PHASE 1full:

Jeans Gravitational Instability Theory

Jeans Instability

Perturbation grows ('collapses') if gravity wins from counteracting pressure, which happens

- if wavelength perturbation

$$\boxed{\lambda > \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}}$$

(in expanding medium: comoving wavelength).

- with corresponding mass

$$\boxed{M > M_J = \frac{4\pi}{3}\,\bar{\rho}(t)\left(\frac{c_s}{a}\right)^3\left(\frac{\pi}{G\bar{\rho}}\right)^{3/2}}$$

- while, as long as wavelength/mass perturbation smaller than Jeans length/mass the perturbation will behave as a sound wave.

Note • Jeans Mass M_J evolves in time:

1. $\bar{\rho}(t)$
2. sound speed: $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$
3. $a(t)$

Jeans Mass !

* **Equation of motion, for a collisional medium:**

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{u}) = 0.$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}_r)\vec{u} = \underbrace{-\frac{1}{\rho}\vec{\nabla}_r p}_{\text{pressure gradient.}} - \vec{\nabla}_r \Phi$$

$$\nabla^2 \Phi = 4\pi G \rho$$

* **In comoving coördinates:**

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot (1+\delta)\vec{v} = 0.$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{a}(\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{\dot{a}}{a}\vec{v} = -\frac{1}{a}\vec{\nabla}\phi - \frac{1}{\rho a}\vec{\nabla} p$$

$$\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta$$

* **for a perfect gas:** **equation of state:**

$$p = p(\rho, S)$$

S: specific entropy per baryon

* **entropy S conserved for each fluid element:**

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{1}{a}(\vec{v} \cdot \vec{\nabla}) S = 0.$$

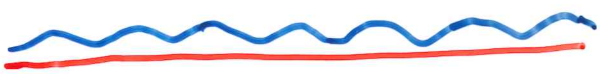
$$\Rightarrow \boxed{\vec{\nabla} p = c_s^2 \vec{\nabla}\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \vec{\nabla} S}$$

with $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_S$ = adiabatic sound speed^2

Longitudinal entropy solutions

$$\frac{dS}{dt} = 0$$

$$\vec{\nabla} S \neq 0.$$

 isothermal.

- $$\ddot{\delta}(\vec{k},t) + 2\frac{\dot{a}}{a}\dot{\delta}(k,t) + \frac{c_s^2}{a^2}(k^2 - k_J^2)\,\delta(\vec{k},t) = -\frac{1}{\rho_m a^2}\left(\frac{\partial p}{\partial S}\right)_\rho k^2 S(\vec{k})$$

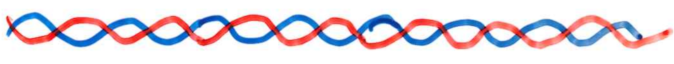
$S(\vec{k})$ constant

- for a photon-baryon fluid:

$$\left.\begin{array}{l} \rho_r \propto T^4 \\ S \propto T^3 \propto \rho_r^{3/4}, \quad T \propto \frac{1}{a} \end{array}\right\} \qquad \frac{\delta S}{S} = \frac{3}{4}\delta_r - \delta_b$$

- possible to make fluctuations:

$$\left.\begin{array}{l} \delta\rho_r + \delta\rho_b = 0. \\ \delta S \neq 0 \end{array}\right\} \qquad \text{"isocurvature"}$$



Qualitative description scenarios:

- adiabatic scenario:
  first structures on scales $M \sim 10^{12} - 10^{14}\, M_\odot$ (clusters, superclusters).
  $\Downarrow$
  galaxies by fragmentation "top-down"
- Isothermal scenario:
  first structures, $M \simeq 10^5 - 10^6\, M_\odot \rightarrow$ hierarchical clustering to larger and larger structures "bottom-up"

- ## Isentropic (adiabatic) longitudinal solutions

$$\vec{\nabla} S = 0$$

(notice: by definition adiabatic: $\frac{dS}{dt} = 0$)

(II) Euler eqn. : $\frac{\partial \vec{v}}{\partial t} + 2\frac{\dot{a}}{a}\vec{v} = -\frac{c_s^2}{a}\vec{\nabla}\delta - \frac{1}{a}\vec{\nabla}\phi$

$$\Rightarrow \boxed{\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = +\frac{c_s^2}{a^2}\nabla^2\delta + 4\pi G\bar{\rho}\,\delta}$$

- Consider solutions in Fourier space:

$$\delta(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\,\hat{\delta}(\vec{k},t)\,e^{-i\vec{k}\cdot\vec{x}}$$

Then :

$$\frac{\partial^2 \hat{\delta}(\vec{k})}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \hat{\delta}(\vec{k})}{\partial t} = -\frac{c_s^2}{a^2}k^2\hat{\delta}(\vec{k}) + 4\pi G\bar{\rho}\,\hat{\delta}(\vec{k})$$

$$\boxed{\ddot{\hat{\delta}}(\vec{k}) + 2\frac{\dot{a}}{a}\dot{\hat{\delta}}(\vec{k}) - \left(4\pi G\bar{\rho} - \frac{c_s^2}{a^2}k^2\right)\hat{\delta}(\vec{k}) = 0.}$$

- Introduce "Jeans length" λ_J :

$$\boxed{\lambda_J \equiv \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}}\,;$$

$$\omega_J^2 \equiv \frac{c_s^2 k^2}{a^2}\left(1 - \frac{\lambda^2}{\lambda_J^2}\right)$$

dispersion relation

Then :

$$\boxed{\ddot{\hat{\delta}}(\vec{k}) + 2\frac{\dot{a}}{a}\dot{\hat{\delta}}(\vec{k}) + \omega_J^2\hat{\delta}(\vec{k}) = 0.}$$

Jean Instability:
Non-Expanding Medium

- template for behaviour in expanding medium.
- In physical coordinates there is not a Hubble drag term à la:

$$\boxed{\ddot{\delta}(\vec{k}) + \omega_J^2 \, \hat{\delta}(\vec{k}) = 0.}$$

$$\Downarrow$$

Jeans length:

$$\lambda_J \equiv \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}$$

$$\omega_J^2 \equiv \frac{c_s^2 k^2}{a^2}\left(1 - \frac{\lambda^2}{\lambda_J^2}\right)$$

Solution:

$$\underbrace{\hat{\delta}(\vec{k}) = \hat{\delta}_+(\vec{k})\, e^{+i\omega t} + \hat{\delta}_-(\vec{k})\, e^{-i\omega t}.}$$

$$\Downarrow$$

Two solution regimes:

① $\lambda > \lambda_J \longrightarrow \omega_J^2 < 0$

$$\Downarrow$$

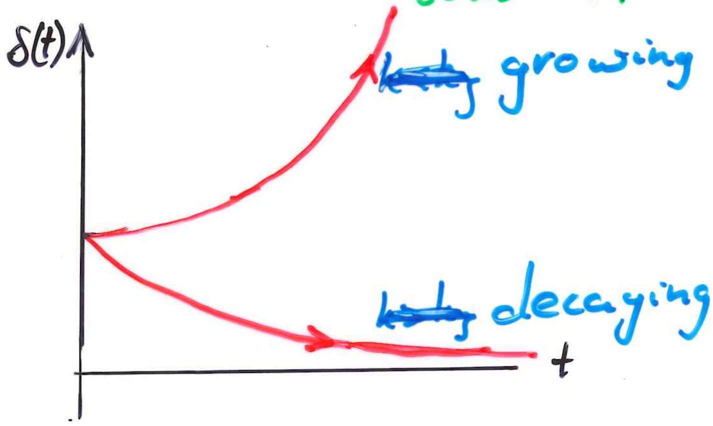
$$\omega^2 = -\omega_J^2 > 0$$

$$\boxed{\hat{\delta}(\vec{k}, t) = \hat{\delta}_+(\vec{k})\, e^{|\omega_J| t} + \hat{\delta}_-(\vec{k})\, e^{-|\omega_J| t}}$$

