Cosmic Structure:

Lecture 8b

the Zeldovich Formalism
understanding the Cosmic Web

Rien van de Weijgaert,
Cosmic Structure Formation, Oct. 2018


Yakov Borisovich Zel’dovich


Lorentz center
Tracing the Cosmic Web
Workshop: 17 – 21 February 2014, Leiden, the Netherlands
Scientific Organizers
• Noam Libeskind, AIP Potsdam
• Rien van de Weygaert, U Groningen
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• Yehuda Hoffman, HUJI Jerusalem
• Fransisco Kitaura, AIP Potsdam
• Sergei Shandarin, KU Lawrence
• Thierry Sousbie, IAP Paris
• Elmo Tempel, U Tartu
Topics
• Large-Scale Distribution of Matter and Galaxies
• Voids, Sheets, Filaments and Clusters
• Geometry, Topology and Multiscale Structure
• Dynamics and Evolution of the Cosmic Web
• Techniques for Characterizing Weblike Patterns
• Galaxy Formation and the Cosmic Web
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Galaxies, Intergalactic gas and dark matter aggregate in a complex network, known as the "cosmic web".
Image: R. Kaehler, O. Hahn, T. Abel. Processing: N. Bos
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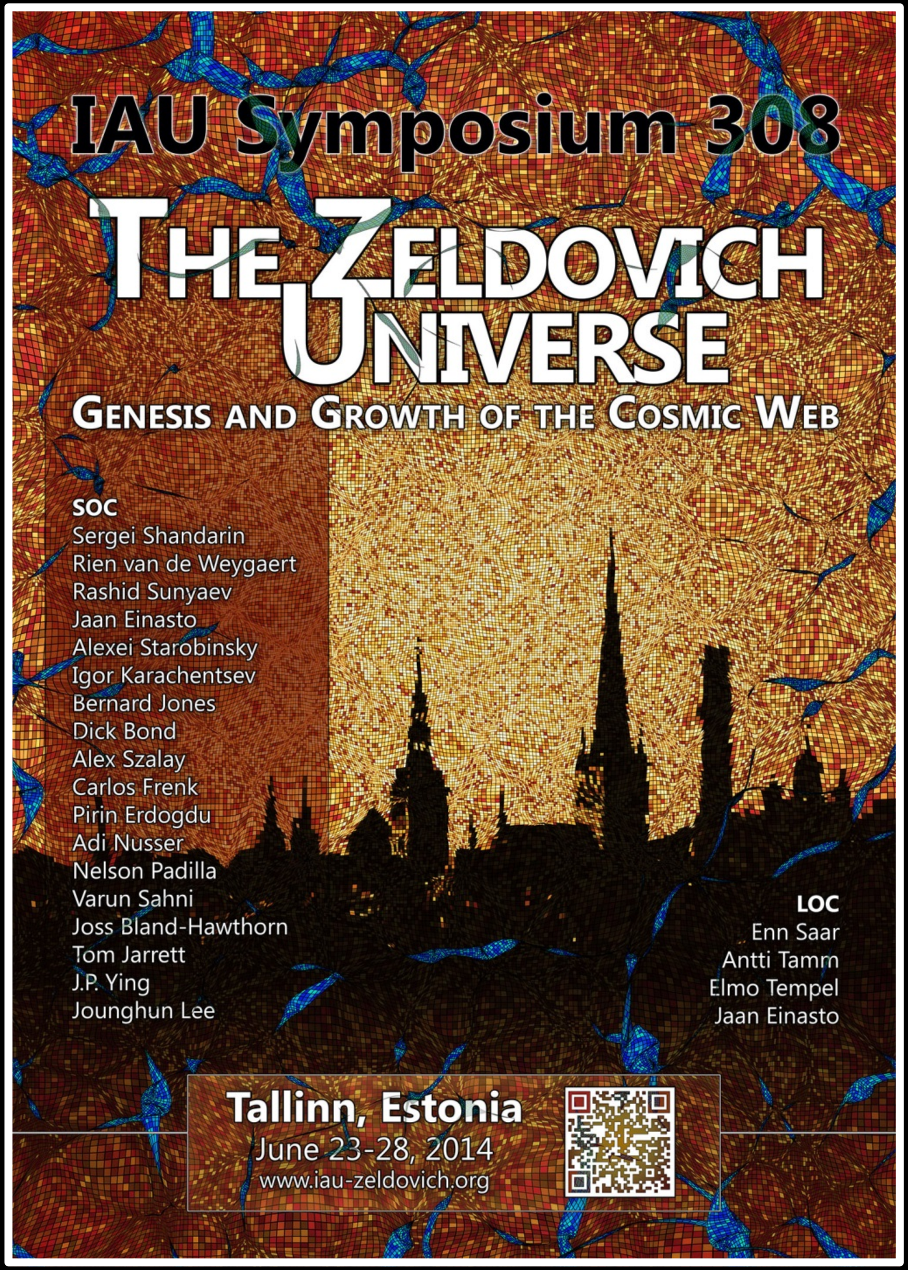
IAU Symposium 308
The Zeldovich Universe
Genesis and Growth of the Cosmic Web
SOC
Sergei Shandarin
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Rashid Sunyaev
Jaan Einasto
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Tallinn, Estonia
June 23-28, 2014
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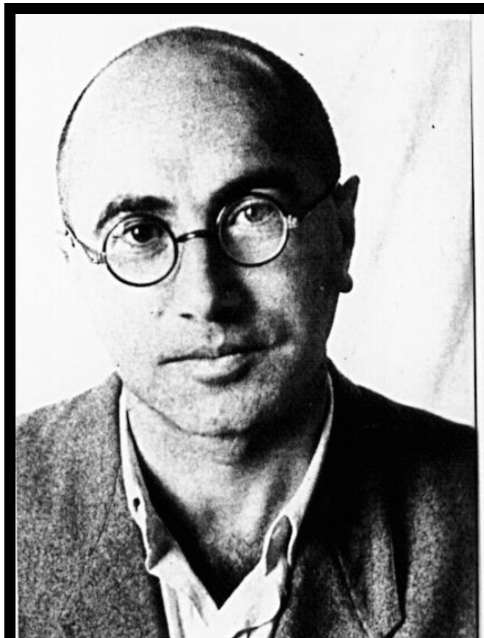
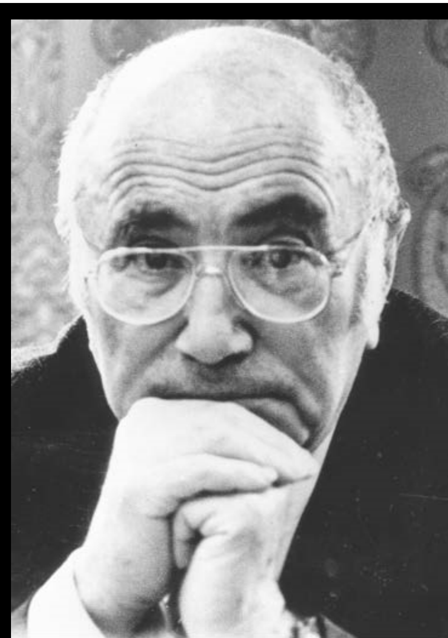




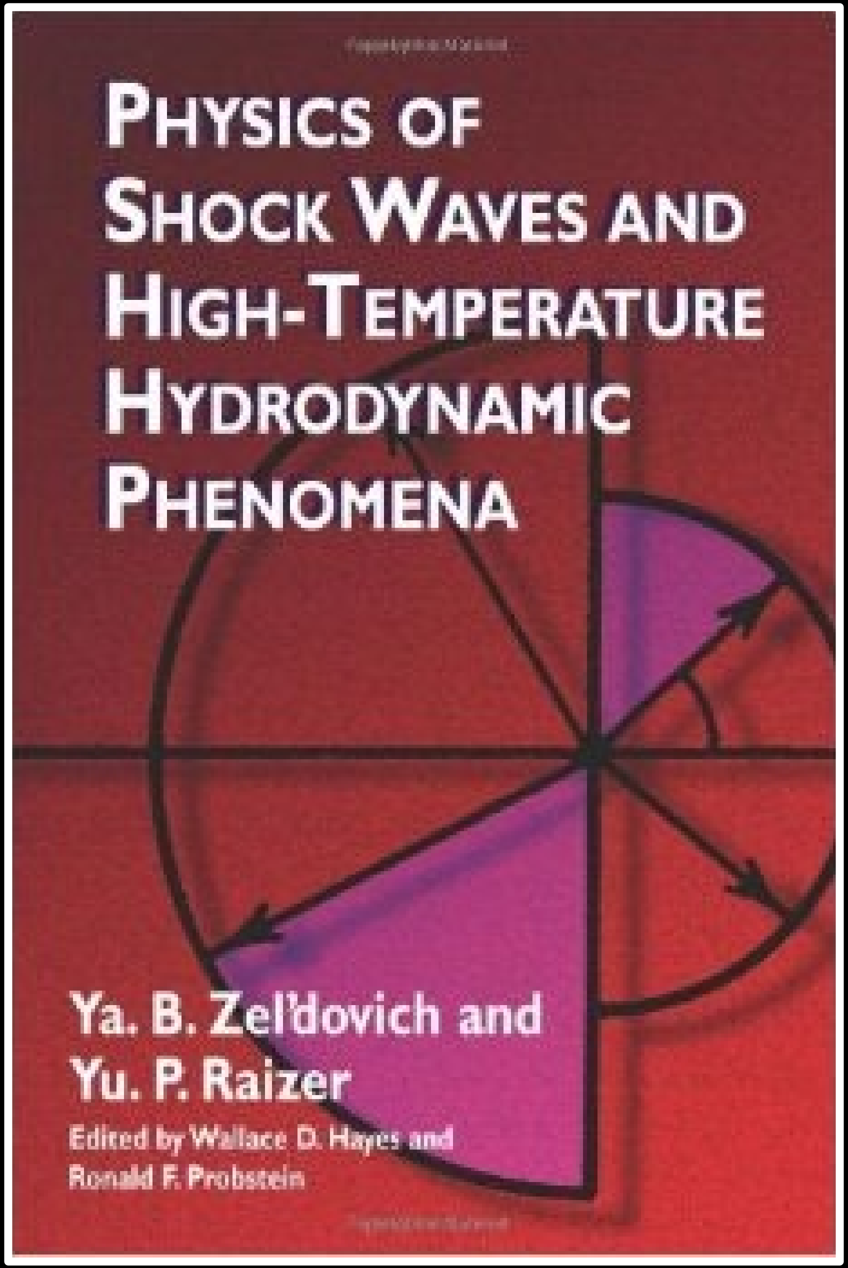
Yakov Borisovich Zel'dovich
Minsk, 1914- Moscow, 1987

stamp Zeldovich,
Russia 2014

monument Zeldovich,
Minsk, Belorussia

Zel’dovich & pope John Paul II

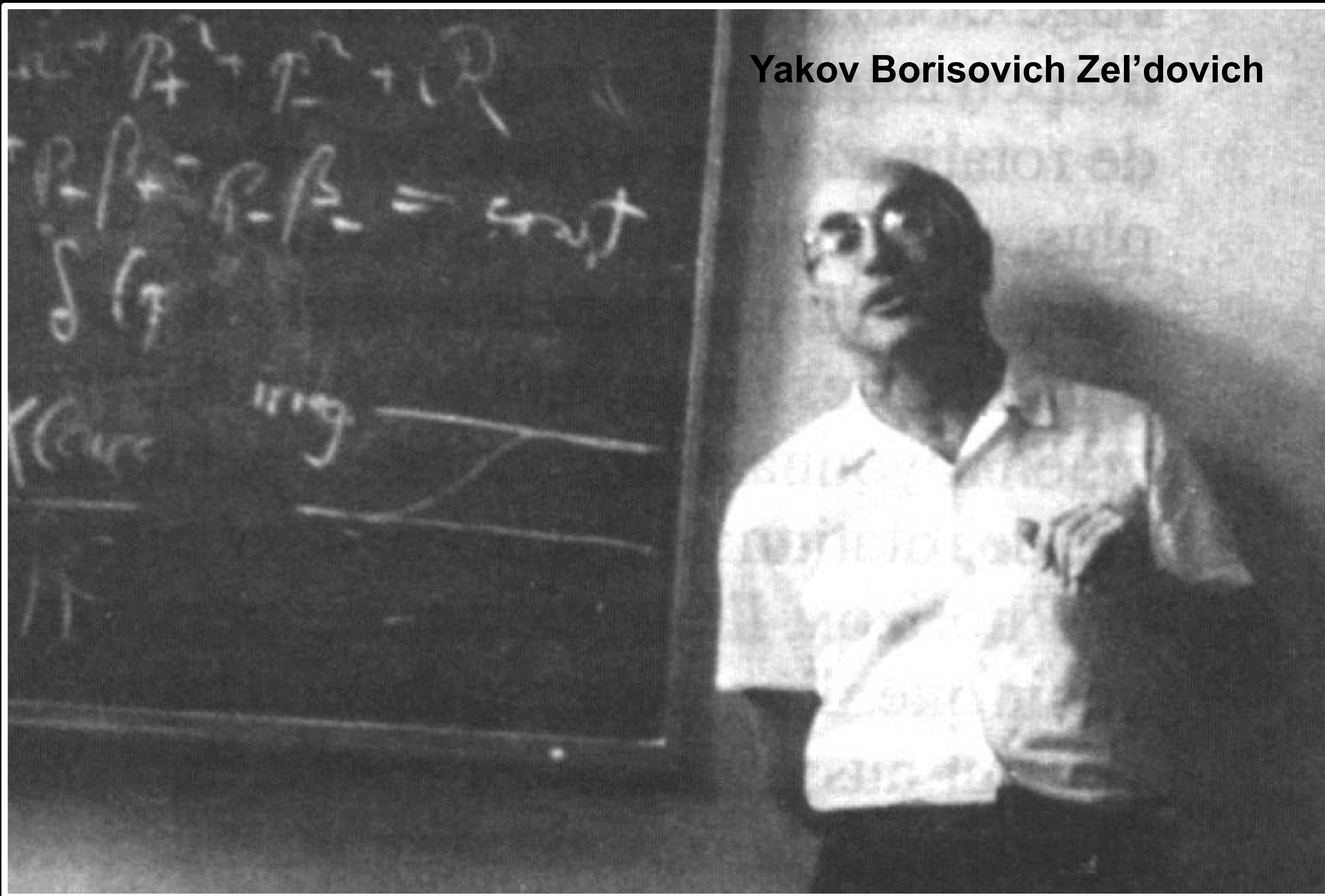
Zel’dovich & Andrei Sacharov



Zeldovich & Raizer
standard book on shock waves ...

Phase-Space Evolution:

Zeldovich & Deformation



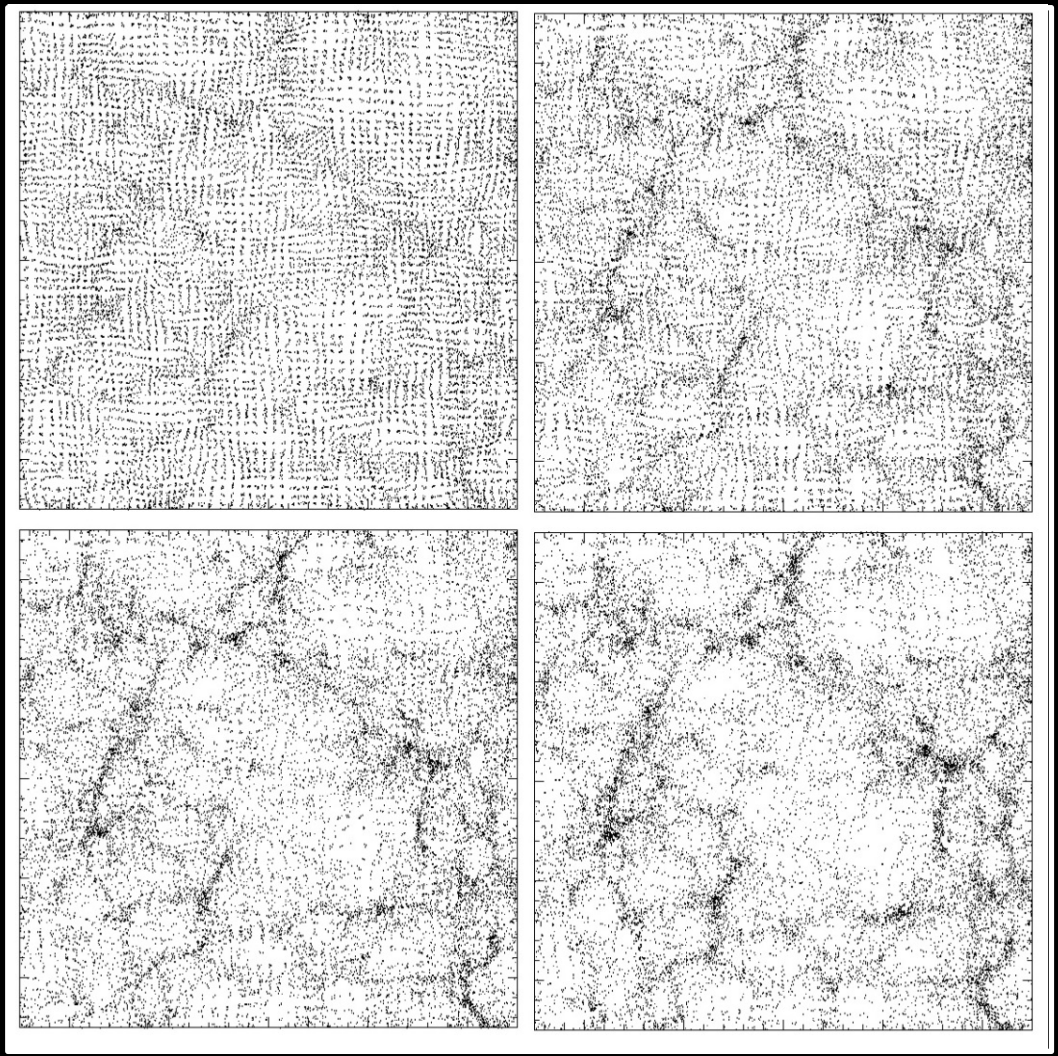
Yakov Borisovich Zel'dovich

Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\,\phi_{lin}\left(\vec{q}\right)$$



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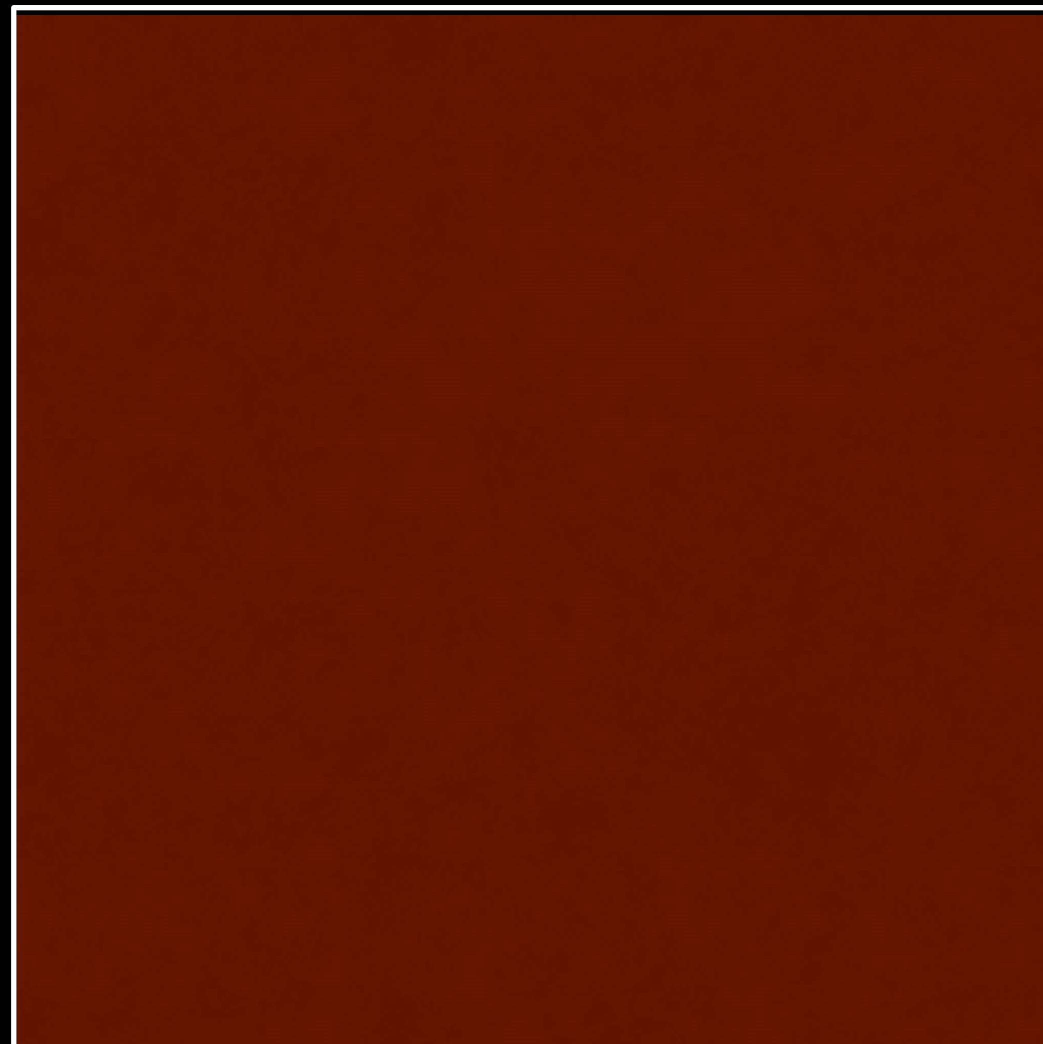
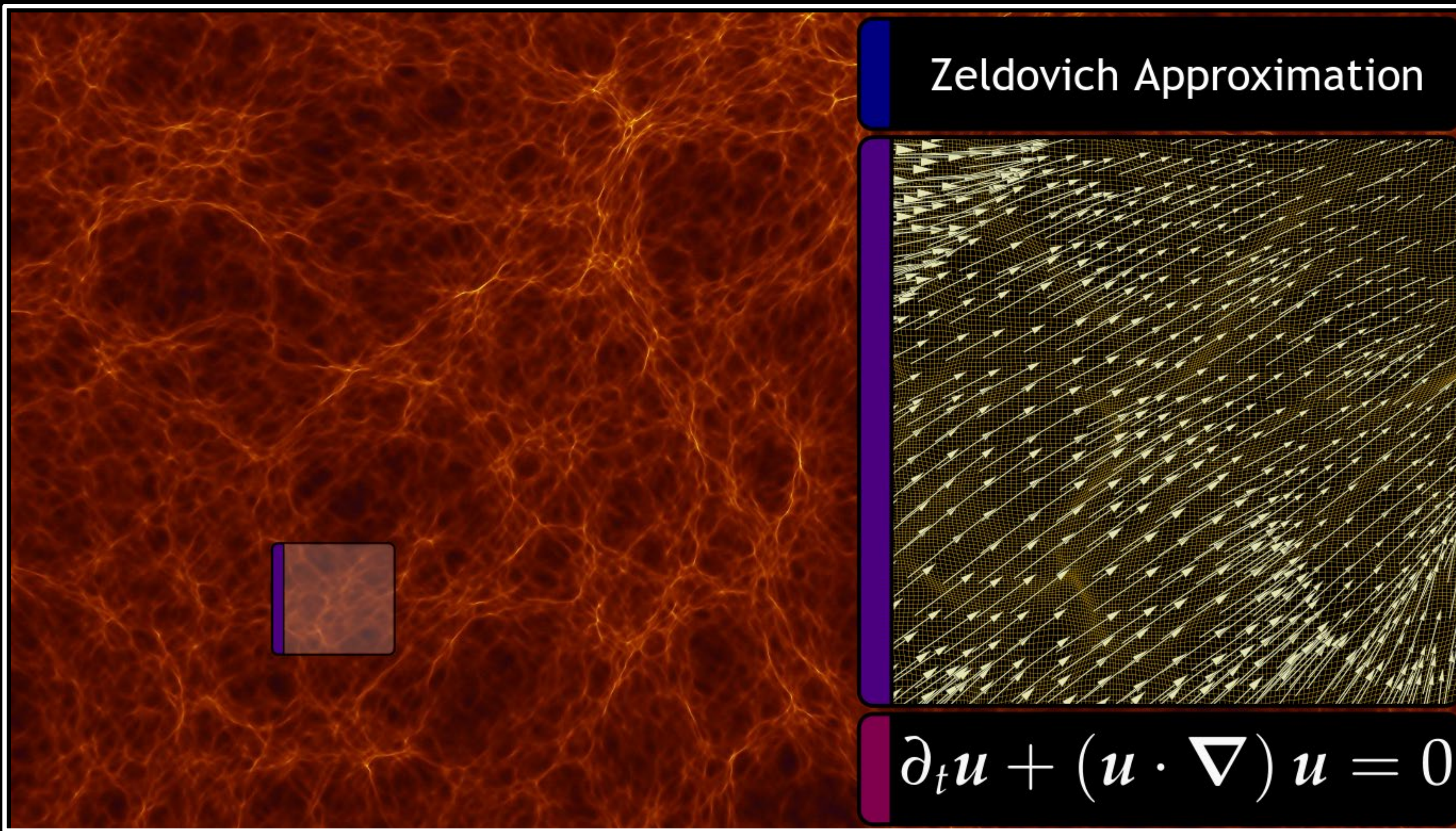


