STARTING CONDITIONS

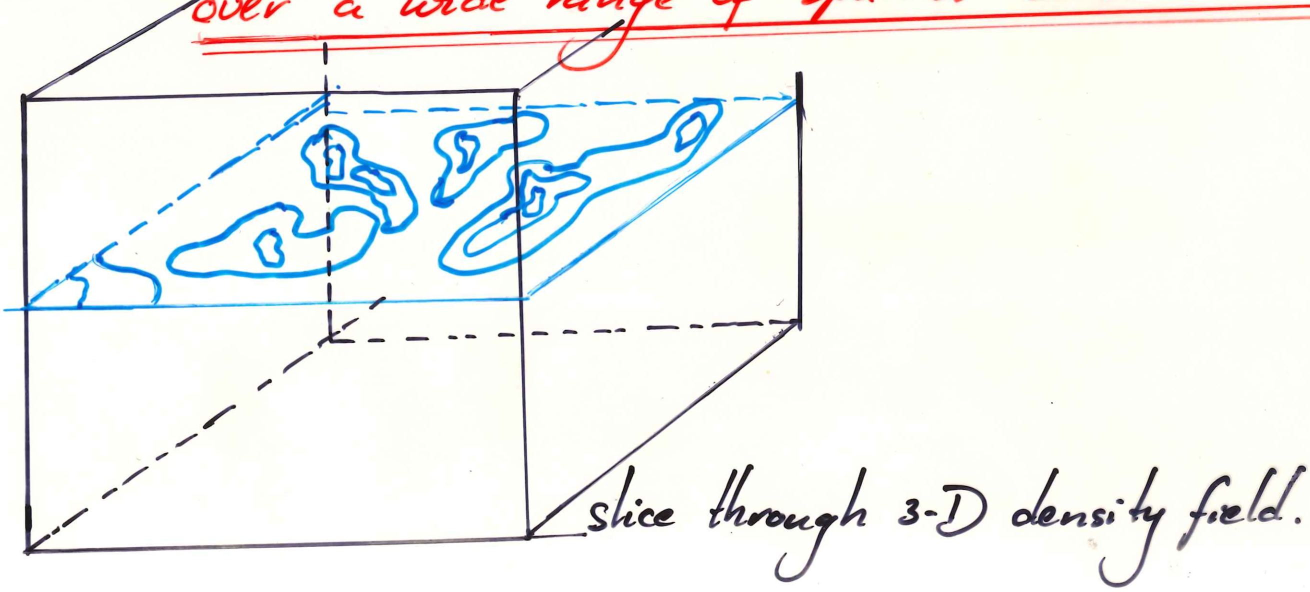
Stochastic Primordial Density Field.

OBSERVED UNIVERSE IS A REALIZATION OF UNDERLYING SPATIAL (!) STOCHASTIC PROCESS.

- Note that as yet there is no first-principle cosmological theory describing the characteristics of the primordial density field, as yet it depends on the proposed structure formation scenario. Most studies work backward. They test a range of structure formation scenarios by investigating the implied repercussions for observable quantities. In principle, if there is no agreement with predictions the scenario is deemed irrelevant.

- How then do the characteristics of the primordial density noise field relate to the product of cosmic evolution?

COSMIC DENSITY FIELD:
unstructured compilation of density fluctuations, over a wide range of spatial scales:



* The exact mode of structure evolution will depend on the character of the density field $\delta(\vec{x},t)$:

- fraction overdense $\leftrightarrow$ underdense regions
  $\updownarrow$
  in general, the distribution function $f(\delta)$ of density fluctuations (i.e. the "1-point" probability function).
- amount and amplitude of small-scale perturbations.
- coherence of the density fluctuation field.
- etc...

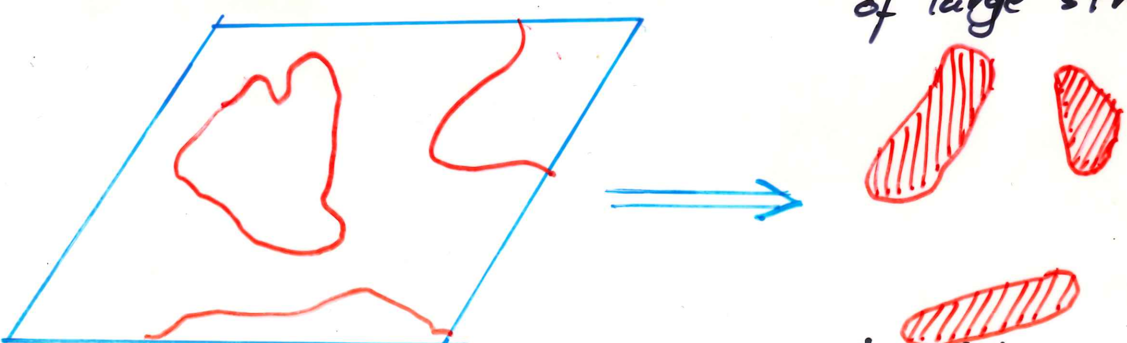
① Bottom-Up:

noisy density field: lot of small-scale power

hierarchical build-up:
1st small scale clumps $\rightarrow$ merging into ever bigger clumps.

② Top-Down:

Monologous collapse of large structures



(followed by secondary fragmentation?)

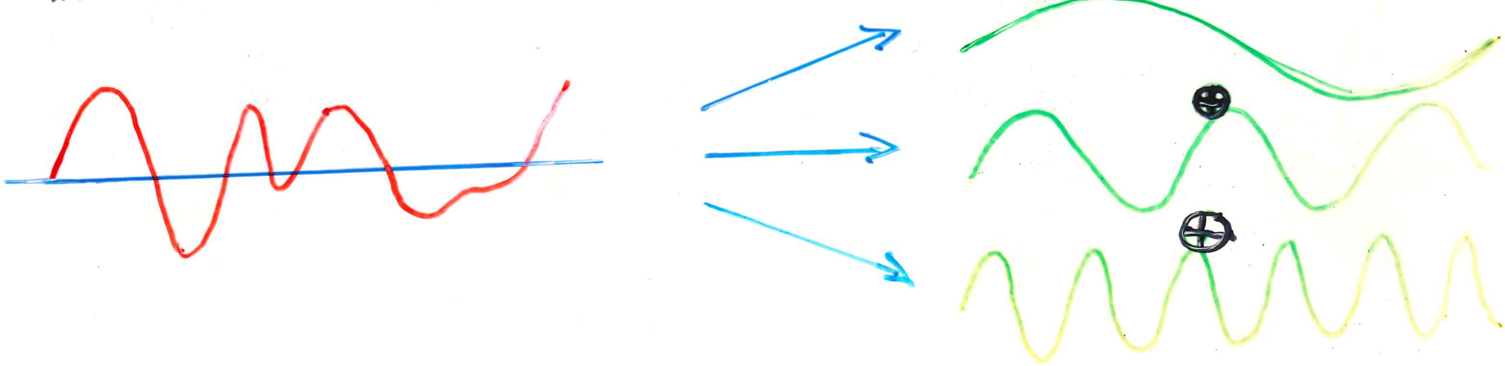
- Thus, for the ensuing evolution, it is good to decompose the stochastic fluctuation field into its various spatial components:

- E.g., 1-D probe through 3-D density field:

The global decomposition into the components on various scales is best done in terms of harmonic waves of wavelength ≡ scale perturbation, which is encapsulated in a Fourier decomposition:

$$\begin{cases} \delta(\vec{x}) = \sum_{\vec{k}} \hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \vec{k} = \dfrac{2\pi}{\lambda_k}\,\hat{e}_k \end{cases} \qquad (\vec{k}:\ \text{wavevector}).$$

in which $\hat{\delta}(\vec{k}) \equiv$ amplitude wave $\vec{k}$



- ## Hierarchical scenario :

small-scale waves have higher amplitude than long waves :

small-scale noise creates most prominent fluctuations

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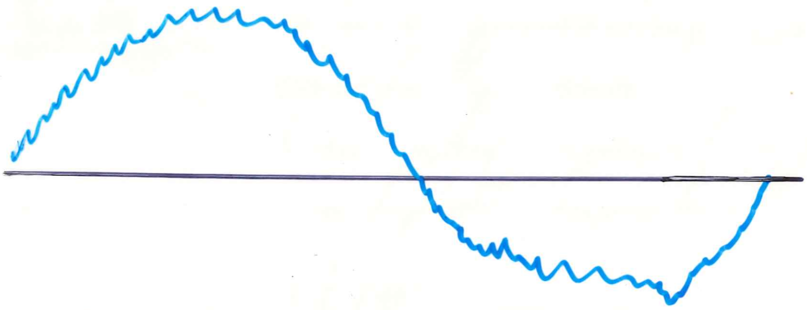
small scale fluctuations collapse first.

- ## Top-Down scenario :

Long waves have higher amplitude than small-scale waves :



large-scale wave collapses before small scale noise condensates into small-scale clumps.

(*) To be able to make more quantitative statements on the structure evolution implied by the different possible primordial density fields, we will therefore have to investigate in more detail the decomposition embodied by the Fourier formulation :

Density Field: *

$$\begin{cases} \delta(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \, \hat{\delta}(\vec{k}) \, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k}) = \int d\vec{x} \, \delta(\vec{x}) \, e^{i\vec{k}\cdot\vec{x}} \end{cases}$$

Fourier Conventions & Definitions.

For each of the relevant dynamical quantities (density δ ; velocity $\vec{v}$; gravity $\vec{g}$; potential ϕ), we can define their respective Fourier components (amplitudes) :

$$\delta(\vec{x},t) \iff \hat{\delta}(\vec{k},t).$$
$$\vec{v}(\vec{x},t) \iff \hat{\vec{v}}(\vec{k},t)$$ ← Vector!
$$\vec{g}(\vec{x},t) \iff \hat{\vec{g}}(\vec{k},t)$$
$$\phi(\vec{x},t) \iff \hat{\phi}(\vec{k},t).$$

- **Note:**
  - before proceeding we need to define carefully our Fourier conventions. (no any special significance, but once adopted important to follow consistently).
  - $\hat{f}(\vec{k}) \Rightarrow f(\vec{x}) \quad : \quad e^{-i\vec{k}\cdot\vec{x}}$
    $f(\vec{x}) \Rightarrow \hat{f}(\vec{k}) \quad : \quad e^{+i\vec{k}\cdot\vec{x}}$
  - $\dfrac{d\vec{k}}{(2\pi)^3}$: "Kaiser" convention : recall any measure "$d\vec{k}$" accompanied by $(2\pi)^3$
  - Arbitrary field $f(\vec{x},t) \iff \hat{f}(\vec{k},t)$:

$$\boxed{\begin{aligned} f(\vec{x},t) &= \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{f}(\vec{k},t)\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{f}(\vec{k},t) &= \int d\vec{x}\, f(\vec{x},t)\, e^{+i\vec{k}\cdot\vec{x}} \end{aligned}}$$

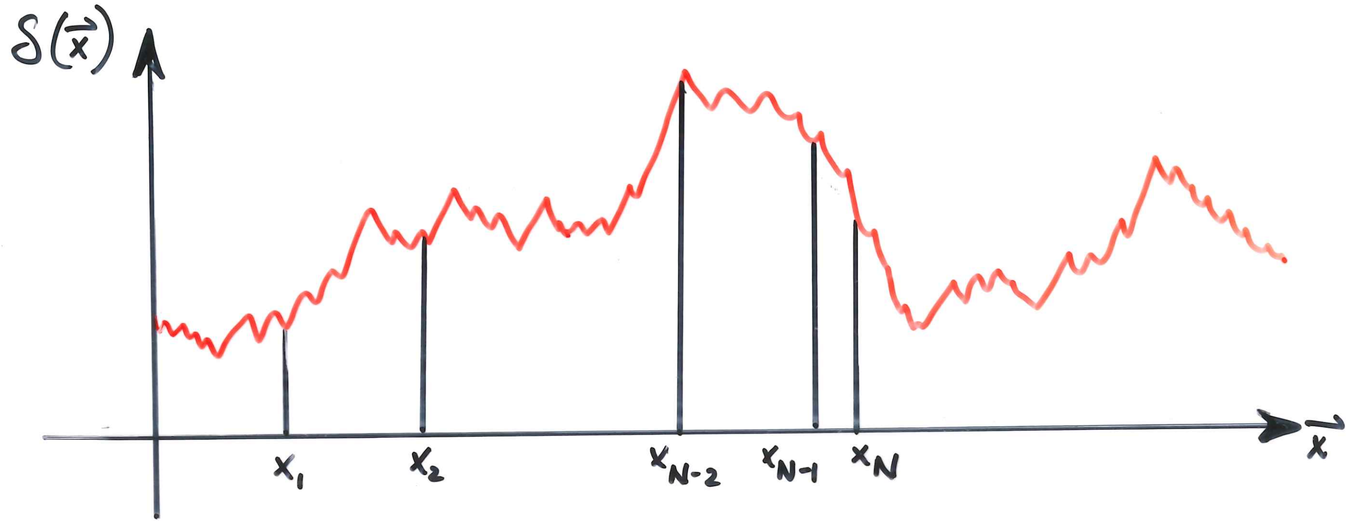
Preliminary Fourier Conventions:

density
$$\begin{cases} \delta(\vec{x}) = \displaystyle\int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k}) = \displaystyle\int d\vec{x}\, \delta(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{cases}$$

velocity
$$\begin{cases} \vec{v}(\vec{x}) = \displaystyle\int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\vec{v}}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\vec{v}}(\vec{k}) = \displaystyle\int d\vec{x}\, \vec{v}(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{cases}$$

grav. potential.
$$\begin{cases} \phi(\vec{x}) = \displaystyle\int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\phi}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\phi}(\vec{k}) = \displaystyle\int d\vec{x}\, \phi(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{cases}$$

Stochastic Density Fields.