Jean instab. cont'd:

$\lambda > \lambda_J$:

$$\boxed{\hat{\delta}(\vec{k},t) = \underbrace{\hat{\delta}_+(\vec{k})\, e^{\omega_J t}}_{\text{growing solution}} + \underbrace{\hat{\delta}_-(\vec{k})\, e^{-\omega_J t}}_{\text{decaying solution.}}}$$

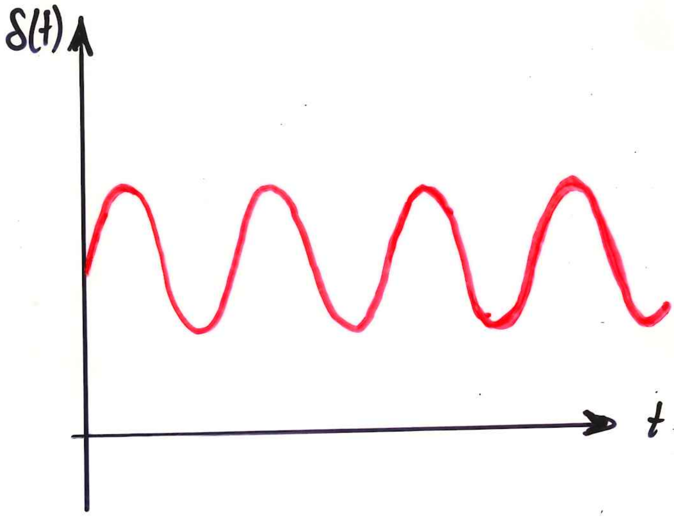


(2) $\lambda < \lambda_J \longrightarrow \omega_J^2 > 0$

$\omega^2 = -\omega_J^2 < 0$

$$\boxed{\hat{\delta}(\vec{k}) = \hat{\delta}_+(\vec{k})\, e^{+i\omega_J t} + \hat{\delta}_-(\vec{k})\, e^{-i\omega_J t}}$$

Oscillating Solution: sound wave



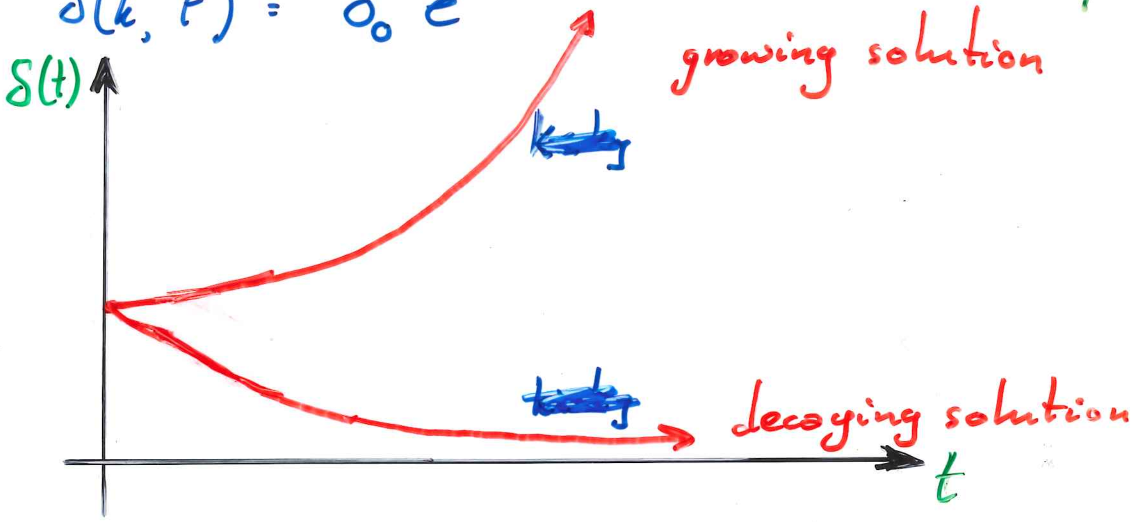
Jeans Instability in Non-expanding Medium

Jeans length: $\boxed{\lambda_J = c_s \sqrt{\frac{\pi}{G\bar{\rho}}}}$

① $\underline{\lambda > \lambda_J}$

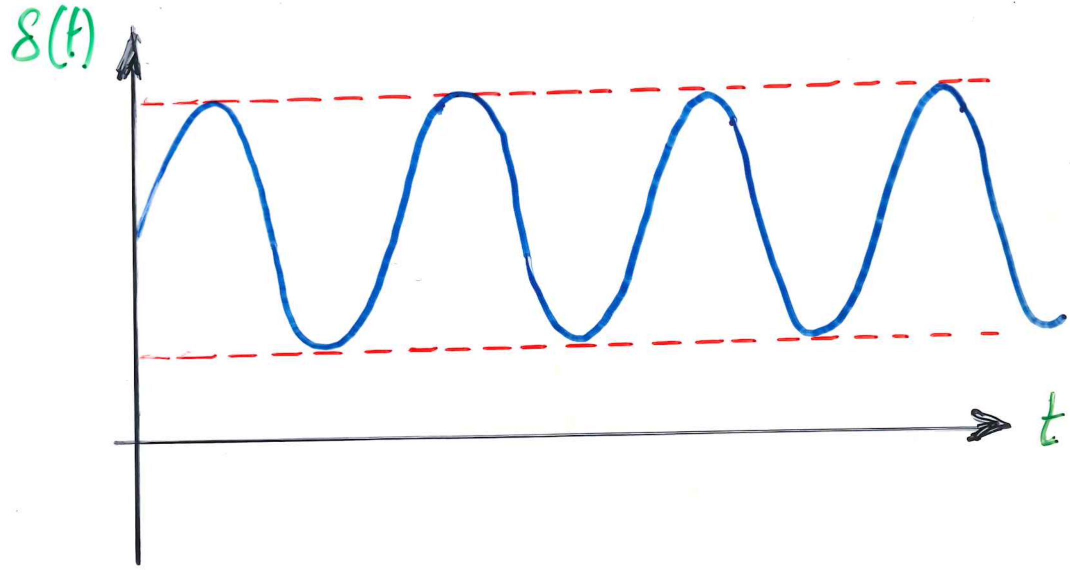
$$\delta(k, t) = \delta_0 \, e^{\pm t\sqrt{4\pi G\bar{\rho} - c_s^2 k^2}}$$

Growing (+)
(Decaying)
fluctuations



② $\underline{\lambda < \lambda_J}$

$$\delta(k, t) = \delta_0 \, e^{\pm it\sqrt{c_s^2 k^2 - 4\pi G\bar{\rho}}}$$



Jeans Instability in an Expanding Medium

- Evolution of a perturbation in an expanding medium gets complicated by the expansion term.