Zel'dovich Approximation



Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

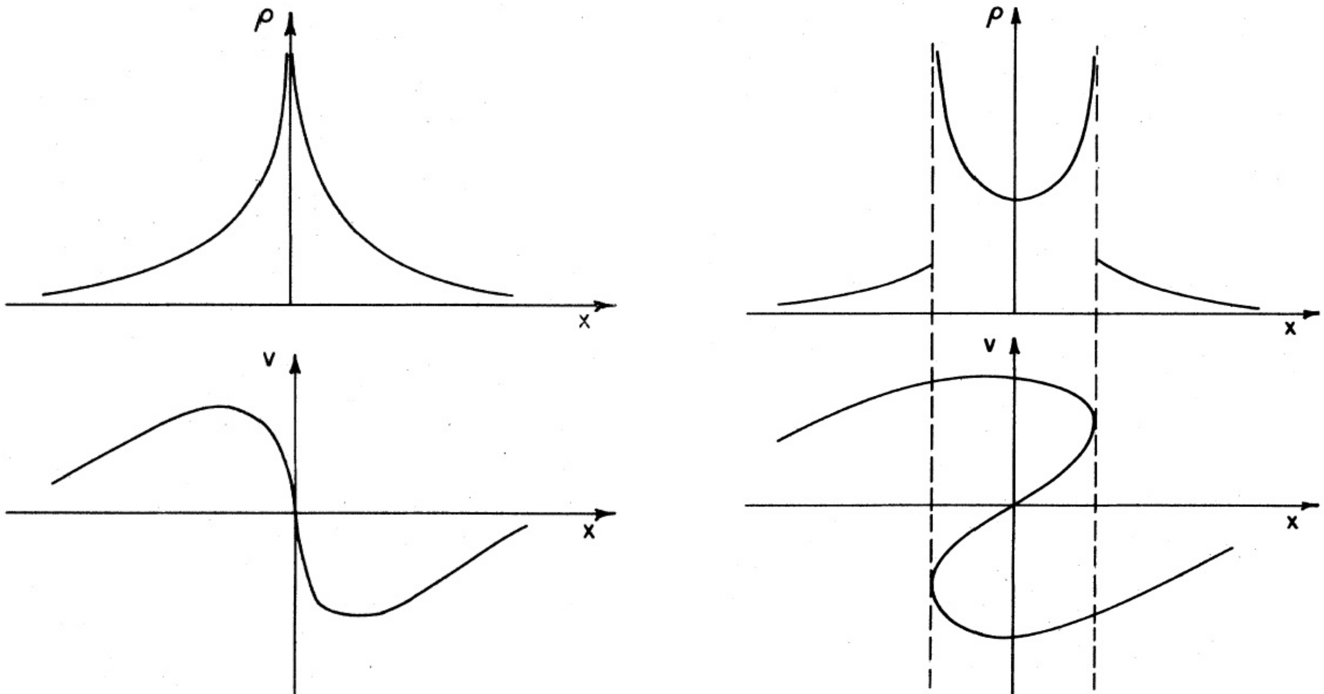
$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$

structure of the cosmic web determined by the spatial field of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

Zeldovich Formalism:

Density Evolution



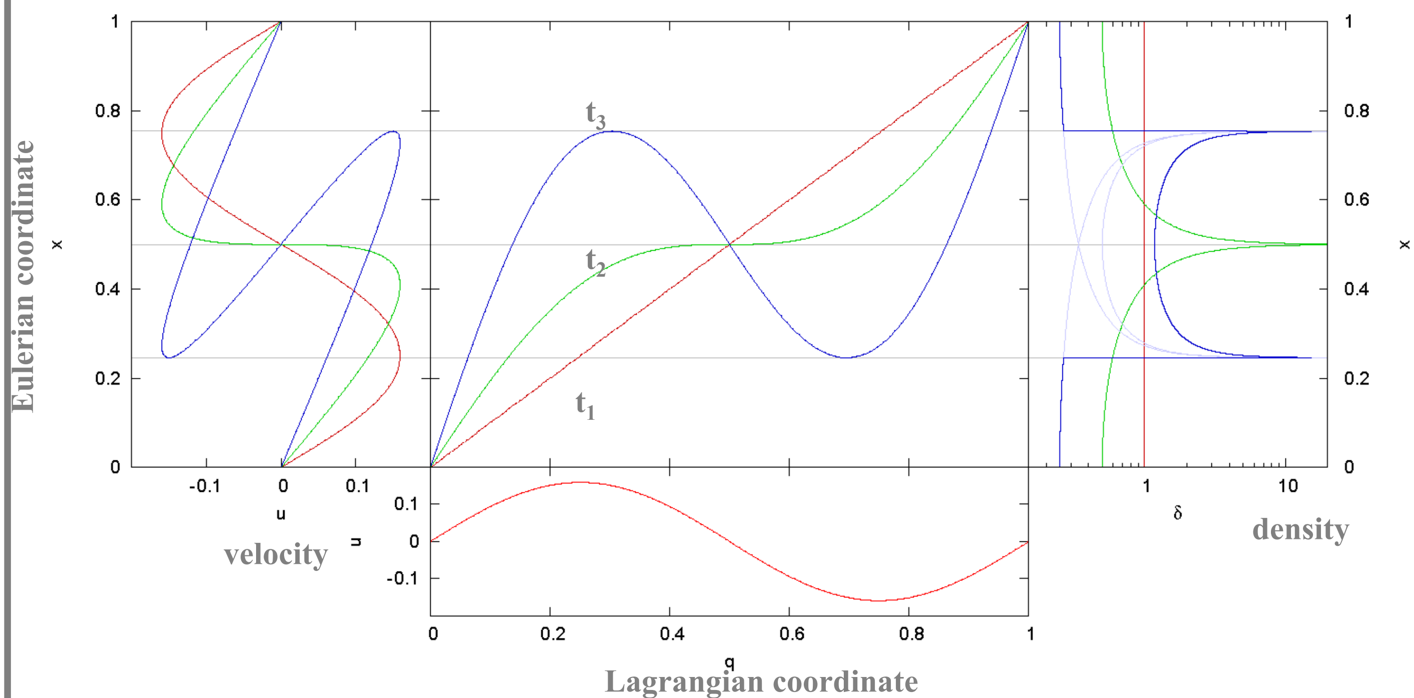
Density Profile through pancake, at moment of formation and shortly thereafter (multistream)

$$\vec{x}(\vec{q},t) = \vec{q} - D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} : \; \lambda_1, \lambda_2, \lambda_3$$

$$\rho\left(\vec{q},t\right) = \frac{\rho_u\left(t\right)}{\left(1 - D(t)\lambda_1\left(\vec{q}\right)\right)\left(1 - D(t)\lambda_2\left(\vec{q}\right)\right)\left(1 - D(t)\lambda_3\left(\vec{q}\right)\right)}$$

Zeldovich Formalism:

Singularities



$$\vec{x}(\vec{q},t) = \vec{q} - D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} : \ \lambda_1, \lambda_2, \lambda_3$$

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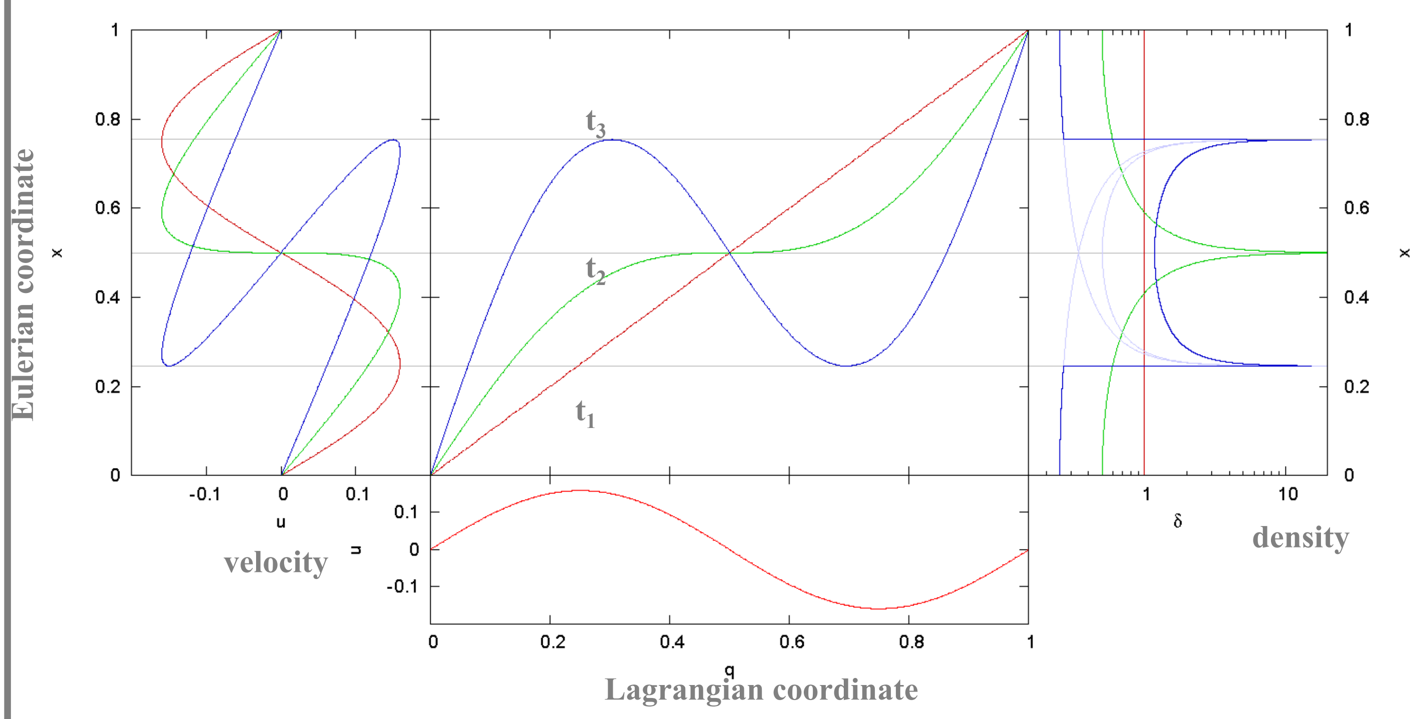
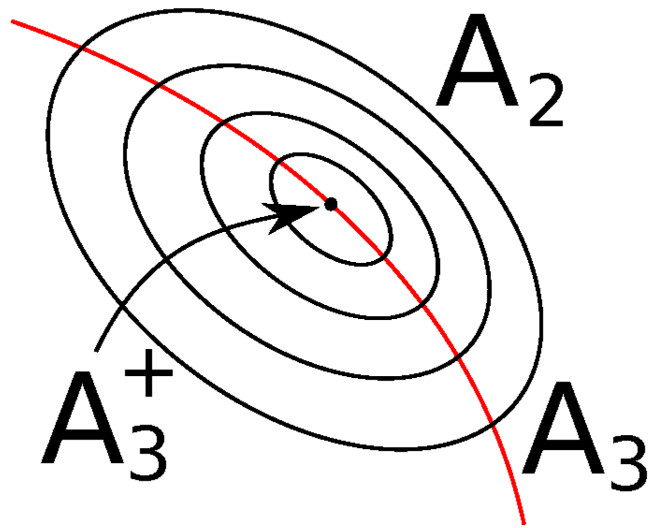
Caustic Formation:

Folds & Cusps



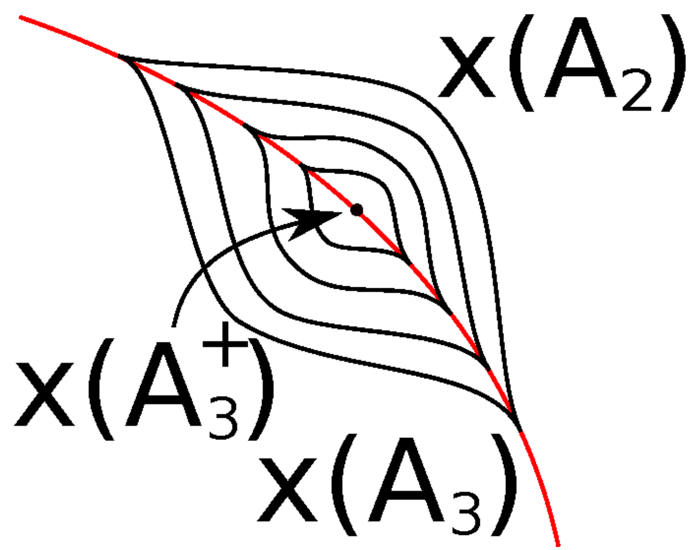


Lagrangian space: A_2 contours

$$\rho(\vec{q},t)=\infty$$

$$\Uparrow$$

$$D(t)\lambda_i=1$$



Eulerian space: folds & cusps

Zel'dovich Morphology

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

$$\lambda_1, \lambda_2, \lambda_3 \qquad \lambda_1 > \lambda_2 > \lambda_3$$

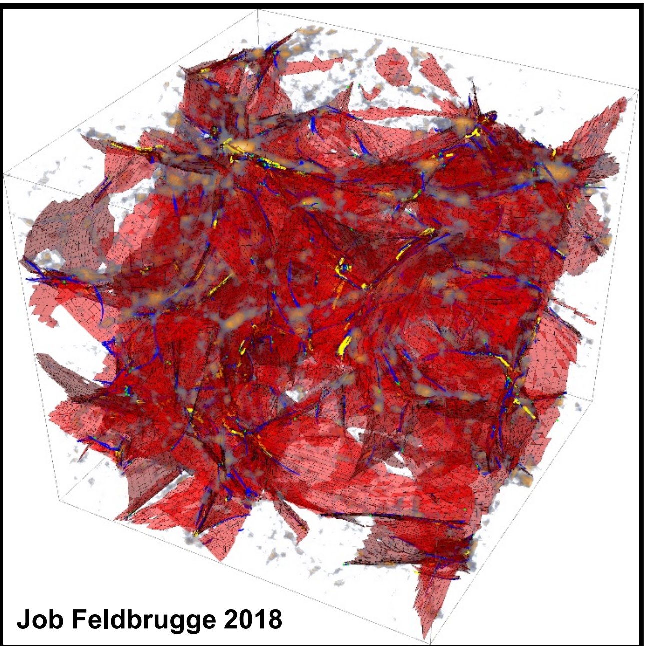
Structure of the cosmic web determined by the spatial field of eigenvalues:

Sequence of formation stages:

λ1 - collapse along first axis:
formation of walls/sheets/pancakes

λ2 - collapse along 2 axes:
formation of elongated filaments

λ3 - possibly – if λ3>0 – collapse along all three axes, into a fully collapsed clump/node



Job Feldbrugge 2018

Zel’dovich Cosmic Web

It is no exaggeration to state that Zeldovich (1970) predicted the existence of the Cosmic Web !

<u>Sequence of formation stages:</u>

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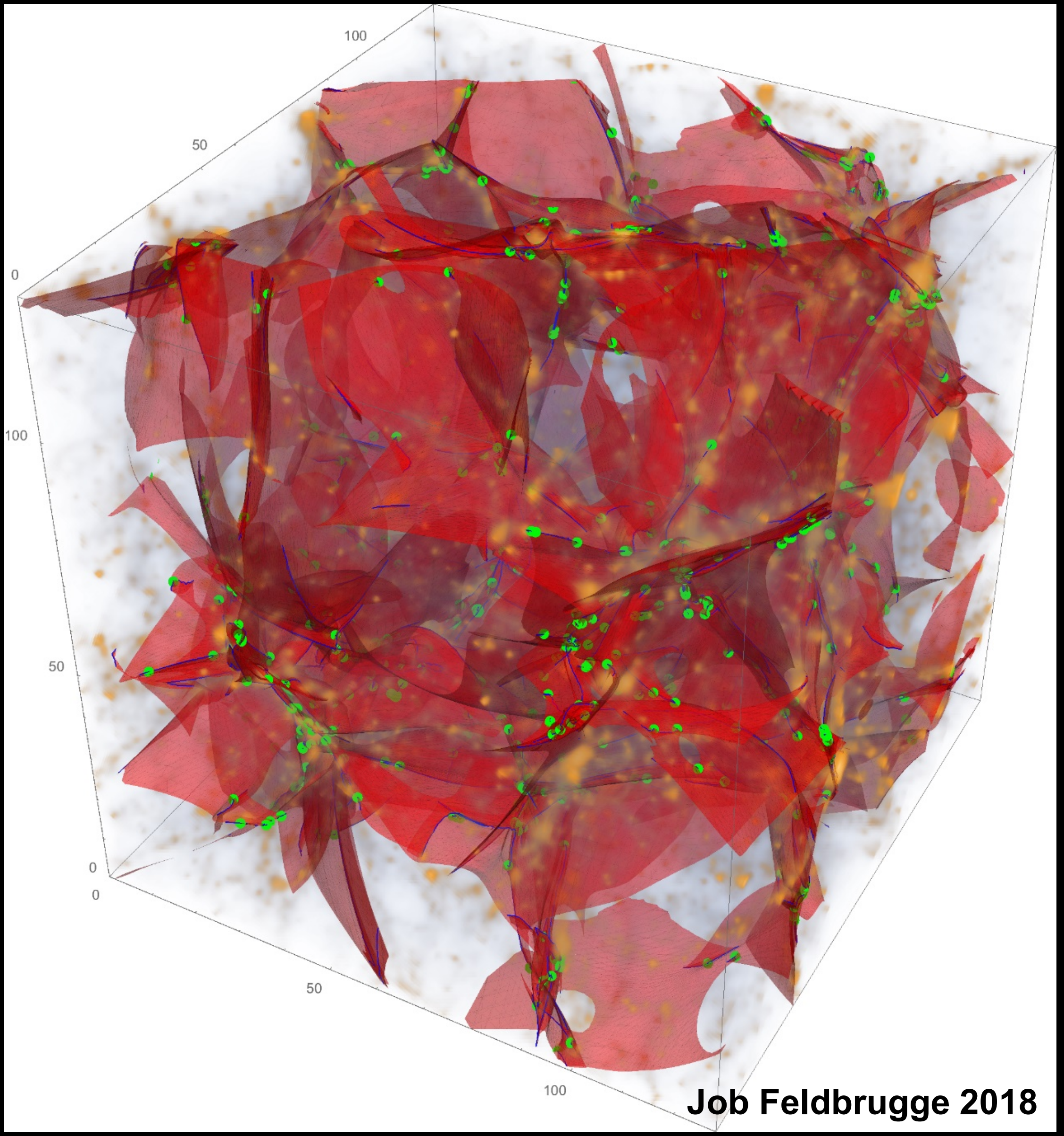

Zeldovich Dynamics

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$

$$\lambda_1, \lambda_2, \lambda_3 \qquad\qquad \lambda_1 > \lambda_2 > \lambda_3$$

By rewriting the Euler equation (in comoving coordinates), we may easily understand dynamical nature of the Zeldovich approximation:

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{a}\vec{\nabla}\phi$$

Define velocity u,
wrt linear growth factor D(t):

$$\vec{u} = \frac{d\vec{x}}{dD} = \frac{\vec{v}}{a\dot{D}}$$

Zeldovich Dynamics

Following some algebraic manipulations, one arrives at the equivalent Euler equation for the normalized velocity u:

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\vec{\nabla}\left(\frac{3\Omega}{2f^2 D}\phi_v + \frac{\phi}{a^2 \dot{D}^2}\right) = -\vec{\nabla} V$$

With velocity potential ϕ_v:

$$\vec{u} = -\vec{\nabla}\phi_v$$

and effective potential V:

$$V = \frac{3\Omega}{2f^2 D}\left(\phi_v + \theta\right)$$

and scaled gravitational potential θ:

$$\theta = \frac{2\phi}{3\Omega a^2 D H^2}$$

Effective & Scaled Potentials

For the Zeldovich approximation we may easily see that the effective potential V=0:

$$V = \frac{3\Omega}{2f^2D}\left(\phi_v + \theta\right) = 0$$

For the Zeldovich approximation:

$$\vec{x} = \vec{q} - D(t)\vec{\nabla}\Psi(\vec{q})$$

with:

$$\Psi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi\left(\vec{x},t\right)$$

so that the scaled gravitational potential θ:

$$\theta = \frac{2\phi}{3\Omega a^2 DH^2} = \Psi\left(\vec{q}\right)$$

The velocity potential ϕ_v we may infer from the velocity corresponding to the Zeldovich approximation:

$$\vec{v} = a\dot{\vec{x}} = -aDH\, f\left(\Omega\right)\vec{\nabla}\Psi\left(\vec{q}\right)$$

$$\vec{u} = \vec{\nabla}\phi_v = \frac{\vec{v}}{a\dot{D}} = -\frac{aDH}{a\dot{D}} f\left(\Omega\right)\vec{\nabla}\Psi\left(\vec{q}\right) = -\vec{\nabla}\Psi\left(\vec{q}\right)$$

from which we see that

$$\phi_v = -\Psi\left(\vec{q}\right)$$

Hence, for the Zeldovich approximation: $\phi_v + \theta = 0 \quad \Rightarrow \quad V = 0$

Zel'dovich ++:

Adhesion Formalism

Zeldovich-Adhesion

We saw that dynamically, the Zeldovich approximation corresponds to a force-free propagation, as evidenced by the Euler equation for the normalized velocity u:

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\vec{\nabla} V = 0$$

The force-free nature of the Zeldovich approximation leads to the ballistic motion, which once a mass element enters a multi-stream nonlinear region ignores the dominant self-gravitational terms, ie. the evolving gravitational potential of high-density structures (such as walls, filaments and clumps).

The adhesion approximation augments this with a (really) artificial term – a non-gravitational term – in terms of a viscosity term (as we know from the Navier-Stokes equation):

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \nu \nabla^2 \vec{u}$$

Zeldovich-Adhesion

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$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \nu \, \nabla^2 \vec{u}$$

This equation, the Navier-Stokes equation for a pressureless medium, goes by the name of

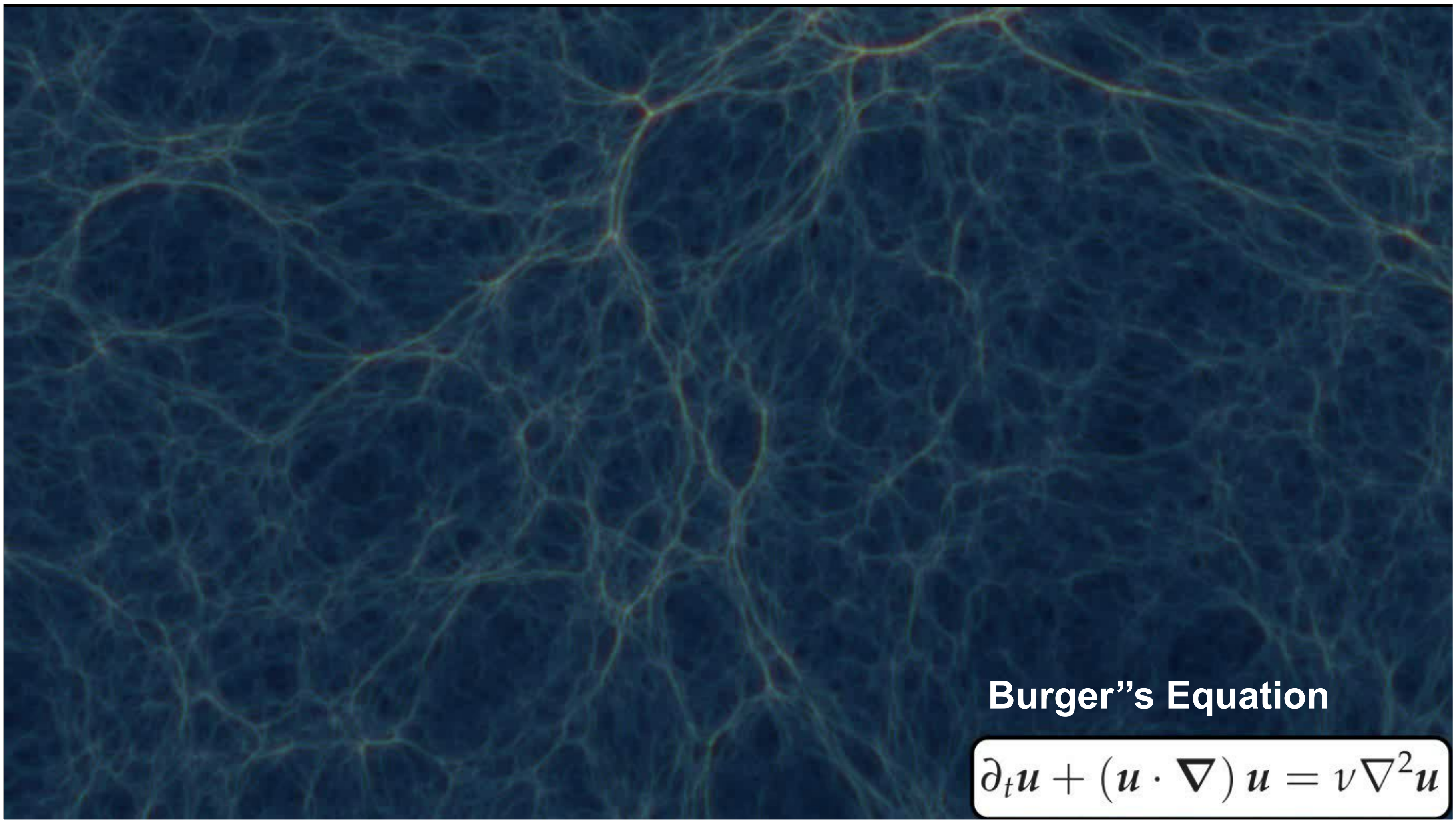
Burger's Equation

after the famous hydrodynamicist. It is one of the few equations that can be fully solved analytically.

The viscosity term here is fully artificial, tries to emulate "selfgravity", and has nothing to do with the physical viscosity we know from hydrodynamics. Basically, it functions as a friction term.