- A cosmological density field is a spatial "noise" process:
  the value of the density $\delta(\vec{x})$ at a specific location $\vec{x}$ is a stochastic quantity:

- a priori, we do not know the value $\delta(\vec{x})$, we may only ask its probability:

$$\mathcal{P}_1 = \mathcal{P}_1(\delta(\vec{x}) = \delta') = \mathcal{P}(\delta)\, d\delta$$

- and, generically, what the density field realization is at N locations (or ... in a region of space):

  N-point probability function:

$$\mathcal{P}_N = \mathcal{P}[\delta(\vec{x}_1), \delta(\vec{x}_2), \ldots, \delta(\vec{x}_N)]\, d\delta_1\, d\delta_2 \ldots d\delta_N$$

- Notice: in the case of complete lack of any spatial coherence,

$$\mathcal{P}_N = \mathcal{P}_1[\delta(\vec{x}_1)]\, \mathcal{P}_1[\delta(\vec{x}_2)] \ldots \mathcal{P}_1[\delta(\vec{x}_N)]$$

The canonic density field pdf :

$\boxed{\text{Gaussian Random Fields}}$

• **Note** : pdf = probability distribution function :

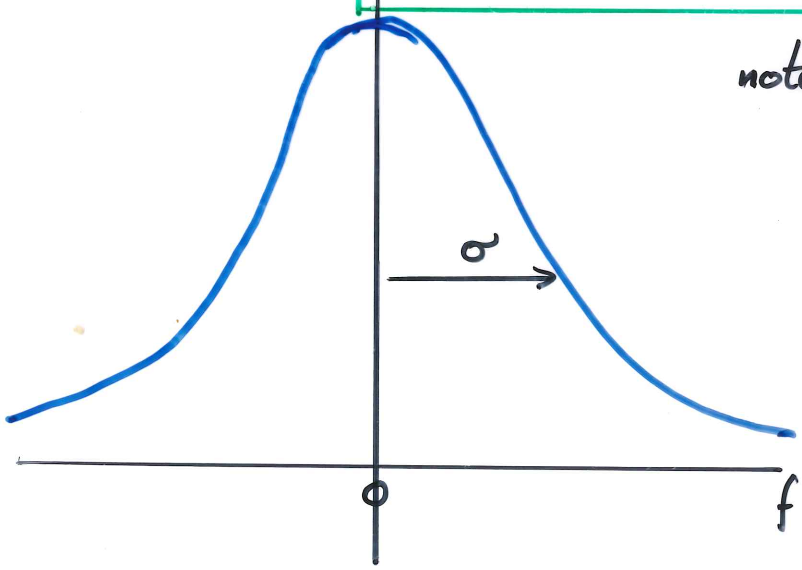
• Gaussian field :

$\mathcal{P}_N$ is a Gaussian distribution

1-point pdf :

$$\boxed{p(f)\,df = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{f^2}{2\sigma^2}\right)df}$$

note : "easy" notation :
$f = \delta$



- if you would measure in a Gaussian field the value δ at a zillion locations, you would find the above curve

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But, what about spatial coherence ?

$\Downarrow$

Look at pdf of values at several locations

Probability Distribution Function:

Density Field δ; R_G: 5.00 h^{-1} Mpc

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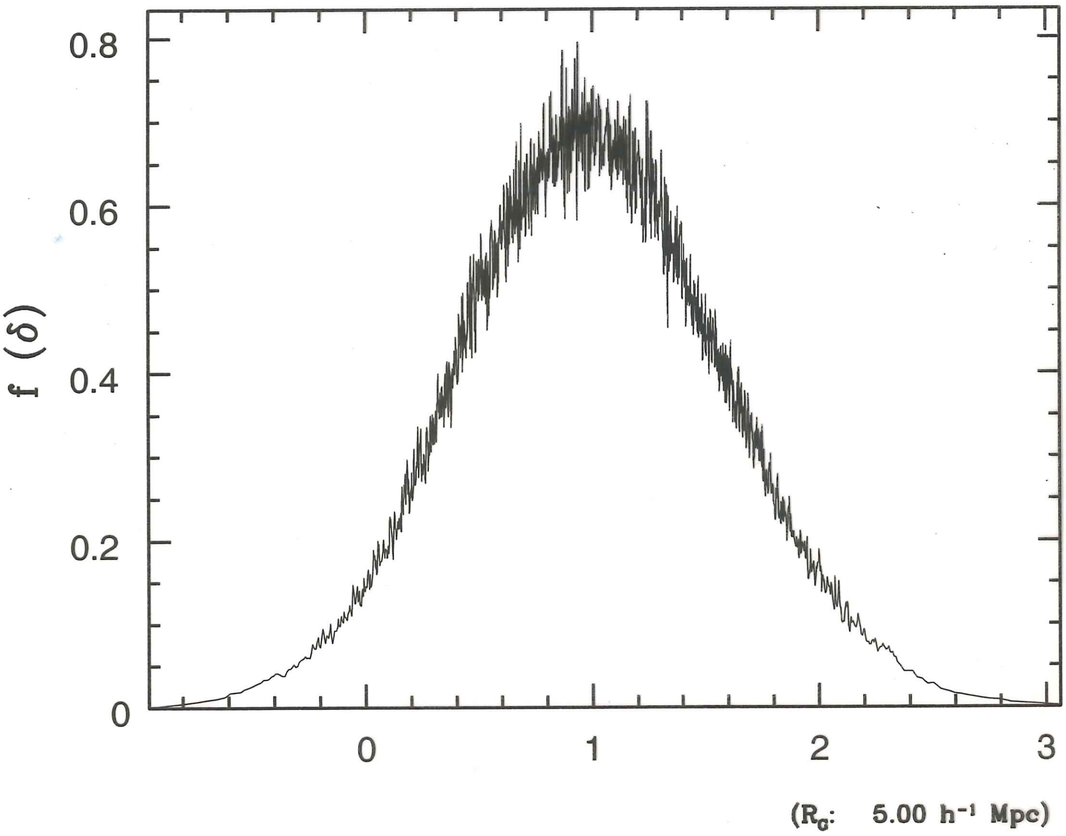
a_{exp} = 1.00; $[\Omega_o, H_o]$ = [1.00, 100.00 km/s/Mpc]

σ_8 = 0.00; $[\Omega_a, H_a]$ = [1.00, 100.00 km/s/Mpc]



Average:	1.00
Median:	0.99
Sigma:	0.58
Skewness:	0.07
Curtosis:	0.05
Minimum:	−1.36
Maximum:	3.59
Low Value %:	50.61
High Value %:	49.39

N-point probability distribution function:

Multivariate Gaussian:

- $$\mathcal{P}_N = \frac{\exp\left[-\frac{1}{2}\sum_{i,j=1}^{M} f_i\,(M_{ij}^{-1})\,f_j\right]}{\left[(2\pi)^N\,(\det M)\right]^{1/2}}\prod_{i=1}^{N} df_i$$

with : M: covariance matrix

- $$M_{ij} = \langle f(\vec{x_i})\, f(\vec{x_j})\rangle \equiv \xi(\vec{x_i}-\vec{x_j}) = \xi(|\vec{x_i}-\vec{x_j}|)$$

ensemble average

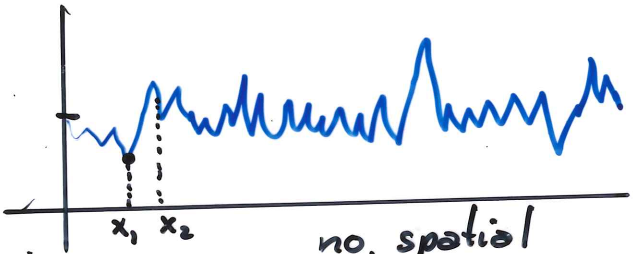
2-pt correlation function

assuming homogeneous, isotropic stochastic process.

statistical characterization by 2-pt correlation function ξ

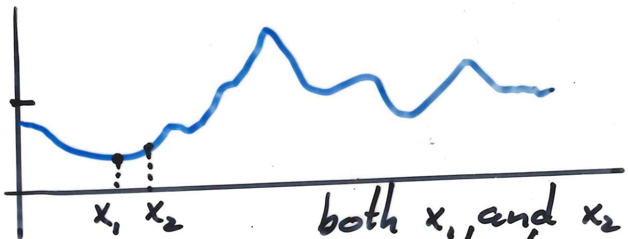
- ξ : 2-pt correlation function

- Quantification of mutual dependence field values at several locations:



no spatial coherence:

$\xi(x_1-x_2)=0$



both x_1 and x_2 in underdense region

$\xi(x_1-x_2)>0$

Random Phases:

* each mode is a complex number:

$$\hat{f}(\vec{k}) = \hat{f}_r(\vec{k}) + i\,\hat{f}_i(\vec{k}) = |\hat{f}(\vec{k})|\, e^{i\phi_k}$$

phase of wave.

