computer assignment we will generate a Gaussian Random Field whose scales are characterized by a power spectra. To generate a random field we start from Fourier space by filling in the grid with normally distributed random numbers and then finally Fourier transforming it to get the field in real space.

Theory

Discrete Fourier transform:

If F_{kl} is the Fourier transform of f_{mn} then for a rectangular MxN grid,

$$F_{kl} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)} \quad (1)$$

$$f_{mn} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F_{kl} \cdot e^{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)} \quad (2)$$

This definition implies the use of normal frequencies instead of angular frequencies (or wavenumbers) and therefore to keep things simple, I suggest you guys to use the parameter $l = k/2\pi$ for all the Fourier components in your code.

Symmetries in Fourier space:

1. We do the following change of the coordinates in the Fourier transform to check what will happen on the other side of the grid,

$$k \rightarrow M - k \text{ and } l \rightarrow N - l \quad (3)$$

The Fourier transform changes into

$$\begin{aligned} F_{M-k, N-l} &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{-2\pi i \left(m - \frac{mk}{M} + n - \frac{nl}{N} \right)} \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{+2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)} \cdot e^{-2\pi i(m+n)} \end{aligned} \quad (4)$$

for m, n integers the last exponential term becomes unity. As the term f_{mn} represents the fluctuations in real space, for a physically acceptable field it can't be imaginary. In other words the above condition can be expressed as,

$$F_{M-k, N-l} = F_{k,l}^* \quad (5)$$

If the frequencies are continuous variable, this condition can be restated as follows,

$$F(l) = F^*(-l) \quad (6)$$

So, we see that only half of the values on the grid are independent and the other half can be filled using this symmetry property. On the line of the grid where either $k = 0$ or $l = 0$ this property tells us,

$$F_{0,N-l} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{2\pi i \frac{nl}{N}} = F_{0,l}^* \quad (7)$$

$$F_{M-k,0} = F_{k,0}^* \quad (8)$$

2. For the points on the halfway of the frequency axis where $k = M/2$ and $l = N/2$ (called Nyquist frequency);

$$F_{M/2,0} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{-\pi i \cdot m} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot (-1)^m \quad (9)$$

$$F_{0,N/2} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{-\pi i \cdot n} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot (-1)^n \quad (10)$$

$$F_{M/2,N/2} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot e^{-\pi i \cdot (m+n)} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} \cdot (-1)^{(m+n)} \quad (11)$$

$$F_{0,0} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} = 0 \quad (12)$$

These points are real valued as f_{mn} is real as well.

The Fourier grid:

If L_x and L_y denote the lengths in the x and y-direction for the simulated field in physical units respectively, the frequency axes in Fourier grid (l_x, l_y) are determined by,

$$-\frac{M}{2} \cdot \frac{1}{L_x} \leq l_x \leq \left(\frac{M}{2} - 1\right) \cdot \frac{1}{L_x} \quad (13)$$

$$-\frac{N}{2} \cdot \frac{1}{L_y} \leq l_y \leq \left(\frac{N}{2} - 1\right) \cdot \frac{1}{L_y} \quad (14)$$

Power spectrum:

For the Gaussian random field we assume that it's power spectrum behaves as a power law,

$$P(l) = A \cdot l^{-n} \quad (15)$$

where n determines the slope of the power spectrum. Note for $l = 0$ the power spectrum diverges, so we put a cut-off at that scale as infinite power at very large scale in a real image is nonphysical. The amplitude A is related to the variance of the fluctuations which can be determined using Parseval's theorem:

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |f_{mn}|^2 = \frac{1}{MN} \sum_{k=-M/2}^{M/2-1} \sum_{l=-N/2}^{N/2-1} |F_{kl}|^2 \quad (16)$$