In its cosmological context, you only want to invoke it close to the emerging multistream regions, so that you take the asymptotic "inviscid" limit, $\nu \rightarrow 0$

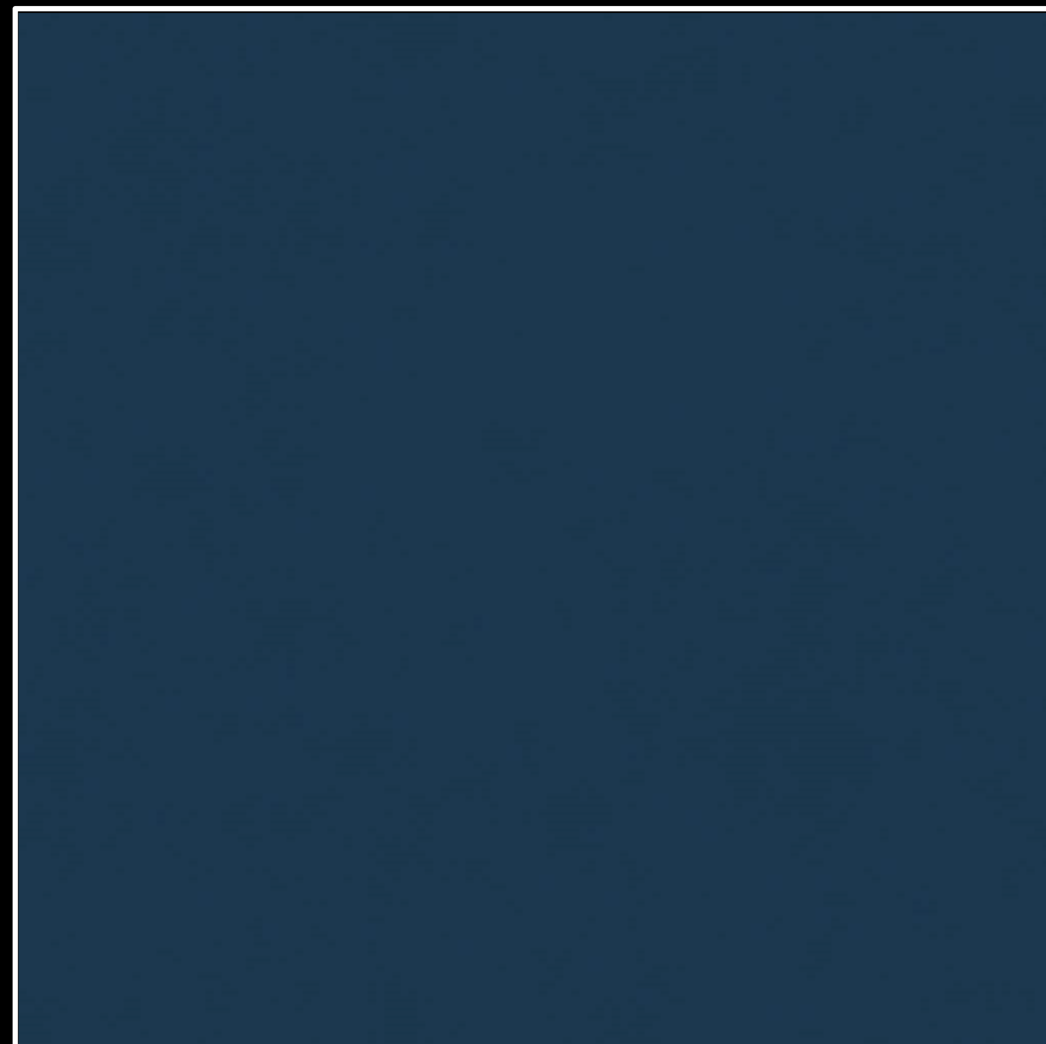
Adhesion Approximation



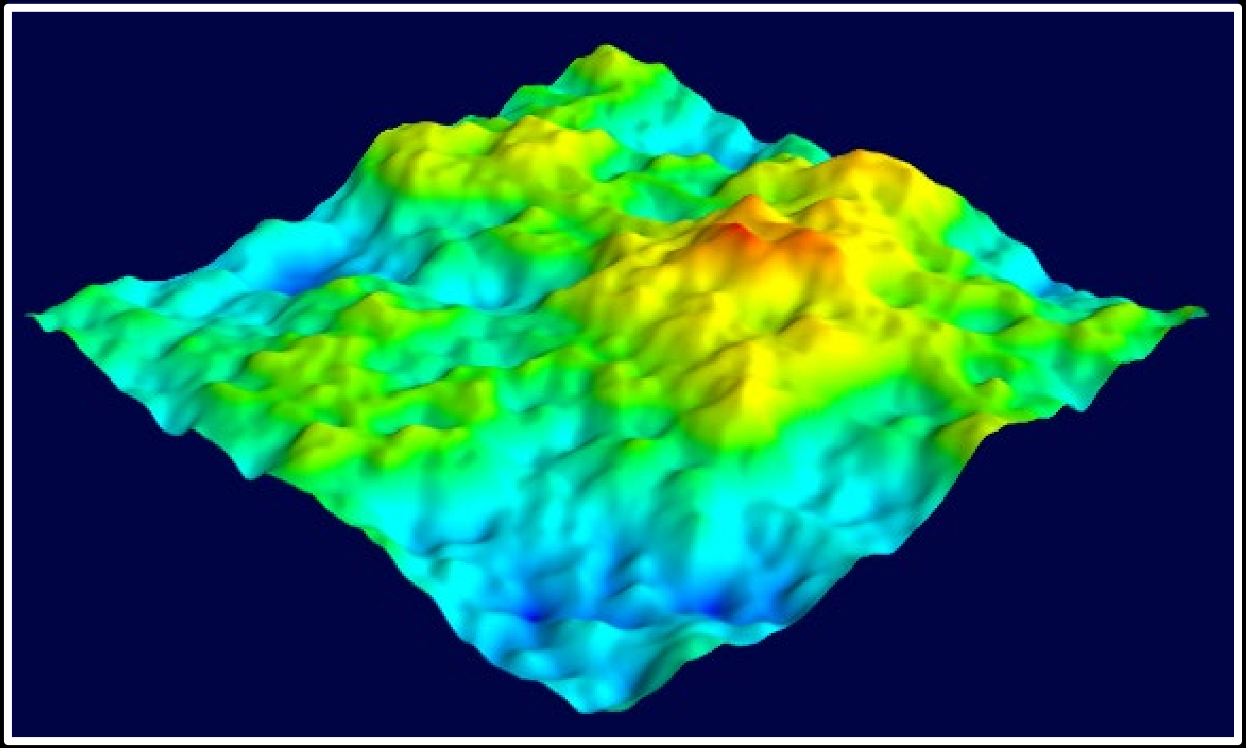
Adhesion Approximation

Gurbatov, Saichev & Shandarin 1987

Hidding 2012



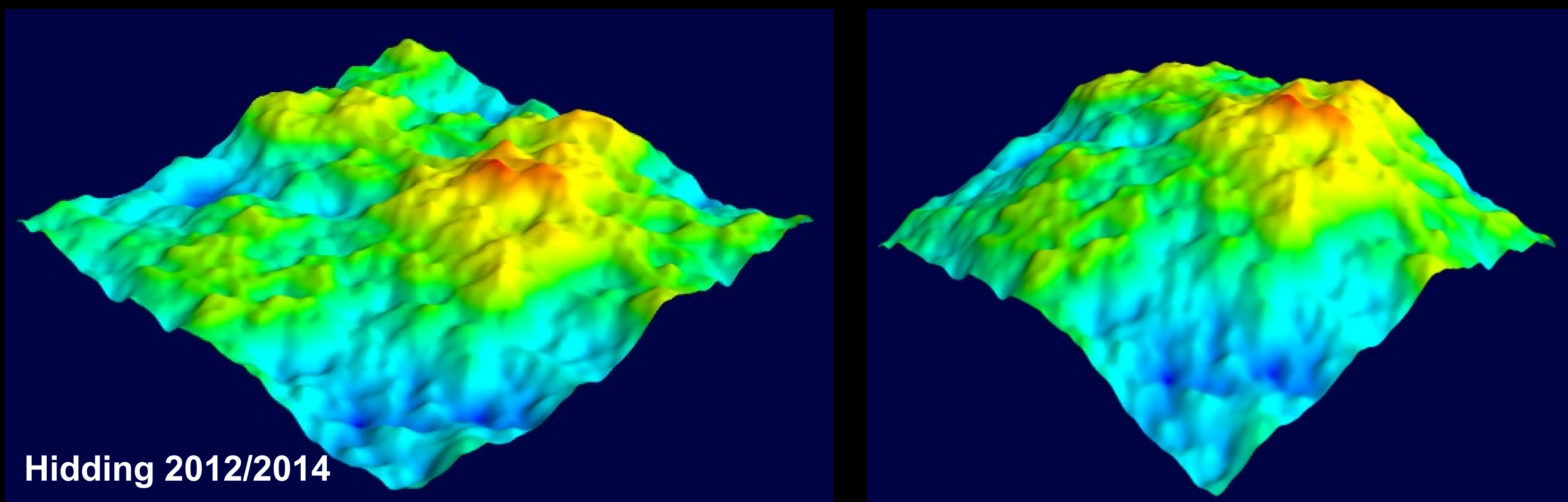
Velocity & Gravity Potential



$$\vec{u}(\vec{q}) = \vec{\nabla}\Phi(\vec{q})$$

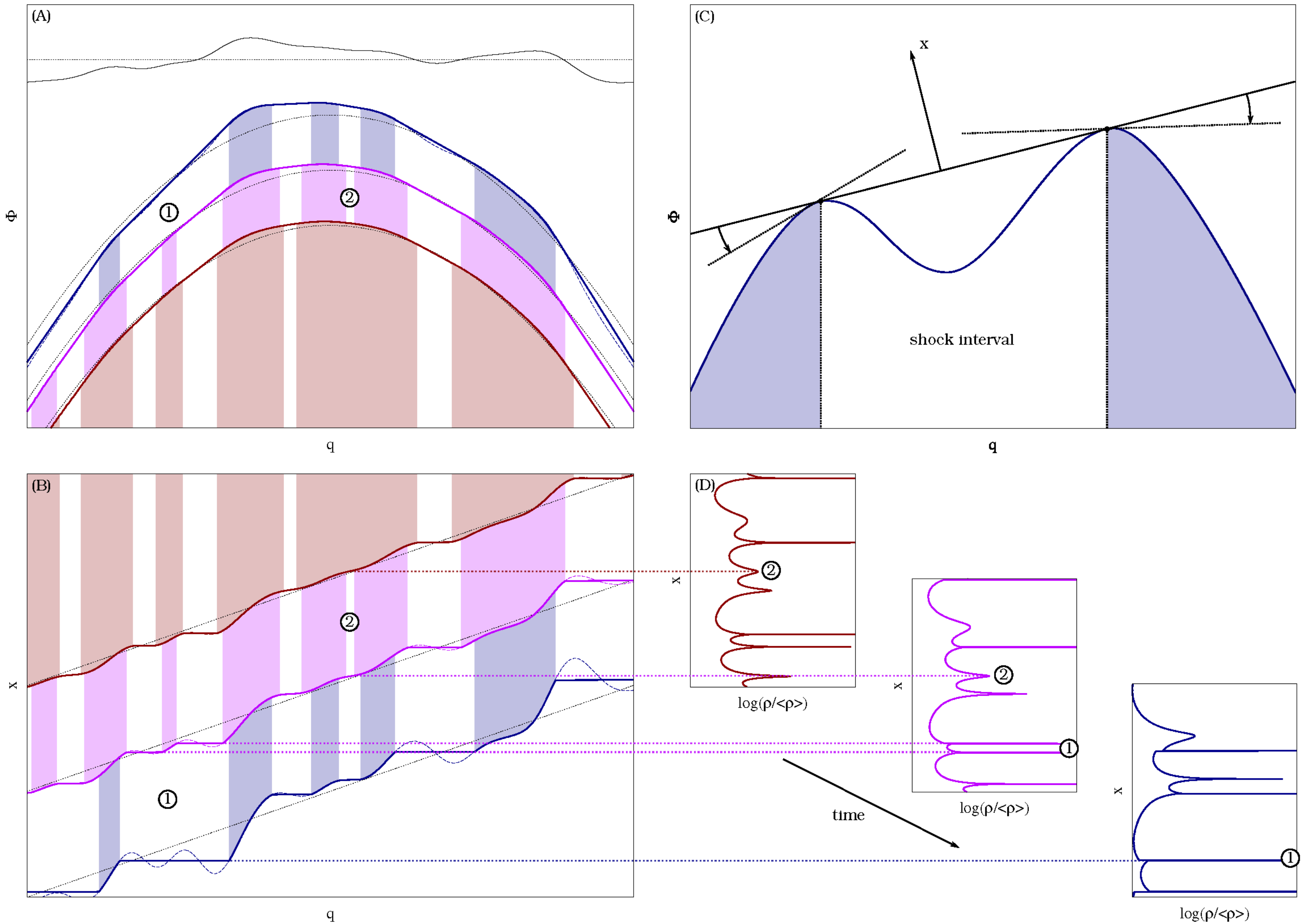
Burger's Equation: Hopf Solution

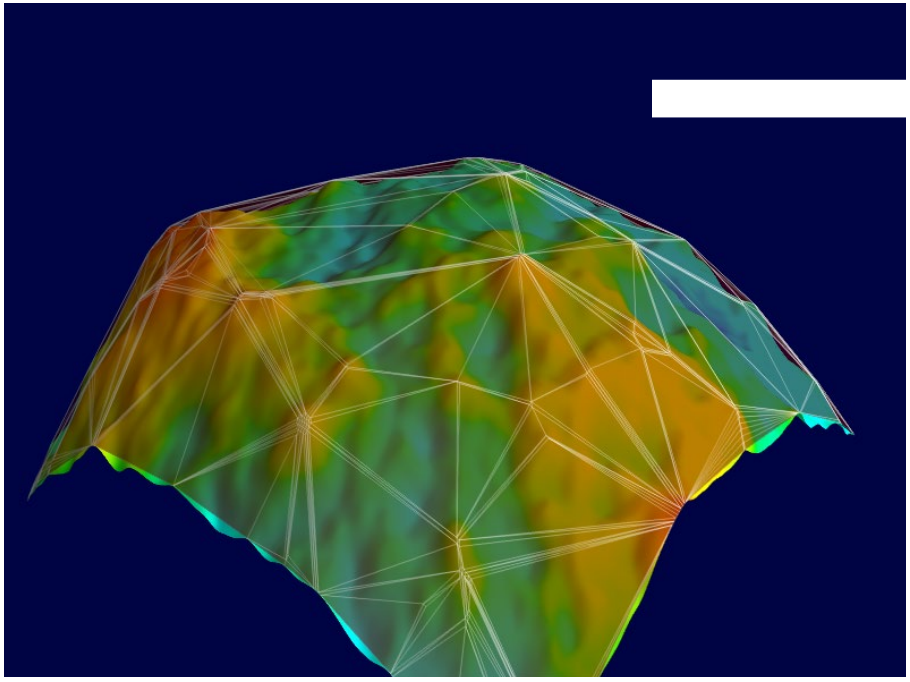
$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \nu \nabla^2 \vec{u}$$

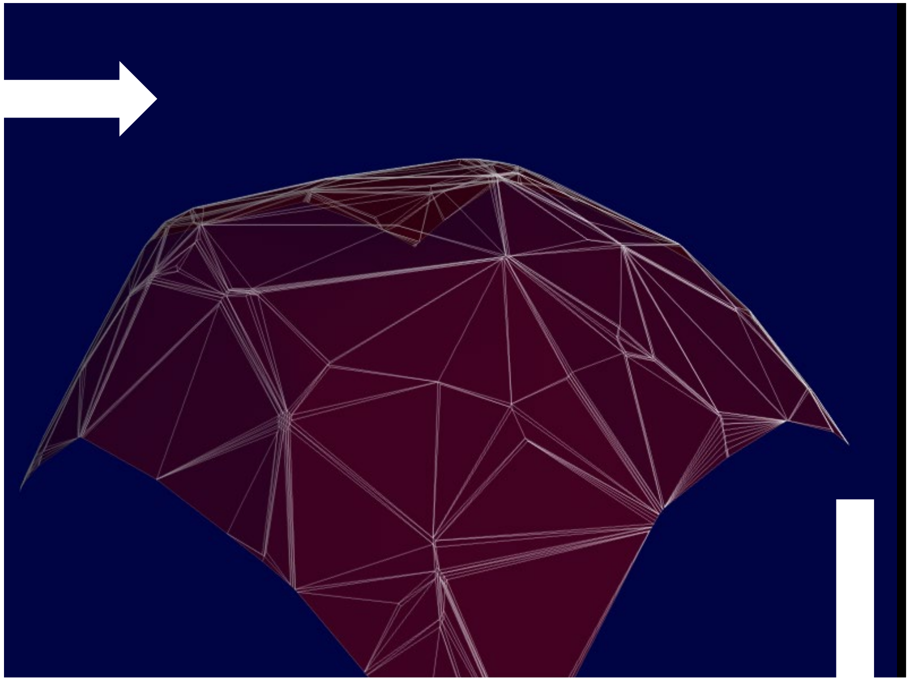


$$\Phi(\vec{x},t) + \frac{x^2}{2} = \max_q \left[\left(t\Phi_0(q) - \frac{q^2}{2} \right) + \vec{x} \cdot \vec{q} \right]$$

Burger's Equation: Hopf Solution



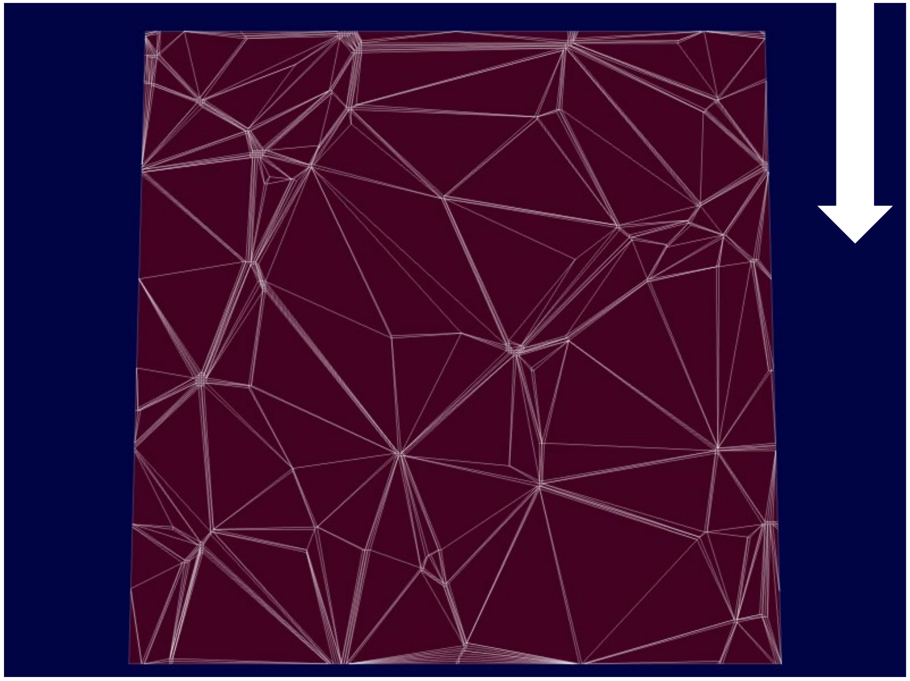




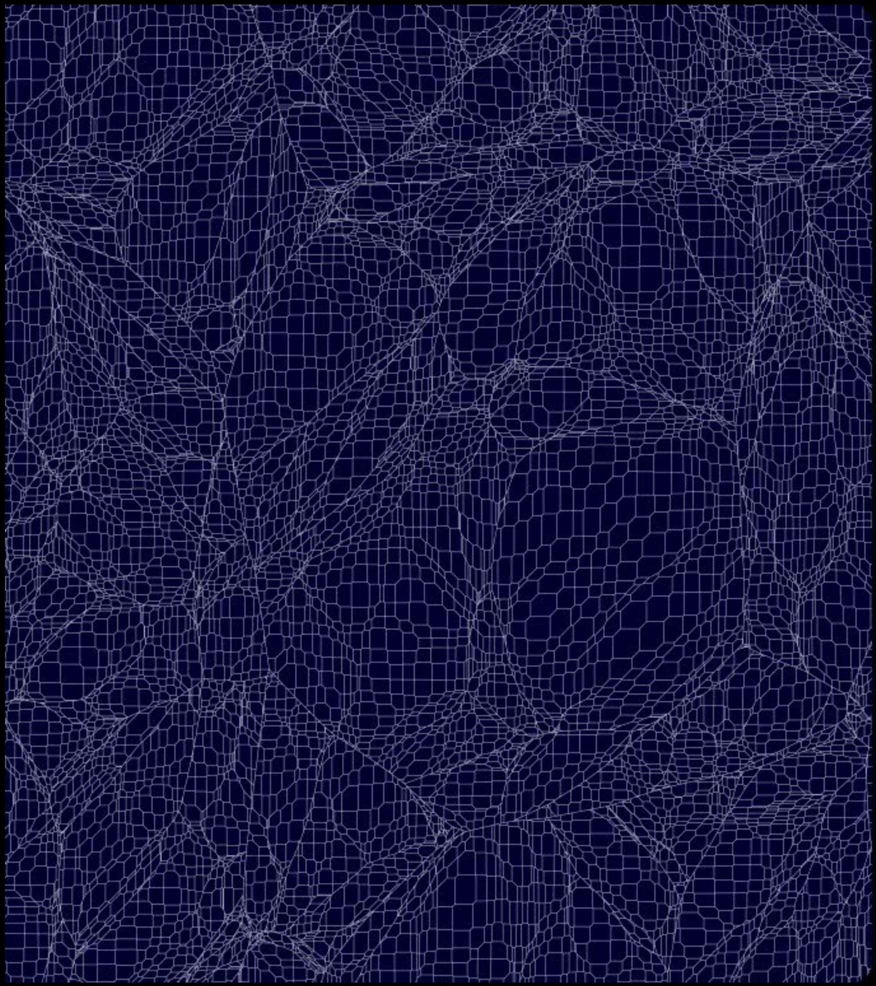
Hidding 2012/2014

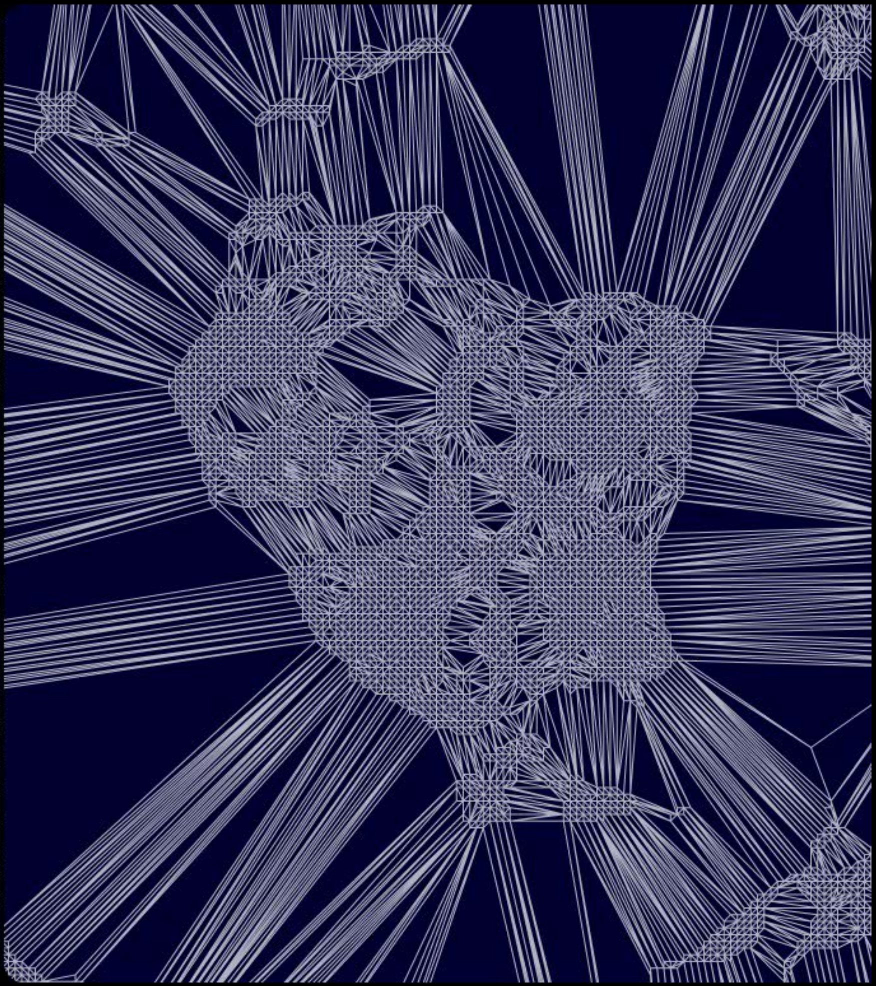
Convex Hull
quadratically lifted potential field