* $$P(\hat{f}(\vec{k})) = P(\hat{f}_r(\vec{k}), \hat{f}_i(\vec{k}))$$

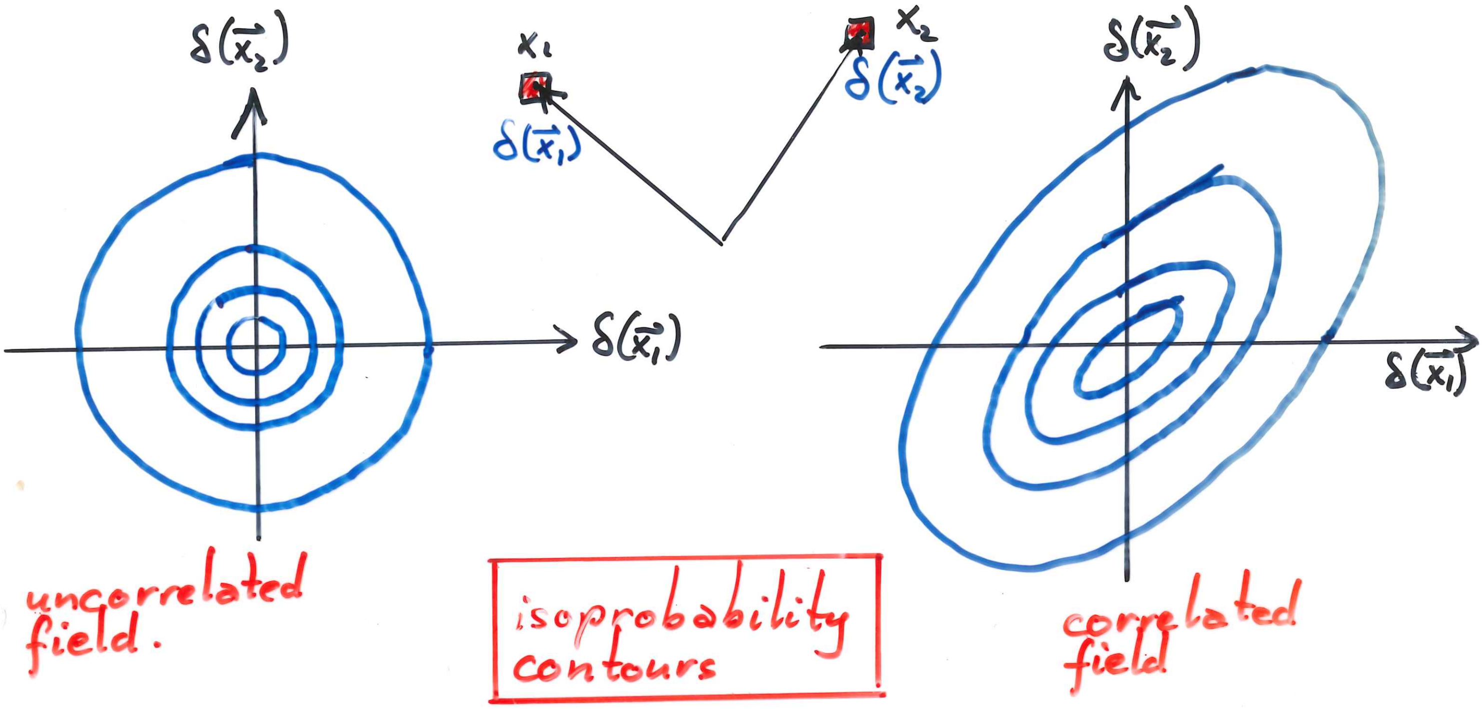
$$= P(\hat{f}_r(\vec{k}))\, P(\hat{f}_i(\vec{k}))$$

$$= e^{-\frac{|\hat{f}(\vec{k})|^2}{2P(k)}} \frac{|\hat{f}(\vec{k})|\, d|\hat{f}(\vec{k})|}{2P(k)} \frac{d\phi_k}{2\pi}$$

Rayleigh distribution

* Strictly, a field is Gaussian only if
  - phases uniformly distributed in $[0, 2\pi]$
  - $|\hat{f}(\vec{k})|$ Rayleigh distributed
  - $|\hat{f}(\vec{k})|$ mutually independent.

Example : 2 point distribution function



Gaussian Random Field

$$\boxed{P(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}}} \quad \Rightarrow \quad \xi(0) = \sigma^2$$

- Probability of field having value y_1 at $\vec{x}_1$ and y_2 at $\vec{x}_2$:

$$\boxed{P(y_1, y_2)\, dy_1 dy_2 = \frac{e^{-(y_1\ y_2)\, M^{-1} \binom{y_1}{y_2}/2}}{(2\pi)(\det M)^{1/2}}\, dy_1\, dy_2}$$

in which:

$$M_{ij} = \begin{pmatrix} \xi(0) & \xi(\vec{x}_1, \vec{x}_2) \\ \xi(\vec{x}_1, \vec{x}_2) & \xi(0) \end{pmatrix}$$

covariance matrix:

$$M_{ij} = \begin{pmatrix} \xi(0) & \xi(\vec{x}_1,\vec{x}_2) \\ \xi(\vec{x}_1,\vec{x}_2) & \xi(0) \end{pmatrix}$$

$$\Rightarrow \quad \det M = \xi^2(0) - \xi^2(r) \quad ; \quad r = |\vec{x}_1 - \vec{x}_2|$$

$$\Rightarrow \quad M^{-1} = \frac{1}{\xi^2(0)-\xi^2(r)} \begin{pmatrix} \xi(0) & -\xi(r) \\ -\xi(r) & \xi(0) \end{pmatrix}$$

$$\Rightarrow \quad y^* M^{-1} y = \frac{1}{\xi^2(0)-\xi^2(r)} (y_1, y_2) \begin{pmatrix} \xi(0)y_1 - \xi(r)y_2 \\ -\xi(r)y_1 + \xi(0)y_2 \end{pmatrix}$$

$$= \frac{1}{\xi^2(0)-\xi^2(r)} \left(\xi(0)y_1^2 + \xi(0)y_2^2 - 2\xi(r)y_1y_2\right)$$

Which leads to $P(y_1, y_2)$:

$$\boxed{P(y_1, y_2) = \frac{1}{\sqrt{2\pi}\,\left(\xi^2(0)-\xi^2(r)\right)^{1/2}} e^{-\frac{\xi(0)y_1^2 + \xi(0)y_2^2 - 2\xi(r)y_1y_2}{2\left(\xi^2(0)-\xi^2(r)\right)}}}$$

Power Spectrum, Odds and Ends

* correct definition:

$$\boxed{(2\pi)^3 P(k_1)\, \delta_D(\vec{k}_1-\vec{k}_2) = \langle \hat{f}(\vec{k}_1)\hat{f}^*(\vec{k}_2)\rangle}$$

* Often you find the following, not so accurate definition:

$$P(k) = \langle |\hat{f}(\vec{k})|^2 \rangle$$

notice though it reveals true meaning $P(k)$:

average contribution diverse wave numbers $\vec{k}$ to fluctuation field.

* from:

$$\mathcal{P}[f] \propto \exp-\left[\int \frac{d\vec{k}}{(2\pi)^3} \frac{|\hat{f}(\vec{k})|^2}{2P(k)}\right]$$

- Power spectrum specifies density field characteristics <u>completely</u> if Gaussian.
- Specifies relative contribution to density field from waves at different wavenumbers $\vec{k}$

$\Downarrow$

Determines character of density field evolution

$\Downarrow$

$\boxed{P(k) \text{ Holy Grail of COSMIC structure formation}}$

Justification for Gaussianity Assumption

① Physical Justification

Inflation

If fluctuations were generated during primordial inflationary phase, from quantum fluctuations that got inflated into macroscopic ripples, then field Gaussian.

② Mathematical Justification

Central Limit Theorem

$$\delta(\vec{x}, t) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{\delta}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

$\Uparrow$

density field superposition of large number of <u>independent vectors</u>.

$\Downarrow$

according to central limit theorem this implies directly: $\boxed{\text{Gaussianity}}$

The Random Field Power Spectrum

- 2-pt (auto)correlation function quantifies mutual dependence field at different locations:
- Look at **correlation function** $\xi(|\vec{x}_1 - \vec{x}_2|)$:

$$- \xi(\vec{x}_1 - \vec{x}_2) = \langle f(\vec{x}_1)\, f(\vec{x}_2) \rangle$$

$$- f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{f}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \quad \xi(\vec{x}_1 - \vec{x}_2) = \left\langle \int \frac{d\vec{k}_1}{(2\pi)^3}\, \hat{f}(\vec{k}_1)\, e^{-i\vec{k}_1\cdot\vec{x}_1} \int \frac{d\vec{k}_2}{(2\pi)^3}\, \hat{f}(\vec{k}_2)^*\, e^{i\vec{k}_2\cdot\vec{x}_2} \right\rangle$$

$$= \iint \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \langle \hat{f}(\vec{k}_1)\, \hat{f}^*(\vec{k}_2) \rangle\, e^{-i\vec{k}_1\cdot\vec{x}_1}\, e^{i\vec{k}_2\cdot\vec{x}_2}$$

$$\Rightarrow \quad \boxed{\xi(\vec{x}_1 - \vec{x}_2) = \int \frac{d\vec{k}}{(2\pi)^3}\, P(\vec{k})\, e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}$$

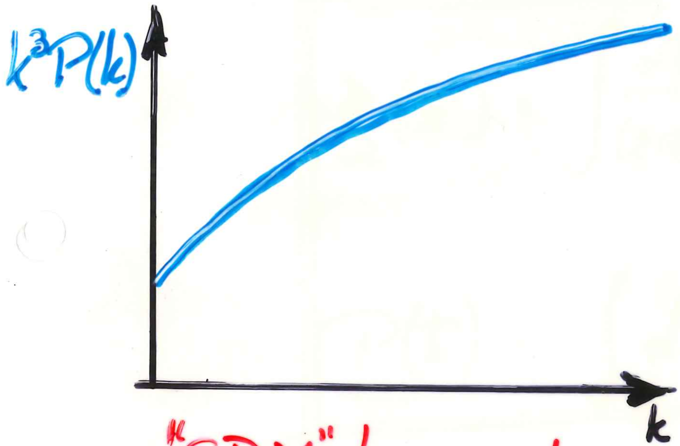
with **Power Spectrum $P(k)$**:

$$\boxed{(2\pi)^3 P(\vec{k}_1)\, \delta_D(\vec{k}_1 - \vec{k}_2) = \langle \hat{f}(\vec{k}_1)\, \hat{f}^*(\vec{k}_2) \rangle}$$

Power Spectrum $\Leftarrow \underset{\text{Transform}}{\text{Fourier}} \Rightarrow$ Correlation Function

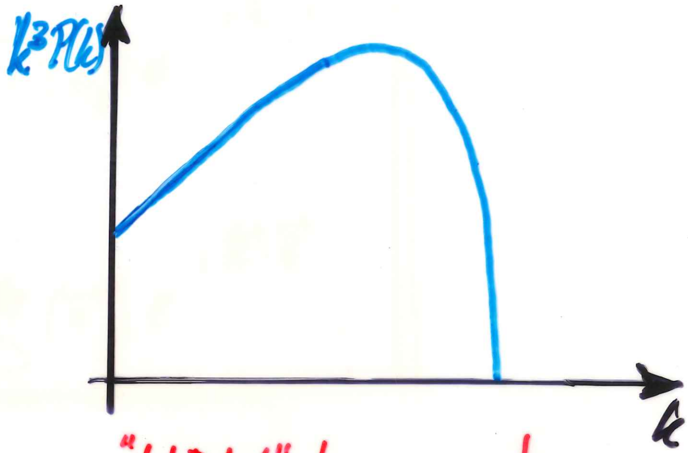
Density Fluctuation Spectrum $P(k)$: Evolutionary Implications

For comparison it is illuminating to assess the spatial structure in a field with a $P(k)$ continuously increasing as $k \to \infty$, versus a $P(k)$ with a cutoff:



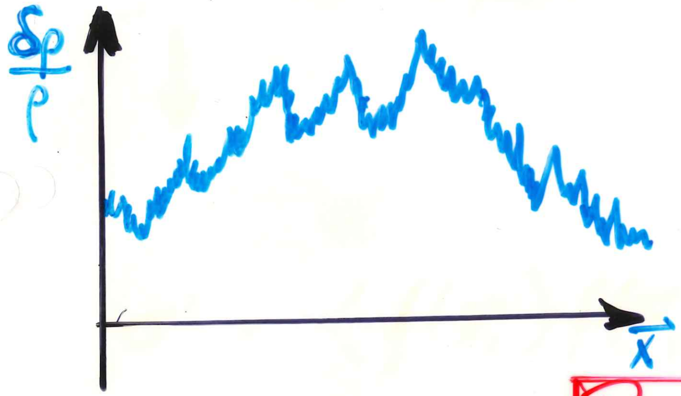
"CDM" type spectrum

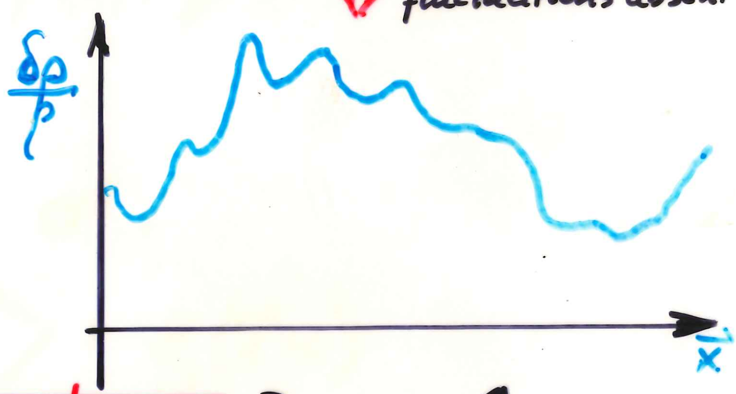
$\Downarrow$ dominant small-scale fluctuations



"HDM" type spectrum, with cutoff.