Perturbation Evolution:

$$\boxed{\ddot{\delta}(\vec{k}) + 2\frac{\dot{a}}{a}\dot{\delta}(\vec{k}) - \left(4\pi G\bar{\rho} - \frac{c_s^2}{a^2}k^2\right)\delta(\vec{k}) = 0}$$

$$\Downarrow$$

① Matter-dominated Universe.

for Einstein-de Sitter: $\Omega = 1$

$$\Downarrow$$

- $a(t) \propto t^{2/3}$; $\frac{\dot{a}}{a} = \frac{2}{3t}$.
- $\bar{\rho} = \frac{1}{6\pi G t^2}$

$$\boxed{\ddot{\delta}(\vec{k}) + \frac{4}{3t}\dot{\delta}(\vec{k}) = \frac{2}{3t^2}\left(1 - \frac{c_s^2 k^2}{4\pi G\bar{\rho}a^2}\right)\delta(\vec{k})}$$

solutions of form

$$\delta(\vec{k}, t) = \delta(\vec{k})\, t^n$$

$$\Longrightarrow \quad \delta(\vec{k}, t) \propto t^{-\frac{1}{6} \pm \frac{5}{6}\sqrt{1 - \frac{6c_s^2k^2}{25\pi G\bar{\rho}a^2}}}$$

Jeans instability expanding medium:

Matter-dominated $\Omega=1$ Universe:

Two different modes of evolution:

① Gravitationally unstable

$$\lambda > \lambda'_J = \frac{\sqrt{24}}{5}\frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}$$

$$\Downarrow$$

- $\hat{\delta}_{\pm}(\vec{k},t) \propto t^{-\frac{1}{6}}\, t^{\pm\frac{5}{6}\sqrt{1-\left(\frac{\lambda'_J}{\lambda}\right)^2}}$

$(\propto t^{2/3}$ if $\lambda \gg \lambda'_J)$

② Oscillatory:

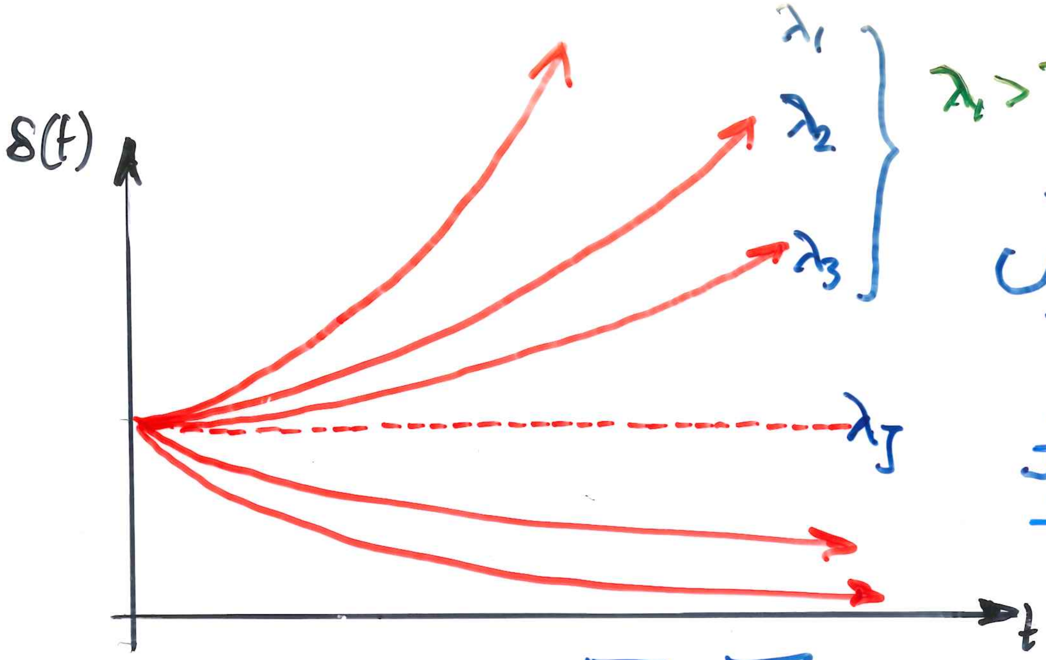
$$\lambda < \lambda'_J$$

$$\Downarrow$$

- $\hat{\delta}_{\pm}(\vec{k},t) \propto t^{-\frac{1}{6}}\, e^{\pm i \ln t \left\{\frac{5}{6}\sqrt{\left(\frac{\lambda'_J}{\lambda}\right)^2 - 1}\right\}}$

Oscillation + Gradual Decay

- Note: - $t^{-1/6}$: gradual decay
  - $e^{\pm i \ln t (\,)}$: oscillatory - with period increasing with $\ln t$

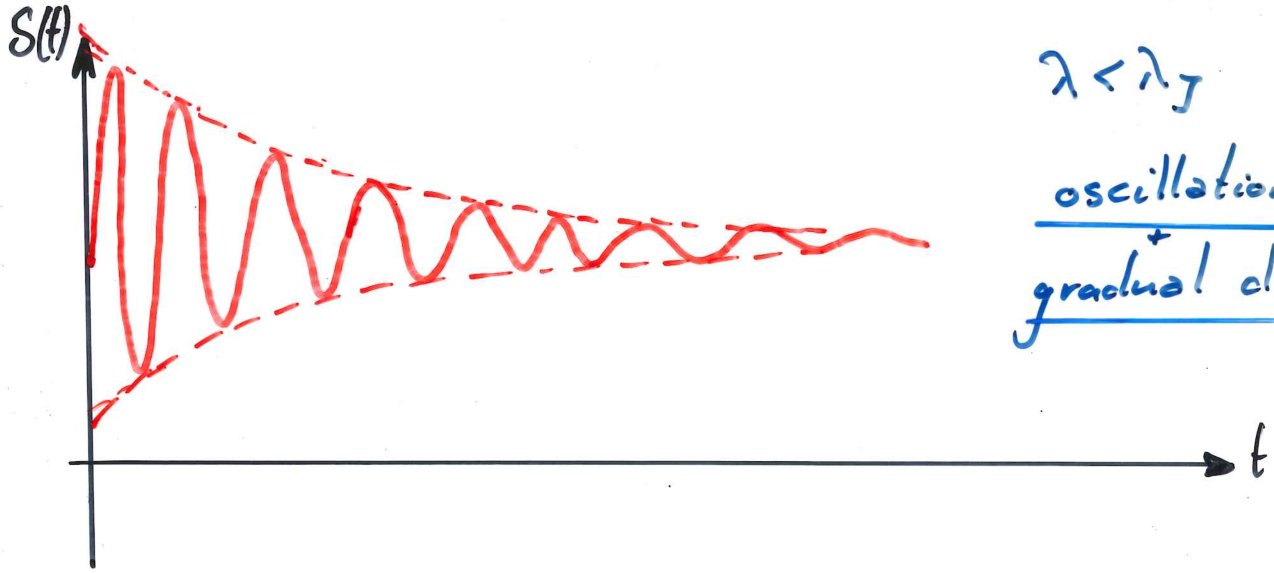


$\lambda_1 > \lambda_2 > \lambda_3$

Jeans Instability

growing & decaying modes.

$$\lambda_J = \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}\ \frac{\sqrt{24}}{5}$$



$\lambda < \lambda_J$

oscillation + gradual decay

$$\boxed{\text{Jeans Instability in Matter-Dominated Universe}}$$

Jeans Instability in an Expanding Medium

Radiation-Dominated Epoch

- The equations of motion change slightly:

$$\begin{cases} (1) & \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho + \dfrac{p}{c^2}\right)\vec{v} = 0. \\ (2) & \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\dfrac{1}{\rho + \frac{p}{c^2}}\vec{\nabla}p - \vec{\nabla}\Phi \\ (3) & \nabla^2\Phi = 4\pi G\left(\rho + \dfrac{3p}{c^2}\right) \end{cases}$$