Delaunay tessellation
generated by maxima potential field



Eulerian – Lagrangian
Voronoi - Delaunay





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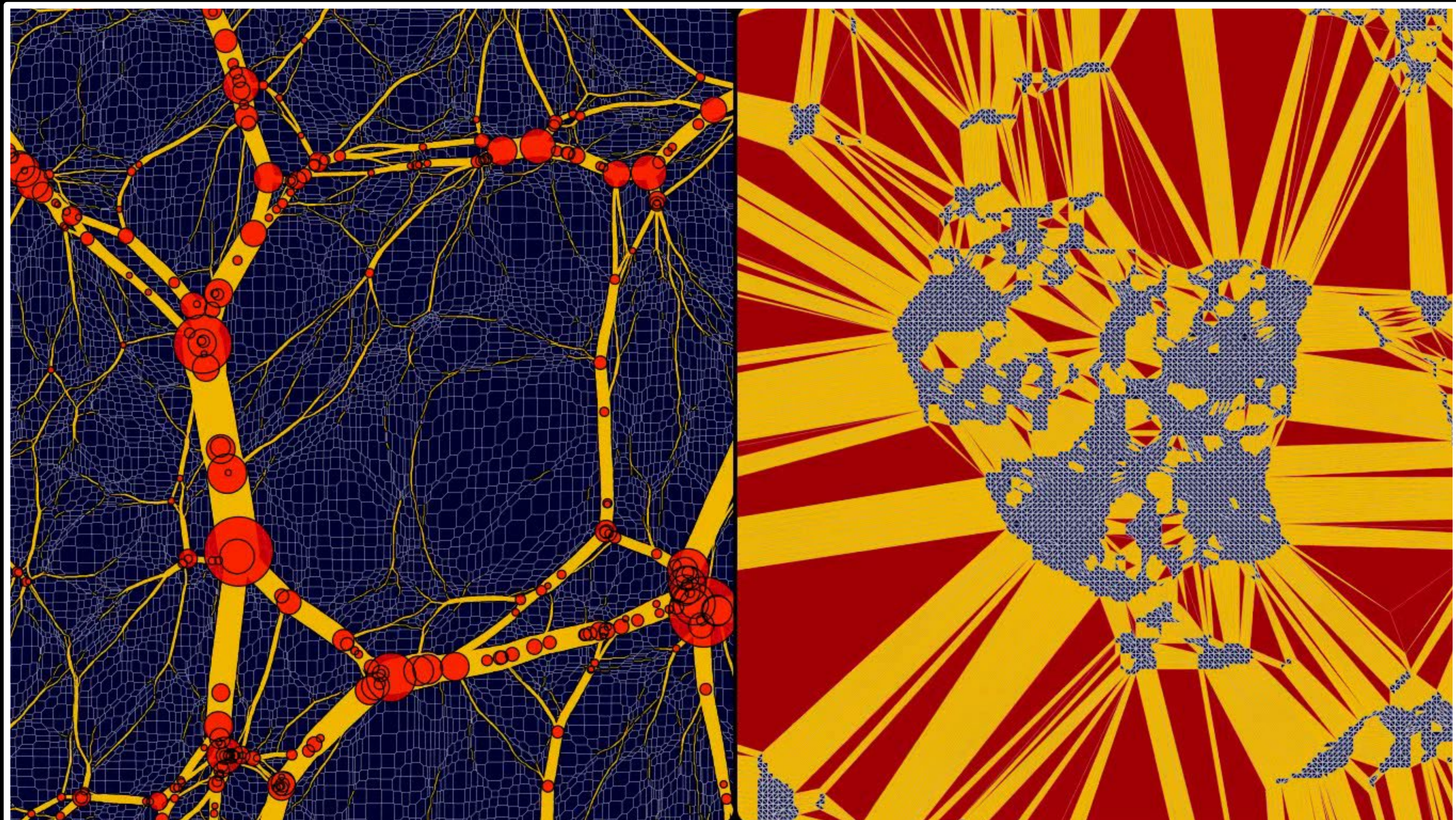
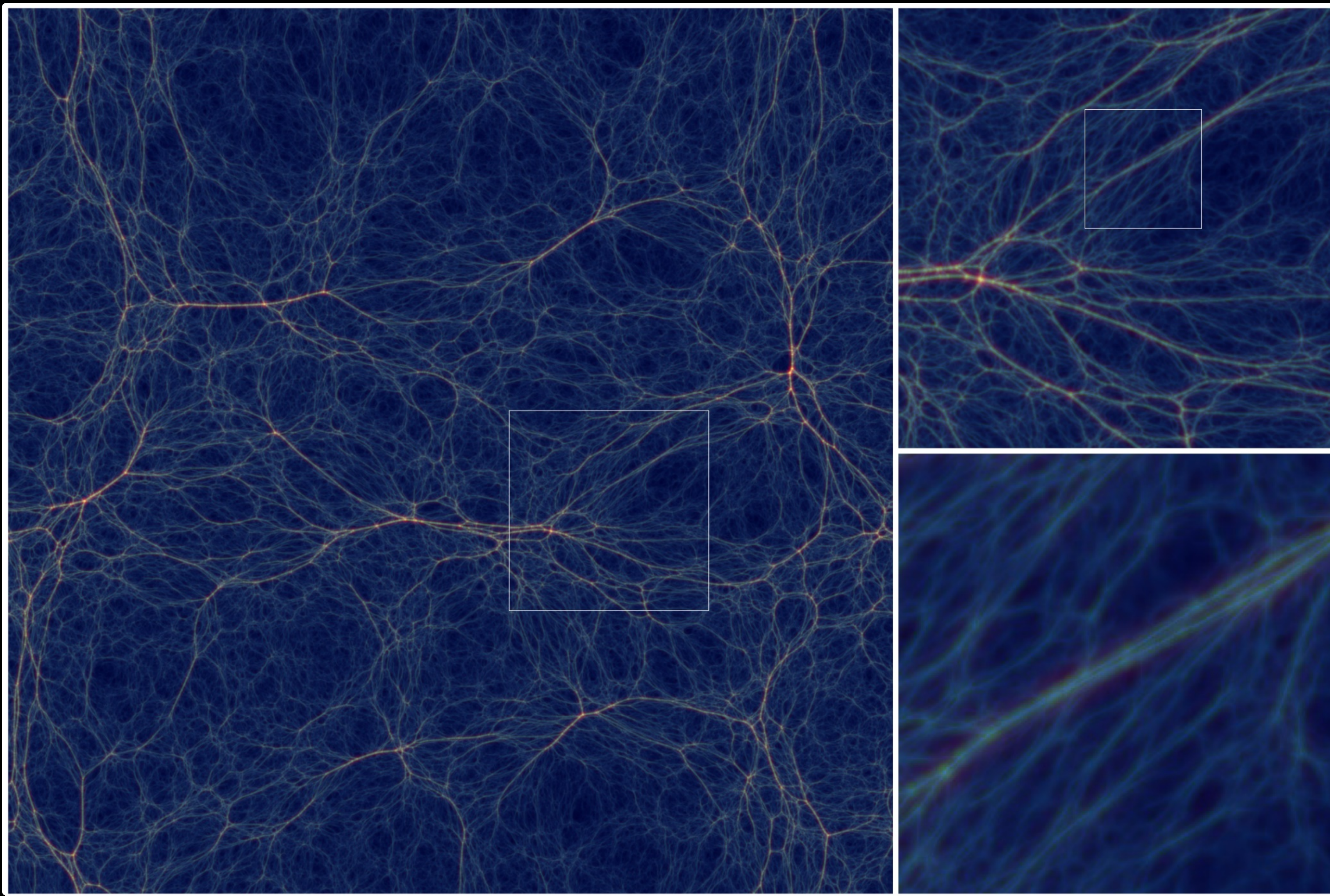
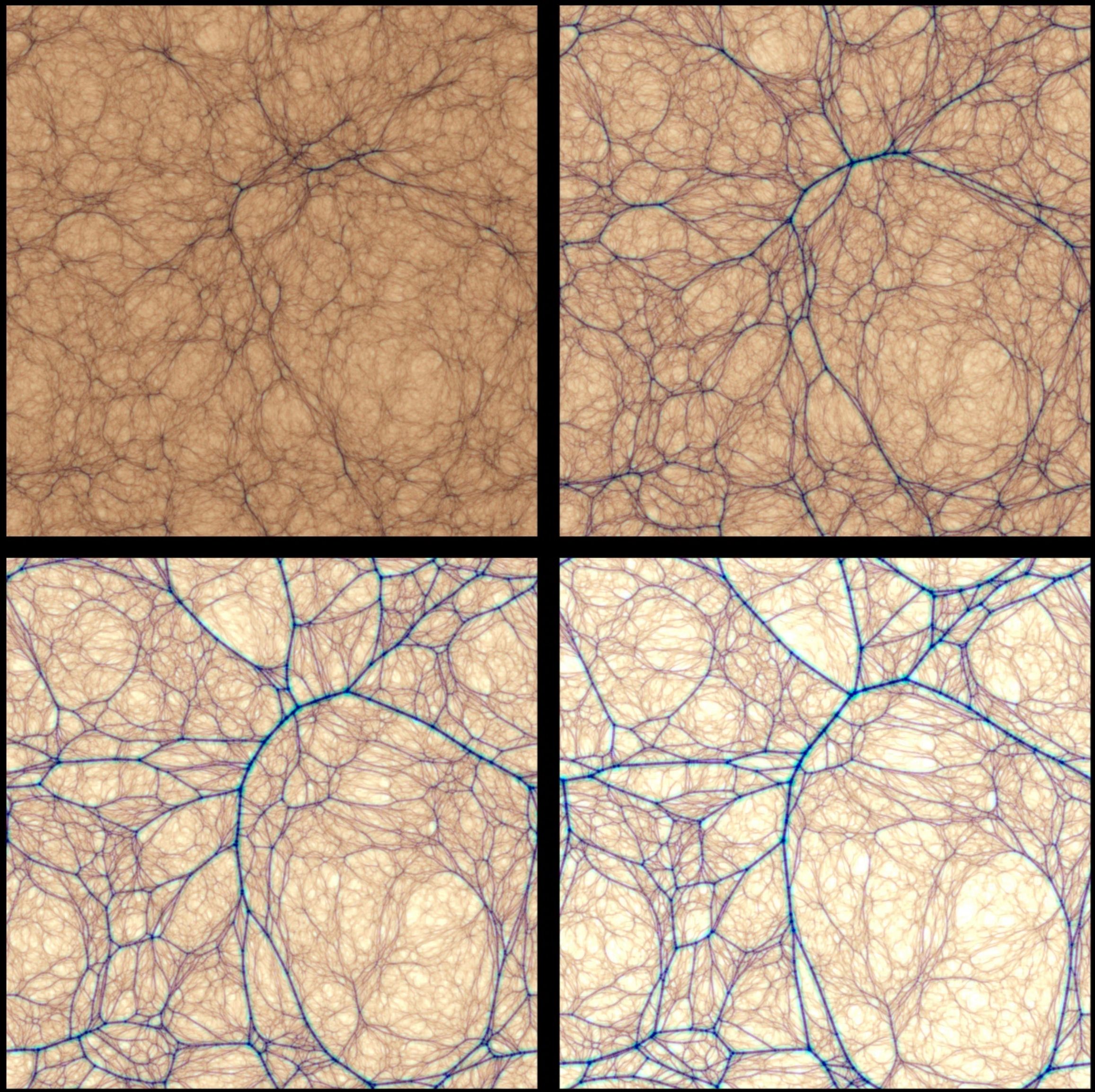


Multiscale Structure



Hierarachical Evolution



The adhesion formalism is ideal for following the hierarchical buildup of the cosmic web:

- Mathematically:
  as a result of the evolving parabolic curvature of the (velocity) potential, more features get embedded in singular valleys enclosed between potential and convex hull.

- Physically:
  - Clearly visible is the merging of small filaments into ever larger arteries.
  - at the same time, we see the continuous merging of small voids into larger voids, the evolving soapsud of void hierarchy.

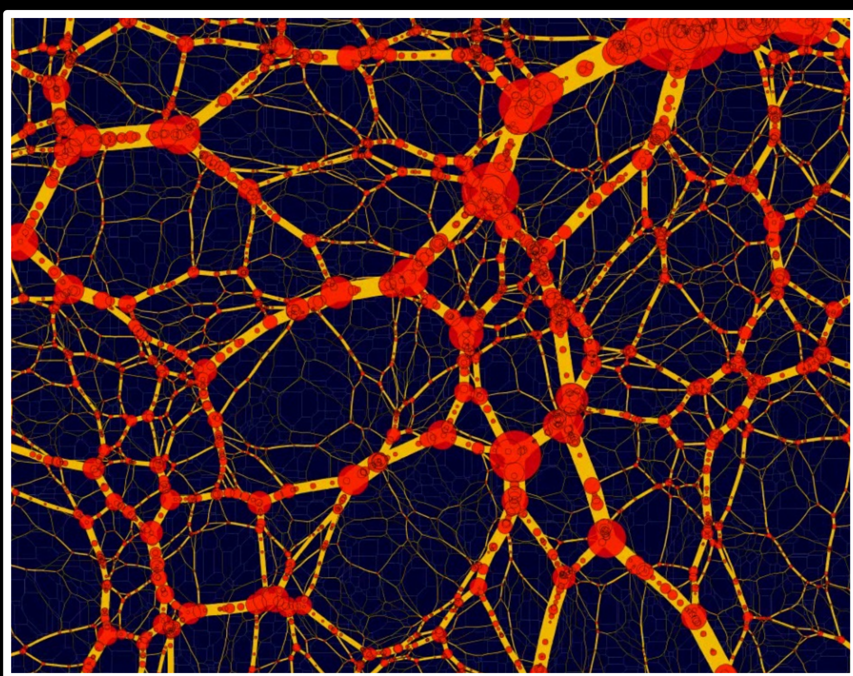
Cosmological Sensitivity

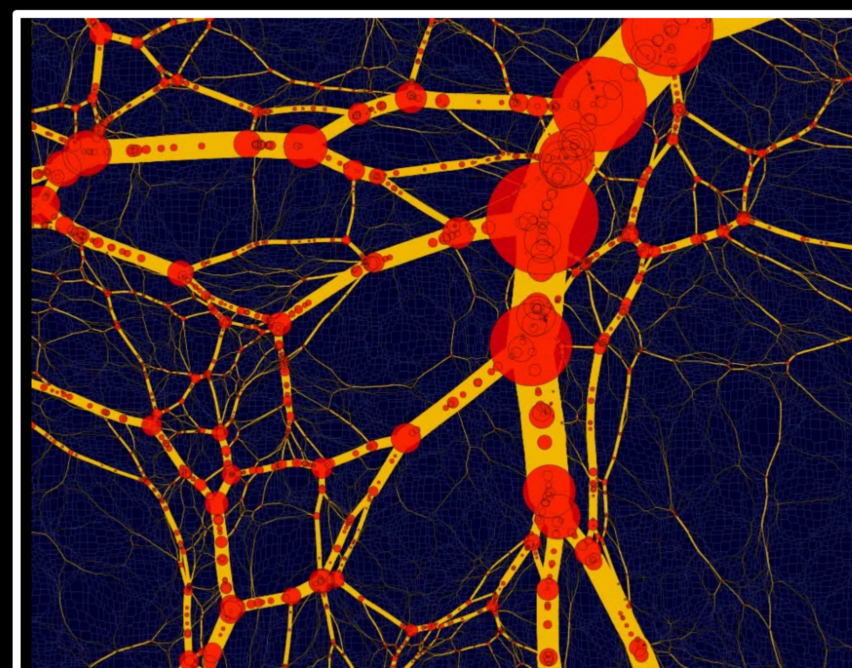
the morphology of the weblike network is
highly sensitive to the underlying cosmology

$P(k) \propto k^{-1.5}$

Hidding 2012/2014

$P(k) \propto k^{-2.0}$



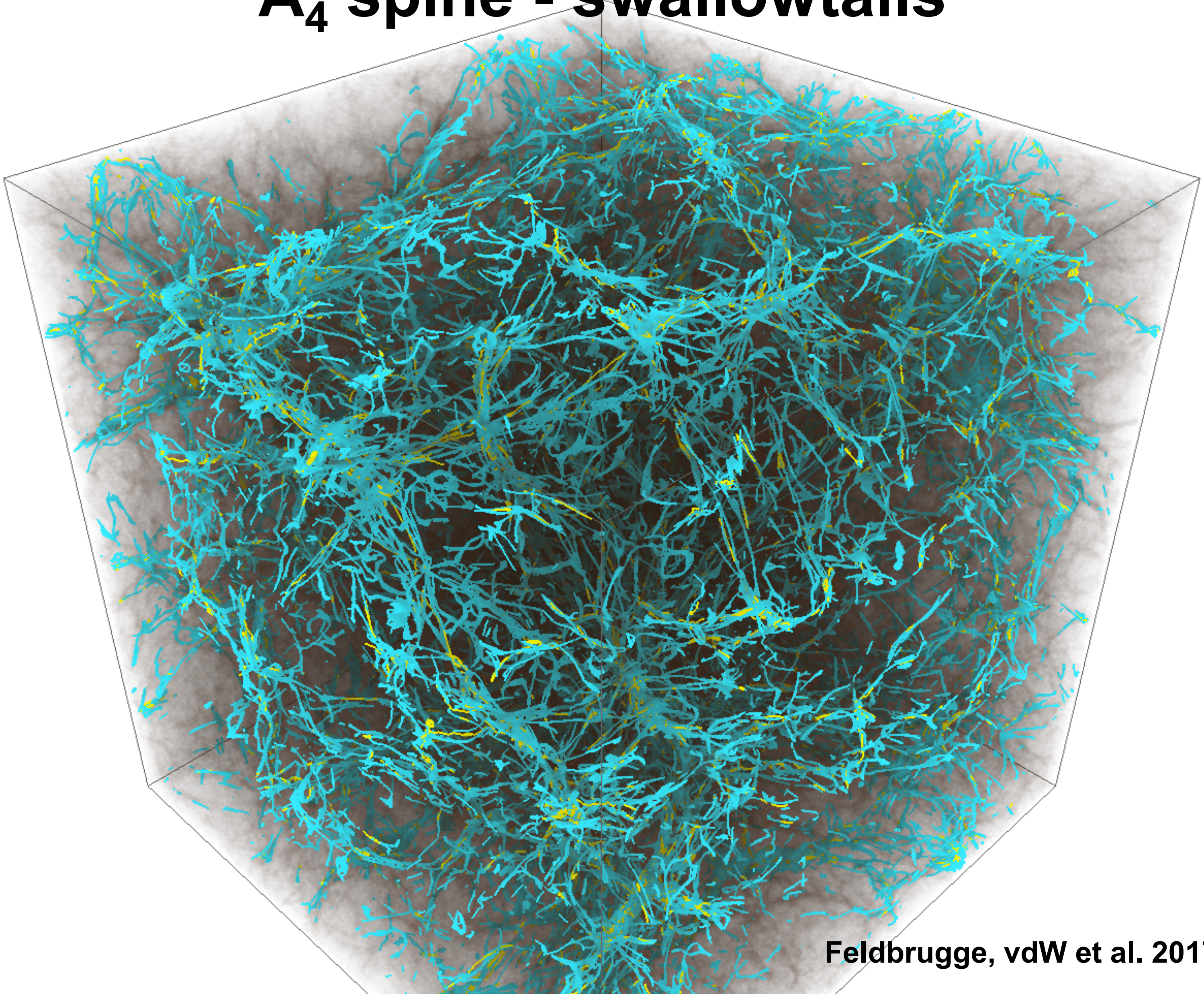


Zel’dovich ++:

Caustics & Catastrophes

Skeleton (3D) Cosmic Web:

A_4 spine - swallowtails



Feldbrugge, vdW et al. 2017b

Zel’dovich Formlism:
Streaming & Caustics

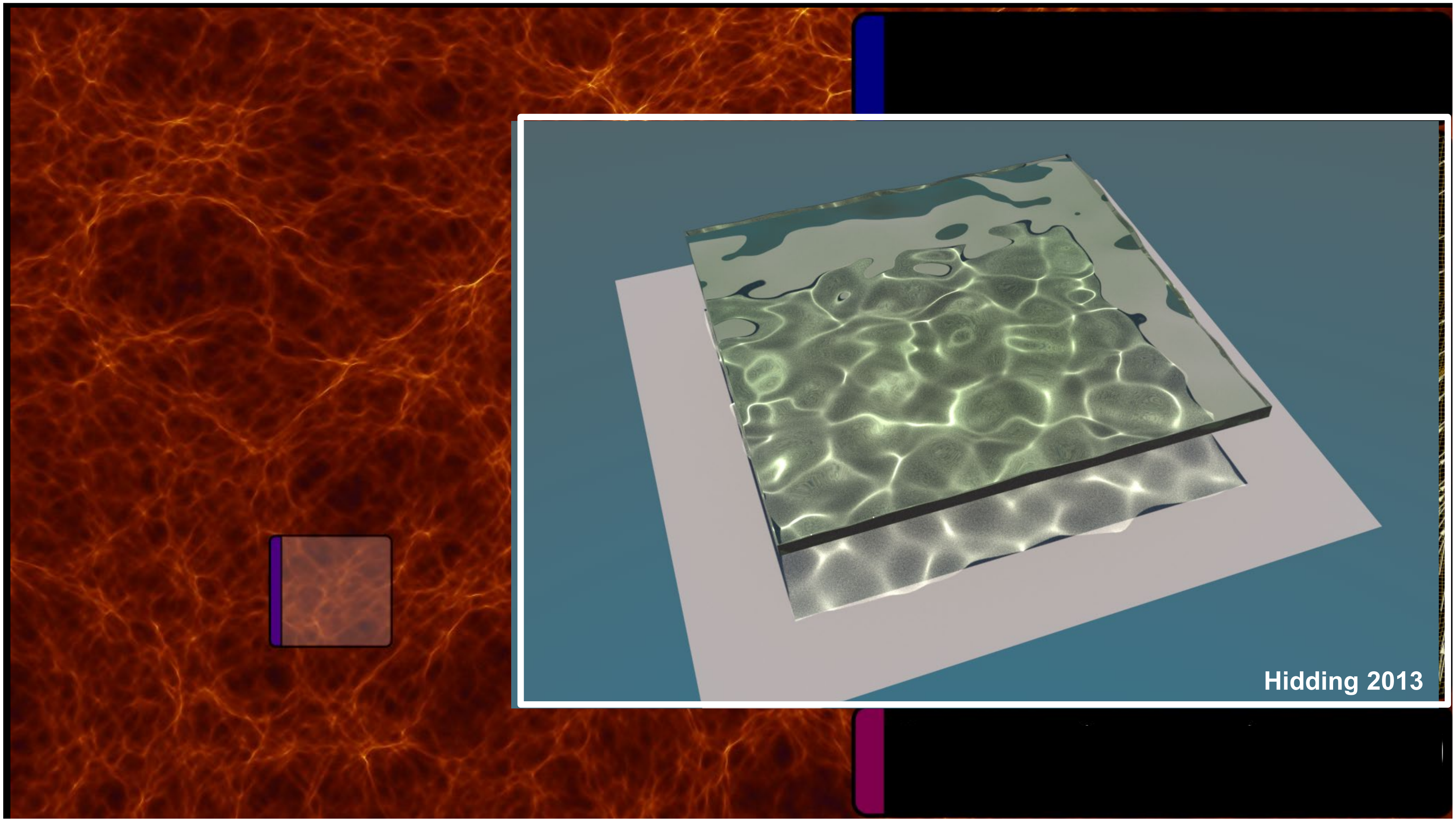


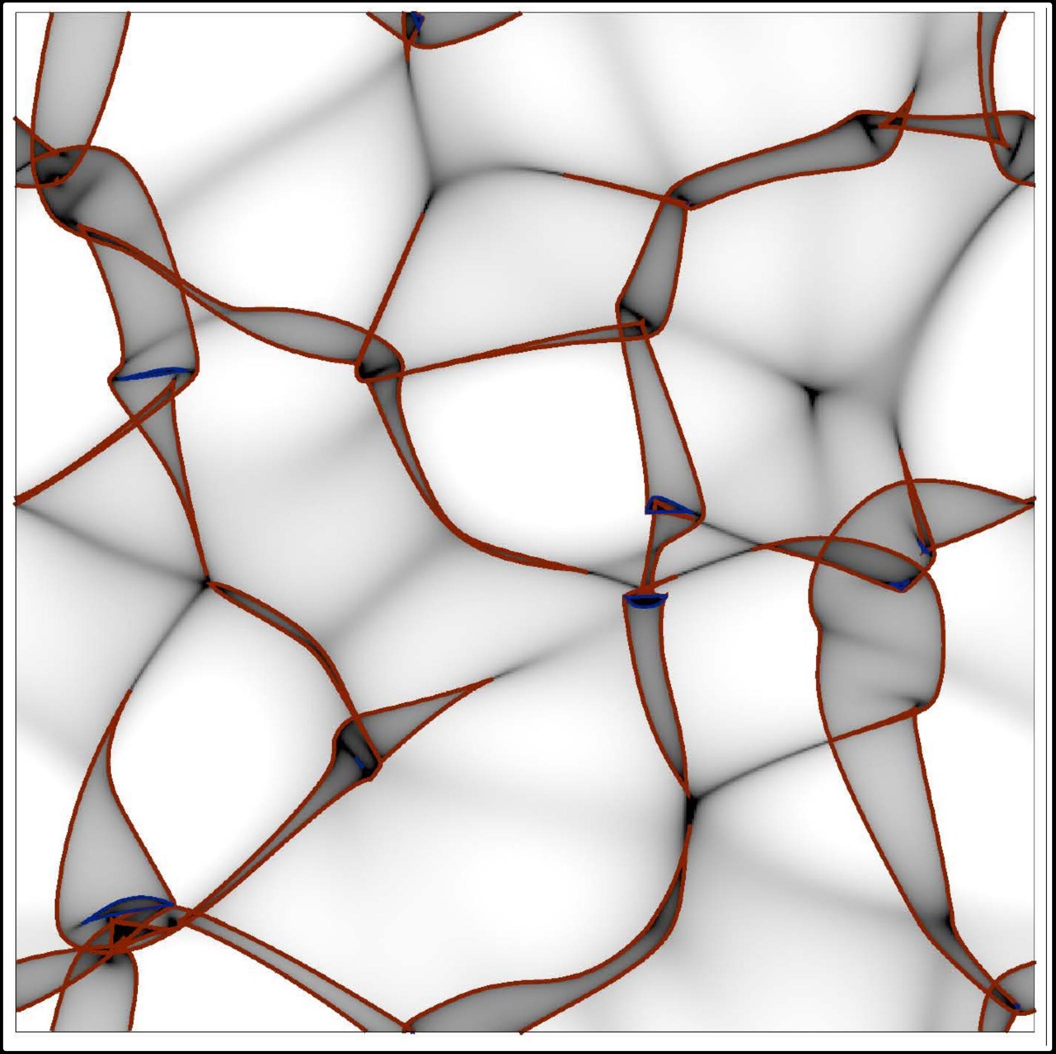
Illustration of the formation of caustics due to
streaming paths of light through deforming medium