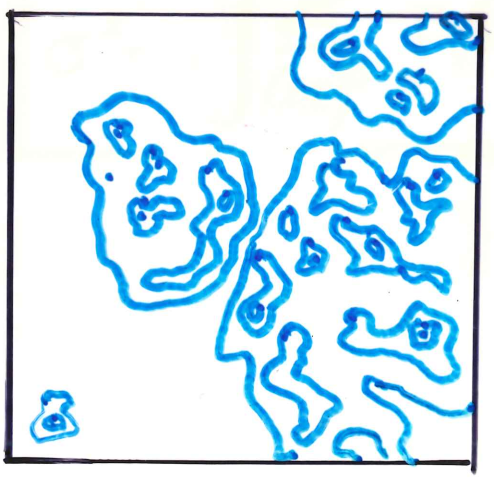
$\Downarrow$ small-scale fluctuations absent

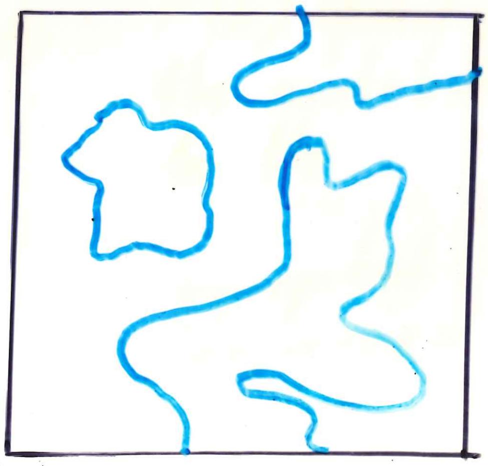




"linear" profiles.

$\boxed{\text{Realizations Density Field:}}$





Power spectrum, cont'd

Power spectrum:

$$\xi(\vec{x}) \overset{FT}{\longleftrightarrow} P(k)$$

- (auto)correlation function $\xi(\vec{x})$ and power spectrum $P(k)$ are each others Fourier transform.

$$\boxed{\begin{aligned} \xi(\vec{x}) &= \int \frac{d\vec{k}}{(2\pi)^3} P(k)\, e^{-i\vec{k}\cdot\vec{x}} \\ P(\vec{k}) &= \int d\vec{x}\, \xi(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} \end{aligned}}$$

- $\xi(\vec{x}) = \langle f(\vec{x}_1 + \vec{x})\, f(\vec{x}_1) \rangle$

$\downarrow$

Thus, density fluctuations σ^2:

$$\sigma^2 = \langle f(\vec{x}_1) f(\vec{x}_1) \rangle = \xi(\vec{x}=0)$$

$\Downarrow$

$$\boxed{\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k)} \Longrightarrow$$

$P(k)$ quantifies the contribution to the total perturbation σ^2 by spatial frequencies $\vec{k}$

Physical spectra:

- Inflation predicts Harrison-Zel'dovich spectrum:

$$P_{prim}(k) = A k$$

- Subsequently, physical processes affect growth of different waves k differently. Expressed in transfer function $T(k)$:

$$T^2(k,a) = \frac{P(k,a)}{P(k,a_i)} :$$

$$\boxed{P(k) = A k\, T^2(k)}$$

- For example:

① Cold Dark Matter:

$$P_{CDM}(k) = \frac{Ak}{(1 + 1.7q + 9.0q^{3/2} + 1.0q^2)^2} \quad ; \quad q = \frac{k}{(\Omega_{CDM} h^2\, \mathrm{Mpc}^{-1})}$$

② Hot Dark Matter:

$$P_\nu(k) = e^{\left[-0.16(kR_\nu) - \frac{(kR_\nu)^2}{2}\right]} \frac{1}{1 + 1.6q + (4.0q)^{3/2} + (0.92q)^2}$$

$$\left(q = \frac{k}{(\Omega_\nu h^2\, \mathrm{Mpc}^{-1})} \quad ; \quad R_\nu = \frac{2.6}{\Omega_\nu h^2}\, \mathrm{Mpc} \right)$$

- Other examples: $P_n(k) = A k^n$

Relations between $\hat{\delta}(\vec{k})$, $\hat{\vec{v}}(\vec{k})$ and $\hat{\phi}(\vec{k})$

In linear regime, equations of motion in Fourier space

(1) $$\frac{d\hat{\delta}(\vec{k})}{dt} - \frac{1}{a}\, i\vec{k}\cdot\hat{\vec{v}}(\vec{k}) = 0$$ continuity equation

(2) $$\frac{\hat{\phi}(\vec{k})}{a^2} = -4\pi G\bar{\rho}\,\frac{\hat{\delta}(\vec{k})}{k^2}$$ Poisson equation

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(1) $\Rightarrow$ $$\hat{\vec{v}}(\vec{k}) = -a\,\frac{i\vec{k}}{k^2}\,\frac{d\hat{\delta}(\vec{k})}{dt} = -Haf\,\frac{i\vec{k}}{k^2}\,\hat{\delta}(\vec{k})$$

$$\vec{v}(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\,\hat{\vec{v}}(\vec{k})\,e^{-i\vec{k}\cdot\vec{x}}$$

$$\boxed{\vec{v}(\vec{x},t) = \underbrace{\int \frac{d\vec{k}}{(2\pi)^3}\, -Haf\,\frac{i\vec{k}}{k^2}\,\underbrace{\hat{\delta}(\vec{k})}_{\text{Gaussian variable}}\,e^{-i\vec{k}\cdot\vec{x}}}_{\text{linear combination of Gaussians: remains a Gaussian}}}$$

$\longrightarrow$ $\vec{v}$ also Gaussian variable

$\Rightarrow$ Gaussian $\vec{v}$ completely specified by its spectrum $P_v(k)$:

$$P_v(k) \propto \langle |\hat{\vec{v}}(\vec{k})|^2 \rangle = (Haf)^2\,\frac{\langle|\hat{\delta}(\vec{k})|^2\rangle}{k^2}$$

$$\boxed{P_v(k) = (Haf)^2\,\frac{P(k)}{k^2}}$$

$\Rightarrow$ velocity power spectrum less dominated by high frequencies $\longrightarrow$ more smooth than $P(k)$

(2): $\Rightarrow$ $\hat{\phi}(\vec{k}) = -\frac{3}{2}\Omega H^2 a^2 \frac{1}{k^2}\hat{\delta}(\vec{k})$.

$$\boxed{\phi(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3} \; -\frac{3}{2}\Omega H^2 a^2 \frac{1}{k^2}\hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}}$$

$\rightarrow$ ϕ also Gaussian variable

$\Rightarrow$ with power spectrum $P_\phi(k)$:

$$\boxed{P_\phi(k) = \left(\frac{3}{2}\Omega H^2 a^2\right)^2 \frac{P(k)}{k^4}}$$

- Summarizing:

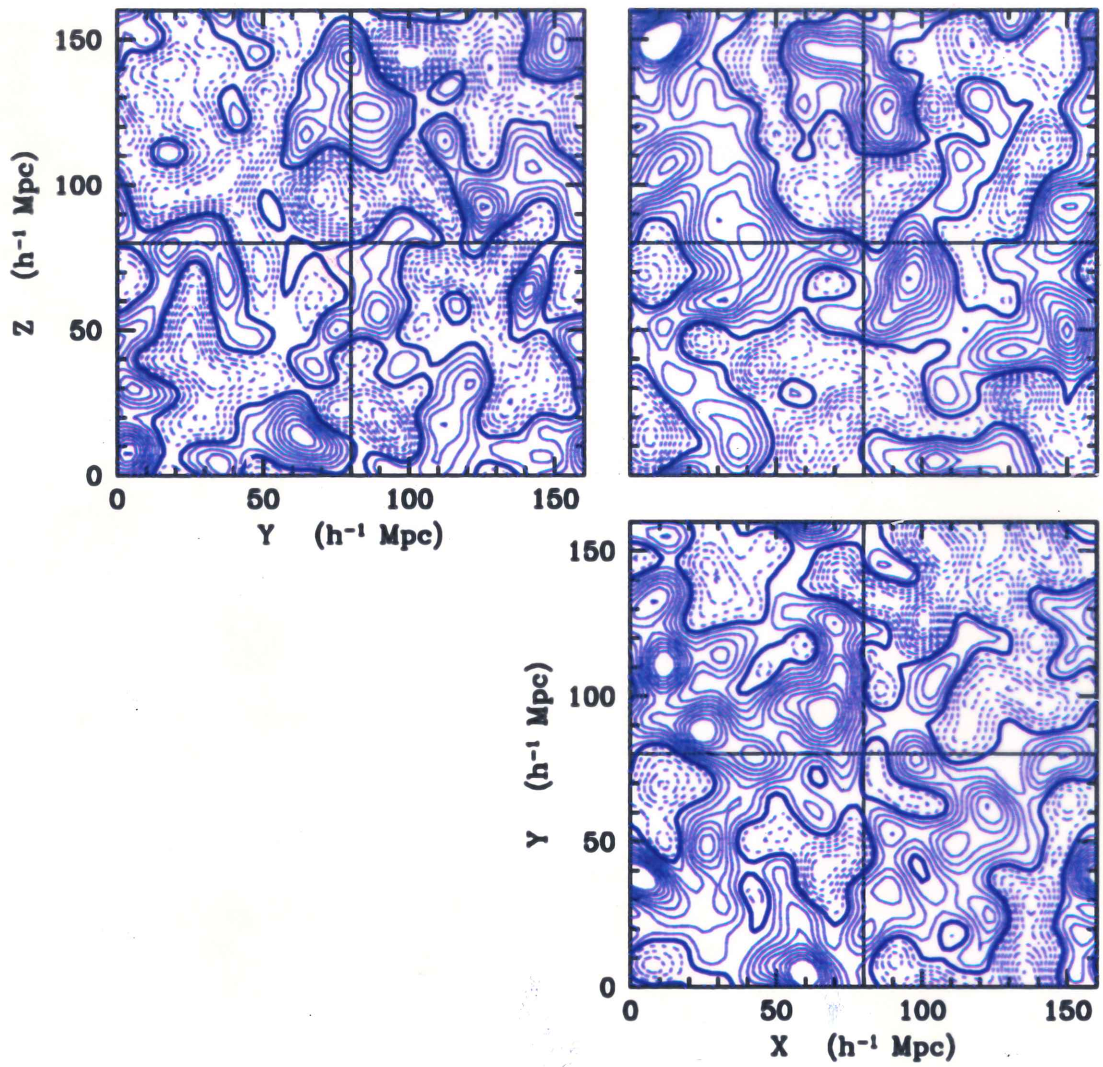
$$\boxed{P(k) \propto k^2 P_v(k) \propto k^4 P_\phi(k)}$$

- This implies:
  - density perturbation spectrum more sensitive to small-scale density field ($\lesssim 10-25\,h^{-1}$ Mpc)
  - velocity perturbation spectrum to medium-scale ($\sim 10-200\,h^{-1}$ Mpc): velocity fields
  - and potential spectrum $P_\phi(k)$ to large-scale fluctuations: MWB

Density Field

(contours)

File:		simulcr1/simulcr1.0		R_s:	5.00 h^{-1} Mpc			
a_{exp} = 1.00;		$[\Omega_0,H_0]$ =		[1.00, 100.00 km/s/Mpc]				
σ_8 = 0.00;		$[\Omega_s,H_s]$ =		[1.00, 100.00 km/s/Mpc]				
Slice:	Width:	160.00 h^{-1} Mpc	X_c =	80.00 h^{-1} Mpc;		T_x =	0.00 h^{-1} Mpc;	
	Thickness:	0.00 h^{-1} Mpc	Y_c =	80.00 h^{-1} Mpc;		T_y =	0.00 h^{-1} Mpc;	
			Z_c =	80.00 h^{-1} Mpc		T_z =	0.00 h^{-1} Mpc	
Field Unit:		$\rho_b = 3\,\Omega H^2/8\pi G$						
Contour Range:		[−0.767, 2.880] =						
		[0.100, 99.900]	percentile					
linear incr.:		0.182				# contours:	20	
reference level:		1.002						