- Following same procedure, and using the following relations for radiation:

$$\begin{cases} \rho + \dfrac{p}{c^2} = \dfrac{4}{3}\rho \\ \rho + \dfrac{3p}{c^2} = 2\rho \\ a(t) = a_0\left(\dfrac{t}{t_0}\right)^{1/2} \end{cases}$$

$$\Downarrow$$

$$\boxed{\ddot{\delta}(k,t) + 2\frac{\dot{a}}{a}\dot{\delta}(k) + \left(\frac{c_s^2 k^2}{a^2} - \frac{32}{3}\pi G\rho\right)\delta(k) = 0}$$

$$\Downarrow$$

$$\boxed{\ddot{\delta}(k) + \frac{\dot{\delta}(k)}{t} - \frac{1}{t^2}\left(1 - \frac{3c_s^2 k^2}{32\pi G\bar{\rho}a^2}\right)\delta(k) = 0}$$

<u>Define</u> : Jeans length λ_J :

$$\boxed{\lambda_J = \frac{c_s}{a}\sqrt{\frac{3\pi}{8G\bar{\rho}}}}$$

<u>Two modes of evolution</u> :

① <u>Gravitationally Unstable Mode:</u>

- $\underline{\underline{\lambda > \lambda_J}}$

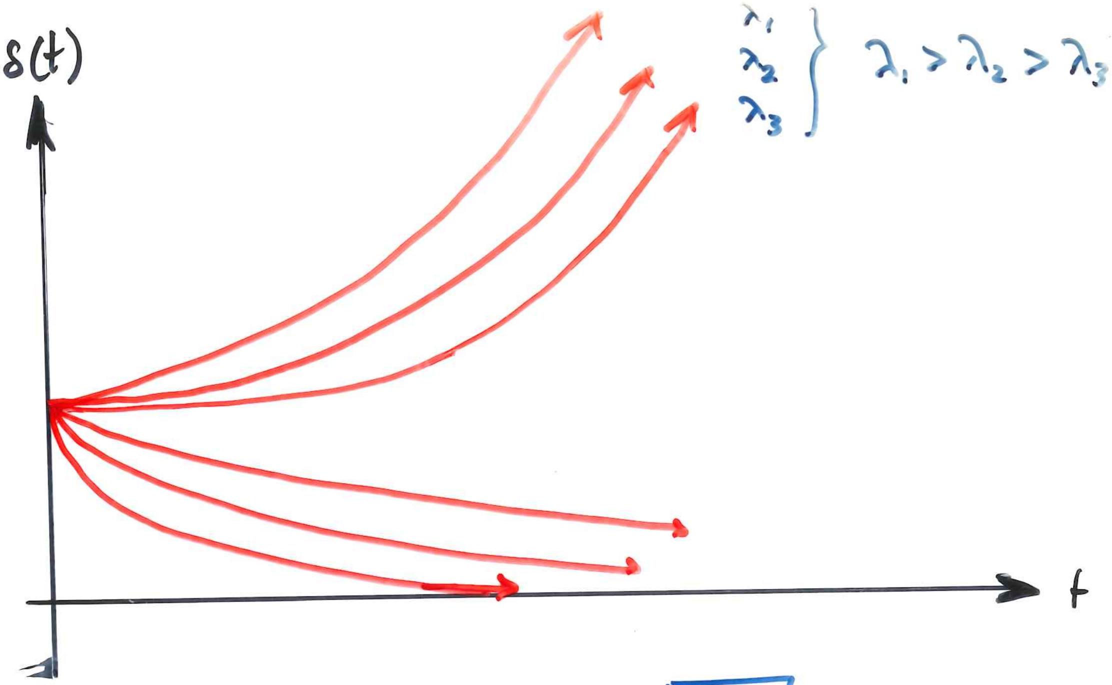
Unstable Mode:
- $\hat{\delta}_\pm(k,t) \propto t^{\pm\sqrt{1-\left(\frac{\lambda_J}{\lambda}\right)^2}}$

$\hat{\delta}_+$: power-law growth.
$\hat{\delta}_-$: power-law decay

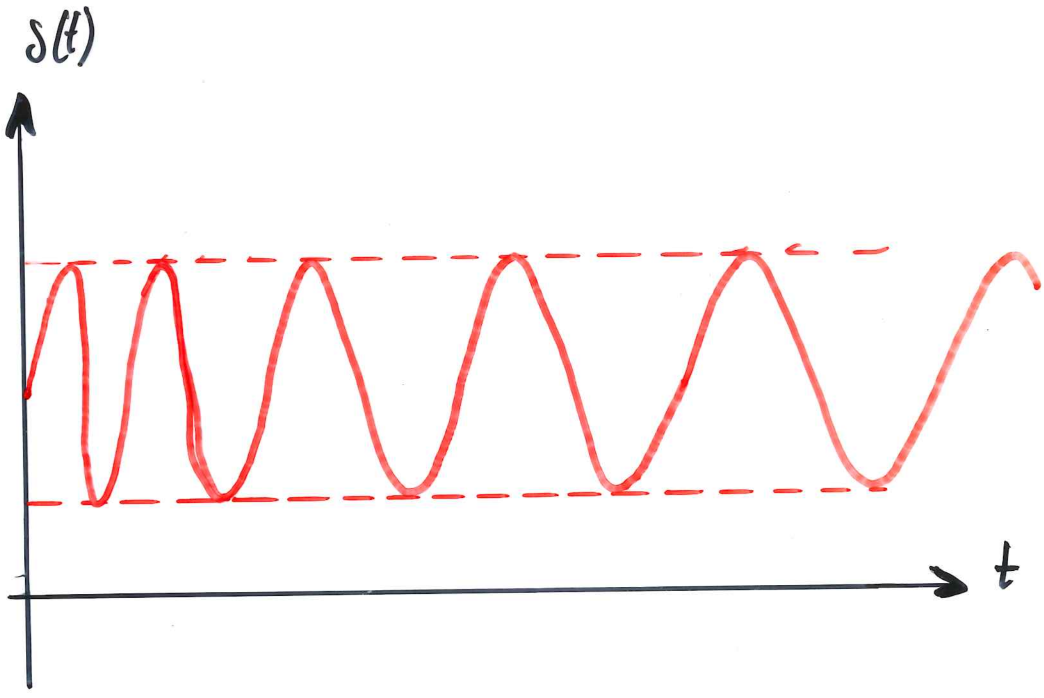
- $\underline{\underline{\lambda < \lambda_J}}$

Oscillatory Mode:
- $\hat{\delta}_\pm(k,t) \propto e^{\pm i \ln t \left(\sqrt{\left(\frac{\lambda_J}{\lambda}\right)^2 - 1}\right)}$

Note
- oscillation with period in t gradually stretching, going along in $\ln t$.
- as opposed to matter-dominated era, no decay term.



$$\lambda_J = \frac{c_s}{a}\sqrt{\frac{3\pi}{8G\bar{\rho}}}$$



$$\boxed{\text{Jeans-Instability in a Radiation-Dominated Universe}}$$

Jeans Mass Scale :

$$\boxed{M_J = \frac{4\pi}{3}\bar{\rho}(t)\,\lambda_J^3(t)}$$

$$\lambda_J = \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}$$

$\Downarrow$

characteristic behaviour determined by evolution of sound speed:

$$\boxed{c_s^{\,2} = \left(\frac{\partial p}{\partial \rho}\right)_S}$$

- Thus, to assess the history of perturbations, we need to assess the evolution of the sound speed $c_s^{\,2}$ in various cosmic epochs.