Zel'dovich Approximation

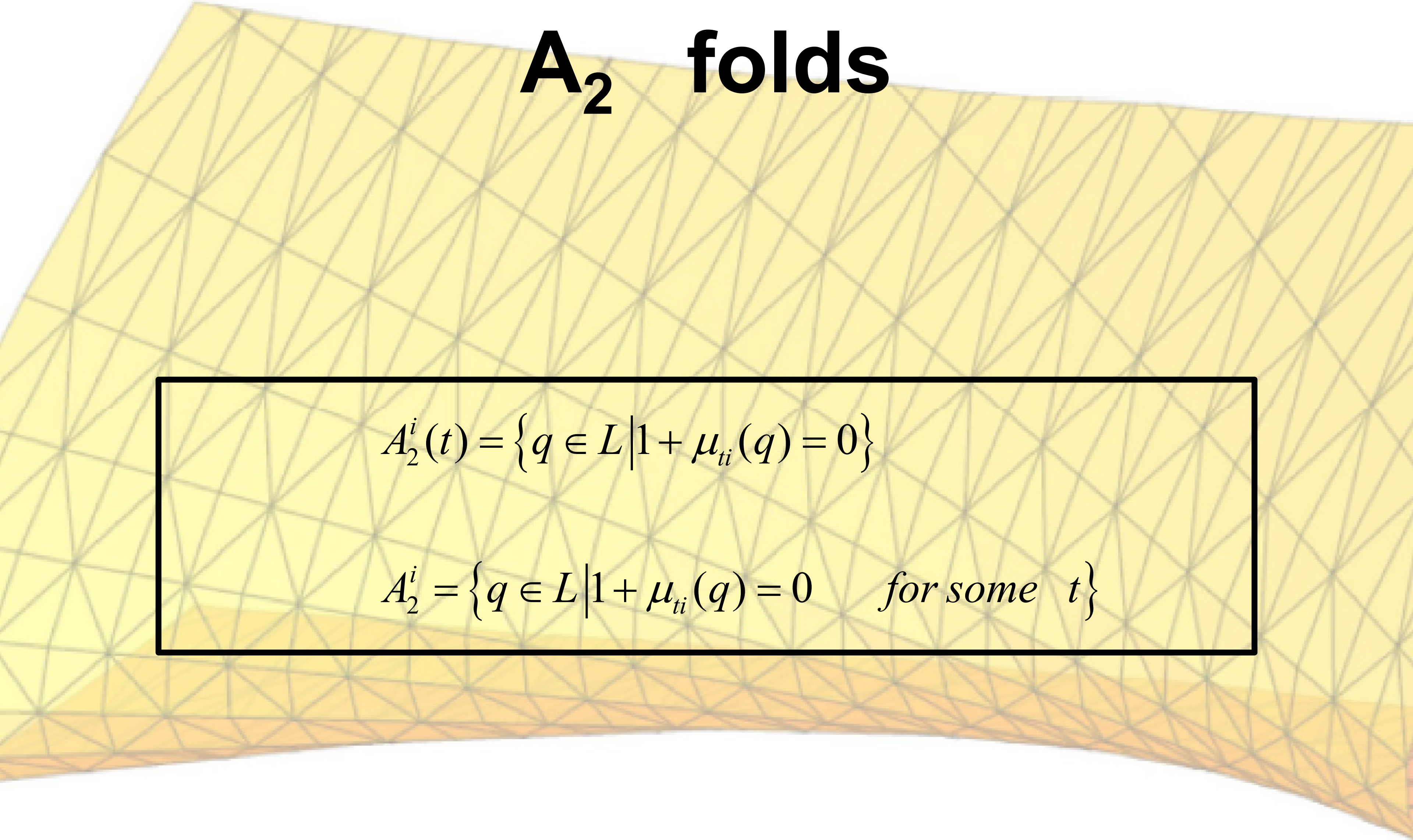
$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi_{lin}(\vec{q})$$



Caustic Conditions:
A_2 folds

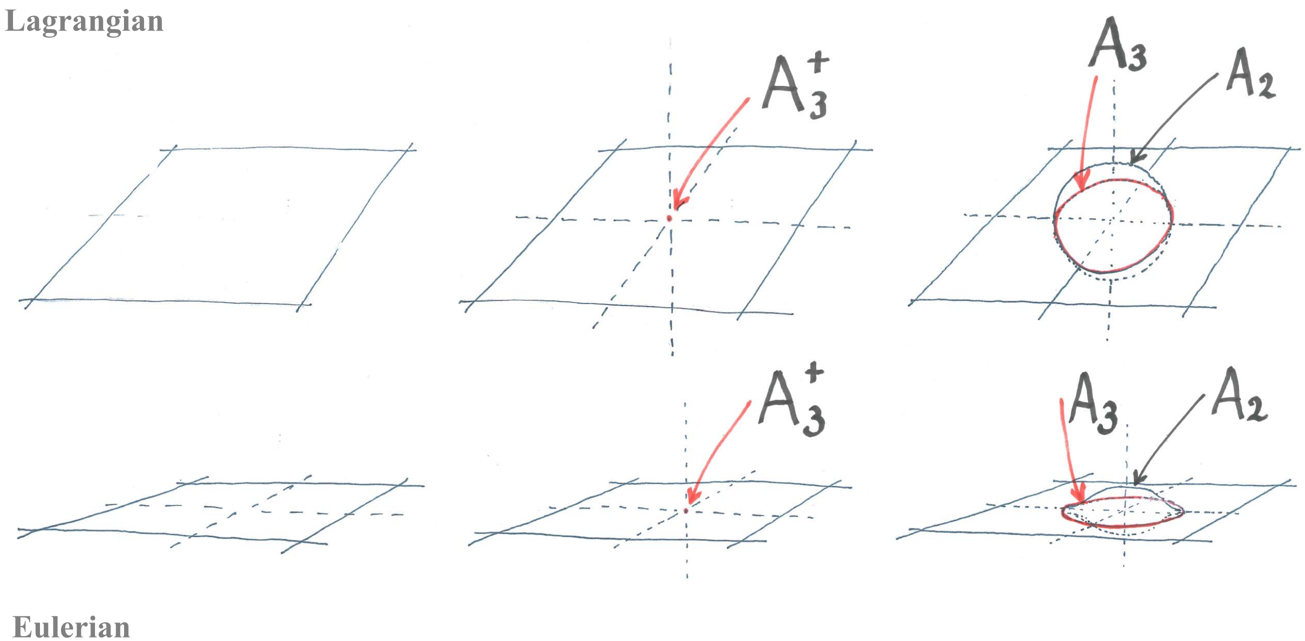


$$A_2^i(t) = \left\{ q \in L \middle| 1 + \mu_{ti}(q) = 0 \right\}$$

$$A_2^i = \left\{ q \in L \middle| 1 + \mu_{ti}(q) = 0 \quad for\ some \quad t \right\}$$

Feldbrugge, vdW et al. 2017a

Feldbrugge, vdW et al. 2017



Emergence of A_3 cusps around A_2 folds

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

(also see Zeeman, Thom)

singularities

D=1

A: λ_1

D=2

A: λ_1

D: λ_1, λ_2

D=3

A: λ_1

D: λ_1, λ_2

E: λ_1, λ_2, λ_3

Caustic Conditions: A_3 cusps

Folding A_2^i manifold into more complex configurations:

For j ≠ i, there is a nonzero tangential vector $\vec{T}$ such that caustic condition

$$\alpha_j = \vec{v}_j^*(q_s)\cdot\vec{T} = 0 \qquad j \neq i$$

$$\vec{T}(\vec{q}) \parallel \vec{v}_i(\vec{q}) \quad \Rightarrow \quad \vec{v}_i(\vec{q}) \perp \vec{n}(\vec{q}) = \vec{\nabla}\mu_i(\vec{q}) \quad \Rightarrow \quad \mu_{ti,i}(\vec{q}) = \vec{n}\cdot\vec{\nabla}\mu_i = 0$$

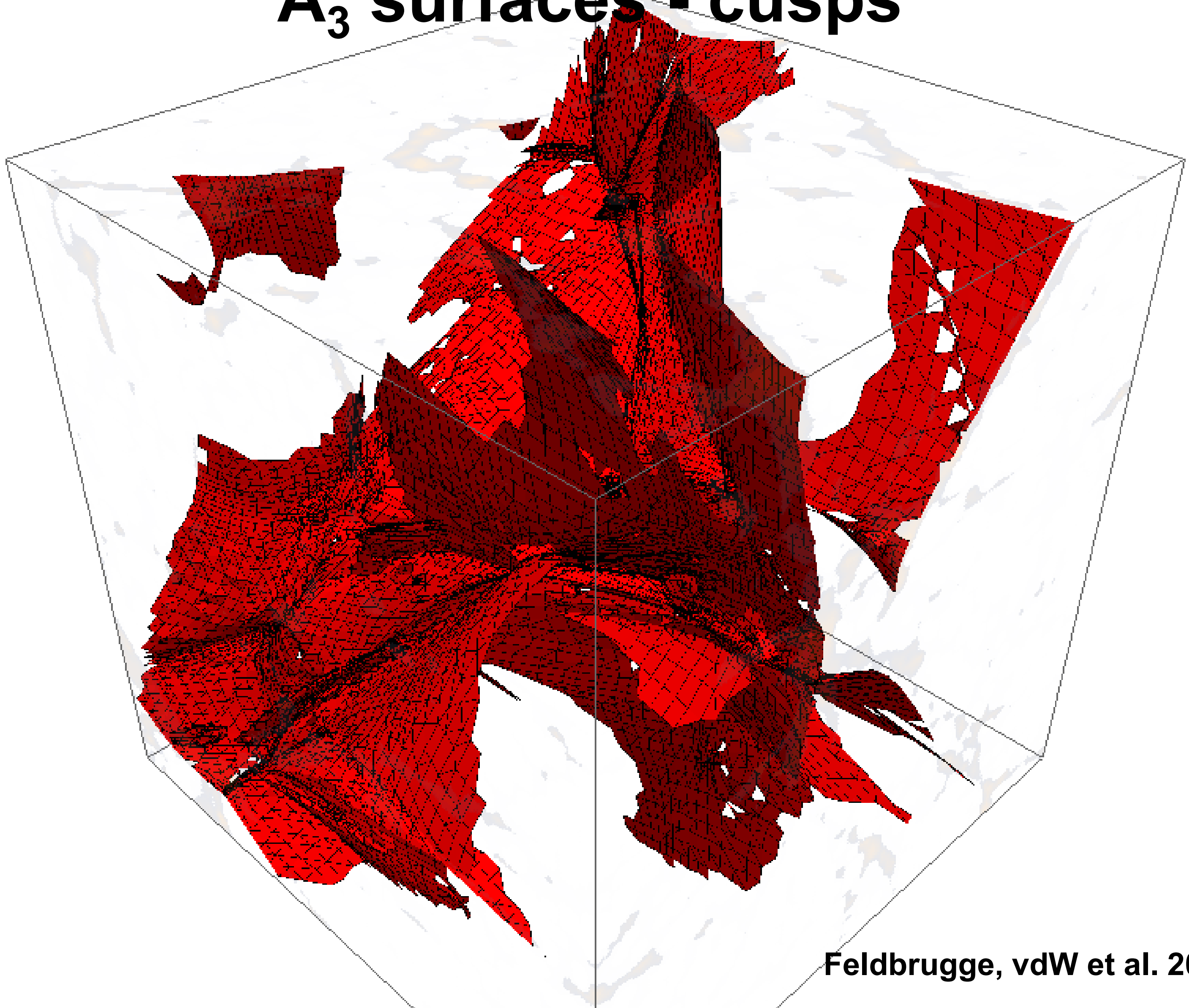
$$A_3^i(t) = \left\{ q \in L \,\middle|\, q \in A_2^i(t) \;\wedge\; 1 + \mu_{ti,i}(q) = 0 \right\}$$

$$A_3^i = \left\{ q \in L \,\middle|\, q \in A_2^i(t) \;\wedge\; 1 + \mu_{ti,i}(q) = 0 \quad \text{for some } t \right\}$$

Feldbrugge, vdW et al. 2017a

Skeleton (3D) Cosmic Web:

A_3 surfaces - cusps



Feldbrugge, vdW et al. 2017b

Zel'dovich Deformation Field

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

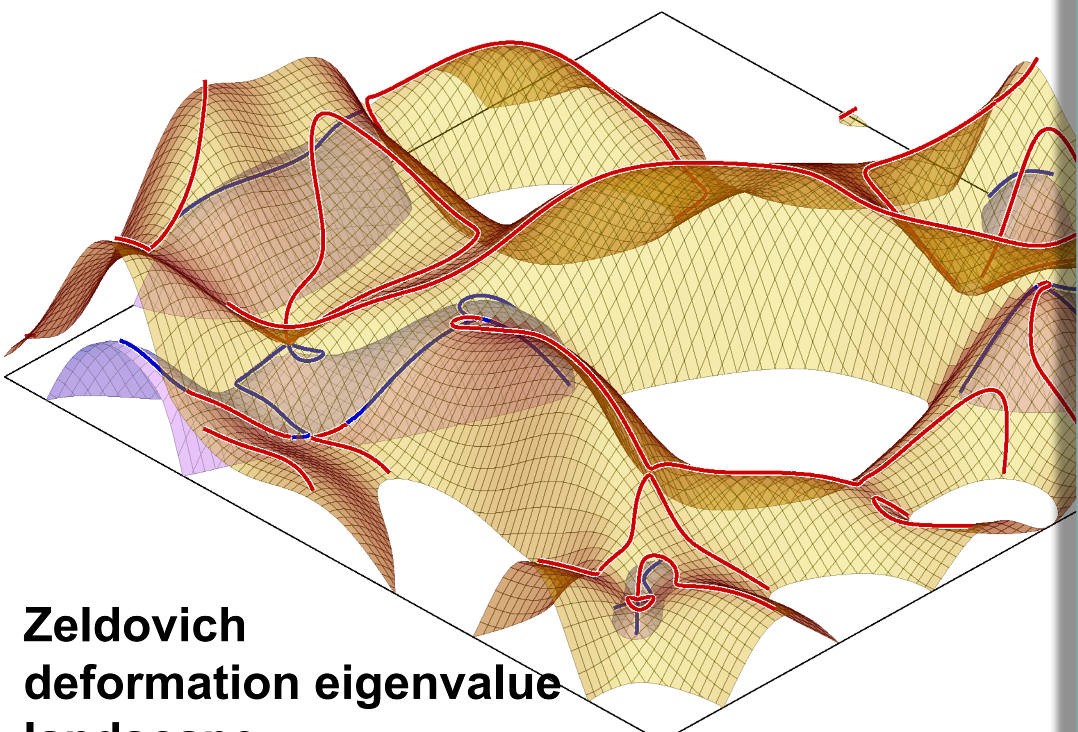
$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

⇩

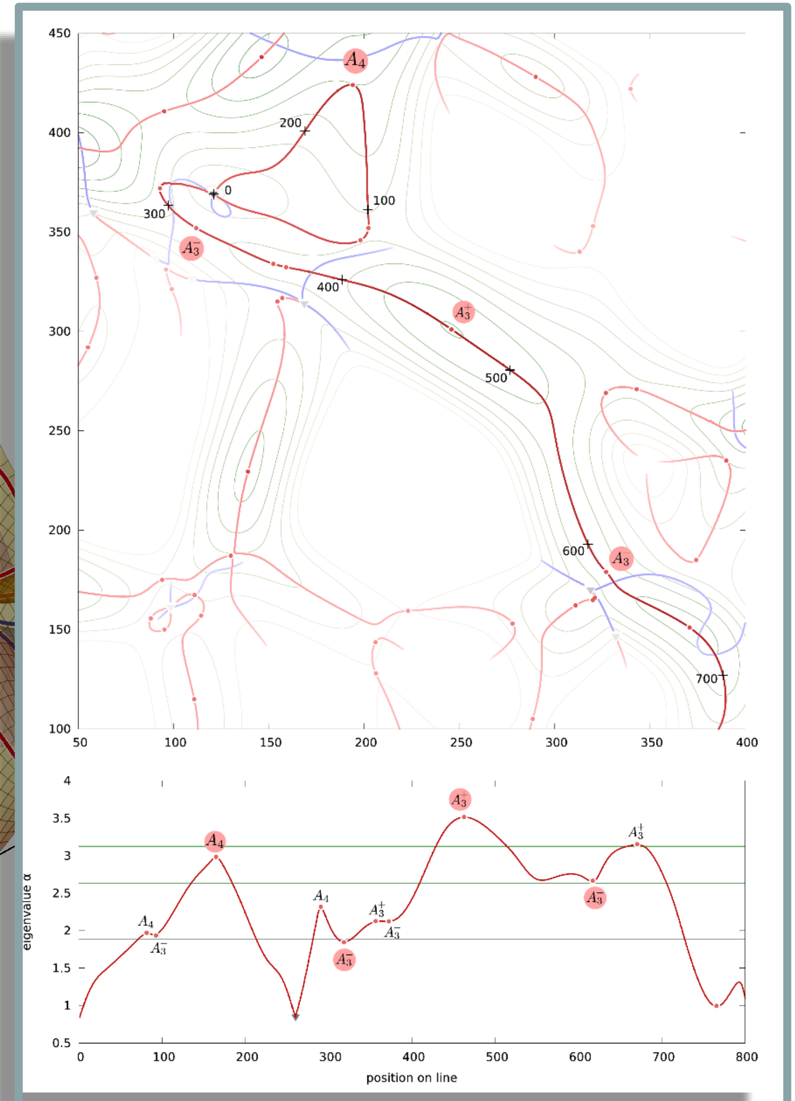
structure of the cosmic web determined by the spatial field of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

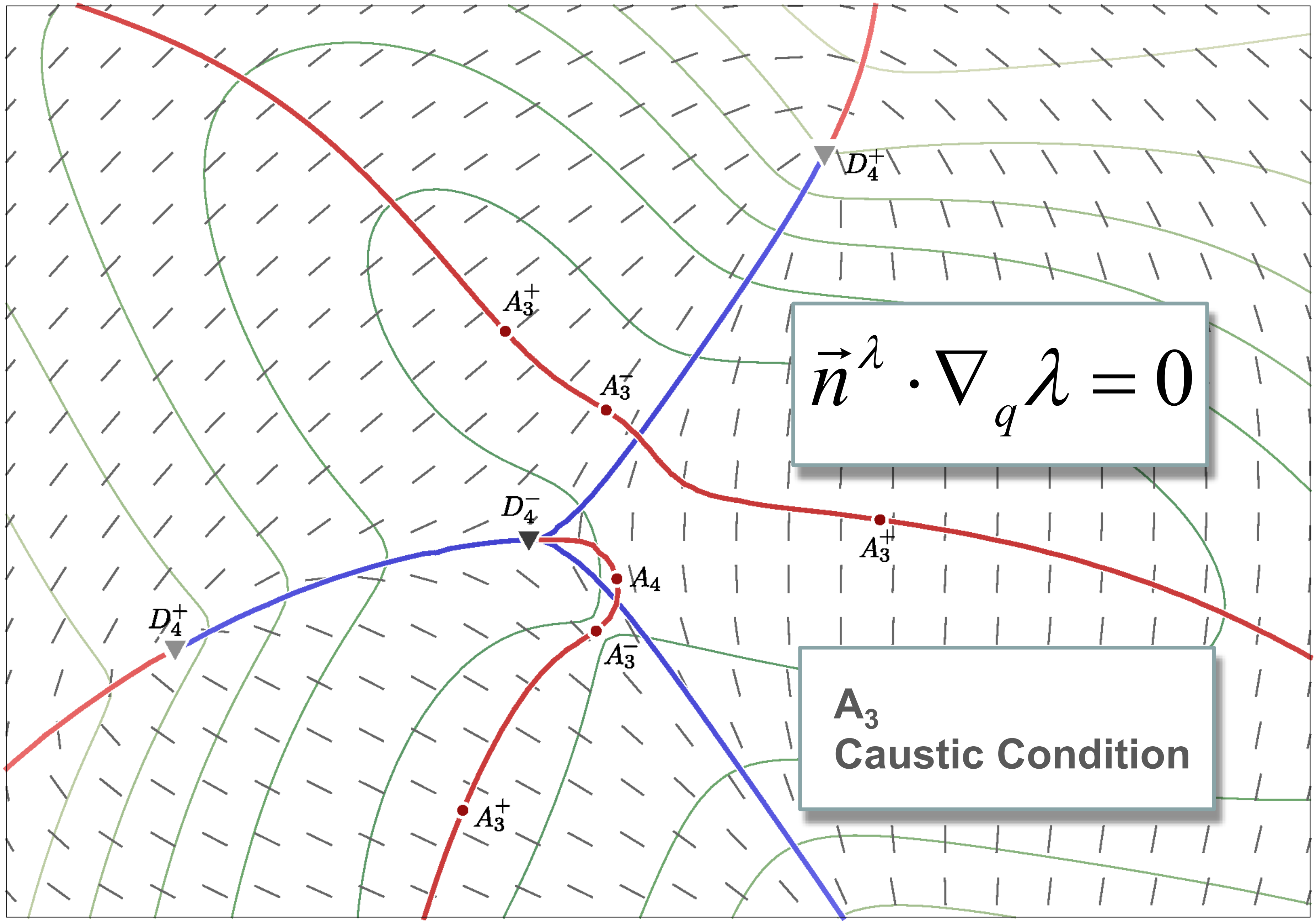
Singularities & Catastrophes:
Deformation Field



Zeldovich deformation eigenvalue landscape



Hidding, Shandarin & vdW 2014

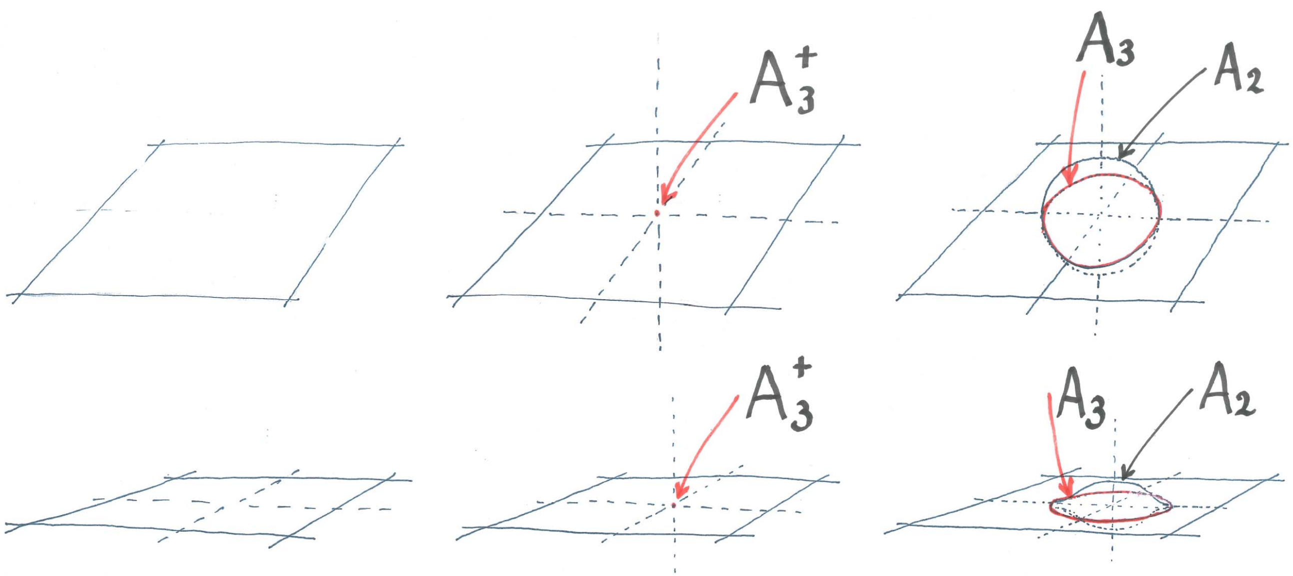


D_4^+
A_3^+
A_3^-
$\vec{n}^{\lambda} \cdot \nabla_q \lambda = 0$
D_4^-
A_3^+
A_4
D_4^+
A_3^-
A_3
Caustic Condition
A_3^+

Caustic Classes

Lagrangian

Feldbrugge, vdW et al. 2017



Eulerian

Emergence of A_3 cusps around A_2 folds (3D)

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

(also see Zeeman, Thom)

