# Cosmic Structure Formation – Assignment 2

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In the last assignment we revised basics of FRW cosmology. In this problem set we will derive equations governing Newtonian perturbation theory. To solve this set of problems, knowledge about the governing equations of fluid dynamics is required. For those who didn't follow the course on astrophysical hydrodynamics few optional problems are added in the end.

## 1 Perturbed fluid equations:

Consider a fluid of mass density $\rho$, pressure $P$ and velocity $\boldsymbol{u}$ in the non-relativistic regime. Let's denote the position vector of a fluid element by $\boldsymbol{r}$ and time by t. The basic equations of motion are:

$$\partial_t \rho + \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}})\boldsymbol{u} = -\frac{\boldsymbol{\nabla}_{\boldsymbol{r}} P}{\rho} - \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi \tag{2}$$

$$\boldsymbol{\nabla}_{\boldsymbol{r}}^2 = 4\pi G \rho \tag{3}$$

The equations are named as *continuity*, *Euler* and *Poisson equations* respectively. The physical coordinates $\boldsymbol{r}$ and the comoving coordinates $\boldsymbol{x}$ are related via the following equation:

$$\boldsymbol{r}(t) = a(t)\boldsymbol{x}$$

1. Derive the following relationships of space and time derivatives between at fixed $\boldsymbol{r}$ and at fixed $\boldsymbol{x}$:

$$\boldsymbol{\nabla}_{\boldsymbol{r}} = a^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \tag{4}$$

$$\left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} - H_0 \boldsymbol{x} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \tag{5}$$

2. For the case of small perturbations around the homogeneous background values (denoted by an overbar) we can decompose a physical quantity $\eta$ as:

$$\eta(t, \boldsymbol{r}) = \bar{\eta}(t) + \delta\eta(t, \boldsymbol{r}) \tag{6}$$

We now introduce a new term called *fractional density perturbation* or *density contrast*:

$$\delta = \frac{\delta \rho}{\bar{\rho}} \tag{7}$$

Show that in zeroth order in perturbations the continuity equation gives us:

$$\frac{\partial \bar{\rho}}{\partial t} + 3H_0 \bar{\rho} = 0 \tag{8}$$

3. Show that at first order in fluctuations we get,

$$\dot{\delta} = -\frac{1}{a}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{9}$$

where $\boldsymbol{v} = a\dot{\boldsymbol{x}}$ is the proper velocity.

4. By following the similar steps show that Euler's equation leads to the following equation:

$$\dot{\boldsymbol{v}} + H_0\boldsymbol{v} = -\frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P - \frac{1}{a}\boldsymbol{\nabla}\delta\Phi \tag{10}$$

5. Show that Poisson equation becomes,

$$\boldsymbol{\nabla}^2\delta\Phi = 4\pi G \tag{11}$$

6. Derive *Jeans equation*:

$$\ddot{\delta} + 2H_0\dot{\delta} - \frac{c_s^2}{a^2}\boldsymbol{\nabla}^2\delta = 4\pi G\bar{\rho}\delta \tag{12}$$

Give a physical explanation of different terms appearing in the equation above.

## 2 Matter, radiation and $\Lambda$ dominated universes:

1. During the matter dominated era Eq.(12) becomes:

$$\ddot{\delta}_m + 2H_0\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \tag{13}$$

Show that the above equation takes the following form:

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0 \tag{14}$$

give proper reasoning why the other term containing $c_s$ has been dropped.

2. Use a power law function as a trial solution $\delta_m \propto t^p$ and show that $\delta_m$ has two solutions:

$$\begin{aligned}\delta_m &\propto t^{-1} \propto a^{-3/2} \\ &\propto t^{2/3} \propto a\end{aligned} \tag{15}$$

Explain the results physically.
[Hint: For matter dominated solution $a \propto t^{2/3}$, $4\pi G\bar{\rho}_m = \frac{3}{2}H_0^2$]

3. Show that during the radiation and $\Lambda$ dominated era Eq.(12) takes the following forms respectively:

$$\ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \tag{16}$$

$$\ddot{\delta}_m + 2H_0\dot{\delta}_m = 0 \tag{17}$$

[Hint: For $\Lambda$ dominated universe assume $H^2 = \text{constant} \gg 4\pi \mathrm{G}\bar{\rho}_\mathrm{m}$ ]

4. Show that in the radiation dominated case $\delta_m$ has two solutions:

$$\begin{aligned}\delta_m &\propto \text{constant} \\ &\propto \ln t \propto \ln a\end{aligned} \tag{18}$$

[Hint: Assume $\ddot{\delta}_m \gg 4\pi G\bar{\rho}_m\delta_m$ (why??)]

