The exercises in this task are meant to give you a working knowledge of Fourier space, amplitudes, phases, power spectra and last but not least random Gaussian fields. Please put the answers into your final report such that the report becomes a self contained story, not just a collection of problem solutions.

Part I: Gaussian Fields.

Structure formation in the Universe started out from a nearly homogeneous distribution of gas, with tiny perturbations of, as far as current observations can tell us, perfect Gaussian nature. Most inflation theories predict deviations from Gaussianity, but we won't delve into that here. In any case it is very important to have a firm understanding of what these Gaussian random fields are. To give a first impression I plotted two examples here.

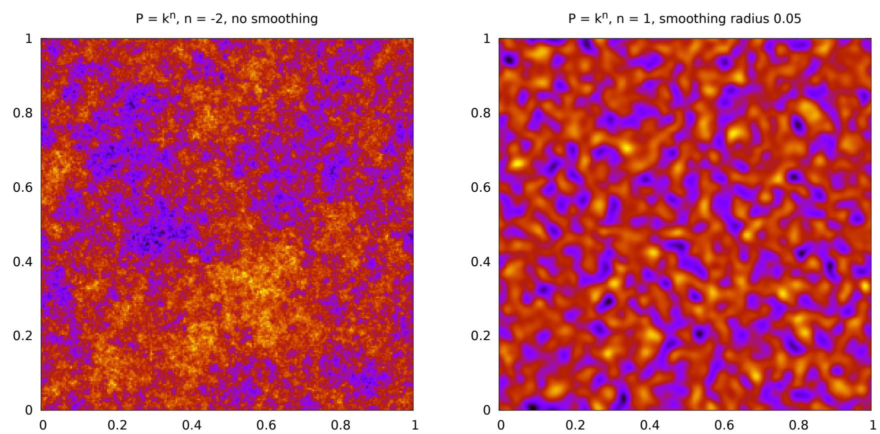

Figure 1: two examples of random Gaussian fields.

From a single image it is in principle impossible to say whether it was generated using a Gaussian process. One of the above images could have been that of a cat or a dog, and you still couldn't tell. We can give a *probability* that it is Gaussian by looking at the phases in Fourier space. Gaussian fields have the property that all of their statistics is found in the *two-point correlation* function. Without showing any proof, it turns out this property implies that the Fourier phases of the random field are all independent (you will find out what this means exactly).

To determine the random nature of a die, we can throw it many times and build statistics of the emerging numbers. Every outcome of the process of throwing the die, we call a *realisation* of this zero-dimensional field. We could also throw two dice at the same time and, asuming mutual independance, learn about the random process governing the dice. Now suppose the dice are rigged with magnets. Their outcome is now correlated. Moreover their correlation depends on the distance they fall appart, since the magnetic force falls off with distance. If the process is Gaussian, we would learn nothing more by adding more dice (higher order correlation functions).

To create a complete Gaussian random field, we can add points one by one. Every time we add a point (throw a dice to some location), we have to multiply the probability functions with respect to all points already present to find the distribution for the new point. If we were to compute a random field on a computer by this method the algorithm would have $O(n!)$ running time! This is why we use Fourier theory to ease our pains.

In reality we have only one Universe, so we have to find a different method of finding correlation statistics than creating multiple 'realisations'. In this reality we use the property that the Universe is homogeneous on the largest scales. We assume that in a different realisation we would be one of our neighbours or one of our neighbour's neighbours (and so on). This substitution of probabilistic averaging with spatial averaging is called the *ergodic theorem*.

We are going to generate Gaussian random fields. We have a field of scalars $f(\boldsymbol{x})$, which has the following Gaussian one-point (probability density) function

$$P(f) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{f^2}{2v}\right],$$

where f expresses the value of $f(\boldsymbol{x})$ at some chosen location $\boldsymbol{x}$. Choosing two locations $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ we can write a bivariate Gaussian two-point function

$$P(f_1, f_2) = \frac{1}{2\pi\sqrt{v^2 - \xi^2}} \exp\left[-\frac{1}{2}\begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} v & \xi \\ \xi & v \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right].$$

We assume isotropy so we can write the two-point corellation as a function of $|\boldsymbol{x_2} - \boldsymbol{x_1}| = |\boldsymbol{r}|$,

$$\xi(|\boldsymbol{r}|) = \langle f(\boldsymbol{x}) f(\boldsymbol{x} + \boldsymbol{r})\rangle,$$

and the variance

$$v = \sigma^2 = \xi(0) = \langle f(\boldsymbol{x}) f(\boldsymbol{x})\rangle.$$

A bivariate Gaussian distribution has the defining property that any linear combination of the two variates should be normally distributed. Also it solves the multi-dimensional equivalent of the central limit theory.

a. Write out the exponent of $P(f_1, f_2)$ in full. To save you some time:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2 - b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$$

What happens when $f_1 = f_2$?