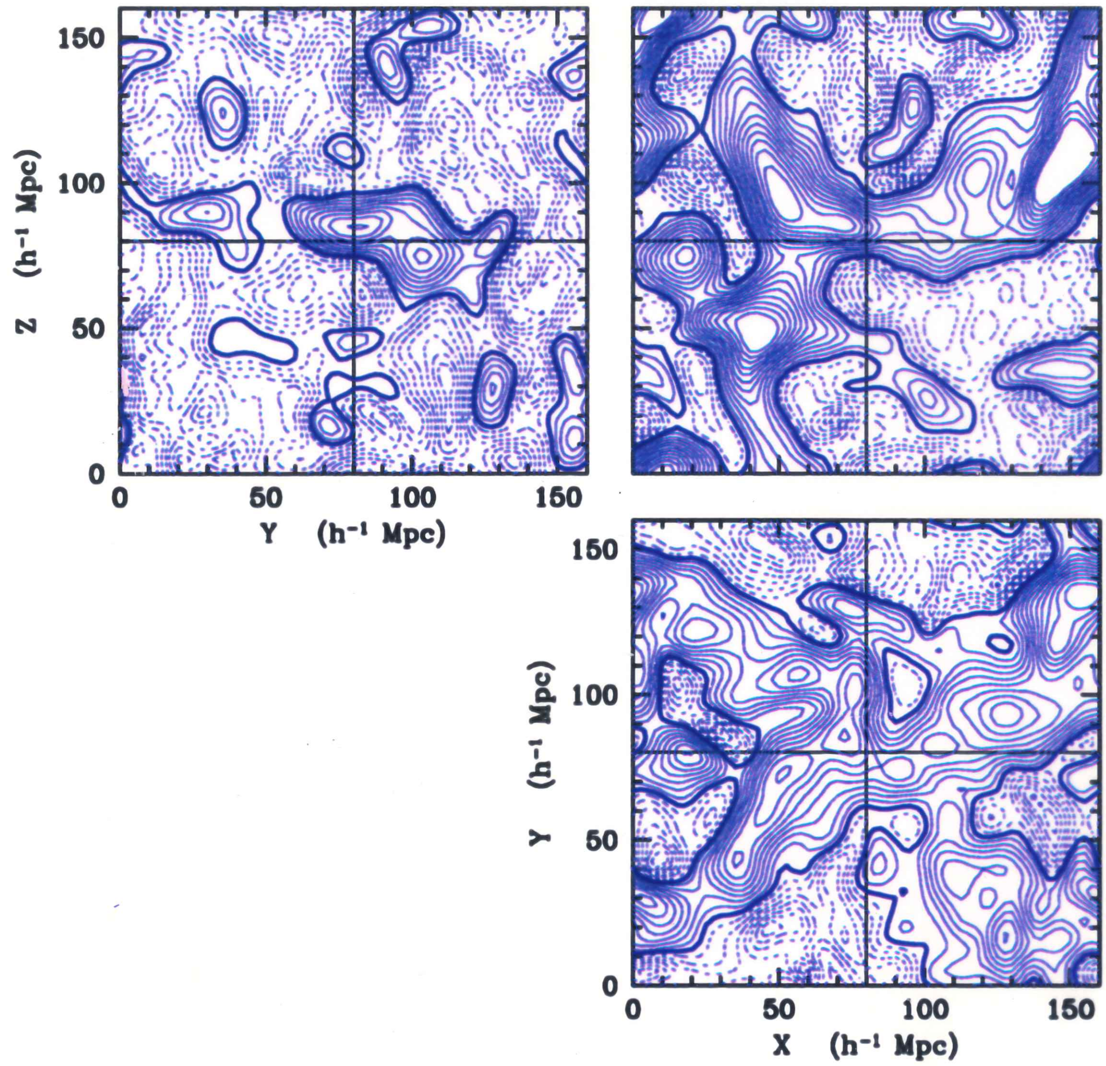


Gravitational Force (contours)

File:		simulcr1/simulcr1.0		R_G: 5.00 h^{-1} Mpc			
a_{exp} = 1.00;		$[\Omega_o, H_o]$ =		[1.00, 100.00 km/s/Mpc]			
σ_8 = 0.00;		$[\Omega_o, H_o]$ =		[1.00, 100.00 km/s/Mpc]			
Slice:	Width:	160.00 h^{-1} Mpc	X_c =	80.00 h^{-1} Mpc;	T_x =	0.00 h^{-1} Mpc;	
	Thickness:	0.00 h^{-1} Mpc	Y_c =	80.00 h^{-1} Mpc;	T_y =	0.00 h^{-1} Mpc;	
			Z_c =	80.00 h^{-1} Mpc	T_z =	0.00 h^{-1} Mpc	

Field Unit: km/s
Contour Range: [42.803,1044.760] =
[0.100, 99.620] percentile

linear incr.: 50.098 # contours: 20
reference level: 439.812



Gravitational Force (vectors)

File: simulcr1/simulcr1.0 R_g: 5.00 h^{-1} Mpc

a_{exp} = 1.00; $[\Omega_o, H_o]$ = [1.00, 100.00 km/s/Mpc]

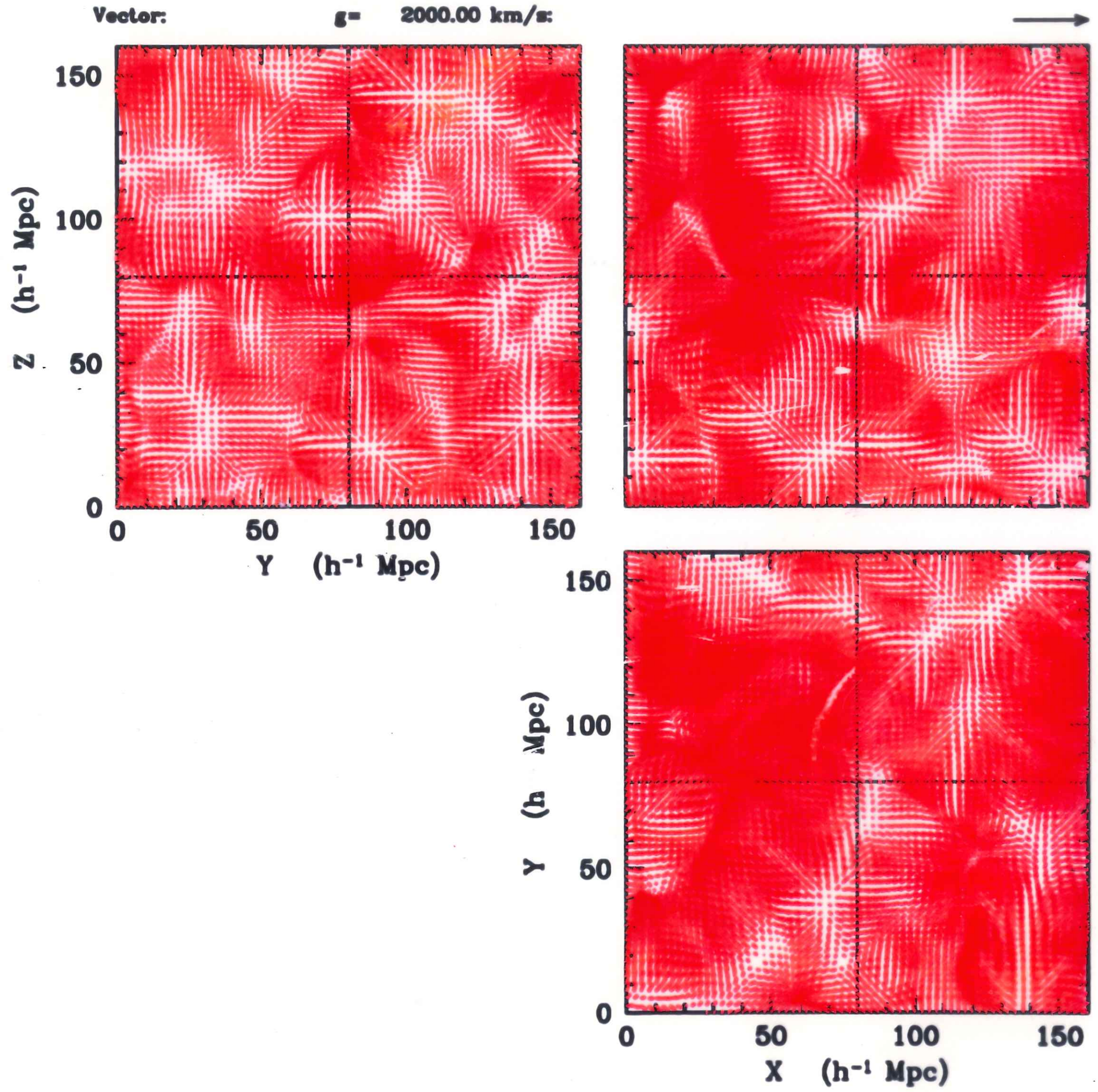
σ_8 = 0.00; $[\Omega_o, H_o]$ = [1.00, 100.00 km/s/Mpc]

Slice: Width: 160.00 h^{-1} Mpc X_c = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;

Thickness: 0.00 h^{-1} Mpc Y_c = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;

Z_c = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc

Vector: g= 2000.00 km/s:

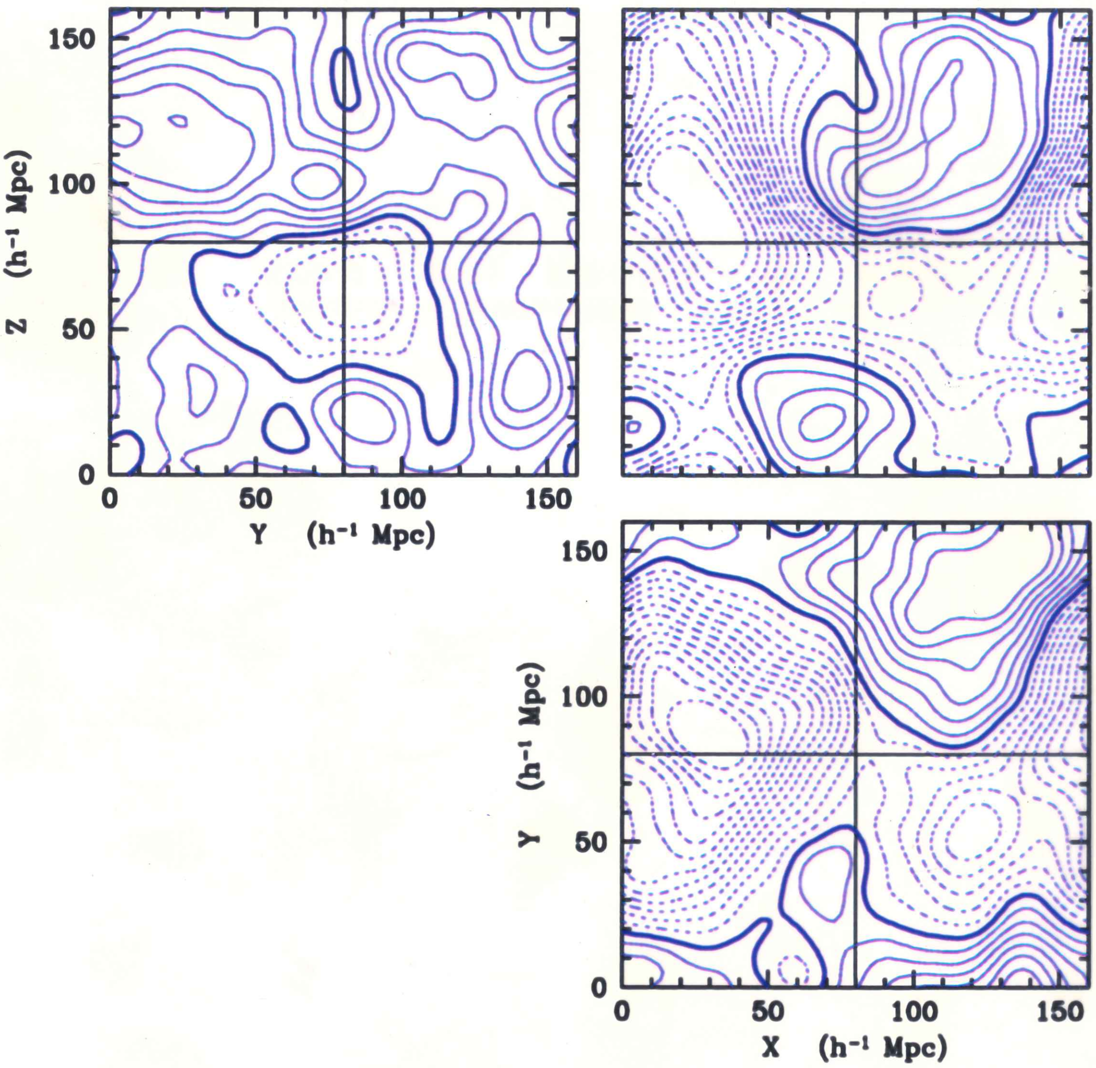


Potential Field

(contours)