D=1

A_2: folds
A_3: cusps

D=2

A_2: folds
A_3: cusps
A_4: swallowtail

D_4: umbilics

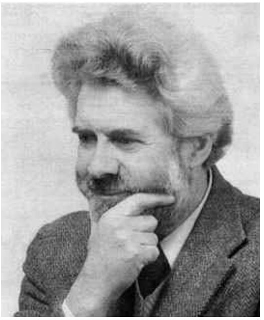
D=3

A_2: folds
A_3: cusps
A_4: swallowtail
A_5: butterfly

D_4: umbilics
D_5

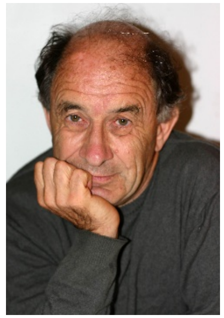
E_5

Leaders of Catastrophe

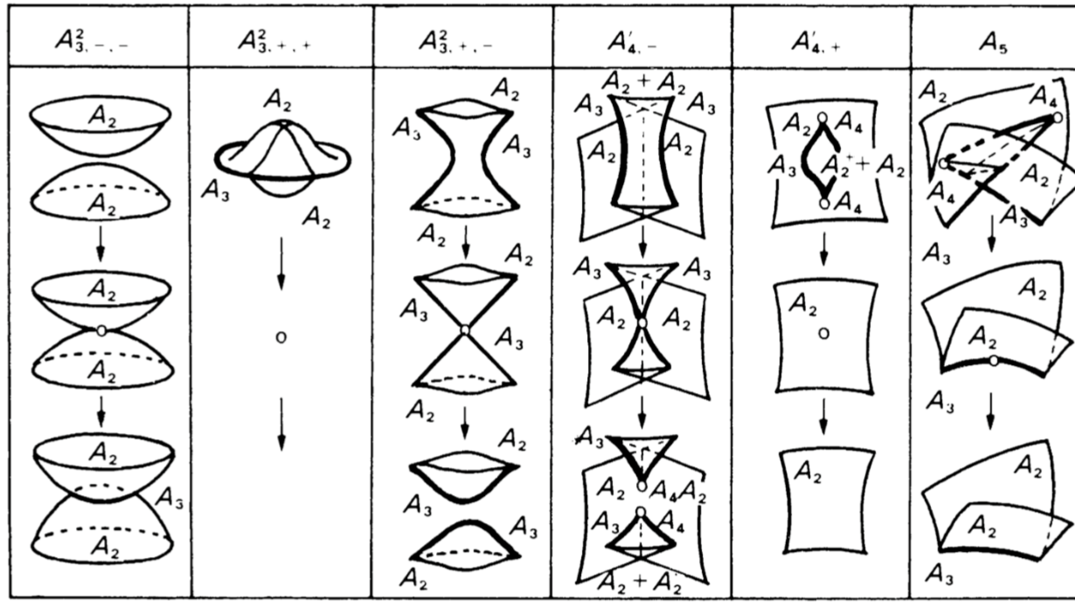


E. Zeeman

R. Thom

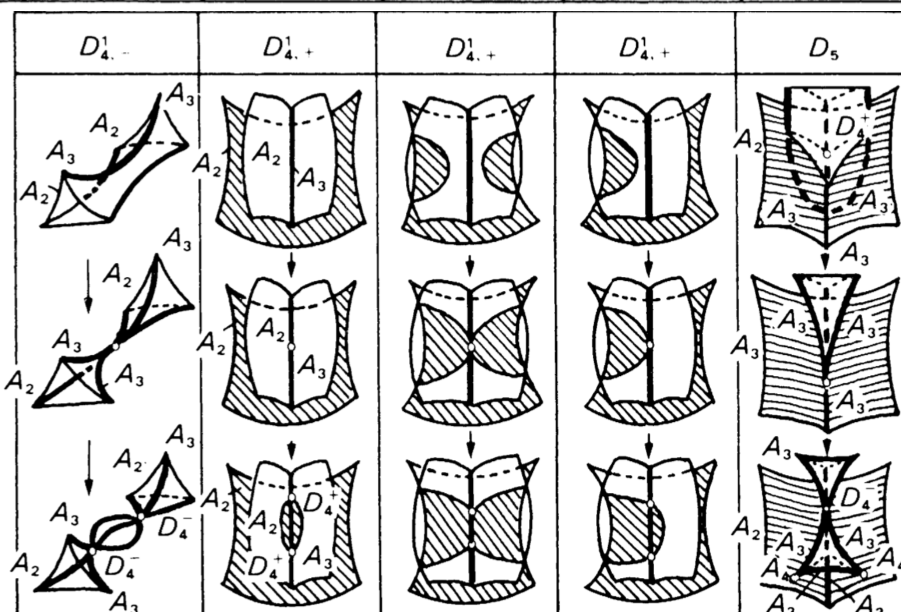


V.I. Arnol'd



Arnold V.I.   (and others):
Caustic classification on basis of Normal Forms:

$$F(q_1) = q_1^5 + \lambda q_1^3 + \mu q_1^2$$

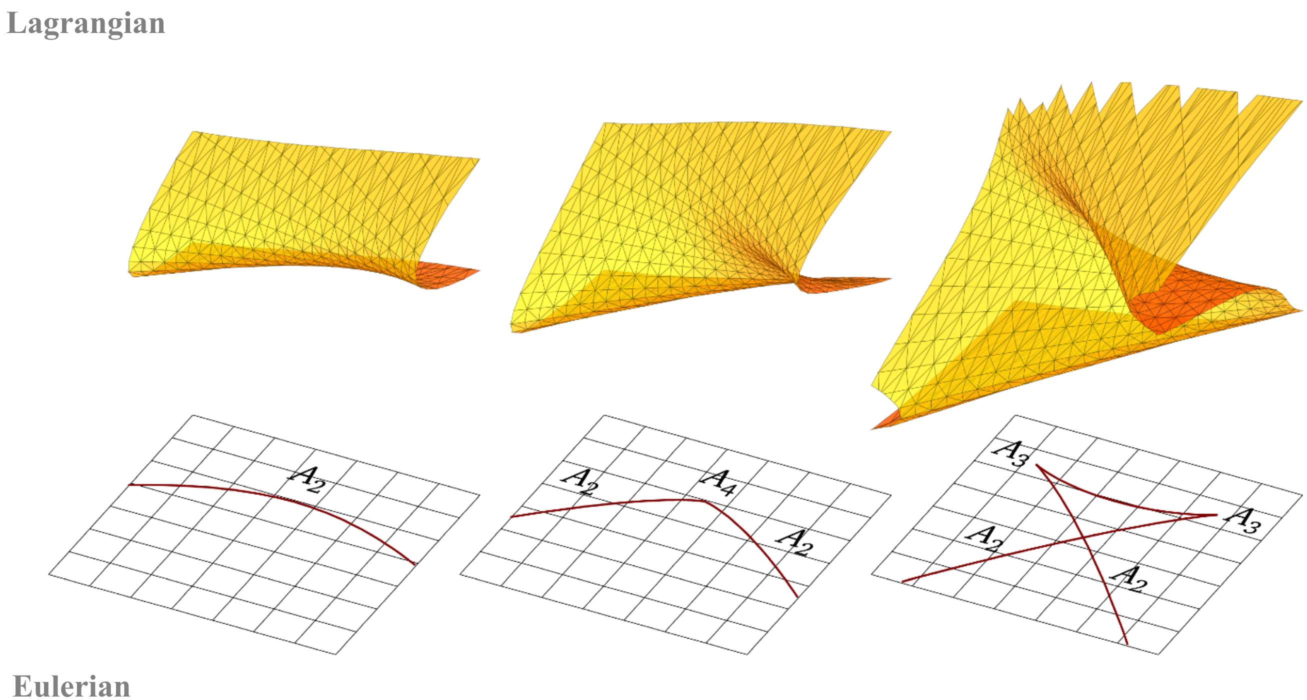


Arnold V.I., 1986,
Catastrophe Theory, Springer

Arnold V.I., Shandarin S.F., Zeldovich Ya.B., 1982
The Large Scale Structure of the Universe I. General Properties. One and Two-dimensional models
Geophys. Astrophys. Fluid Dynamics, 20, 1-2

Arnold V.I., 1986,
Evolution of singularities of potential flows in collisionless-free media and the metamorphosis of caustics in three-dimensional space

Hidding, Shandarin, vdW 2014



Formation of a A_4 swallowtail (2D)

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

(also see Zeeman, Thom)

D=1

A_2: folds
A_3: cusps

D=2

A_2: folds
A_3: cusps
A_4: swallowtail

D_4: umbilics

D=3

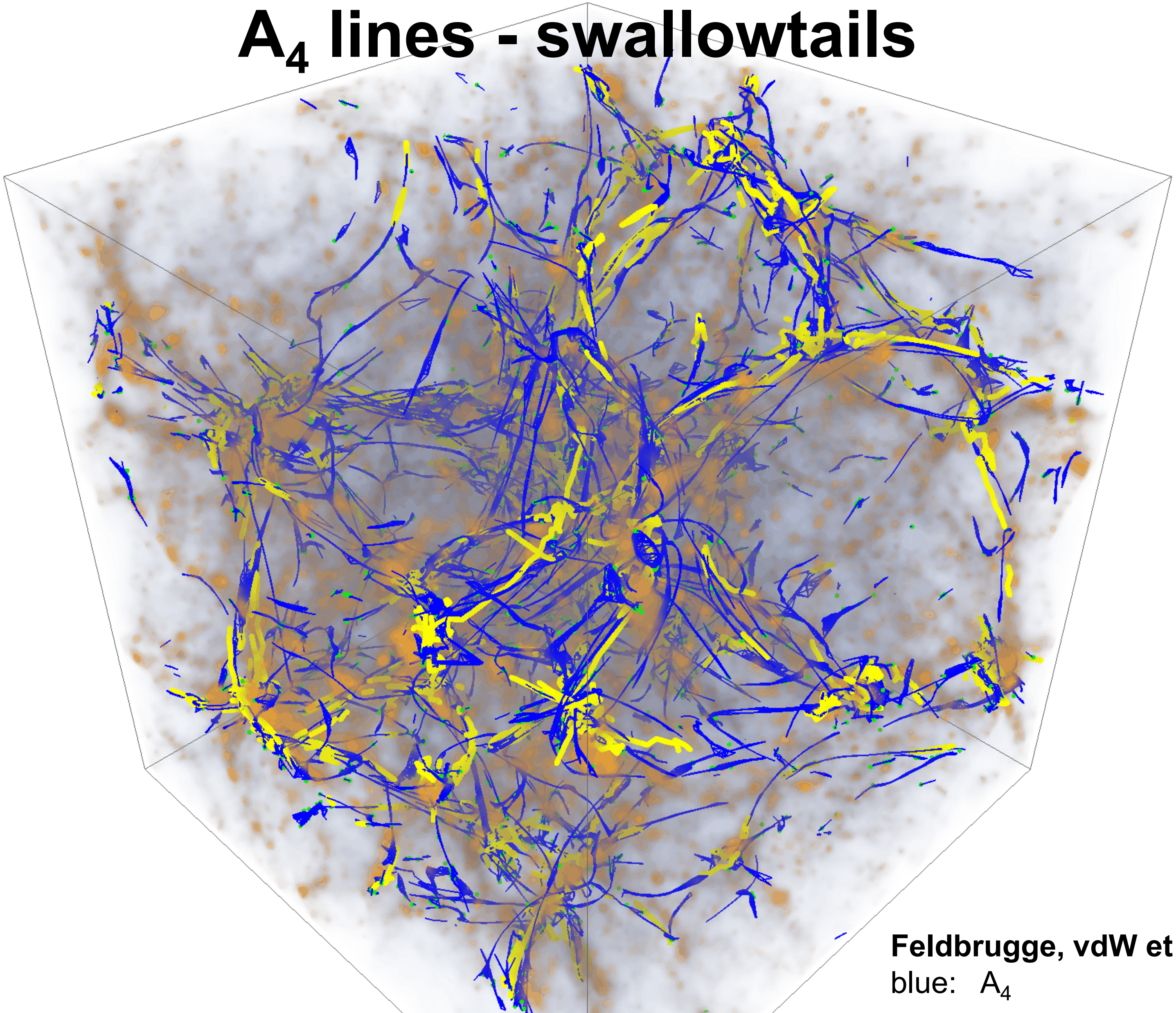
A_2: folds
A_3: cusps
A_4: swallowtail
A_5: butterfly

D_4: umbilics
D_5

E_5

Skeleton (3D) Cosmic Web:

A_4 lines - swallowtails



Feldbrugge, vdW et al. 2018

blue: A_4

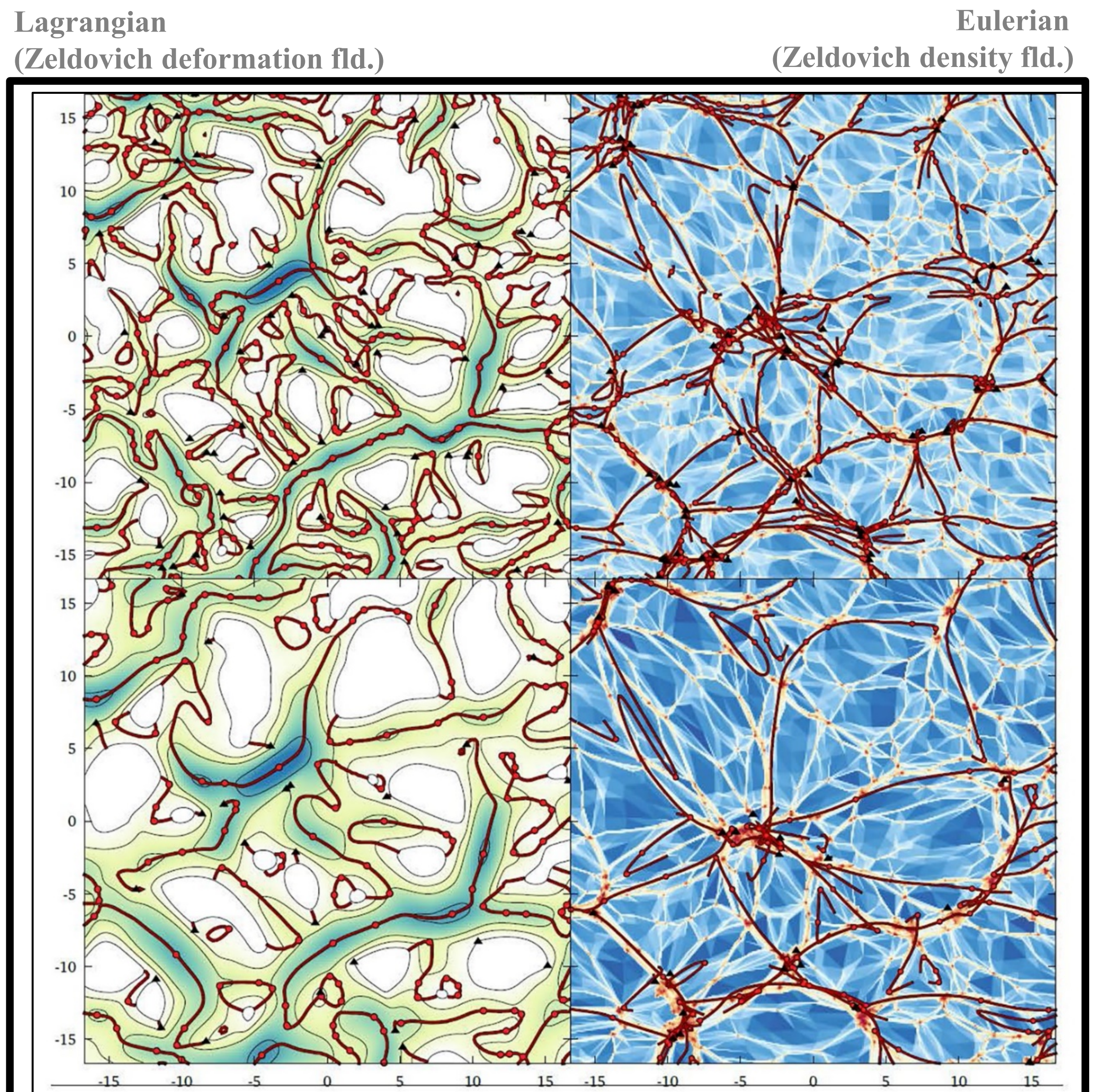
Caustic Skeleton & Cosmic Web

Skeleton (3D) Cosmic Web:
catastrophic connections

Singularity class	Singularity name	Feature in the 2D cosmic web	Feature in the 3D cosmic web
A_2	fold	collapsed region	collapsed region
A_3	cusp	filament	wall or membrane
A_4	swallowtail	cluster or knot	filament
A_5	butterfly	not stable	cluster or knot
D_4	hyperbolic/elliptic	cluster or knot	filament
D_5	parabolic	not stable	cluster or knot

Feldbrugge, vdW et al. 2017b

Skeleton (2D) Cosmic Web:
catastrophic connections



Feldbrugge, vdW et al. 2016

2D Zeldovich density field (log density)

A3 - cusp - red sheets - filaments
A4 - swallowtail - blue lines - nodes

Skeleton (3D) Cosmic Web:

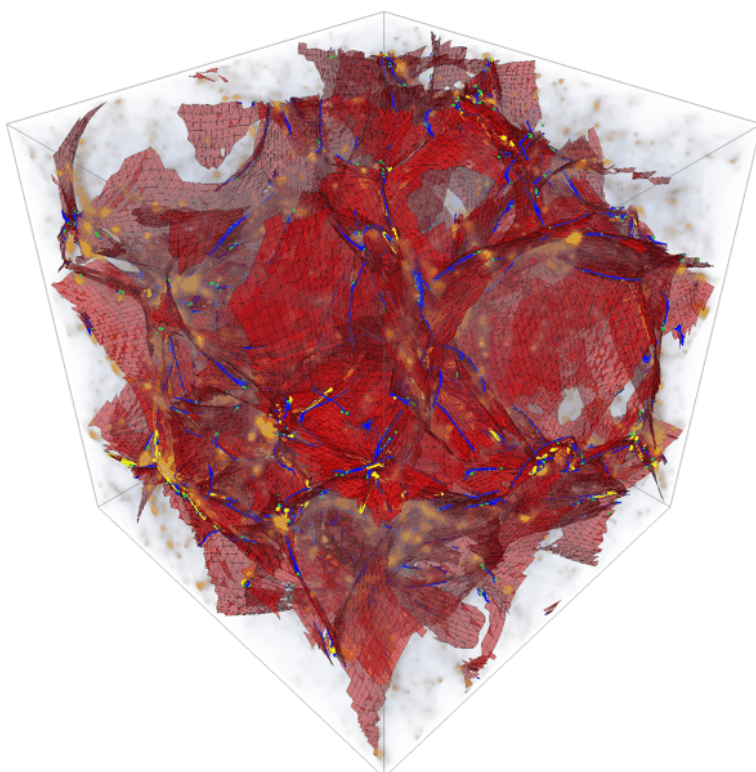
Wall/Membrane formation:

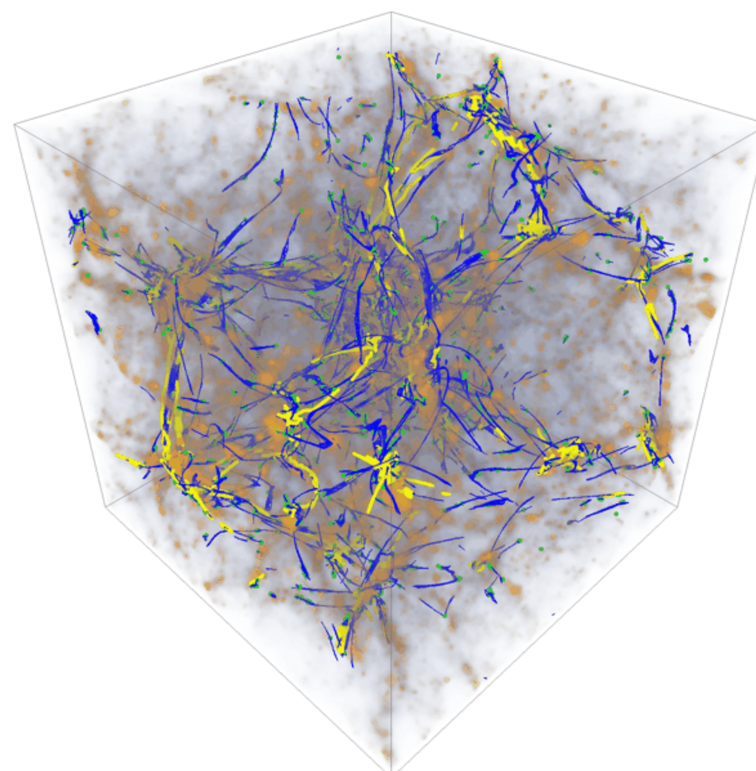
A_2 (cusp) membranes (red):
- collapse along 1 direction

Filament formation:
not necessary to collapse along 2 directions !

A_4 (swallowtail) filaments (blue):
- collapse along 1 direction
- at edges & intersections A_3 sheets

D_4 umbilic filaments (yellow)
- collapse along 2 directions
- higher density filamentary extensions nodes





Feldbrugge, vdW et al. 2018

the Spherical Model

Spherical Model

The spherical model (Gunn & Gott 1972) describes the evolution of a spherical mass distribution. It forms THE reference point for all further evaluations of structure formation.

- **Because of Birkhoff's theorem we may see the evolution of each individual mass shell as due only to the integrated mass distribution within its radius.**

- **As long as two mass shells are not crossing – e.g. due to the faster infall of an outer shell into an overdensity -- the motion of a shell – with radius r -- is simply that of an individual spherical shell attracted by a point mass M(r), with M(r) the integrated mass within radius r.**

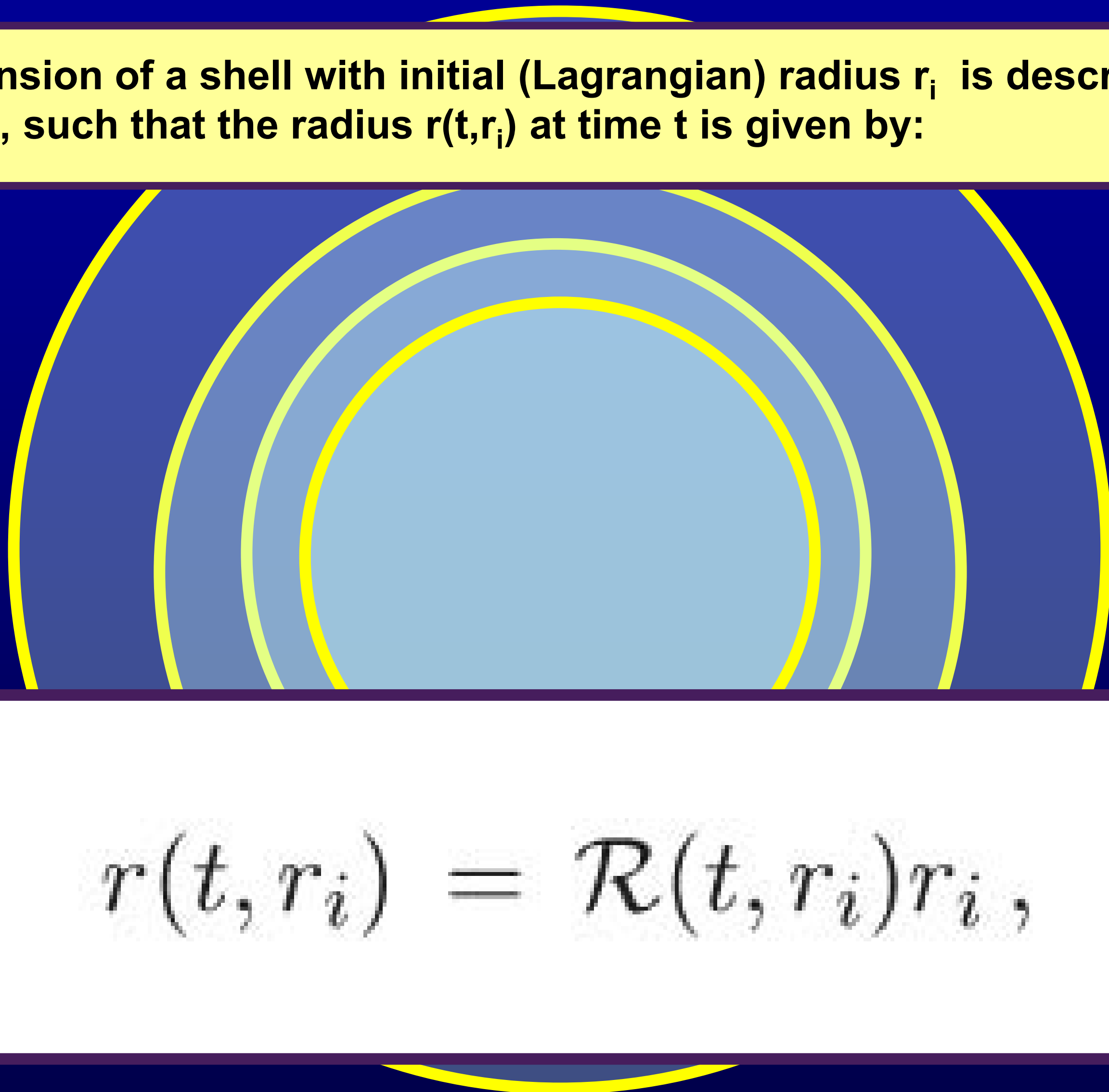
- **Perhaps not surprisingly, the equations of motion for the mass shells are the same as that of Friedmann-Robertson-Walker universes for an equivalent density parameter $\Omega(r)$.**

- **These equations of motion for each mass shell can be solved analytically for any decently behaving mass profile (i.e. the mass profile should be sufficiently centrally concentrated to prevent shell crossing).**

- **The spherical model is equally valid for overdensities as well as for underdensities.**

Spherical Model

Contraction/Expansion of a shell with initial (Lagrangian) radius r_i is described by a scale factor $R(t,r_i)$, such that the radius $r(t,r_i)$ at time t is given by:



$$r(t, r_i) = \mathcal{R}(t, r_i) r_i \, ,$$

Spherical Model

The motion is fully determined by the average mass density Δ(r,t) within a radius r,

$$\Delta(r,t) = \frac{3}{r^3}\int_0^r \left[\frac{\rho(y,t)}{\rho_u(t)} - 1\right] y^2\, dy = \frac{3}{r^3}\int_0^r \delta(y,t)\, y^2\, dy\,,$$

$$1+\Delta_{ci} = \Omega_i\left[1+\Delta(t_i,r_i)\right]$$

$$\alpha_i = \left(\frac{v_i}{H_i r_i}\right)^2 - 1\,.$$

and by the the peculiar velocity $v_{pec,i}$ of the shell. For this we usually take the peculiar velocity predicted by linear theory for the growing mode.

$$v_{pec,i} = -\frac{H_i r_i}{3} f(\Omega_i)\Delta(r_i,t_i)\,,$$

$$\alpha_i = -\frac{2}{3} f(\Omega_i)\Delta(r_i,t_i)\,.$$

It is convenient to describe the density perturbation with respect to a EdS Universe, in terms of Δ_i and the velocity perturbation with respect to the Hubble expansion in terms of parameter α_i .