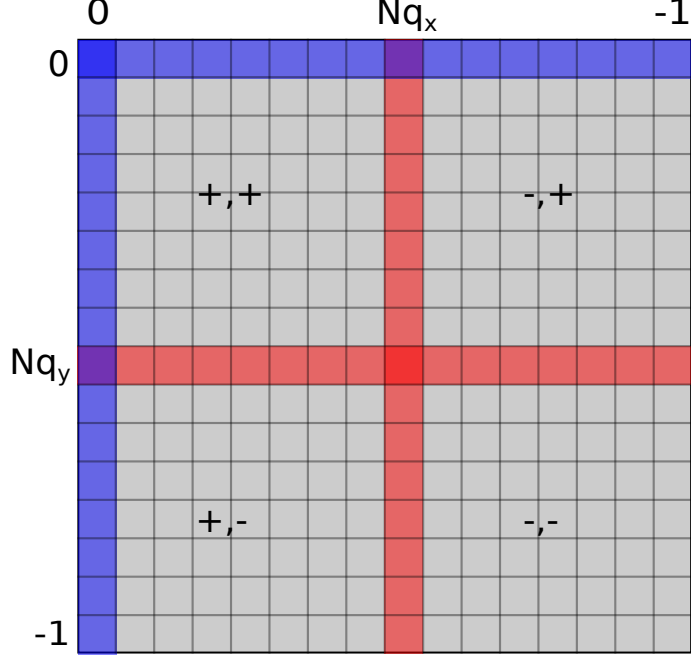


Figure 1: Discrete Fourier space grid for an array of size N . The Fourier modes are stored from 0 to $N/2$, then $-N/2 + 1$ to -1 .

We will denote $MN = N_{\text{pix}}$ as total number of pixels in the simulated image. Now, assuming that Parseval's theorem holds for the variances as well,

$$\begin{aligned}
 \sum_{m,n} \sigma_{mn}^2 &= \frac{1}{N_{\text{pix}}} \sum_{k,l} \sigma_{kl}^2 \\
 N_{\text{pix}} \cdot \sigma_{\text{fluct}}^2 &= \frac{1}{N_{\text{pix}}} \sum_{k,l} \sigma_{kl}^2 \\
 &= \frac{1}{N_{\text{pix}}} \sum_{k,l} P_{kl} \\
 \Rightarrow \sigma_{\text{fluct}}^2 &= \frac{1}{N_{\text{pix}}^2} \sum_{k,l} P_{kl}
 \end{aligned}$$

which solves the normalization constant or the amplitude A as,

$$A = \frac{\sigma_{\text{fluct}}^2 N_{\text{pix}}^2}{2 \sum l^{-n}} \quad (17)$$

A factor of two is added in the denominator because half of the grid is generated and the rest is calculated as a complex conjugate of the first part. However in Fourier space, a point and its complex conjugate are not independent. Therefore the result is a factor two increase of the variance for which we needed this correction.

Box-Muller transform:

The random field is created by using Gaussian random numbers at each point in the grid, where the square root of the power spectrum at that frequency gives the standard deviation of the distribution. A fast