Fourier Space.

Properties of Gaussian random fields are most easily computed in Fourier space

$$f(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \hat{f}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}},$$

$$\hat{f}(\boldsymbol{k}) = \int d^3\boldsymbol{x} f(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

b. The main advantage of Fourier space is the Fourier convolution theorem

$$\int d^3\boldsymbol{x} f(\boldsymbol{x}) g(\boldsymbol{y} - \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \hat{f}(\boldsymbol{k}) \hat{g}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}.$$

Prove it.

c. Show that for a real-valued field f, $\hat{f}(k) = \hat{f}^*(-k)$. Turning this relation around, what does this tell you about the Fourier transforms of even and/or odd functions?

bonus. Show that, using the Fourier convolution theorem, we can express the correlation function as

$$\xi(|\boldsymbol{r}|) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} P(|\boldsymbol{k}|) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}},$$

where $P(k)$ is the *power spectrum.*

$$(2\pi)^3 P(k) \delta(\boldsymbol{k} - \boldsymbol{k}') \equiv \langle \hat{f}(\boldsymbol{k})^* \hat{f}(\boldsymbol{k}') \rangle$$

d. Calculate $P(f_1, f_2)$ for $P(k) = 1$. We call this *white noise*; what is special about this case?

e. Suppose we have a field of white noise (call this field $g(\boldsymbol{x})$). How can we take this noise, and give it the power spectrum we want? (hint: write $\hat{f}$ in terms of $\hat{g}$ and $P(k)$). Do we sample the space of Gaussian random field realisations with this particular power spectrum evenly using this method?

Discrete Fourier space.

$$f(x_n) = \sum_{j=0}^{N} \hat{f}_j e^{-ik_j x_n}$$

Working in Fourier space takes some fingerspitzengefühl. We will be working with Fourier transforms of real-valued arrays. They follow a symmetry

$$\hat{f}(k) = \hat{f}^*(-k).$$

In the context of DFT this introduces two special frequencies: the zero-mode and the Nyquist frequency. The zero-mode gives you the average over the whole field, while the Nyquist frequency is the smallest scale mode: half the sampling frequency. In general the array of Fourier coefficients is stored as follows

$$k_i = \frac{2\pi}{N} \left[0, 1, 2 \ldots, \mathrm{Nq}, 1 - \mathrm{Nq}, 2 - \mathrm{Nq} \ldots, -2, -1\right].$$

Both the zero and Nyquist mode have no counterpart in negative frequencies on the discrete Fourier space. There is only one mean value, as there is only one sequence going $-1, +1, -1, +1, \ldots$. So the symmetry relation given above forces their imaginary parts to vanish.

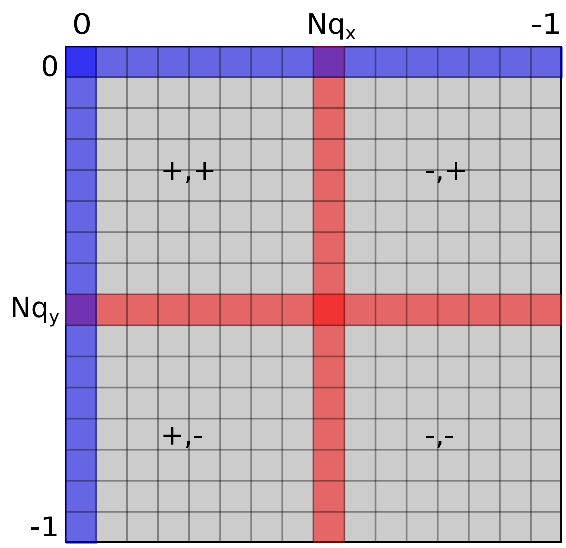

Figure 2: Discrete Fourier space: for an array of size N, the Fourier modes are stored from 0 to $N/2$, then $-N/2+1$ to -1. The zero- and Nyquist-modes are special, they obey a different symmetry: there is no $k = -0$ nor $k = -\mathrm{Nq}$.

e. Generate 2-D Gaussian random fields with $P(k) \propto k^n$ for $n = 1, 0, -1, -2$, using the same white noise realisation.

Take a distinct 2-D data set; you will be given a suitable text file containing pixel values, but you may use a different picture if you don't like cats (in the light of verifiability, supply the data along with your report if you choose to use your own data)

f. Compute the power spectrum of the image, as if it were Gaussian (include error-bars in your plot). What is the source of the uncertainty?

g. Randomize the Fourier amplitudes of this image; you may vary the power spectrum or leave it the same.

h. Randomize the Fourier phases. What do you conclude from this exercise?