Spherical Model

The solutions for the scale factor of overdense/underdense shells can be written in the same parameterized form, by means of shell angle Θ, as we know from the solutions for FRW universes,

$$\mathcal{R}(\Theta_r) = \begin{cases} \frac{1}{2} \dfrac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})} \; (\cosh\Theta_r - 1) & \Delta_{ci} < \alpha_i, \\ \\ \frac{1}{2} \dfrac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)} \; (1 - \cos\Theta_r) & \Delta_{ci} > \alpha_i \, , \end{cases}$$

with time dependence specified by

$$t(\Theta_r) = \begin{cases} \frac{1}{2} \dfrac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}} \; (\sinh\Theta_r - \Theta_r) & \Delta_{ci} < \alpha_i \\ \\ \frac{1}{2} \dfrac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}} \; (\Theta_r - \sin\Theta_r) & \Delta_{ci} > \alpha_i \end{cases}$$

Spherical Model

The corresponding peculiar velocity of the shell

$$v_{pec}(r,t) = v(r,t) - H_u(t)r(t)\,,$$

can be inferred from

$$v_{pec}(r,t) = H_u(t)r(t)\left\{\frac{g(\Theta_r)}{g(\Theta_u)} - 1\right\}$$

with

$$g(\Theta) = \begin{cases} \dfrac{\sinh\Theta\,(\sinh\Theta - \Theta)}{(\cosh\Theta - 1)^2} & \text{open}\,, \\ \dfrac{2}{3} & \text{critical}\,, \\ \dfrac{\sin\Theta\,(\Theta - \sin\Theta)}{(1 - \cos\Theta)^2} & \text{closed} \end{cases}$$

Evolution

Spherical Tophat Halo

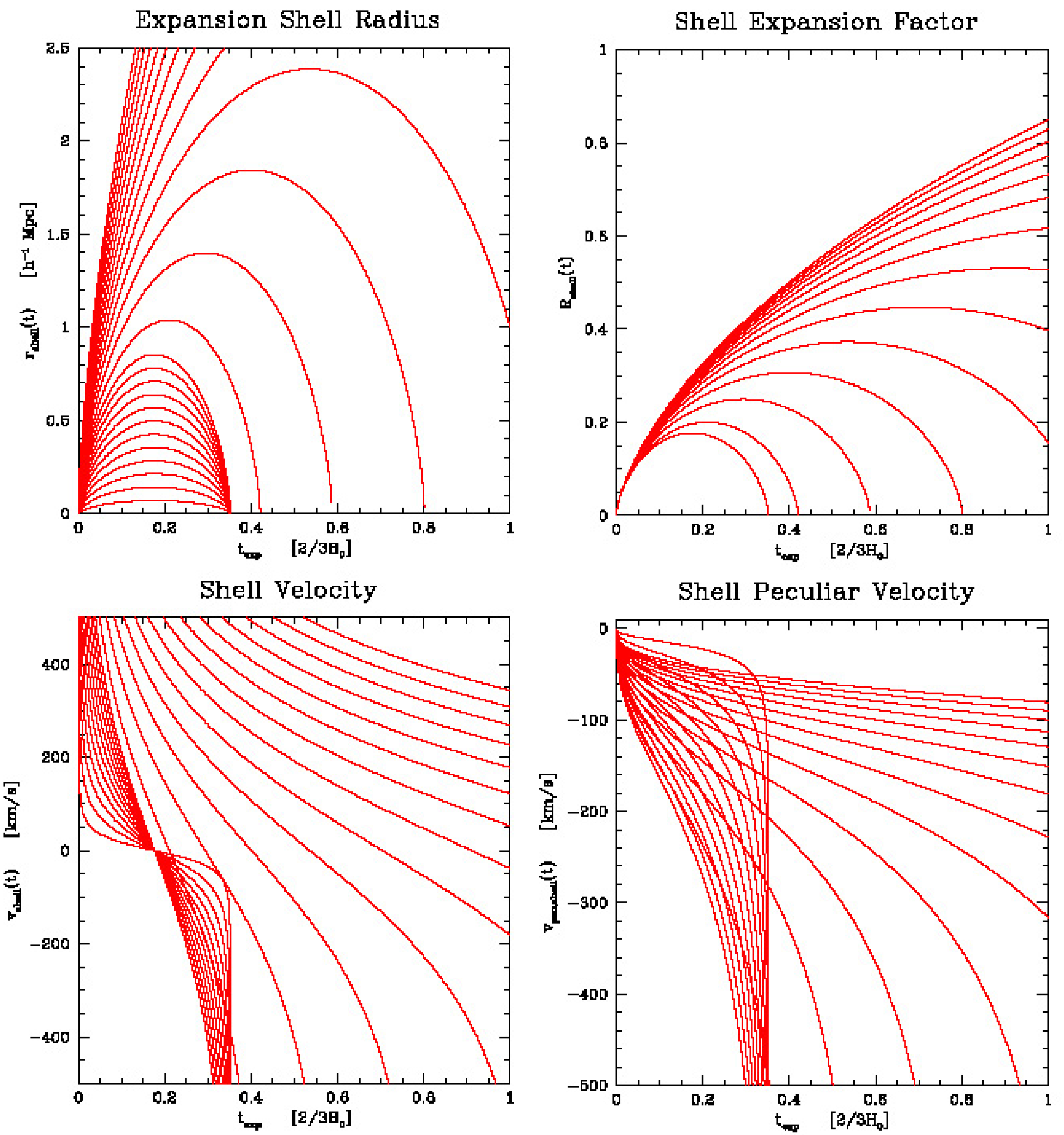
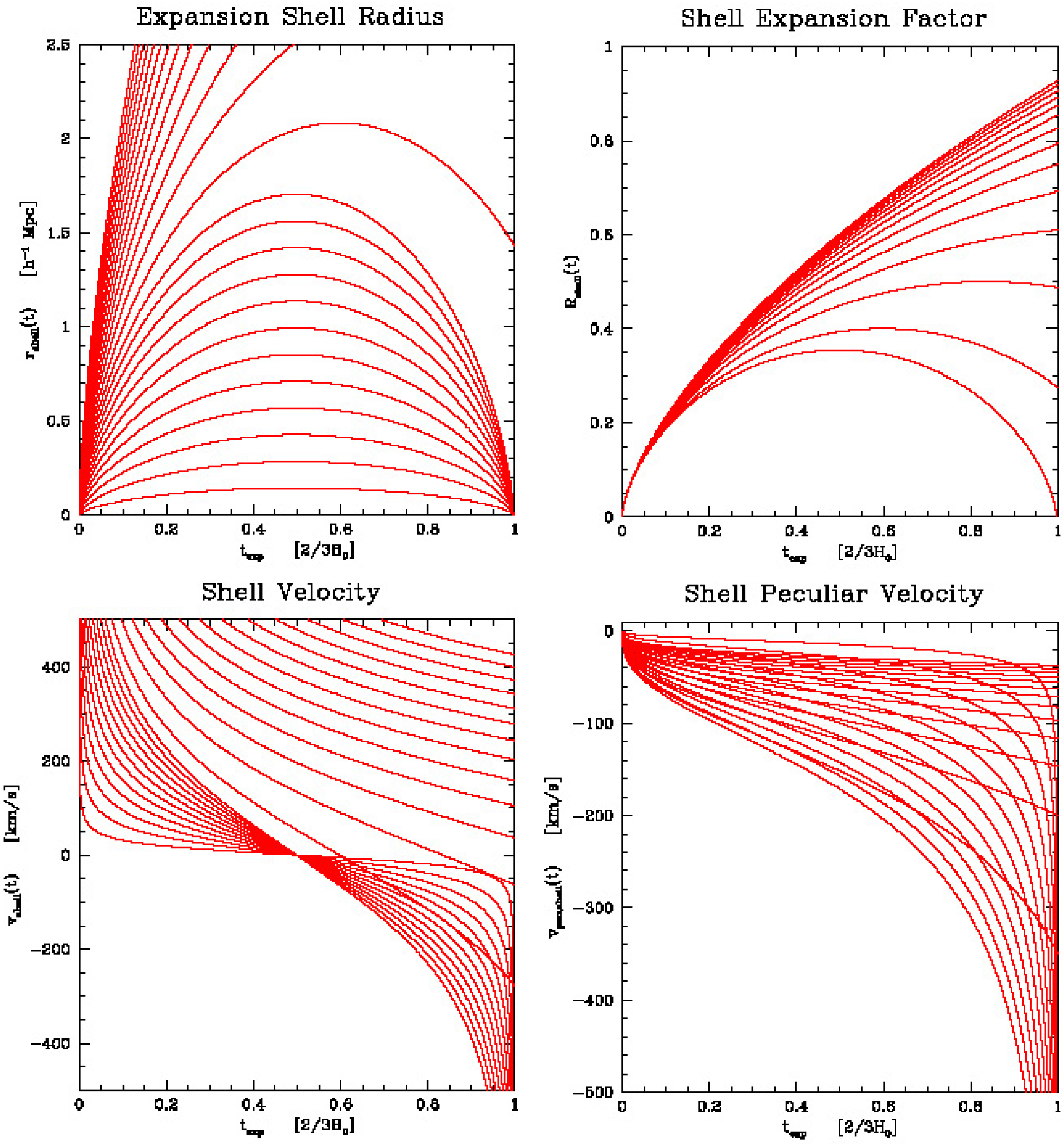


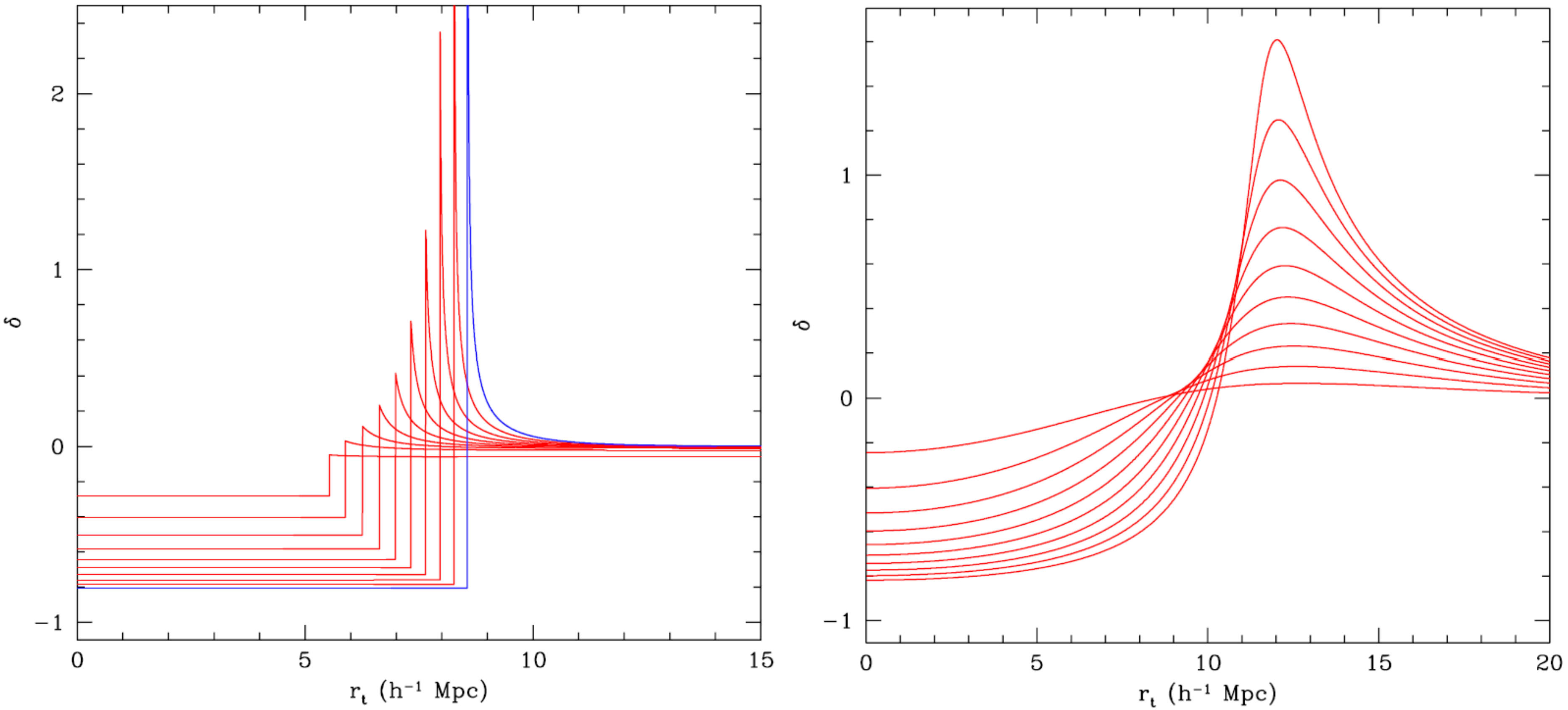
Figure 7. Spherical Peak 2

Evolution

Spherical Tophat Halo



Spherical Void Evolution



Spherical Model

Having determined the evolution of the radius and velocity of each spherical shell of the density perturbation, we may then proceed to derive the corresponding evolution of the density profile of the shell. Here we limit ourselves to the integrated density profile $\Delta(r,t)$,

$$1 + \Delta(r,t) = \frac{1+\Delta_i(r_i)}{\mathcal{R}^3} \frac{a(t)^3}{a_i^3},$$

whose solution can be specified in terms of a density function $f(\theta)$,

$$1 + \Delta(r,t) = f(\Theta_r)/f(\Theta_u),$$

Spherical Model

whose solution can be specified in terms of a density function f(θ),

$$f(\Theta) = \begin{cases} \dfrac{(\sinh\Theta - \Theta)^2}{(\cosh\Theta - 1)^3} & \text{open}, \\ 2/9 & \text{critical}, \\ \dfrac{(\Theta - \sin\Theta)^2}{(1 - \cos\Theta)^3} & \text{closed}, \end{cases}$$

At maximum expansion of an overdense shell, Θ=π, defining the turnaround radius of the matter concentration, we thus find that the integrated overdensity of the shell is

$$1 + \Delta(r, t_{ta}) = (3\pi/4)^2$$

~ 5.6

Spherical Model

In the "imaginary" situation in which the overdensity would have continued to evolve linearly, it would have reached an overdensity dictated by the linear growth factor D(t) for the corresponding background Universe. For the situation of an Einstein-de Sitter Universe, with

$$D \propto (t/t_0)^{2/3}$$

a mass overdensity reaches its turnaround at a linear overdensity

$$\Delta_{lin}(z_{\rm ta}) = \delta_{\rm ta} = (3/5)(3\pi/4)^{2/3} \approx 1.062.$$

The consequences of this finding are truely wonderful: the cosmologist may resort to the primordial density field, search for the peaks in this Gaussian field, and assuming they are spherical (which they are not at all), and identify the ones that reach turnaround at some redshift z. Even more useful is the equivalent case for final collapse.

Spherical Model

Collapse, ie. $\Delta=\infty$, happens when the density fluctuation would have reached a linear overdensity of

$$\Delta_{lin}(z_c) = \delta_c = \left(\frac{3}{5}\right)\left(\frac{3\pi}{2}\right)^{2/3} \approx 1.686\,.$$

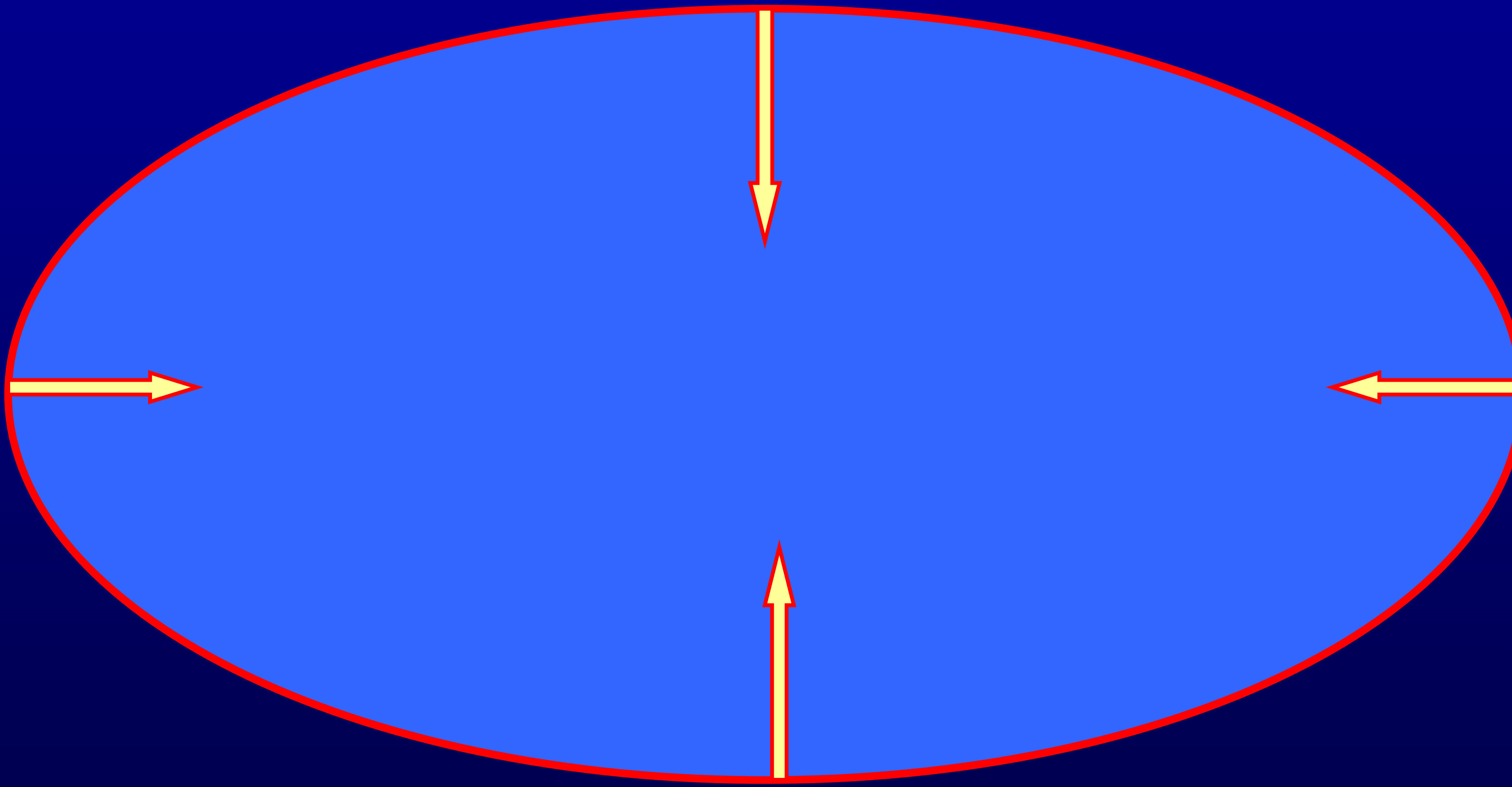
The fact that this is a universal value, valid for any (spherical) density peak, makes it into one of the most crucial numbers in the theory of structure formation. We may thus find the collapse redshifts z_{coll} for any primordial density peak,

$$D(z_{\rm coll})\,\Delta_{lin,0} = \delta_c\;.$$

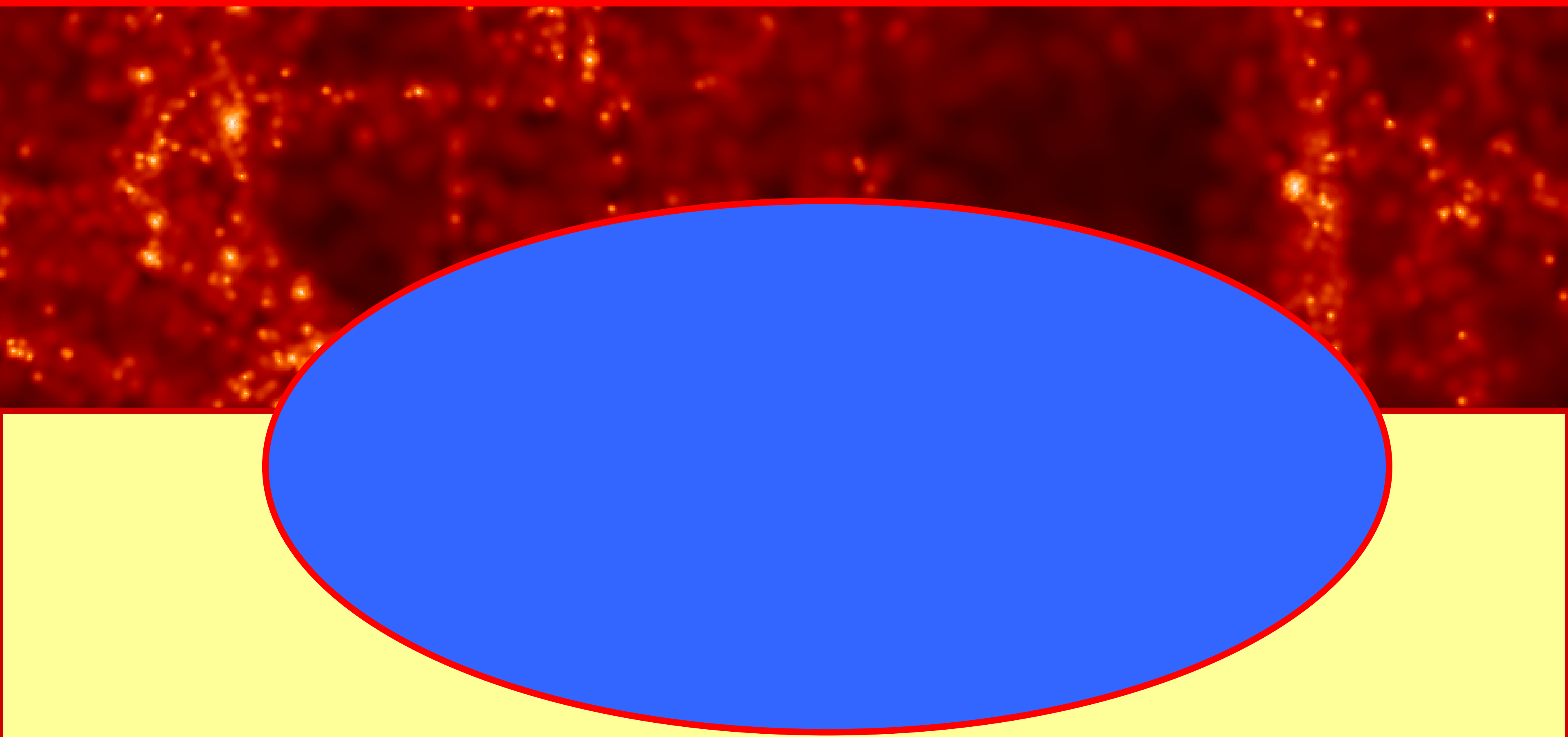
$$1 + z_{coll} = \frac{\Delta_{lin,0}}{1.686}.$$

the Ellipsoidal Model

Homogeneous Ellipsoids

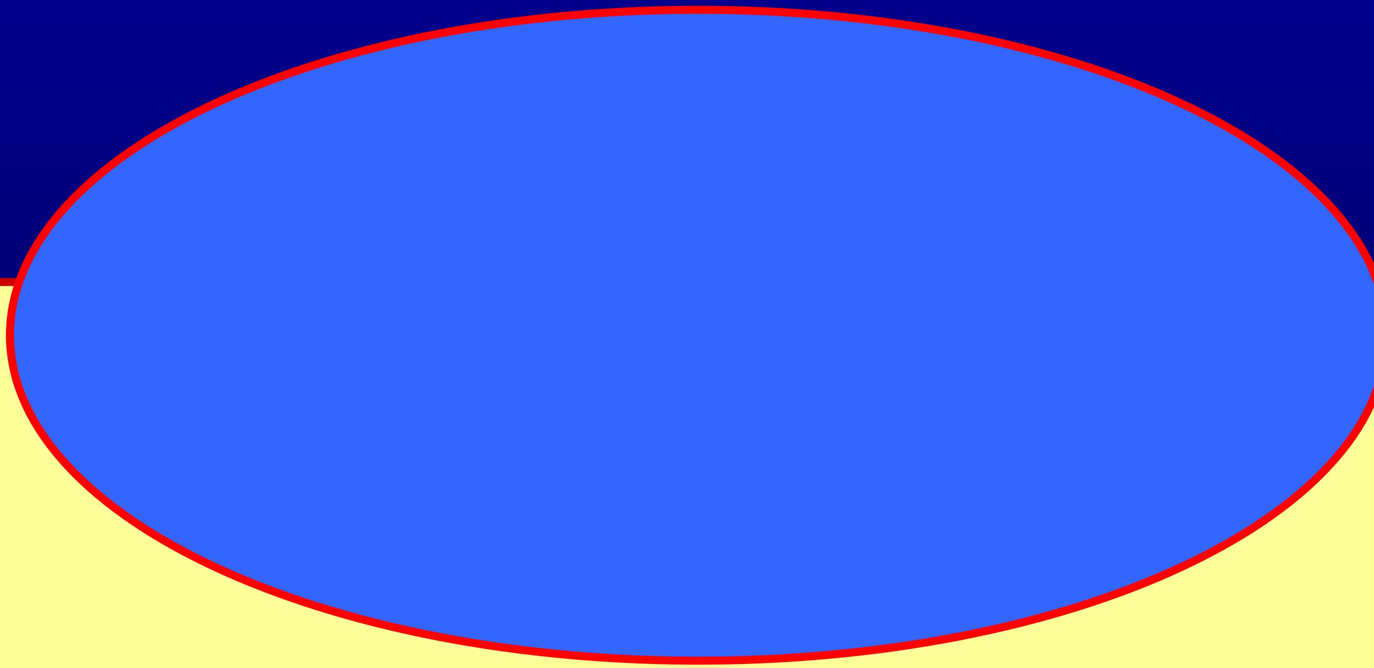


Homogeneous Ellipsoids



$$\Phi^{(tot)}(\mathbf{r}) = \Phi_b(\mathbf{r}) + \Phi^{(int,ell)}(\mathbf{r}) + \Phi^{(ext)}(\mathbf{r})$$