Jeans' Instability :

Sound Velocity & Cosmological Evolution

$$\boxed{c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s}$$

For instructive purposes, it is good to restrict ourselves at the outset to a 2-component medium, consisting of:

1. radiation
2. baryonic matter.

① Radiation :

- pressure : $P_r = \frac{a}{3} T^4$
- energy density : $\rho_r = \frac{a}{c^2} T^4 \quad \left(= \frac{3}{c^2} P_r \right)$

② Baryonic Matter :

- pressure : $P_m = \frac{\rho k T}{m_p}$
- density : $T = cst. \, \rho^{\gamma - 1}$

γ = gas

I Pre-Recombination:

$$P_{rad} \gg P_b \text{ , i.e. } \rho \ll \rho_I = \frac{aT^3}{3k}$$

Gas pressure negligible.

$$\boxed{c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\left(\frac{\partial p}{\partial T}\right)_{rad}}{\left(\frac{\partial \rho}{\partial T}\right)_{rad} + \left(\frac{\partial \rho}{\partial T}\right)_m}}$$

- radiation:

$$\left(\frac{\partial p}{\partial T}\right)_r = \frac{4}{T}\, p_r = \frac{4}{3}\,\frac{\rho_r c^2}{T}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_r = 4\,\frac{\rho_r}{T}$$

- (baryonic) gas:
  - $$\left(\frac{\partial \rho}{\partial T}\right)_m = \frac{1}{\gamma - 1}\,\frac{\rho_m}{T}$$

Before recombination: equilibrium gas-radiation:

$$T_m = T_{rad} \Rightarrow \gamma = \frac{4}{3}\ !$$

$$\Downarrow$$

$$\left(\frac{\partial \rho}{\partial T}\right)_m = 3\,\frac{\rho_m}{T}$$

pre-recombination sound velocity:

$$\boxed{c_s^2 = \frac{c^2}{3}\,\frac{4\rho_r}{4\rho_r + 3\rho_m}}$$

We distinguish 2 regimes in this pre-recombination period:

① Radiation-Dominated.

$t \ll t_{eq}$: $\rho_m \ll r \Longrightarrow$

- $c_s^2 \approx \dfrac{c^2}{3} \Longrightarrow \boxed{c_s = \dfrac{c}{\sqrt{3}}}$

② Matter-Dominated

$t_{eq} < t < t_{rec}$: before recombination.

- pressure: radiation
- inertia : $\rho_m \gg \rho_r \Longrightarrow \rho \approx \rho_m$

- $\boxed{c_s^2 = \dfrac{4c^2}{9}\,\dfrac{\rho_r}{\rho_m}}$

- $\rho_r = \dfrac{a}{c^2} T^4 = \dfrac{a}{c^2} T_0^4 (1+z)^4$.
- $\rho_m(z) = \rho_{m,0}(1+z)^3 = \Omega_m \rho_c (1+z)^3$

sound velocity, cont'd:

$$c_s^2 = \frac{4c^2}{g}\frac{\rho_r}{\rho_m} \Longrightarrow$$

$$\boxed{c_s^2 \doteq \frac{4aT_0^4(1+z)}{g\,\Omega_m \rho_c}} \Rightarrow \boxed{c_s \approx \frac{10^6\sqrt{z}}{\sqrt{\Omega_m h^2}}\,\text{m/s}}$$

II Post-Recombination

Influence of radiation negligible (since decoupling):

$$\boxed{c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_m = \sqrt{\frac{\gamma k T}{m_p}}}$$

$\Downarrow$ Hydrogen gas: $\gamma = \frac{5}{3}$

$$\boxed{c_s \simeq 5\times 10^5\sqrt{\frac{1+z}{1+z_{rec}}}\ \text{m/s}}$$

for as long as $T_m \simeq T_r$, which is true for $z \gtrsim 300$, due to residual ionization. (Compton diffusion allows energy exchange).

Evolution of Jeans mass

Jeans mass: $\quad M_J = \frac{\pi}{6} \rho_m (a\lambda_J)^3$

$$\lambda_J \simeq c_s \sqrt{\frac{\pi}{G\rho}}.$$

$$\left(M \propto \rho \lambda^3 \rightsquigarrow , \; M_{pc} \triangleq 10^{11} \frac{1}{\Omega h^2} M_\odot \right)$$

$\Rightarrow$ M_J dependent on evolution sound speed:

adiabatic: $p = p_r + p_m \simeq p_r \simeq \rho_r c^2/3$

$$c_s = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2} \simeq \frac{c}{\sqrt{3}} \left(1 + \left(\frac{\partial \rho_m}{\partial \rho_r}\right)_s\right)^{-1/2} = \frac{c}{\sqrt{3}} \left(1 + \frac{3\rho_m}{4\rho_r}\right)^{-1/2}$$

$$\simeq \frac{c}{\sqrt{3}} \qquad z \gg z_{eq}$$

$$\simeq 0.83 \frac{c}{\sqrt{3}} \qquad z = z_{eq}$$

$$\simeq \frac{c}{\sqrt{3}} \left(\frac{4\rho_r}{3\rho_m}\right)^{1/2} \simeq \frac{c}{\sqrt{3}} \left(\frac{1+z}{1+z_{eq}}\right)^{1/2} \simeq 2 \times 10^8 \left(\frac{1+z}{1+z_{eq}}\right)^{1/2} \mathrm{m/s}$$

isothermal: gas of monatomic particles of mass m_p and temperature $T_m \simeq T_r = T_{0r}(1+z)$, proton mass:

$$c_s = \sqrt{\frac{\partial p_m}{\partial \rho_m}} = \sqrt{\frac{\gamma k_B T}{m_p}}$$

hydrogen: $\gamma = 5/3$:

$$c_s \simeq \sqrt{\frac{k_B T_{rec}}{m_p}} \sqrt{\frac{1+z}{1+z_{rec}}} \simeq 5 \times 10^5 \sqrt{\frac{1+z}{1+z_{rec}}} \; \mathrm{m/s}.$$

$$\left(T_{rec} \simeq T(z_{rec}) \simeq 4000 \; \mathrm{K}\right)$$

$z \geq z_{eq}$:

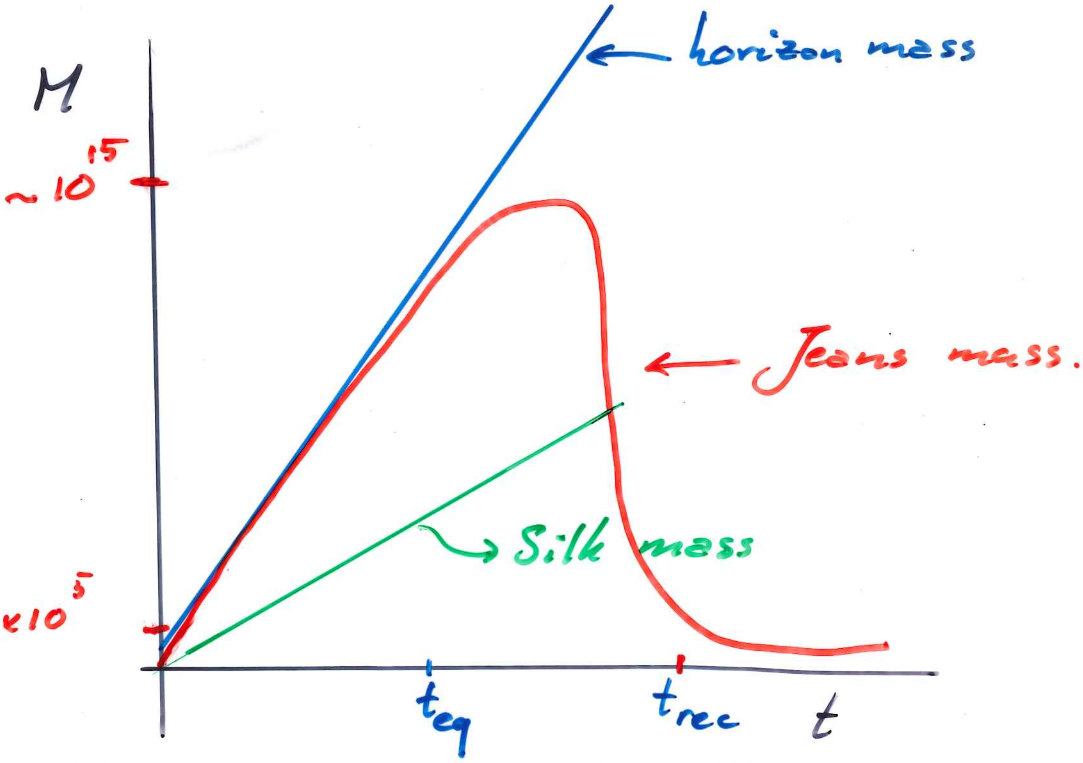
$$M_J \simeq 3.5 \times 10^{15} \frac{1}{\Omega h^2} \left(\frac{1+z}{1+z_{eq}} \right)^{-3} M_\odot$$

$z_{eq} > z > z_{rec}$:

$$M_J \simeq \frac{\pi}{6} \rho_m \left[\frac{c}{\sqrt{3}} \left(\frac{1+z}{1+z_{eq}} \right)^{1/2} \left(\frac{\pi}{G\bar{\rho}} \right)^{1/2} \right]^3 \simeq \text{constant}$$

$z < z_{rec}$

$$M_J \simeq 5 \times 10^5 \left(\frac{1+z}{1+z_{rec}} \right)^{3/2} M_\odot :$$



③ Silk Mass : Photon Diffusion.

Dissipation of adiabatic waves by photon diffusion

$$M_D = \frac{\pi}{6} \rho_m \lambda_D^3 \simeq 0.5 \left(\frac{m_p c}{\sigma_T G^{1/2}} \right)^{3/2} \left(\rho_{0c}^2 \rho_{0r}^3 \Omega^2 \right)^{-1/4} (1+z)^{-9/2}$$

$$\simeq 7 \times 10^{10} (\Omega h^2)^{-5} \left(\frac{1+z}{1+z_{eq}} \right)^{-9/2} M_\odot$$

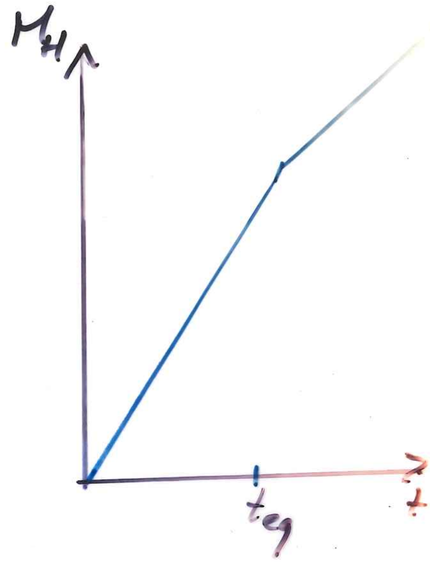
- Horizon mass:

mass in sphere of diameter $d = R_H$:

$$M_H = \frac{4\pi}{3}\bar{\rho}(t)\left(\frac{R_H}{2}\right)^3 = \frac{\pi}{6}\bar{\rho}R_H^3$$

Concentrate on baryonic mass.

- Before equivalence:



$$\left.\begin{array}{l} M_{Hb} \simeq \frac{\pi}{6}\rho_m(2ct)^3 \\ 1+z \propto t^{-1/2} \\ \rho_m(z) = \rho_m(z_{eq})\left(\frac{1+z}{1+z_{eq}}\right)^3 \end{array}\right\}$$

$$M_{Hb} \simeq 5\times10^{14}\frac{1}{(\Omega h^2)^2}\left(\frac{1+z}{1+z_{eq}}\right)^{-3}$$

After equivalence:

$$\left.\begin{array}{l} M_{Hb} \simeq \frac{\pi}{6}\rho_m(3ct)^3 \\ 1+z \propto t^{-2/3} \end{array}\right\}$$

$$M_{Hb} \simeq 5\times10^{14}\frac{1}{(\Omega h^2)^2}\left(\frac{1+z}{1+z_{eq}}\right)^{-3/2}$$

Additional Prerecombination Universe Mass Scales

- **Silk Mass:**
  Photon Diffusion
- **Meszaros Effect:**
  "Stagspansion" Dark Matter Perturbations

Photon Diffusion : Silk Damping

- ## Before recombination :

  - Plasma of photons, electrons and protons
    (disregard He nuclei)
  - thermal contact via :

    ① photons $\longleftrightarrow$ electrons : Thomson scattering

    ② electrons $\longleftrightarrow$ protons : Coulomb interaction

  level of thermal equilibrium set by :

  ① mean free time : $\tau_\gamma = \dfrac{\lambda_T}{c} = \dfrac{1}{n_e \sigma_T c}$

  n_e : electron density

  σ_T : Thomson cross-section :

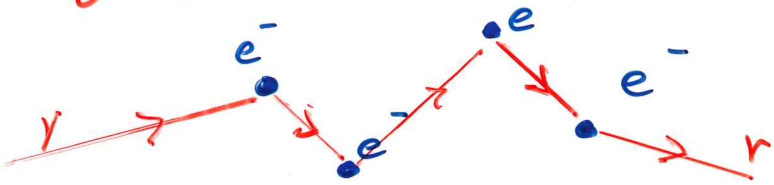
  $$\sigma_T = \frac{8\pi e^4}{3\, m_e^2} = 6.665 \times 10^{-29}\ m^2$$

  ② Coulomb collision time : $\tau_e \lesssim \tau_\gamma \left(\dfrac{kT}{m_e}\right)^{3/2}$

  (negligible wrt τ_γ)

- $\Rightarrow$ finite τ_γ (wrt. dynamic timescale $\approx$ Hubble time)
  causes imperfect coupling :

  photons manage to move out of region through finite number of collisions.



Photon Diffusion !

