Today, we are going to practice our understanding of (stochastic) random fields, and the description of random fields. To this end, we will explore extensively the Fourier description of fields $f(\mathbf{x})$ in terms of their Fourier components $\hat{f}(\mathbf{k})$,

$$f(\mathbf{x}) \;=\; \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \hat{f}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{1}$$

Generals

The statistical nature of a random field $f(\mathbf{x})$ is defined by its set of N-point joint probabilities. For a Gaussian random field, this takes the simple form:

$$\mathcal{P}_N \;=\; \frac{\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} f_i\,(\mathsf{M}^{-1})_{ij}\,f_j\right]}{\left[(2\pi)^N\,(\det \mathsf{M})\right]^{1/2}} \prod_{i=1}^{N} \mathrm{d}f_i \tag{2}$$

where $\mathcal{P}_N$ is the probability that the field f has values in the range $f(\mathbf{x}_j)$ to $f(\mathbf{x}_j) + \mathrm{d}f(\mathbf{x}_j)$ for each of the $j = 1, \ldots, N$ (with N an arbitrary integer and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ arbitrary locations in the field). (We have assumed zero mean in this expression, as would be the case for $\delta, \mathbf{g}$ and $\mathbf{v}$.)

The matrix M^{-1} is the inverse of the $N \times N$ covariance matrix M,

$$M_{ij} \;\equiv\; \langle f(\mathbf{x}_i) f(\mathbf{x}_j)\rangle \;=\; \xi(\mathbf{x}_i - \mathbf{x}_j)\ , \tag{3}$$

in which the brackets $\langle \ldots \rangle$ denote an ensemble average over the probability distribution. In effect, M is the generalization of the variance σ^2 in a one-dimensional normal distribution. The equation above shows that a Gaussian distribution is fully specified by the matrix M, whose elements consist of specific values of the autocorrelation function $\xi(r)$, the Fourier transform of the power spectrum $P_f(k)$ of the fluctuations $f(\mathbf{r})$,

$$\xi(\mathbf{r}) \;=\; \xi(|\mathbf{r}|) \;=\; \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\; P_f(k) \mathrm{e}^{-\mathbf{i}\mathbf{k}\cdot\mathbf{r}}\,, \tag{4}$$

with the power spectrum $P_f(k)$ defined as

$$(2\pi)^3 P_f(k_1)\, \delta_D(\mathbf{k}_1 - \mathbf{k}_2) = \langle \hat{f}(\mathbf{k}_1) \hat{f}^*(\mathbf{k}_2)\rangle\,, \tag{5}$$

Notice that the identity of $\xi(\mathbf{r})$ and $\xi(|\mathbf{r})|)$ is assumed, not required.

A homogeneous and isotropic Gaussian random field f is statistically fully characterized by the power spectrum $P_f(k)$.

Question 1.: 2-point Gaussian distribution function

We wish to determine the distribution function for the field values at 2 points, $\mathbf{x}_1$ and $\mathbf{x}_2$, in a Gaussian random field with correlation function $\xi(r)$. To this end, we are going to evaluate its expression from the general formula for the N-point Gaussian distribution function in eqn. 2.

a. Write down the general expression for the probability density $P(f_1, f_2)$ for a 2-point field realization (f_1, f_2), still employing the notation M for the correlation matrix.

The key concept Gaussian distribution is the 2-point correlation function

$$\xi(\mathbf{x}_1 - \mathbf{x}_2) \;\equiv\; \langle f(\mathbf{x}_1) f(\mathbf{x}_1) \rangle \;=\; \xi(|\mathbf{x}_1 - \mathbf{x}_2|) \tag{6}$$

whose values are determining the correlation matrix M, ie.

$$M_{ij} = \xi(|\mathbf{x}_1 - \mathbf{x}_2|)\,. \tag{7}$$

A Gaussian distribution is FULLY specified by this second order moment (and first order moment, the mean field value $\langle f \rangle$, taken to be $\langle f \rangle = 0$).

b. Express the 2×2 correlation matrix M in terms of the correlation function values

$$\begin{aligned} \xi(\mathbf{x}_1 - \mathbf{x}_2) &\equiv \xi(r) \\ \xi(0) &= \sigma^2\,, \end{aligned} \tag{8}$$

where $\sigma^2 = \langle f^2 \rangle$ is the variance of the field.