File:		simulcr1/simulcr1.0		R_G:	5.00 h^{-1} Mpc
a_{exp} = 1.00;		$[\Omega_o, H_o]$ =	[1.00, 100.00 km/s/Mpc]		
σ_8 = 0.00;		$[\Omega_o, H_o]$ =	[1.00, 100.00 km/s/Mpc]		
Slice:	Width: 160.00 h^{-1} Mpc		X_c = 80.00 h^{-1} Mpc;		T_x = 0.00 h^{-1} Mpc;
	Thickness: 0.00 h^{-1} Mpc		Y_c = 80.00 h^{-1} Mpc;		T_y = 0.00 h^{-1} Mpc;
			Z_c = 80.00 h^{-1} Mpc		T_z = 0.00 h^{-1} Mpc

Field Unit:	1.5 ΩH^2			
Contour Range:	[−251.995, 143.336] =			
	[0.100, 99.900]	percentile		
linear incr.:	19.767		# contours:	20
reference level:	0.000			



Power Spectra: Cosmic Physics

- Inflation predicts a

$$\boxed{\text{Harrison-Zel'dovich Spectrum.}}$$

$$\Longrightarrow \boxed{P_{prim}(k) = A k.}$$

- <u>Harrison-Zel'dovich Spectrum:</u>

  <u>"constant curvature spectrum"</u>

$$\sigma_\phi^2 = \int \frac{d\vec{k}}{(2\pi)^3} P_\phi(k)$$

$$= \int \frac{k^2 dk}{2\pi^2} \frac{P(k)}{k^4} \qquad \Longleftarrow \quad P(k) \propto k^n$$

$$\propto A \int \frac{d \log k}{2\pi^2} \frac{k^n}{k}$$

$$\propto A \int \frac{d \log k}{2\pi^2} k^{n-1}$$

$\Longrightarrow$ $\boxed{\text{if} \quad n=1}$: contributions to potential perturbations "scale-free", all equal per logarithmic bin.

• Power-law Power Spectra

$$\boxed{P(k) \propto A k^n}$$

$$n(k) = \frac{d \log P}{d \log k} :$$

① $n > -3$

hierarchical clustering.

② $n \leq 1$

If $n > 1$ then ultraviolet divergences: small-scale perturbations diverge.

• Post-horizon Power Spectrum

After a perturbation with wavenumber k enters the horizon:

- it can start growing under force of gravity.
- physical processes start to effect growth of primordial perturbations and modify it.

• baryons } pressure :
• photons } Jeans damping & oscillation.
• Silk damping : photon free-streaming
• dark matter : stagspansion (logarithmic growth rad. dom.)

- dark matter : free streaming

$\Downarrow$

- free streaming length: $\ell \propto \frac{1}{n_s \sigma_s}$

n_s : density DM species

σ_s : cross-section

$\Downarrow$

- all waves with:

$$k \gtrsim k_s \approx \frac{1}{\ell} \propto n_s \sigma_s$$

- damped.

$\Longrightarrow$ signatures in spectrum of

$$k_s \simeq n_s \sigma_s \propto \Omega_s h^2$$

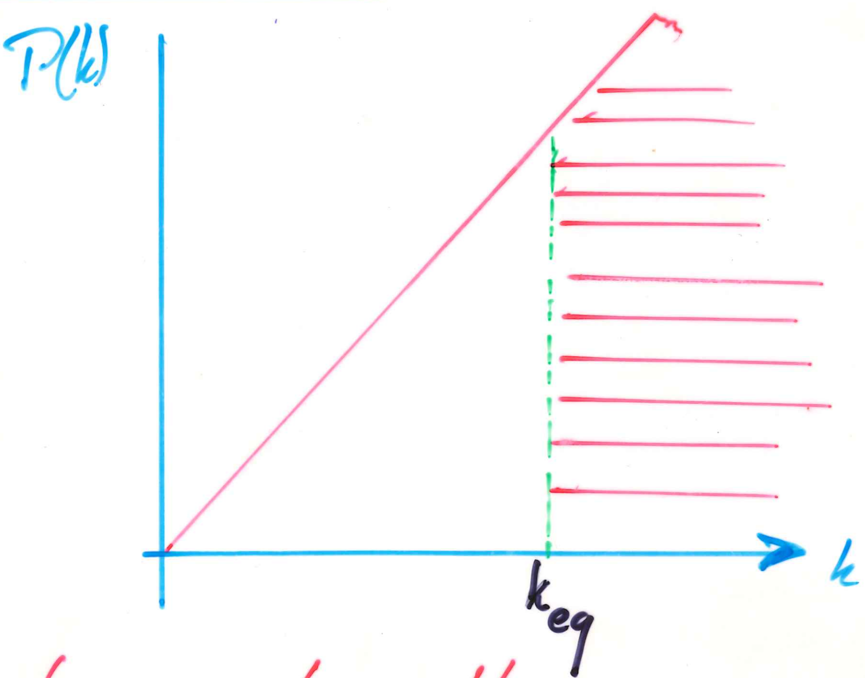
$\Downarrow$

$$\frac{k}{\Omega_s h^2} \approx \frac{1}{\Omega_s h} h^{-1} k \equiv \frac{k}{\Gamma_s} \qquad k : [h^{-1} \mathrm{Mpc}]$$

$$q \equiv \frac{k}{\Gamma_s}$$

- **Recall:**
  - $g_{00} = g^{0}_{00} + h_{00} \quad \propto \frac{\Phi}{c^2}$
  - $h_{00} \propto \frac{\phi}{c^2} \implies$ equal contributions to potential give equal contributions to curvature $h_{\mu\nu}$.

- **Harrison-Zel'dovich**



- fluctuations enter with same amplitude the horizon, then (after $t > t_{eq}$) start growing with $D(t) \propto a(t)$ $(\Omega \approx 1)$:

$$\hat{\delta}(k,t) \propto \frac{D(t)}{D(t_{hor})} \hat{\delta}(k, t_{hor})$$

$$\Downarrow$$

$$\hat{\delta}(k,t) \propto k$$

Identifying Protostructures in a Primordial Density Field.

Filtering Stochastic Density Fields.

Power Spectrum $P(k)$

$$\boxed{\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k)}$$

$\Downarrow$ isotropic random field.

- $\sigma^2 = \int \frac{k^2 dk}{2\pi^2} P(k) = \int \frac{d \log k}{2\pi^2} k^3 P(k)$

$\Downarrow$

- $\boxed{\underline{k^3 P(k)} \text{ is power per logarithmic frequency band } d \log k}$

To appreciate the meaning of the power spectrum in terms of physical structures, it is better to discuss the fluctuations in terms of a <u>physical scale R</u> or <u>physical mass scale M</u>.

- Recall that σ^2 per se does not provide information on contributions by various scales
- While σ^2 may even diverge, if the integral is infinite:
- To discuss density fluctuations, CRUCIAL to specify SCALE !

Fluctuation Mass Scales

* In order to identify (embryonic) structures in a primordial cosmological density field we look at the field filtered on scale R:

$$\boxed{\delta_f(\vec{r}) = \int d\vec{x}\, \delta(\vec{x})\, W_f(\vec{x},\vec{r};R)}$$

(see accompanying page on 'filtering random field'.

* The corresponding Mass Scale $\overline{M}_f$

$$\boxed{\overline{M}_f = \int d\vec{x}\, \bar{\rho}\, W_f(\vec{x},R)}$$

For example, for a regular Tophat Filter W_{TH}, we deal with a straight "mean" mass $\overline{M}_f$:

$$\boxed{\overline{M}_f = \frac{4\pi}{3}\bar{\rho} R^3}$$

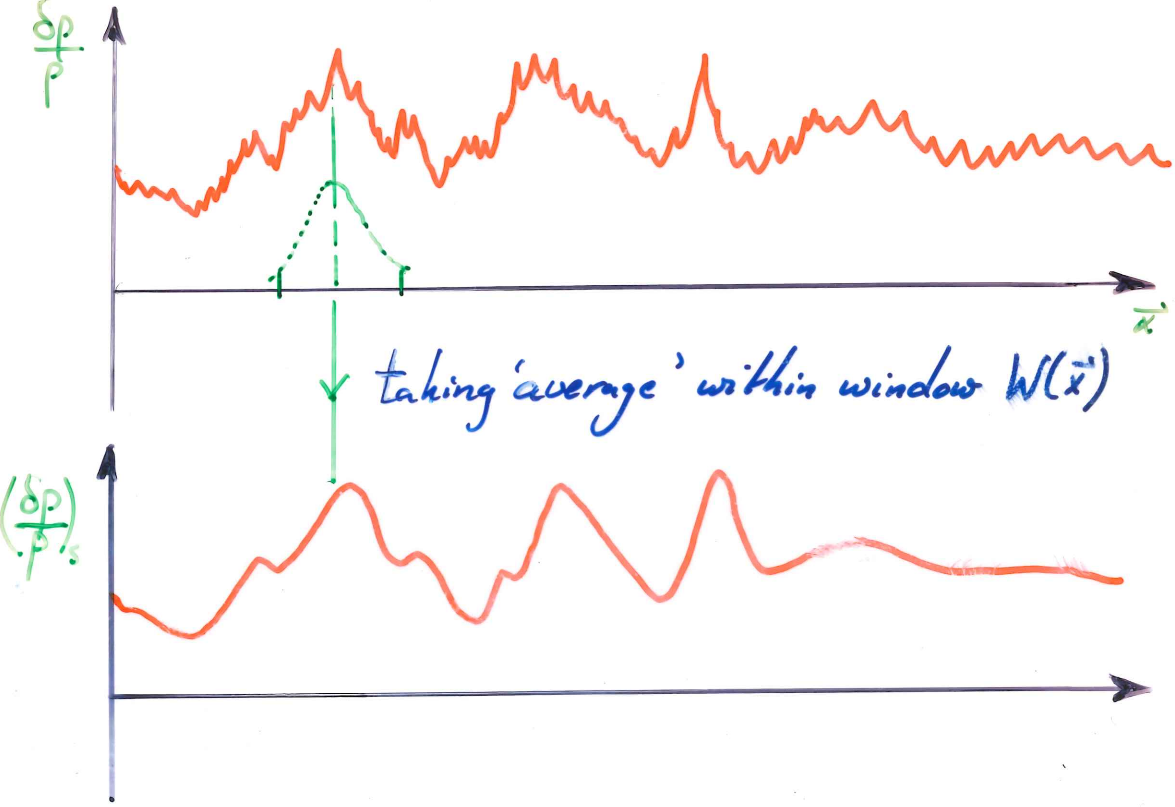
* Field fluctuations on mass scale M_f are then:

$$\sigma_M^2 = \frac{\langle (M_f - \overline{M}_f)^2 \rangle}{\overline{M}_f^{\,2}} = \frac{\langle \{ \int d\vec{x}\, \delta(\vec{x}) W_f(\vec{x},R) \}^2 \rangle}{\{ \int d\vec{x}\, W_f(\vec{x},R) \}^2}$$

so that for a normalized filter W_f: $\int d\vec{x}\, W_f = 1$,

$$\boxed{\sigma_M^2 = \int \frac{d\log k}{2\pi^2}\, k^3 P(k)\, \hat{W}_f(kR_f)}$$