Photon diffusion, cont'd:

- in time t: N collisions: $N = \frac{t}{\tau_\gamma}$

$$\Downarrow$$

Diffusion: net radial drift: $\boxed{\lambda_D \sim \sqrt{N}\, c\tau_\gamma}$

$$\downarrow$$

$$\lambda_D = c(At)^{1/2} \quad ; \quad A \simeq \frac{4\pi^2}{6}\frac{1}{\sigma_T n_e c}$$

- $\Longrightarrow$ By diffusing out over distance λ_D, all fluctuations over that scale get damped (in particular, as most of its energy density in radiation):

corresponding mass: Silk Mass

$$\boxed{M_D = \frac{4\pi}{3}\rho_B \lambda_D^3}$$

$$\lambda_D^3 \propto \left(\frac{t}{n_e}\right)^{3/2} \propto (\Omega_B h^2)^{-3/2}(1+z)^{-15/2}$$

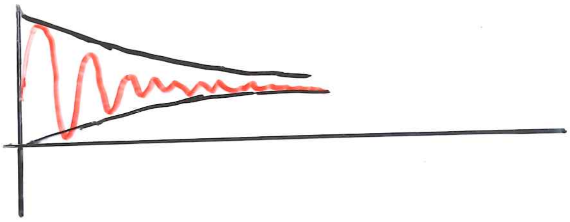
- radiation-dom:
  - $t = \left(\frac{3c^2}{32\pi G\sigma T_0^4}\right)^{1/2}\frac{1}{(1+z)^2} = \frac{2.4\times10^{19}}{(1+z)^2}$ s
  - $n_e = \frac{\rho_B(z)}{m_p} = \frac{\Omega_{B,0}\rho_{c,0}(1+z)^3}{m_p}$

$n_e = n_p$ = baryon density / proton mass

$$n_e = 11\,\Omega_B h^2 (1+z)^3 \ \mathrm{m}^{-3}$$

$$\rho_B(z) = \Omega_B \rho_{crit,0}(1+z)^3 \propto (\Omega_B h^2)(1+z)^3$$

Silk damping, cont'd:



- radiation dom. :

$$\boxed{M_D = \frac{4\pi}{3}\lambda_D^3 \rho_B = 2.4\times10^{26}(\Omega_B h^2)^{-1/2}(1+z)^{-9/2}\, M_\odot}$$

- after $t_{equality}$ $\quad (1+z < 1+z_{eq} \simeq 4.3\times10^4\,\Omega_0 h^2)$

$$a(t) = \left(\frac{9\Omega_0 H^2}{4}\right)^{1/3} t^{2/3} \Rightarrow$$

$$t = \frac{2}{3H_0\Omega_0^{1/2}}\frac{1}{(1+z)^{3/2}} = \frac{2.06\times10^{17}}{(\Omega_0 h^2)^{1/2}(1+z)^{3/2}}$$

$$\Rightarrow \lambda_D^3 \propto \left(\frac{t}{n_e}\right)^{3/2} \propto (\Omega_0 h^2)^{-\frac{9}{4}}(1+z)^{-\frac{27}{4}}$$

$$\boxed{M_D = 2.0\times10^{23}(\Omega_B h^2)^{-5/4}(1+z)^{-15/4}}$$

- Damping continues until recombination:

$\underline{z=1000}$: $\quad \boxed{M_D = 10^{12}(\Omega_B h^2)^{-5/4}\, M_\odot}$

$\Downarrow$

more detailed calculations (Peebles 1981):

$$M_D = 1.3\times10^{12}(\Omega_B h^2)^{-3/2}\, M_\odot$$

$\Rightarrow$ all perturbations with $M \lesssim 10^{12}\, M_\odot$ damped out through diffusion of photons.

- Damping behaviour:

$$\delta(k,t) \propto e^{-\Gamma(k)t}$$

$$\boxed{\Gamma(k) = \frac{k^2 \tau_\gamma}{6}\left(1 - \frac{5}{6R} + \frac{1}{R^2}\right)}$$

$$R \equiv 1 + \frac{3}{4}\frac{\rho_m}{\rho_\gamma}$$

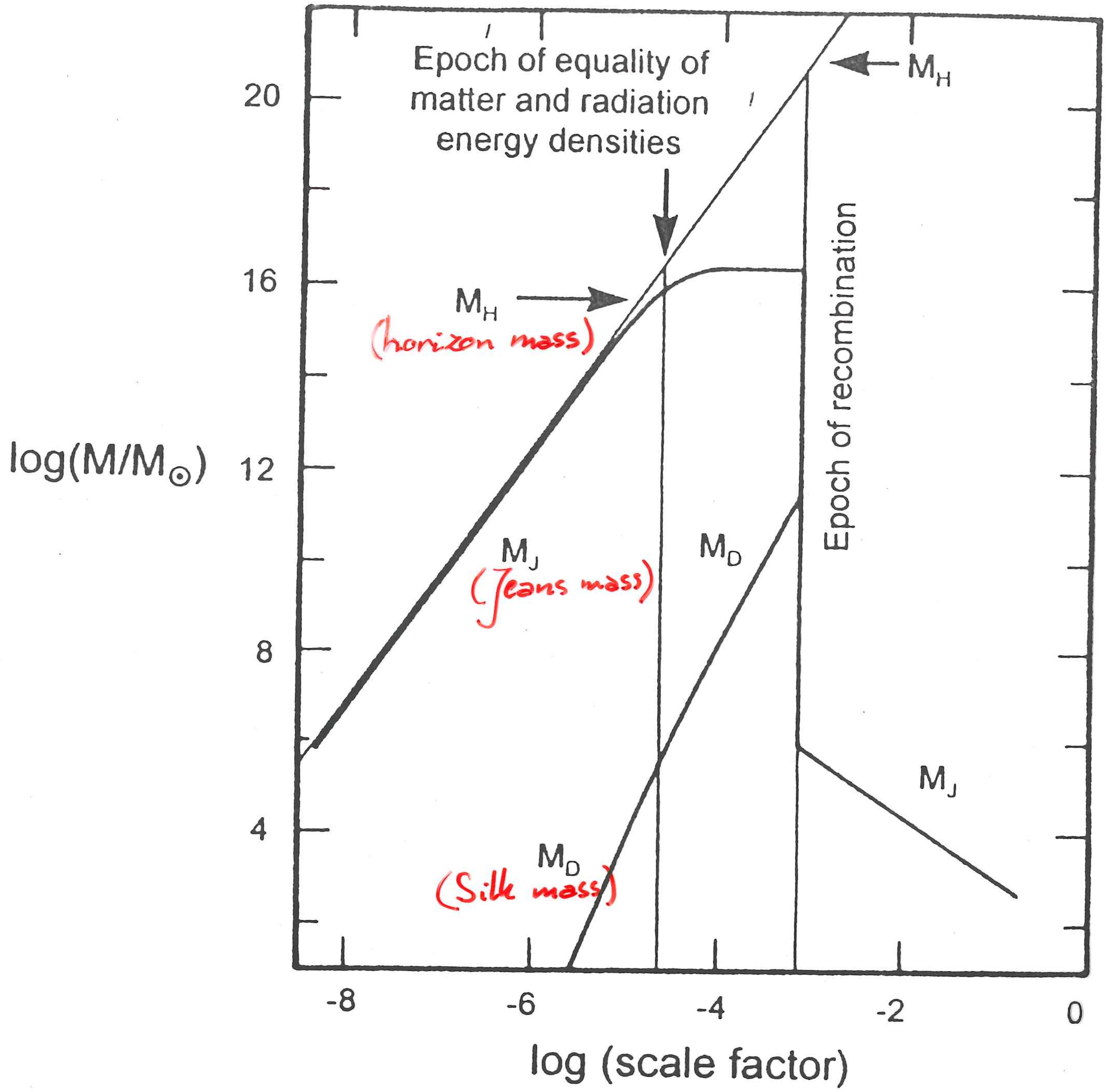


Fig. 12.1. The evolution of the Jeans' mass and the baryonic mass within the particle horizon with scale factor. Also shown is the evolution of the mass scales which are damped by photon diffusion.

(from Longair, 1998).