c. Show that the determinant of the correlation matrix M is given by

$$\det M \;=\; (\sigma^4 - \xi^2(r)) \tag{9}$$

d. Subsequently, whow that the inverse M^{-1} of the correlation matrix is

$$M^{-1} \;=\; \frac{1}{(\sigma^4 - \xi(r)^2)} \begin{pmatrix} \sigma^2 & -\xi(r) \\ -\xi(r) & \sigma^2 \end{pmatrix} \tag{10}$$

e. Using the inferred expressions for the determinant and inverse of the 2-point correlation matrix M, show that the complete 2-point distribution function is given by

$$P(f_1, f_2) \;=\; \frac{1}{2\pi\; (\sigma^4 - \xi(r)^2)^{1/2}}\; \exp - \frac{\sigma^2 f_1 + \sigma^2 f_2 - 2\xi(r) f_1 f_2}{2(\sigma^4 - \xi(r)^2)}$$

f. If the spatial correlations in the field are zero, ie. $\xi(r) = 0$, show that the the 2-point Gaussian function $\mathcal{P}(f_1, f_2)$ is the product of two 1-point Gaussian distribution functions. In other words, argue that in this situation the field values f_1 and f_2 are mutually independent.

Question 2.:
Density, Velocity and Potential Power spectrum

In this assigment we derive the expression for the velocity power spectrum $P_v(k)$ and potential power spectrum $P_\phi(k)$ in terms of the density power spectrum $P_\delta(k)$.

a. Rewrite the linearized continuity, Euler and Poisson equation in equations in expressions for the Fourier components of density, velocity and potential field, $\delta(\mathbf{k})$, $\hat{\mathbf{v}}(\mathbf{k})$ and $\hat{\phi}(\mathbf{k})$,

$$\begin{aligned}
\delta(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \hat{\delta}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \\
\mathbf{v}(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \hat{\mathbf{v}}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \\
\phi(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \hat{\phi}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,,
\end{aligned} \tag{11}$$

showing that the Fourier versions of the linearized continuity/energy equation, the Euler equation and the Poisson equation are

$$\begin{aligned}
\frac{d\hat{\delta}(\mathbf{k})}{dt} - \frac{1}{a}\, i\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{k}) &= 0\,, \\
\frac{d\hat{\mathbf{v}}(\mathbf{k})}{dt} + \frac{\dot{a}}{a}\,\mathbf{v}(\mathbf{k}) &= \frac{1}{a}\, i\mathbf{k}\cdot\hat{\phi}(\mathbf{k})\,, \\
\frac{\hat{\phi}(\mathbf{k})}{a^2} &= -4\pi G\rho_u\, \frac{\hat{\delta}(\mathbf{k})}{k^2}
\end{aligned} \tag{12}$$

In order to arrive at this expression, insert the Fourier definitions (12) in the linear fluid equations,

$$\begin{aligned}
\frac{\partial\delta}{\partial t} + \frac{1}{a}\,\boldsymbol{\nabla}_x\cdot\boldsymbol{v} &= 0 \\
\frac{\partial\boldsymbol{v}}{\partial t} + \frac{\dot{a}}{a}\,\boldsymbol{v} &= -\frac{1}{a}\,\boldsymbol{\nabla}_x\phi \\
\nabla_x^2\phi &= 4\pi G\, a^2\,\rho_u\,\delta
\end{aligned} \tag{13}$$

and use the virtuous circumstance that spatial derivatives correspond to mere multiplications in Fourier space,

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\ \hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \\ \nabla f(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\ -i\mathbf{k}\,\hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \\ \nabla^2 f(\mathbf{x}) &= -\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\ k^2\,\hat{f}(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{x}}\,. \end{aligned} \tag{14}$$

b. What important conclusion can you draw from the Fourier expression of the fluid equations in the linear regime, ie. from expressions 13.

c. From the Fourier expressions of the Poisson equation (see eqn. 13), in combination with the linear relation between peculiar velocity and gravity,

$$\mathbf{v}(\mathbf{x},t) = \frac{2f(\Omega)}{3H\Omega}\mathbf{g}\,, \qquad \mathbf{g}(\mathbf{x},t) = -\frac{\nabla\phi}{a}\,, \tag{15}$$

in which $f(\Omega)$ is the Peebles (velocity) growth factor,

$$f(\Omega) = \frac{a}{D}\frac{\mathrm{d}D}{\mathrm{d}a}\,. \tag{16}$$

show that the relation between the Fourier potential component $\hat{\phi}(\mathbf{k})$, Fourier velocity component $\hat{\mathbf{v}}(\mathbf{k})$ and density perturbation component $\hat{\delta}(\mathbf{k})$ are given by

$$\begin{aligned} \hat{\phi}(\mathbf{k}) &= -\frac{3}{2}\Omega H^2 a^2\ \frac{\hat{\delta}(\mathbf{k})}{k^2}\,, \\ \hat{\mathbf{v}}(\mathbf{k}) &= -(Haf)\ \frac{i\mathbf{k}}{k^2}\ \hat{\delta}(\mathbf{k}\,.) \end{aligned} \tag{17}$$