Filtering of random field



- $$\boxed{f_G(\vec{x}) = \int d\vec{y}\, f(\vec{y})\, W_G(\vec{y}, \vec{x})}$$

$W_G(\vec{y}, \vec{x})$: filter function

- How does this translate into Fourier space:

$$f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{f}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$$

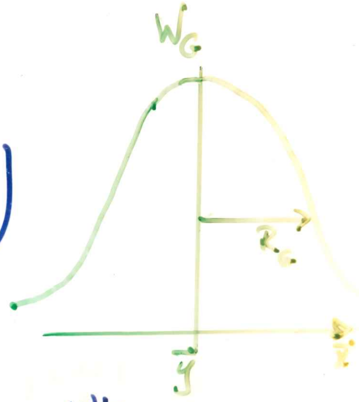
$$W_G(\vec{y}, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{W}(\vec{k})\, e^{-i\vec{k}\cdot(\vec{y}-\vec{x})}$$

$$\Rightarrow \boxed{f_G(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{f}(\vec{k})\, \hat{W}^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}}$$

Two most important filters:

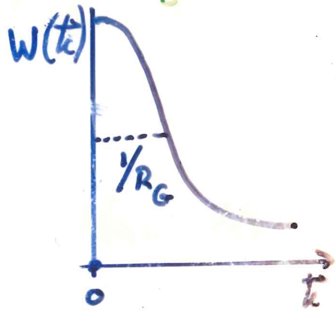
- **Gaussian filter:**

$$W_G(\vec{y},\vec{x}) = \frac{1}{(2\pi R_G^2)^{3/2}} \exp\left(-\frac{|\vec{y}-\vec{x}|^2}{2R_G^2}\right)$$



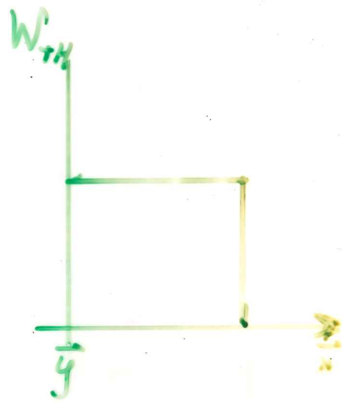
R_G: filter scale.

$$\Rightarrow \quad \boxed{\hat{W}(\vec{k}) = e^{-\frac{k^2 R_G^2}{2}}}$$



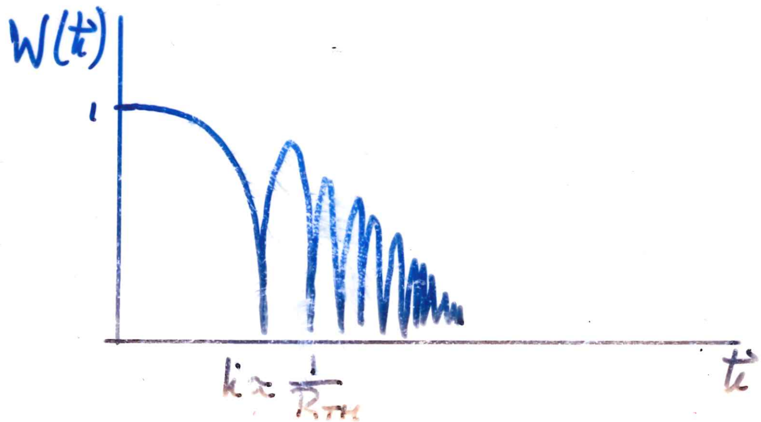
- **Tophat filter:**

$$W_{TH}(\vec{y},\vec{x}) = \begin{cases} \frac{1}{\frac{4\pi}{3} R_{TH}^3} & |\vec{y}-\vec{x}| \leq R_{TH} \\ 0 & \text{else} \end{cases}$$



R_{TH}: filter scale.

$$W(\vec{k}, R_{TH}) = \frac{3\left[\sin(kR_{TH}) - (kR_{TH})\cos(kR_{TH})\right]}{(kR_{TH})^3}$$



fluctuations, mass scale M, cont'd:

$$\boxed{\sigma_M^2 = \int \frac{dk}{2\pi^2} k^2 P(k) \hat{W}^2(kR_f)}$$

- To understand the ramifications of a particular power spectrum for the scale-dependence of mass fluctuations, evaluate a

① Power-law power spectrum: $P(k) = Ak^n$

② Gaussian filter $W_G(kR)$

(the Gaussian filter behaves distinctly better than the tophat filter:)

$$W_G(kR_G) \to 0 \qquad k < \frac{1}{R_G}$$

- $$\sigma_M^2 = \int \frac{dk}{2\pi^2} k^2 P(k) e^{-k^2 R_G}$$

$$\Rightarrow \quad \sigma_M^2 = A \int \frac{dk}{2\pi^2} k^{(2+n)} e^{-k^2 R_G}$$

$$\Longrightarrow \quad \boxed{\sigma_M^2 = \frac{A}{2\pi^2} \frac{1}{n+3} \Gamma\left(\frac{n+3}{2}\right) R_G^{-(n+3)}}$$

- $M_G = 4.37 \times 10^{12}\, \Omega_0 h^{-1} R_G^3\, M_\odot$

$R_G \propto M_G^{1/3}$

$$\Longrightarrow \quad \boxed{\sigma_M^2 \propto M_G^{-\frac{n+3}{3}}}$$

fluctuations, mass scale M, cont'd:

$$\boxed{\sigma_M \propto M^{-\frac{n+3}{6}}} \equiv M^{-\alpha_M}$$

- The exponent $\alpha = \frac{3+n}{6}$: Mass index

- Poisson distribution :

  random distribution N mass particles in volume V:

  $$\sigma_M = \frac{\delta N}{N} \simeq N^{-1/2} \propto M^{-1/2} \Longrightarrow n=0$$

  Spectral index $n=0$: white noise

- Hierarchical clustering : $\boxed{n > -3}$

  - As long as $n > -3$:

    $$\sigma_{M_1} < \sigma_{M_2} \quad \text{for} \quad M_1 > M_2$$

    Fluctuations LARGER for SMALLER scales:
    structure on smaller scales will grow faster,
    condense out earlier than on large scales:
    Hierarchical Clustering

  - In principle this may be viewed on all scales for any spectrum $P(k)$:

    $$\boxed{n(k) \equiv \frac{d\log P}{d\log k}} \Longrightarrow n(k) > -3$$

    Hierarchical Clustering.

Mass scale M_f and Filter scale R_f.

$$\boxed{M_f = \int d\vec{x}\; W_f(\vec{x}, R_f)\, \bar{\rho}}$$

$$\Downarrow$$

① **Tophat Filter**:

$$\boxed{W_{TH} = \begin{cases} 0 & x > R_{TH} \\ \dfrac{1}{\frac{4\pi}{3} R_{TH}^3} & x \leq R_{TH} \end{cases}}$$

$$\Rightarrow \quad M_{TH} = \frac{4\pi}{3} \bar{\rho} R_{TH}^3$$

$$\bar{\rho} = \Omega_0 \rho_{crit} = \Omega_0 \cdot \frac{3H^2}{8\pi G} = \Omega_0 h^2 \; 1.8791 \cdot 10^{-29} \, g/cm^3$$

$$= 2.78 \times 10^{11} \, \Omega_0 h^2 \; M_\odot / Mpc^3$$

$$\Longrightarrow \quad \boxed{M_{TH} = 1.16 \times 10^{12} \, \Omega_0 h^2 \, R_{TH}^3 \; M_\odot} \quad (R_{TH} : h^{-1} Mpc)$$

② **Gaussian Filter**:

$$\boxed{W_G = \frac{1}{(2\pi R_G^2)^{3/2}} \, e^{-\frac{x^2}{2R_G^2}}}$$

$$\Rightarrow \quad M_G = (2\pi)^{3/2} \bar{\rho} R_G^3$$

$$\boxed{M_G = 4.37 \times 10^{12} \, \Omega_0 h^2 R_G^3 \; M_\odot} \quad (R_G : h^{-1} Mpc)$$

fluctuations, mass scales M, cont'd:

- **Mass scales & filter scales:**

	Ω_0	h	R_G (h^{-1} Mpc)
	1.0	0.5	0.97
"Galaxy": $M \approx 10^{12} M_\odot$	0.3	0.5	1.45
	0.1	0.5	2.09
	1.0	0.5	4.5
"Cluster": $M \approx 10^{14} M_\odot$	0.3	0.5	6.7
	0.1	0.5	9.7

Mass scales, cont'd:

- **Filter-defined mass scales:**

Notice that these are artificial, user-defined mass scales, which are used to identify (a posteriori) a filter scale in the primordial density field.
Because of our lack of full understanding how particular fully evolved nonlinear objects emerge out of the (linear) pristine Universe, and how they relate to particular perturbations in the primordial Universe, the use of filter-defined mass scales is a useful approximation. However,

- does not take into account the presence of characteristic mass-scales in spectrum
- ignores a lot of the detailed physics of nonlinear structure formation:
- assumes that small-scale internal structure does not really affect the emerging object.
- But works rather good for a lot of approximate considerations...

- **Natural Filters**

Of considerable more fundamental importance is the question of which Characteristic Mass Scales are imprinted by the pre-recombination Universe onto the various fluctuations:

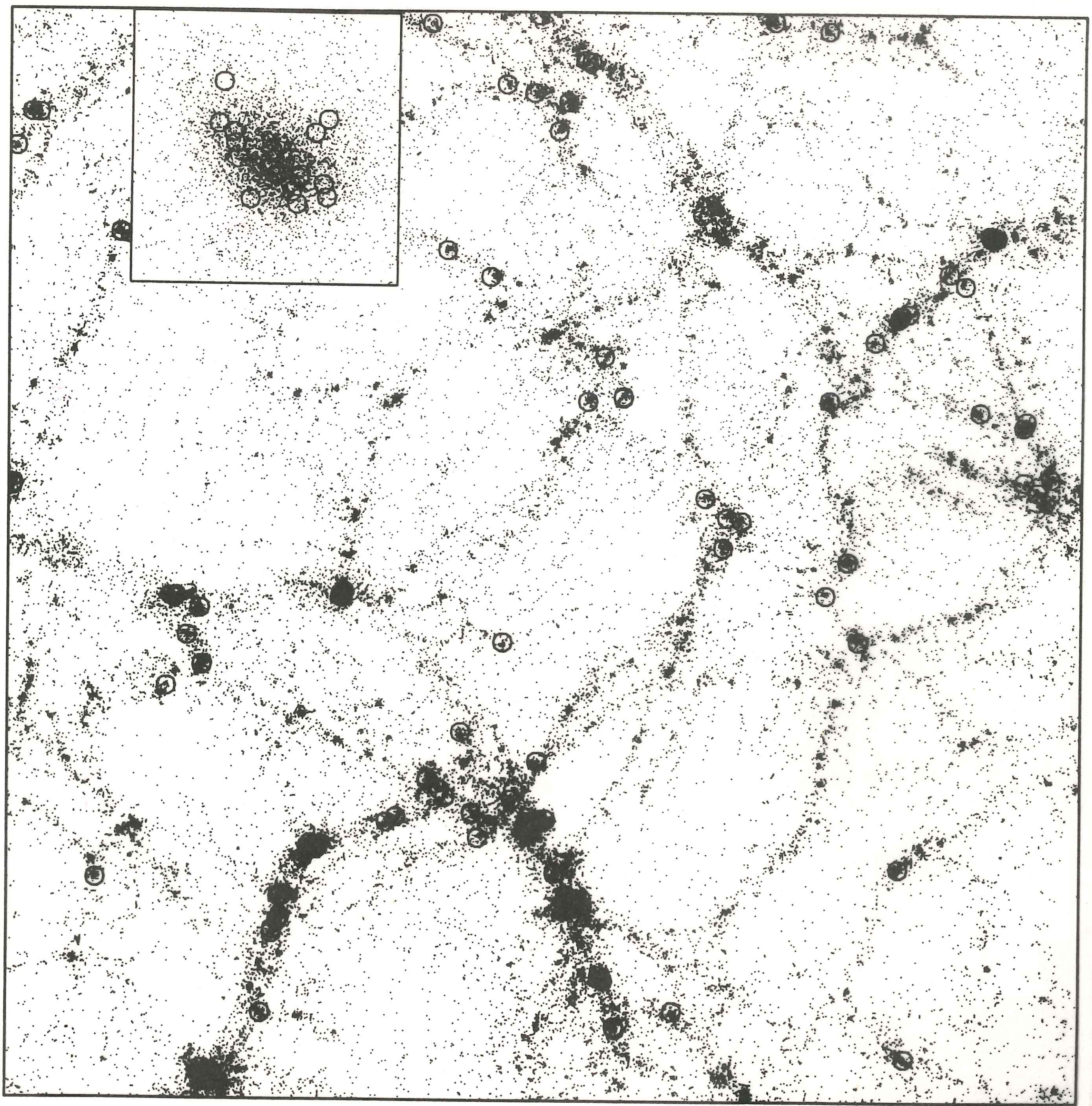
Which "Natural Filters" can we identify?