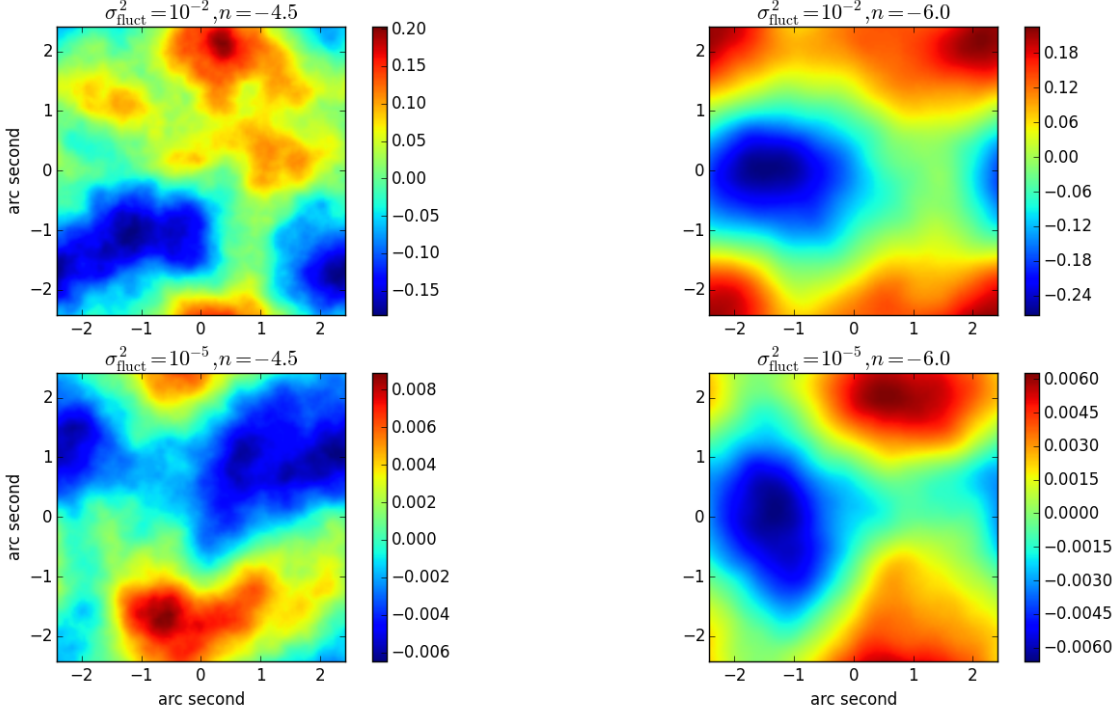


Figure 2: Examples of realizations of Gaussian random field for two different combinations of slope and variance in power spectrum.

method for generating normally distributed numbers is to use the polar form of the Box-Muller transform. This generates two independent Gaussian random numbers. The complex values in the Fourier space grid will be of the form $F(l) = f_1(l) + if_2(l)$ where f_1 and f_2 are the numbers generated from the Box-Muller transform. It works by generating uniformly distributed random numbers u and v with values between -1 and 1. The transform from a uniform to a Gaussian distribution is then made by,

$$\begin{aligned}
 f_1 &= u \sqrt{\frac{-2 \ln(s)}{s}} \\
 f_2 &= v \sqrt{\frac{-2 \ln(s)}{s}}
 \end{aligned}
 \tag{18}$$

where $s = u^2 + v^2 \leq 1$. These relations give a distribution with unit variance, but for the fluctuations a non-unit variance is necessary. To solve this problem one only has to multiply the numbers by the preferred standard deviation (or the square root of the power spectrum value in this case) and get,

$$\begin{aligned}
 f_1(l) &= u \sqrt{P(l)} \sqrt{\frac{-2 \ln(s)}{s}} \\
 f_2(l) &= v \sqrt{P(l)} \sqrt{\frac{-2 \ln(s)}{s}}
 \end{aligned}
 \tag{19}$$

Where $f_1(l)$ and $f_2(l)$ are the real and imaginary parts of the complex number $F(l)$ respectively.

Problems

1. Generate 2-D Gaussian random fields with 121x121 points with $L_x = L_y = 4.84$ and with power spectrum

$$P(l) = A \cdot l^{-n}$$

where the normalization A is determined in the **Theory** section before. Take $n = -1, 0, 1$ and $\sigma_{\text{fluct}}^2 = 1$. Note that this is the resolution of deep-HST images.

2. Make a 2-D power spectrum of the images that you generated.
3. Calculate a 1-D azimuthally averaged power spectrum from the 2-D maps.
4. Make 10 realizations for each of the three cases and calculate the average 1-D power spectrum.
5. The error in each bin j of the average power spectrum is calculated by the root mean square deviation from the mean,

$$\text{rms}_j^2 = \frac{\sum_{i=1}^N (P_{ij} - \langle P \rangle_j)^2}{N - 1}$$

where in our case $N = 10$. Plot the 1-D average power spectrum with these rms as error bars.

6. The above is the error for a single measurement and error for N observations is determined by dividing the error for a single measurement by a factor of \sqrt{N} . Plot the power spectrum with both error bars.
7. Now fit a two parameter power law with the average power spectrum and determine the amplitude and power exponent. Notice that the amplitude is same as σ_{fluct}^2 but with our normalization, it will contain a factor of N_{pix}^2 .

References

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3. G. Marsaglia and T. A. Bray. A convenient method for generating normal variables. SIAM Review, 6(3):pp. 260-264, 1964. ISSN 00361445. URL <http://www.jstor.org/stable/2027592>.
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