5. Similarly solve for the $\Lambda$ dominated case and show that it has the following solutions:

$$\begin{aligned}\delta_m &\propto \text{constant} \\ &\propto e^{-2H_0 t} \propto a^{-2}\end{aligned} \tag{19}$$

# Fluid dynamics

1. **Convective operator and convective derivative:**
For a vector field the $\boldsymbol{A}$ convective operator is defined as $(\boldsymbol{A} \cdot \boldsymbol{\nabla})$. In fluid dynamics we often use the Lagrangian description, where we follow the fluid elements when describing macroscopic properties of the fluid. Instead of talking e.g. about a density $\rho_{\text{Euler}}(\boldsymbol{x}, t)$ at a fixed position $\boldsymbol{x}$, we consider the density $\rho_{\text{Lagrangian}}(\boldsymbol{x}(t), t)$ of a specific fluid element that moves in the flow.

When we want to take the time derivative of a macroscopic property of a fluid element, like the density, we need to use the Lagrangian derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \tag{20}$$

where $\boldsymbol{u}$ is the flow velocity.

(a) Apply chain rule of partial derivative and derive the result. Interpret in the context of convection of a property of the fluid, e.g. temperature.

2. **Momentum conservation: Euler equation**
The Euler equation describes the rate of change of the momentum of a fluid with time. Remember that the time derivative of the momentum has units of force. For an inviscid (frictionless) fluid of volume $V$ the Euler equation has three components:

- External volume forces $\boldsymbol{f}$ (force per unit mass, i.e. units of acceleration) affect every fluid volume element $dV$ in the volume $V$. By multiplying with the density and integrating over the whole volume we get the momentum rate of change due to volume forces:

$$\int_V \rho \boldsymbol{f} dV \,. \tag{21}$$

- The pressure of the fluid also contributes to the rate of change of momentum of the fluid, because pressure is force per unit area and hence momentum transfer per unit time through a unit area element. This adds a term to the Euler equation that goes like

$$-\int_S p\boldsymbol{n} dS \,. \tag{22}$$

- Finally, there is momentum of fluid elements moving into and out of the volume $V$, i.e. the momentum flux:

$$-\int_S (\rho \boldsymbol{u})\boldsymbol{u} \cdot \boldsymbol{n} dS \,. \tag{23}$$

Together these form the Euler equation:

$$\frac{d}{dt}\int_V \rho \boldsymbol{u} dV = \int_V \rho \boldsymbol{f} dV - \int_S p\boldsymbol{n} dS - \int_S (\rho \boldsymbol{u})\boldsymbol{u} \cdot \boldsymbol{n} dS \,. \tag{24}$$

(a) Derive the following form of the Euler equation:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{f} \,. \tag{25}$$

3. **Hydrostatics:**
This question gives some examples of static fluids (no velocities). In that case, Euler's equation becomes

$$\boldsymbol{\nabla} P = \rho \mathrm{g} \,, \tag{26}$$

where $P$ is the pressure of the fluid and $\rho$g the force acting on the fluid.

(a) Consider a very large, very massive fluid such that self-gravity becomes important (any idea what we are talking about here?). The Poisson equation – familiar from mechanics, stellar dynamics and/or stellar evolution courses – relates the gravitational potential $\phi$ to the density $\rho$

$$\nabla^2\phi(\boldsymbol{r}) = 4\pi G_N \rho(\boldsymbol{r})\,, \tag{27}$$

where $G_N$ is Newton's constant. What is $\boldsymbol{g}$ in this scenario? Use the assumption of spherical symmetry to find (one of) the equations of stellar structure

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G_N \rho\,. \tag{28}$$

N.B. It's not only in stars that this equation is used. Another example are clusters of galaxies. Then $\rho$ is the total matter density: baryonic (stellar, gas) and dark.

**Comments:** In order to solve this, we need a relation between $P$ and $\rho$, called the *equation of state*. Often this is a polytropic relation

$$P \propto \rho^{1+1/n}\,, \tag{29}$$

where $n$ is called the polytropic index. Low mass white dwarf stars are well approximated as $n = 1.5$ polytropes, red giants with $n = 3$, the giant planets Jupiter and Saturn with $n = 1$ and the small planets with a constant density (no relation between $P$ and $\rho$).

# References


1. John Peacock, Cosmological Physics, Cambridge University Press, 1998
2. Landau L.D. and Lifshitz E. M., Course of Theoretical Physics, Vol. 6: Fluid Mechanics