$\Longrightarrow$ brings us to a variety of Primordial Physical Processes $\rightarrow$ Horizon Mass, Jeans Mass, Silk damping mass

Peak Statistics

* Crucial result by Kaiser (1984):

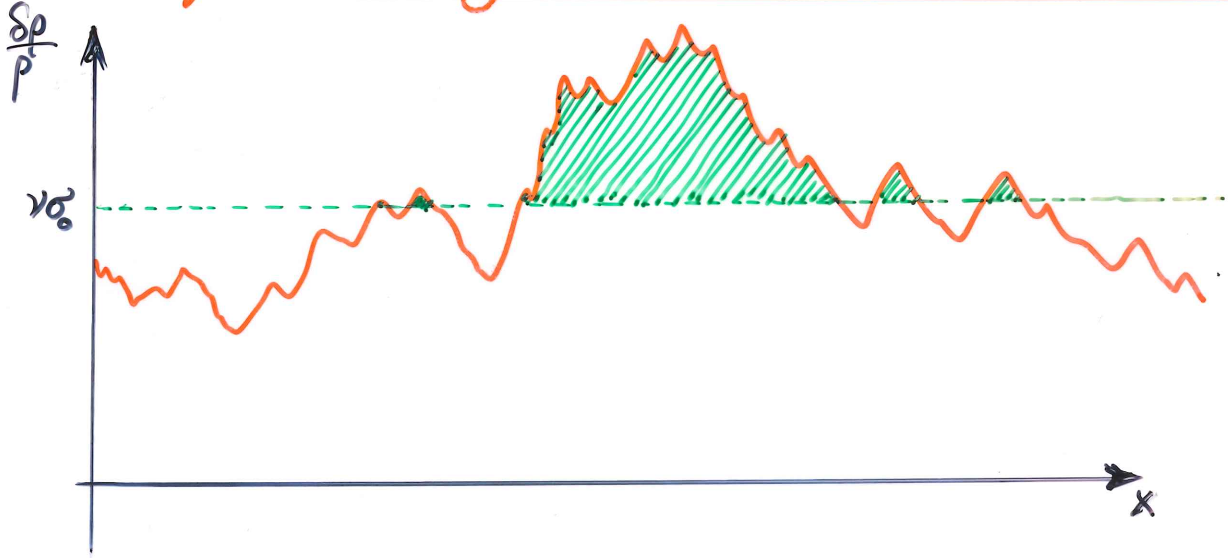
Peaks have enhanced clustering with respect to background fluctuation field



Simulation by C. Park of CDM $\Omega = 1$ Universe, with $\delta > 2.5\,\sigma_0\,(0.6\,h^{-1}\,\mathrm{Mpc})$ peaks indicated.

* Why do peaks in Gaussian field cluster stronger?

- Nice and insightful application of Gaussian random field theory:



- Calculate correlation function $\xi_{>\nu}(r)$ of matter with density $\delta > \nu\sigma_0$

- Kaiser, N. ; Ap. J. 284, L9-L12, 1 Sept. 1984.

- Gaussian random field:

$$\boxed{P(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}}} \qquad \xi(0) = \sigma^2$$

- Probability P_1 of a randomly chosen point to have $\delta > \nu\sigma$:

$$P_1 = \int_{\nu\sigma}^{\infty} P(y)\, dy$$

- Probability P_2 of 2 points having $\delta_1 > \nu\sigma$ and $\delta_2 > \nu\sigma$:

$$P_2 = \int_{\nu\sigma}^{\infty}\int_{\nu\sigma}^{\infty} P(y_1, y_2)\, dy_1\, dy_2$$

· with :

$$\boxed{P(y_1, y_2)\, dy_1\, dy_2 = \frac{e^{-(y_1\ y_2)\, M^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} / 2}}{(2\pi)\,(\det M)^{1/2}}\, dy_1\, dy_2}$$

· in which :

$$M_{ij} = \begin{pmatrix} \xi(0) & \xi(r_{ij}) \\ \xi(r_{ij}) & \xi(0) \end{pmatrix}$$

$$\Rightarrow \det M = \xi^2(0) - \xi^2(r)$$

$$\Rightarrow M^{-1} = \frac{1}{\xi^2(0) - \xi^2(r)} \begin{pmatrix} \xi(0) & -\xi(r) \\ -\xi(r) & \xi(0) \end{pmatrix}$$

$$\Rightarrow y^* M^{-1} y = \frac{1}{\xi^2(0) - \xi^2(r)} (y_1\ \ y_2) \begin{pmatrix} \xi(0)\, y_1 - \xi(r)\, y_2 \\ -\xi(r)\, y_1 + \xi(0)\, y_2 \end{pmatrix}$$

$$= \frac{1}{\xi^2(0) - \xi^2(r)} \left(\xi(0)\, y_1^2 + \xi(0)\, y_2^2 - 2\xi(r)\, y_1 y_2 \right)$$

* So that,

$$\boxed{P(y_1, y_2) = \frac{1}{\sqrt{2\pi}\, (\xi^2(0) - \xi^2(r))^{1/2}}\, e^{-\frac{\xi(0)\, y_1^2 + \xi(0)\, y_2^2 - 2\xi(r)\, y_1 y_2}{2(\xi^2(0) - \xi^2(r))}}}$$

Correlation function for points with $\delta > \nu\sigma$:

$$\boxed{P_2 = P_1^2 \left(1 + \xi_{>\nu}(r)\right)}$$

⇐ definition
see Peebles 1980

* i.e.

$$1 + \xi_\nu(r) = \frac{\displaystyle\iint_{\nu\sigma}^{\infty} P(y_1, y_2)\, dy_1\, dy_2}{\left(\displaystyle\int_{\nu\sigma}^{\infty} P(y)\, dy\right)^2}$$

* **Use the following results:**

(1.) $$\int_{\nu\sigma}^{\infty} P(y)\, dy = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\nu\sigma}^{\infty} e^{-y^2/2\sigma^2}\, dy = \frac{1}{2}\, \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

(2.) $$\iint_{\nu\sigma}^{\infty} P(y_1, y_2)\, dy_1\, dy_2 = \frac{1}{2\pi} \iint_{\nu\sigma}^{\infty} \frac{1}{\xi(0)\left(1 - \frac{\xi^2(r)}{\xi^2(0)}\right)^{1/2}}\, e^{-\frac{y_1^2 + y_2^2 - 2\frac{\xi(r)}{\xi(0)} y_1 y_2}{2\xi(0)\left[1 - \xi^2(r)/\xi^2(0)\right]}}\, dy_1\, dy_2$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{\nu}^{\infty} dy\, e^{-\frac{1}{2}y^2}\, \mathrm{erfc}\left[\frac{\nu - y\,\xi(r)/\xi(0)}{\left\{2\left[1 - \xi^2(r)/\xi^2(0)\right]\right\}^{1/2}}\right]$$

* Then :

$$\boxed{1 + \xi_{>\nu}(r) = \sqrt{\frac{2}{\pi}} \left(\mathrm{erfc}\, \frac{\nu}{\sqrt{2}}\right)^{-2} \int_{\nu}^{\infty} dy\, e^{-\frac{y^2}{2}}\, \mathrm{erfc}\left[\frac{\nu - y\,\xi(r)/\xi(0)}{\left\{2\left[1 - \xi^2(r)/\xi^2(0)\right]\right\}^{1/2}}\right]}$$

• Use the fact that $\xi(r) \ll \xi(0)$:

$$\operatorname{erfc}\left\{\frac{\nu - y\,\xi(r)/\xi(0)}{\left\{2\left[1-\xi^2(r)/\xi^2(0)\right]\right\}^{1/2}}\right\} \approx \operatorname{erfc}\left(\frac{\nu - y\,\xi(r)/\xi(0)}{\sqrt{2}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\int_\nu^\infty e^{-\frac{\left(x - y\,\xi(r)/\xi(0)\right)^2}{2}}\,dx$$

• from which it follows that:

$$1+\xi_{>\nu}(r) = \frac{\iint_\nu^\infty dx\,dy\; e^{-\frac{x^2 - 2xy\frac{\xi(r)}{\xi(0)} + y^2}{2}}}{\left(\int_\nu^\infty dy\; e^{-\frac{y^2}{2}}\right)^2}$$

using: $1+\frac{\xi^2(r)}{\xi^2(0)} \approx 1$.

• ⇒

$$\boxed{\xi_{>\nu}(r) = \frac{\iint_\nu^\infty dx\,dy\; e^{-\frac{x^2+y^2}{2}}\left(e^{xy\frac{\xi(r)}{\xi(0)}} - 1\right)}{\left(\int e^{-\frac{1}{2}y^2}\,dy\right)^2}}$$

• $x^2+y^2 \geq 2xy \gg 2xy\,\frac{\xi(r)}{\xi(0)}$

⇒ in regime where $xy \gg 1$: $x^2+y^2 > xy \gg 1$.

$e^{-\frac{x^2+y^2}{2}} \downarrow 0$ faster than $e^{2xy\frac{\xi(r)}{\xi(0)}} \uparrow \infty$

⇒ integral contributed mainly in regime where $xy \lesssim 1$:

$$e^{xy\frac{\xi(r)}{\xi(0)}} - 1 \approx xy\,\frac{\xi(r)}{\xi(0)}.$$

$$\Rightarrow \iint_\nu^\infty dx\,dy\; e^{-\frac{x^2+y^2}{2}}\left(e^{xy\,\xi(r)/\xi(0)}-1\right)$$

$$\approx \iint_\nu^\infty dx\,dy\,\frac{\xi(r)}{\xi(0)}\,xy\,e^{-\frac{x^2+y^2}{2}} = \frac{\xi(r)}{\xi(0)}\left(\int_\nu dx\,x e^{-\frac{x^2}{2}}\right)^2 = \frac{\xi(r)}{\xi(0)}\left(e^{-\frac{\nu^2}{2}}\right)^2$$

- $$\Rightarrow \boxed{\xi_{>\nu}(r) = \frac{\xi(r)}{\sigma^2}\left(e^{\frac{\nu^2}{2}}\int_\nu^\infty e^{-\frac{y^2}{2}}dy\right)^{-2}}$$

- Take limit $\nu \gg 1$:

$$\int_\nu^\infty e^{-\frac{y^2}{2}}dy = \frac{e^{-\nu^2/2}}{\nu} - \underbrace{\int_\nu^\infty \frac{e^{-y^2/2}}{y^2}dy}_{<\frac{1}{\nu^3}e^{-\frac{\nu^2}{2}} \ll \frac{1}{\nu}e^{-\frac{\nu^2}{2}} \text{ if } \nu\gg 1} \approx \frac{e^{-\nu^2/2}}{\nu} \quad \text{if } \nu \gg 1.$$

- $$\Rightarrow \boxed{\xi_{>\nu}(r) \approx \left(\frac{\nu^2}{\sigma^2}\right)\xi(r)}$$