d. Given the definitions of the density power spectrum $P_\delta(k)$, the velocity power spectrum $P_v(k)$ and the potential power spectrum $P_\phi(k)$,

$$\begin{aligned} (2\pi)^3 P_\delta(k_1)\,\delta_D(\mathbf{k}_1-\mathbf{k}_2) = \langle\hat{\delta}(\mathbf{k}_1)\hat{\delta}^*(\mathbf{k}_2)\rangle \qquad &\Longleftrightarrow \qquad P_\delta(k) \propto \langle\hat{\delta}(\mathbf{k})\hat{\delta}^*(\mathbf{k})\rangle \\ (2\pi)^3 P_v(k_1)\,\delta_D(\mathbf{k}_1-\mathbf{k}_2) = \langle\hat{\mathbf{v}}(\mathbf{k}_1)\hat{\mathbf{v}}^*(\mathbf{k}_2)\rangle \qquad &\Longleftrightarrow \qquad P_v(k) \propto \langle\hat{\mathbf{v}}(\mathbf{k})\hat{\mathbf{v}}^*(\mathbf{k})\rangle \\ (2\pi)^3 P_\phi(k_1)\,\delta_D(\mathbf{k}_1-\mathbf{k}_2) = \langle\hat{\phi}(\mathbf{k}_1)\hat{\phi}^*(\mathbf{k}_2)\rangle \qquad &\Longleftrightarrow \qquad P_\delta(k) \propto \langle\hat{\phi}(\mathbf{k})\hat{\phi}^*(\mathbf{k})\rangle \end{aligned} \tag{18}$$

show that they are related as,

$$\begin{aligned} P_v(k) &= (Haf)^2 \frac{P_\delta(k)}{k^2} \\ P_\phi(k) &= (\frac{3}{2}\Omega H^2 a^2)^2 \frac{P_\delta(k)}{k^4} \end{aligned} \tag{19}$$

e. What can you conclude on the basis of the previous results with respect to the relative scale sensitivity of the perturbation quantities $\delta(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$ and $\phi(\mathbf{x})$.

Question 3.: Nonlinear evolution and mode coupling

In this task, we will investigate what is happening to the evolving mass distribution once the fluctuations start to grow nonlinear. We do this on the basis of an assessment of the evolution of the Fourier components of the density, velocity and potential field. We will see that nonlinear evolution leads to a mixing of Fourier modes, ie. the modes do not develop independently anymore,

a. Following the analysis in the previous question, infer the Fourier expressions for the fully nonlinear fluid equations,

$$\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a}\boldsymbol{\nabla}_x \cdot (1+\delta)\boldsymbol{v} &= 0 \\ \frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{a}\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\right)\boldsymbol{v} + \frac{\dot{a}}{a}\boldsymbol{v} &= -\frac{1}{a}\boldsymbol{\nabla}_x\phi \\ \nabla_x^2\phi &= 4\pi G\, a^2\, \rho_u\, \delta \end{aligned} \tag{20}$$

and show that

$$\begin{aligned} \frac{d\hat{\delta}(\mathbf{k})}{dt} - \frac{1}{a}\, i\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{k}) - \frac{1}{a}\int \frac{d\mathbf{k}'}{(2\pi)^3}\, i\hat{\delta}\,\mathbf{k}'\cdot\hat{\mathbf{v}}(\mathbf{k}-\mathbf{k}') &= 0 \\ \frac{d\hat{\mathbf{v}}(\mathbf{k})}{dt} + \frac{\dot{a}}{a}\mathbf{v}(\mathbf{k}) - \frac{1}{a}\int \frac{d\mathbf{k}'}{(2\pi)^3}\left[i\hat{\mathbf{v}}(\mathbf{k}')\cdot(\mathbf{k}-\mathbf{k}')\right]\hat{\mathbf{v}}(\mathbf{k}-\mathbf{k}') &= \frac{1}{a}\, i\mathbf{k}\cdot\hat{\phi}(\mathbf{k}) \\ \frac{\hat{\phi}(\mathbf{k})}{a^2} &= -4\pi G\rho_u \frac{\hat{\delta}(\mathbf{k})}{k^2} \end{aligned} \tag{21}$$

b. Describe the physical reason(s) for the emerging mode coupling as the evolving density field becomes nonlinear. You may use illustrations to explain how the different modes turn nonlinear.

c. What will be the main manifestation of mode coupling: the transfer of power from large to small scales (low to high frequencies), or from small to large scales (high to low frequencies). Why do you expect so ?