Homogeneous Ellipsoids



$$\Phi_b(\mathbf{r}) \;=\; \frac{2}{3}\pi G\rho^b \left(r_1^2 + r_2^2 + r_3^2\right)$$

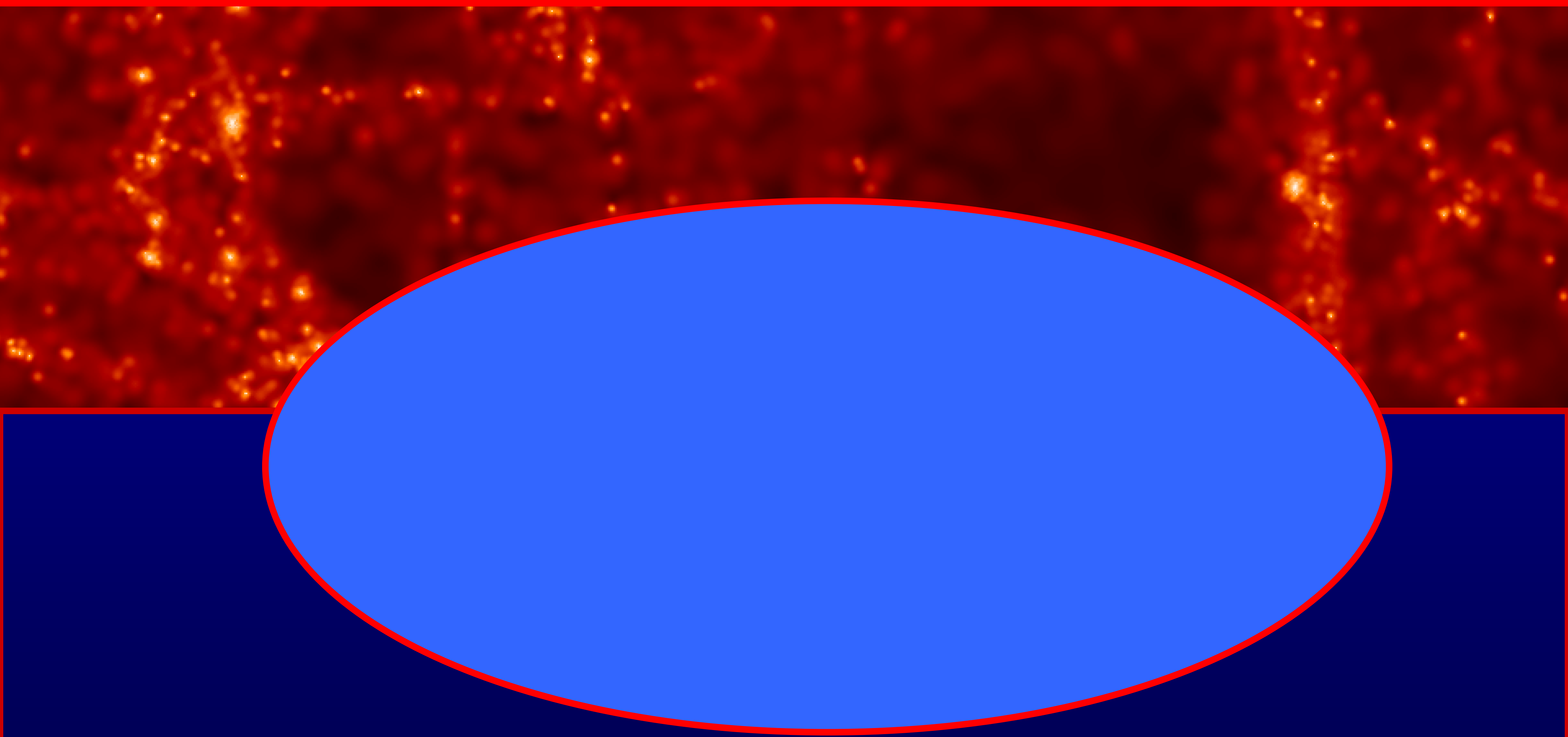
Homogeneous Ellipsoids

$$\Phi^{(int,ell)}(\mathbf{r}) = \frac{1}{2}\sum_{m,n}\Phi^{(int,ell)}_{mn} r_m r_n$$

$$= \frac{2}{3}\pi G(\rho^{ell} - \rho^{b})\left(r_1^2 + r_2^2 + r_3^2\right) + \frac{1}{2}\sum_{m,n} T^{(int)}_{mn} r_m r_n$$

$$T^{(int)}_{mn} \equiv \frac{\partial^2 \Phi^{(int,ell)}}{\partial r_m \partial r_n} - \frac{1}{3}\nabla^2\Phi^{(int,ell)}\ \delta_{mn}$$

Homogeneous Ellipsoids



$$\Phi^{(ext)}(\mathbf{r}) = \frac{1}{2}\sum_{m,n} T^{(ext)}_{mn} r_m r_n \quad \Longleftarrow \quad T^{(ext)}_{mn}(t) \equiv \frac{\partial^2 \Phi^{(ext)}}{\partial r_m \partial r_n}$$

Homogeneous Ellipsoids

$$\Phi^{(int,ell)}(\mathbf{r}) = \pi G \left(\rho^{ell} - \rho^{b}\right) \sum_m \alpha_m r_m^2$$

$$T_{mn}^{(int)} = 2\pi G(\rho^{ell} - \rho^{b}) \left(\alpha_m - \frac{2}{3}\right) \delta_{mn}$$

$$\alpha_m = c_1 c_2 c_3 \int_0^\infty (c_m^2 + \lambda)^{-1} \prod_{n=1}^{3} \frac{1}{\sqrt{c_n^2 + \lambda}} \mathrm{d}\lambda$$

Homogeneous Ellipsoids

$$\frac{d^2 r_m}{dt^2} = -\frac{4\pi}{3} G \rho^b r_m(t) - \sum_n \Phi_{mn}^{(int,ell)} r_n(t) - \sum_n T_{mn}^{(ext)} r_n(t)$$

Homogeneous Ellipsoids

$$r_m(t) \;=\; \sum_k R_{mk}(t) r_{k,i}$$

$$\frac{d^2 R_{mk}}{dt^2} \;=\; -\frac{4\pi}{3}\pi G R_{mk} \;-\; \sum_n \Phi_{mn}^{(int,ell)} R_{nk} \;-\; \sum_n T_{mn} R_{nk}$$

$$\frac{d^2 R_{mk}}{dt^2} \;=\; -2\pi G \left[\alpha_m \rho^{ell} + (\frac{2}{3} - \alpha_m)\; \rho^b \right] R_{mk} \;-\; T_{mm}^{(ext)} R_{mk}$$

Homogeneous Ellipsoids

$$R_{mn}(t_i) = R_m(t_i)\delta_{mn}$$

$$\frac{d^2R_{mk}}{dt^2} = -2\pi G\left[\alpha_m\rho^{ell} + (\frac{2}{3} - \alpha_m)\ \rho^b\right]R_{mk} - T_{mm}^{(ext)}R_{mk}$$

⇓

$$\frac{d^2R_m}{dt^2} = -2\pi G\left[\alpha_m\rho^{ell} + (\frac{2}{3} - \alpha_m)\ \rho^b\right]R_m - T_{mm}^{(ext)}R_m$$

Homogeneous Ellipsoids

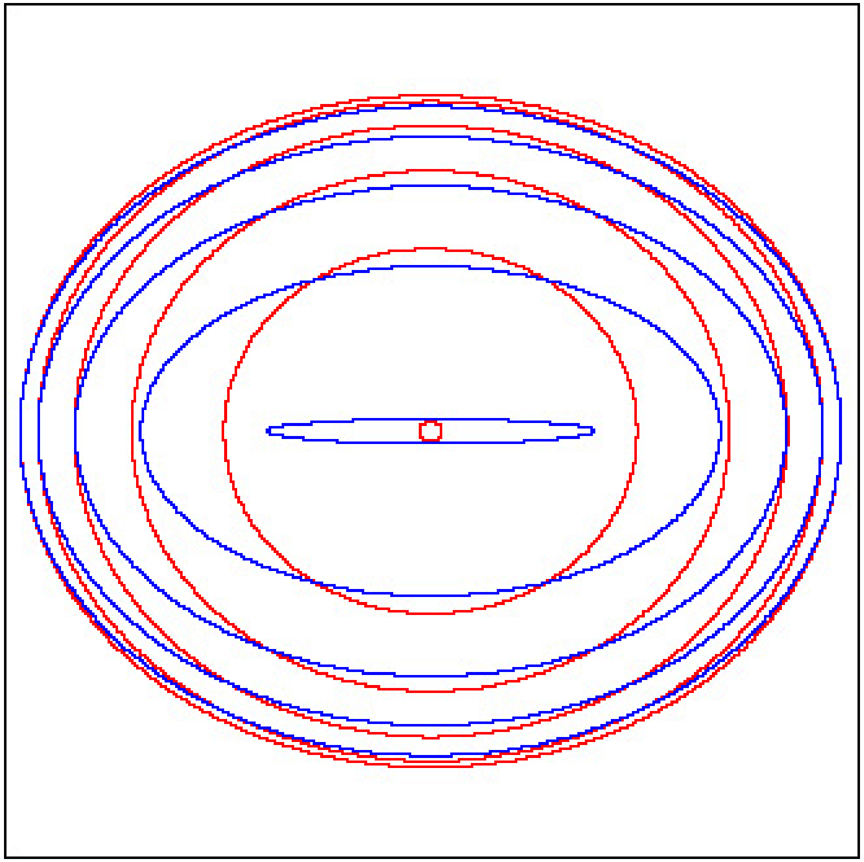
$$v_{pec,m}(t_i) = \frac{2f(\Omega_i)}{3H_i\Omega_i} g_{pec,m}(t_i)$$
$$= -\frac{1}{2} H_i f(\Omega_i) \left[\alpha_{m,i}\delta_i + \frac{4T^{(ext)}_{mm,\circ}}{3\Omega_0 H_0^2} D_i \right] r_{m,i}$$

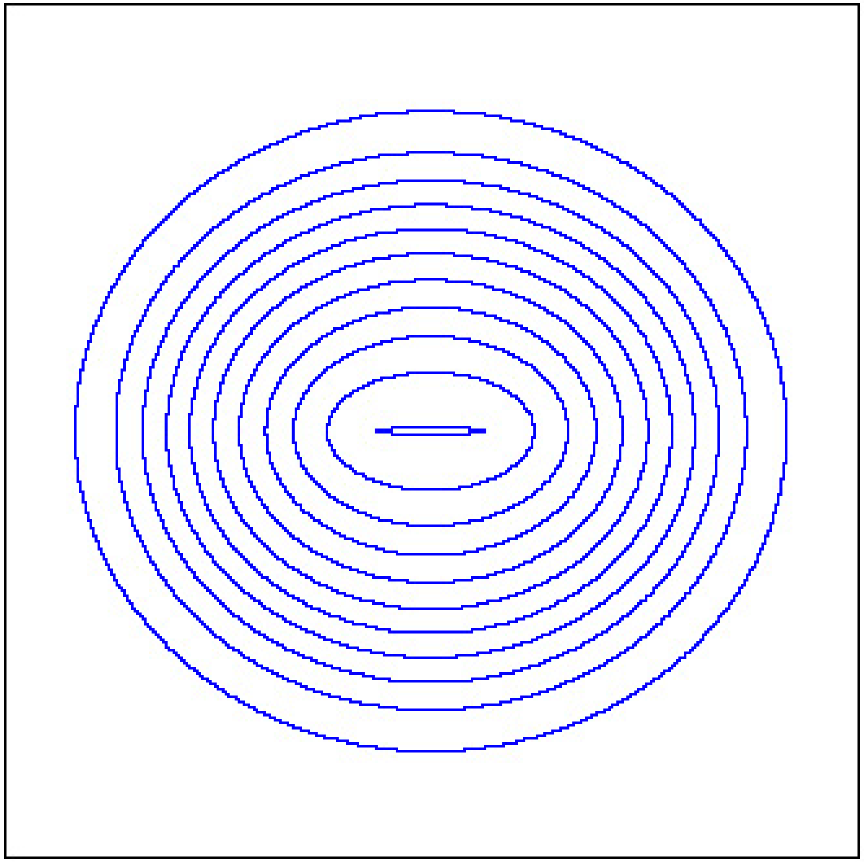
$$\frac{d^2 R_m}{dt^2} = -2\pi G \left[\alpha_m \rho^{ell} + \left(\frac{2}{3} - \alpha_m\right) \rho^b \right] R_m - T^{(ext)}_{mm} R_m$$

⇓

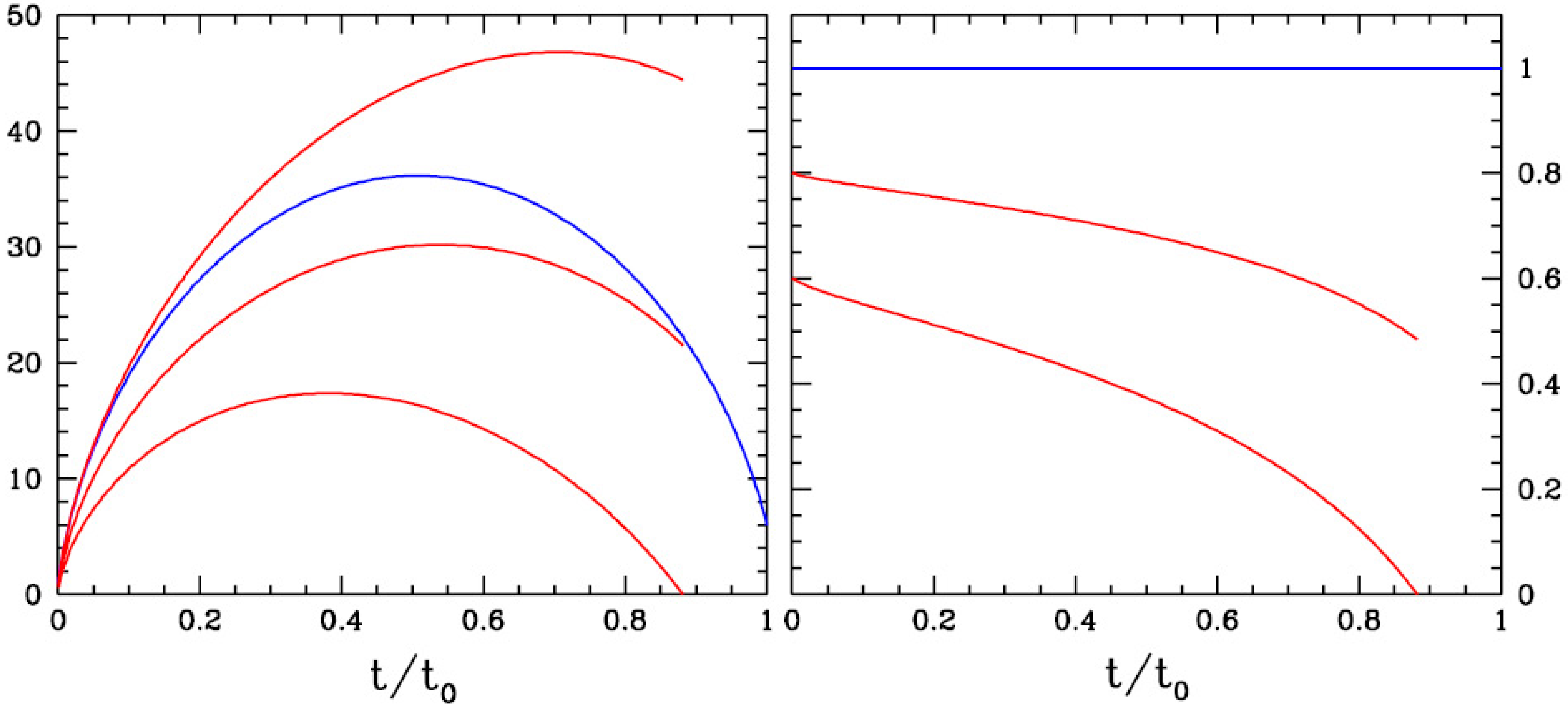
$$c_m(t) = R_m(t) c_{m,i} \Longrightarrow (c_1, c_2, c_3)$$

Homogeneous Ellipsoids

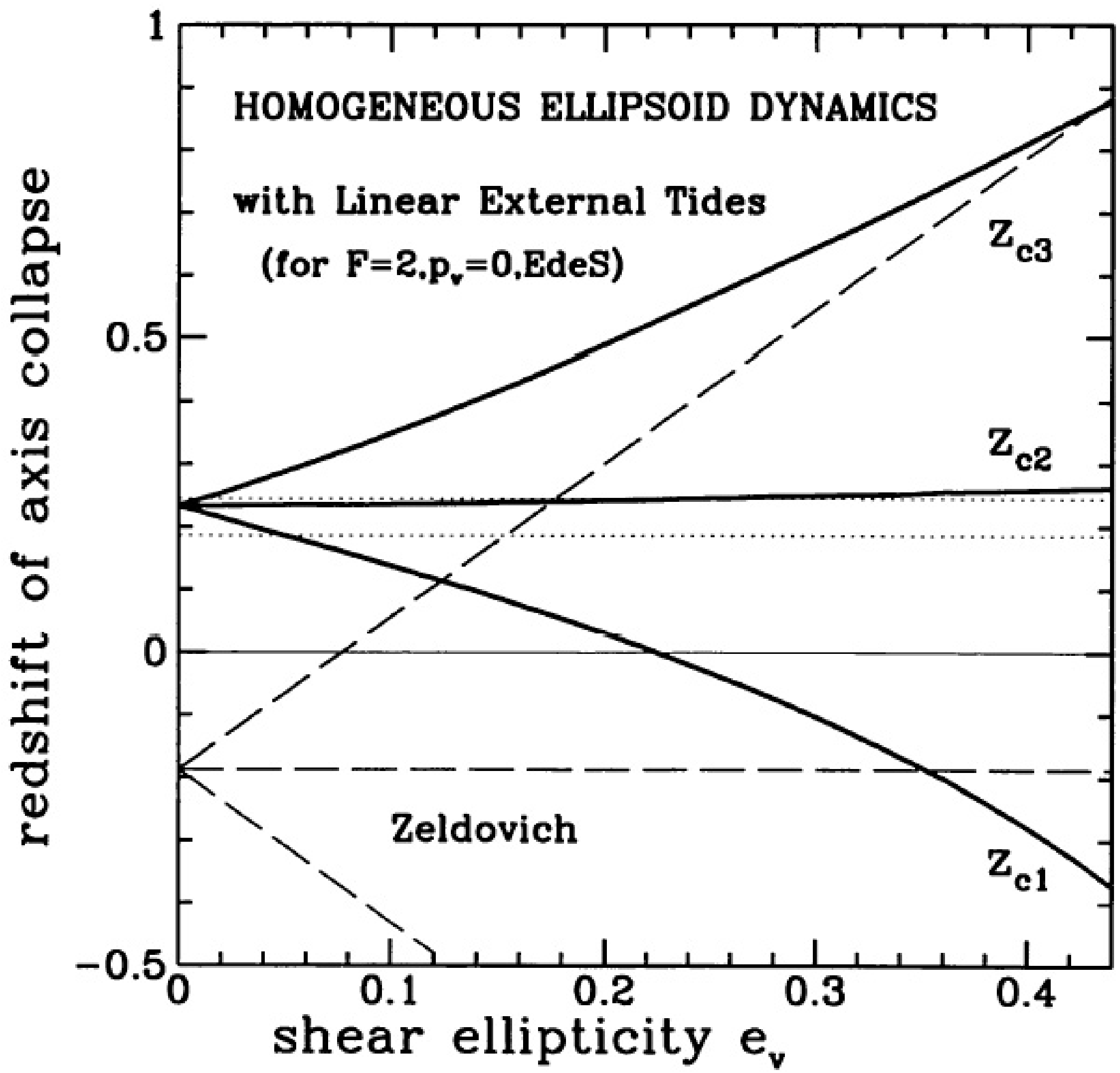




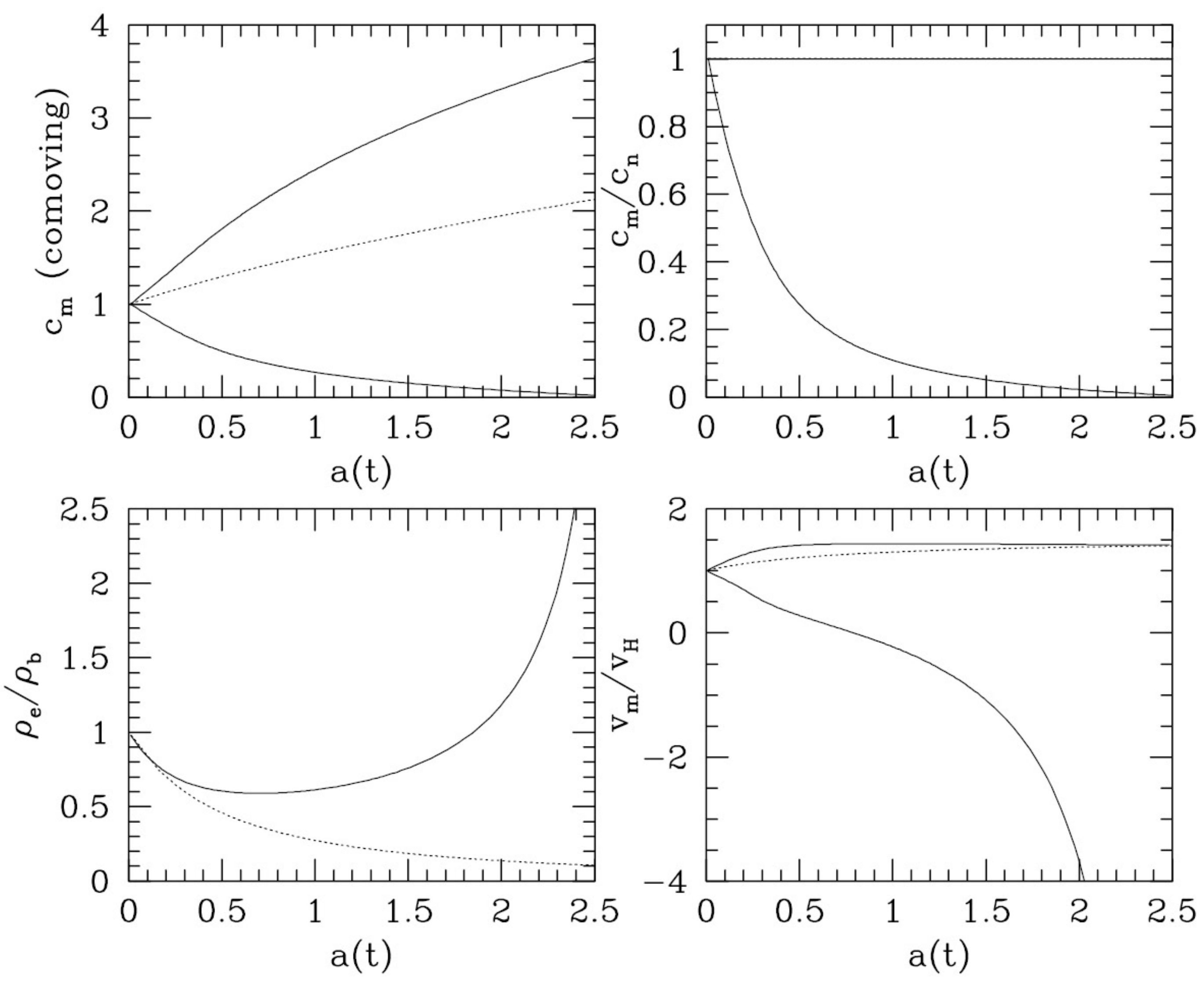
Homogeneous Ellipsoids



Homogeneous Ellipsoids



Homogeneous Ellipsoids



Homogeneous Ellipsoids