④ Non-baryonic Matter

- Possibly 80% of matter non-baryonic:

  weakly interacting particles

  ① Hot Dark Matter (HDM)

  collisionless particles with large velocity dispersion. → a "top-down" scenario

  ② Cold Dark Matter (CDM)

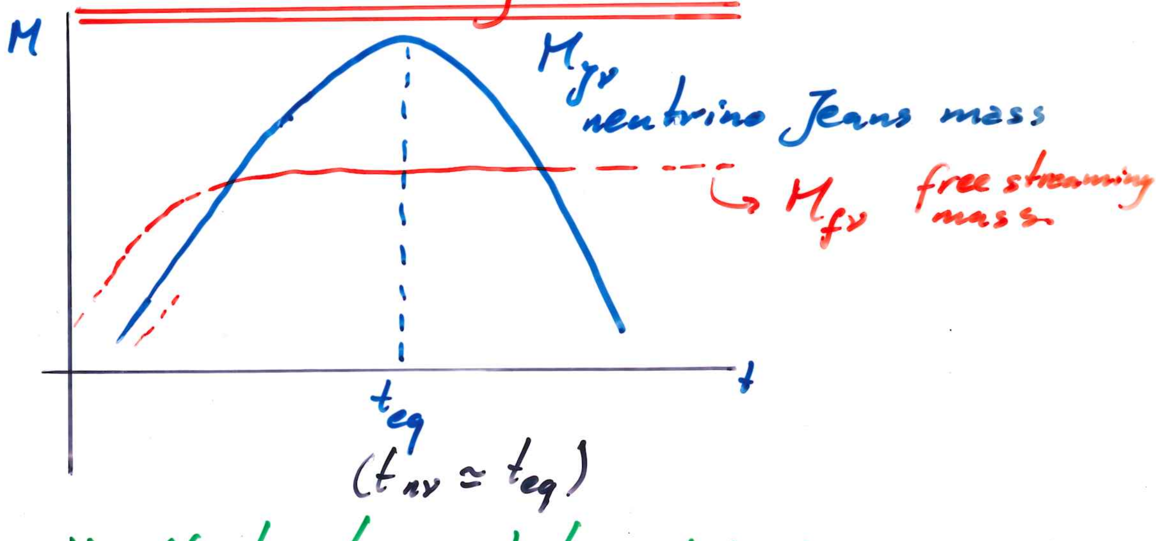
  collisionless particles with very small velocity dispersion.

  decouple when no longer relativistic.

  → "bottom-up" scenario

- Relevant Mass Scales set by 2 effects:

  ① Free Streaming Mass



All fluctuations below $M < M_{f\nu}$ wiped out.

② Meszaros effect

- In a mixture of relativistic and non-relativistic components:

  how does non-relativistic component evolve in sea of relativistic particles:

  for isocurvature perturbations:

  even when $\lambda > \lambda_J$, growth of non-relativistic particles 'frozen' until z_{eq}:

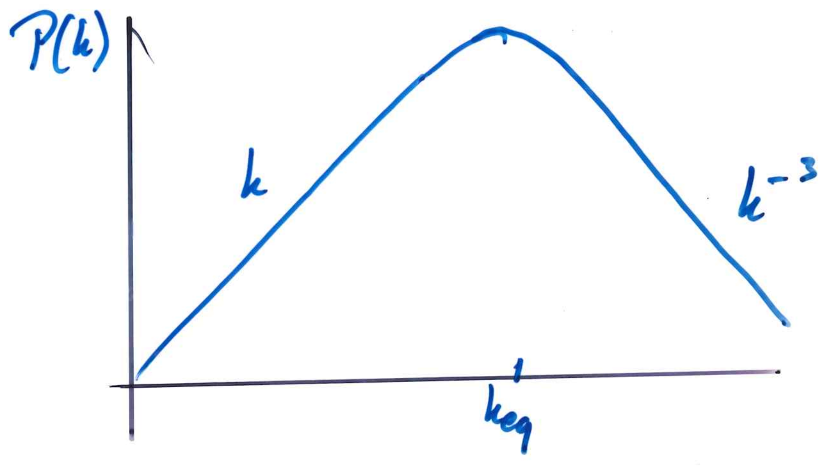
  'Stagnation' or 'Meszaros effect':

  Important for CDM (free streaming of no importance)

  ⇝

  once fluctuations enter horizon they can't grow (for $k \gg k_{eq}$):

  $$P(k) \propto P_{prim}(k) \times k^{-4} \log^2 k.$$



Mészaros Effect

Evolution of (nonbaryonic = only gravitational interacting) dark matter perturbations in a Radiation dominated Universe (smooth radiation background)

- smooth radiation background : $\delta_r = 0$
- nonbaryonic dark matter : $p_m = 0$.

- **Evolution of fluctuations** :

$\delta = \delta_m$: $\rho_m \delta_m + \rho_r \delta_r = \rho_m \delta_m . = \rho_m \delta$

$$\boxed{\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G \rho_m \delta}$$

- **parameterize time** :

$$y = \frac{a}{a_{eq}} = \frac{\rho_m}{\rho_r} \qquad \left(\rho_m \propto a^{-3} \,;\, \rho_r \propto a^{-4}\right)$$

- **FRW Universe** :

$k=0$ (radiation regime) :

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\left(\rho_m + \rho_r\right) \\ \ddot{a} &= -\frac{4\pi G}{3} a \left(\rho_m + \rho_r + \frac{3p}{c^2}\right) \\ &= -\frac{4\pi G}{3} a \left(\rho_m + 2\rho_r\right) \end{aligned}}$$

$: H^2 = \frac{8\pi G}{3}\rho_r(1+y)$

$: \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_r (y+2)$

- notation: $\dot{Q} \equiv \frac{d}{dt}Q \quad ; \quad Q' = \frac{d}{dy}Q$

- $\frac{d^2\delta}{dt^2} = \frac{d^2\delta}{dy^2}\left(\frac{dy}{dt}\right)^2 + \frac{d\delta}{dy}\frac{d^2y}{dt^2} = \delta'' H^2 y^2 + \delta' \ddot{y}$
- $4\pi G \rho_m \delta = \frac{3}{2}\frac{H^2}{1+y} y \delta$
- $\frac{d\delta}{dt} = \frac{d\delta}{dy}\frac{dy}{dt} = \delta' H y$

$$\Rightarrow \quad \boxed{\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0.}$$

① Growing solution:

- $\delta'' = 0$: $\quad (2+3y)\delta' - 3\delta = 0.$

$$\boxed{\delta_{DM} = \frac{2}{3} + y}$$

- for $y = \frac{a}{a_{eq}} \gg 1$: $\quad \delta \propto a$

~~if~~ when radiation of scant influence, growing mode solution EdS Universe.

- for $y \ll 1$: $\quad \delta = cst = \frac{2}{3}$

② Decaying solution.

- $y \ll 1.$ (see Padm. pg. 165):

$$\boxed{\delta_{dec} = \left(1 + \frac{3y}{2}\right)\log\left\{\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right\} - 3(1+y)^{1/2}}$$

$$\delta \overset{y \ll 1}{\approx} -\log y + Cst \Rightarrow \boxed{\delta \propto -\log y}$$

$\Rightarrow$ gradual decay between horizon entering and t_{eq}

Mesz., cont'd:

General Solution:

$$\delta_{gen}(y) = A\delta_1(y) + B\delta_2(y)$$

$$= \boxed{\begin{matrix} \frac{2}{3}A + B\log\left(\frac{4}{y}\right) \\ Ay + \frac{4}{5}By^{-3/2} \end{matrix}} \quad \begin{matrix} y \ll 1. \\ y \gg 1 \end{matrix}$$

Note: this describes the evolution of CDM fluctuations as they enter the horizon:

upon entering the horizon they remain constant: $\delta(k) \propto k^{-3}$