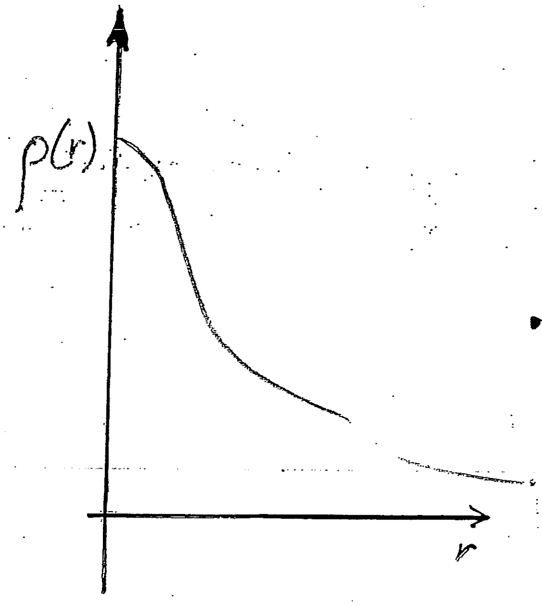
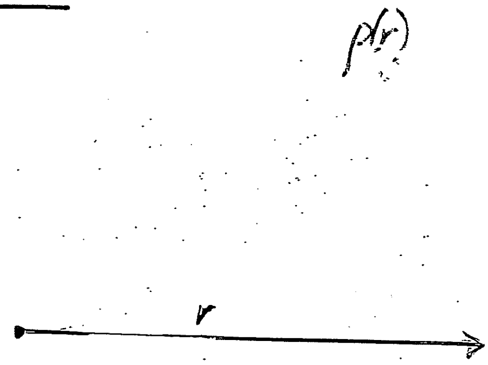
② Systems of High Symmetry

- Why useful:
  - these overidealized situations provide:
    - qualitative insight
    - interpretational tool for more realistic situations
    - reference point for N-body simulations.

① Spherical Model

- a purely spherical mass distribution:





- Equation of motion of shell at radius r:

(1) $$\frac{d^2 r}{dt^2} = -\frac{GM(r)}{r^2}$$

(2) $$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = E = cst$$ (energy of shell).

- ~~Equations of motion analytically solvable,~~ on the condition that:

(1) There is no shell crossing (i.e. $M(r(r_i)) = cst.$)
(2) No internal 'nonradial' substructures
(3) Isolated object, no external influences.

- Evolution completely determined by

$$\Delta(r,t) = \frac{3}{r^3}\int_0^r \left[\frac{\rho(y,t)}{\bar{\rho}(t)} - 1\right] y^2 dy$$

$$= \frac{3}{r^3}\int_0^r \delta(y,t)\, y^2 dy$$

i.e. by the internal mass distribution.

- Evolution of mass shell described by its motion from initial radius r_i to radius $r(t)$

$$r(t,r_i) = R(t,r_i)\, r_i$$

($r(t) = a(t)\, x$: physical radius).

- Introduce parameters Δ_{ci} and α_i :

- $1 + \Delta_{ci} = \Omega_i \left[1 + \Delta(t_i, r_i) \right]$ – density defic... wrt. $\Omega = 1$ Univ...

- $\alpha_i = \left(\frac{v_i}{H_i r_i}\right)^2 - 1$ – measure of kinetic energy of shell.

Three cases can be distinguished:

) Open shell: $E > 0$ $\alpha_i > \Delta_{ci}$

) Critical shell: $E = 0$ $\alpha_i = \Delta_{ci}$

) Closed shell: $E < 0$ $\alpha_i < \Delta_{ci}$

Solution for $R(t, r_i)$:

I) Open Shell: (parametrize by development angle Φ_r.

$$\begin{cases} R(t,r_i) = \frac{1}{2} \frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})} \left(\cosh \Phi_r - 1\right) \\ H_i t = \frac{1}{2} \frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}} \left(\sinh \Phi_r - \Phi_r\right) \end{cases}$$

II) Critical Shell:

$$R(t,r_i) = \left\{ \frac{3}{2} H_i \left(1+\Delta_{ci}\right)^{1/2} t \right\}^{2/3}$$

) Closed Shell:

$$\begin{cases} R(t,r_i) = \frac{1}{2} \frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)} \left(1 - \cos \Phi_r\right) \\ H_i t = \frac{1}{2} \frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}} \left(\Phi_r - \sin \Phi_r\right) \end{cases}$$

How to determine development angle $\Phi_r(t)$

- Assume given an expansion factor $a(t)$ of Universe

$\Rightarrow$ <u>solve for t</u> :

$$
a(\Phi_u) = \begin{cases} \frac{1}{2}\,\frac{\Omega_0}{1-\Omega_0}\left(\cosh\Phi_u - 1\right) & \Omega_0 < 1 \\ \left(\frac{3}{2}H_0 t\right)^{2/3} & \Omega_0 = 1 \\ \frac{1}{2}\,\frac{\Omega_0}{\Omega_0 - 1}\left(1 - \cos\Phi_u\right) & \Omega_0 > 1 \end{cases}
$$

($\Omega_0 = 1$: $\Rightarrow$ $H_0 t$ follows directly)
(otherwise invert to obtain Φ_u)

$$
H_0 t = \begin{cases} \frac{1}{2}\,\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\left(\sinh\Phi_u - \Phi_u\right) & \Omega_0 < 1 \\ \frac{2}{3}\,a^{3/2} & \Omega_0 = 1 \\ \frac{1}{2}\,\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left(\Phi_u - \sin\Phi_u\right) & \Omega_0 > 1 \end{cases}
$$

Then invert for each individual shell, the equations:
(for critical shell $H_i t$ is directly in equation)

$$
H_i t = \begin{cases} \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}}\left(\sinh\Phi_r - \Phi_r\right) & \alpha_i > \Delta_{ci} \\ \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}}\left(\Phi_r - \sin\Phi_r\right) & \alpha_i < \Delta_{ci} \end{cases}
$$

- Having determined $R(t, r_i)$ and Φ_r :

$$\boxed{1+\Delta(r,t) = \frac{1+\Delta_i(r_i)}{R^3}\left(\frac{a(t)}{a_i}\right)^3}$$

Density deficit within radius

- Which, specifically, can be worked out as:

$$\boxed{1+\Delta(r,t) = \frac{f(\Phi_r)}{f(\Phi_u)}}$$

with Φ_r the development angle of the shell and Φ_u the development angle of the Universe,

$$f(\Phi) = \begin{cases} \dfrac{(\sinh\Phi - \Phi)^2}{(\cosh\Phi - 1)^3} & \text{open} \\ \dfrac{2}{9} & \text{critical} \\ \dfrac{(\Phi - \sin\Phi)^2}{(1-\cos\Phi)^3} & \text{closed.} \end{cases}$$

While for density $\delta(r,t)$ in the shell :

$$\boxed{\delta(r,t) = \frac{1+\delta_i}{R^3\left[1 + r_i \dfrac{R'}{R}\right]}\left(\frac{a(t)}{a_i}\right)^3 - 1}$$

$$\cdot \quad R' \quad \frac{dR}{}$$

- The velocity of each shell is then :

$$
v(r,t) = \begin{cases}
H_i r_i \, (\alpha_i - \Delta_{ci})^{1/2} \, \dfrac{\sinh \Phi_r}{\cosh \Phi_r - 1} & \alpha_i > \Delta_{ci} \\[2ex]
H_i r_i \left(\dfrac{2(1+\Delta_{ci})}{3H_i\,(t-t_0)} \right)^{1/3} & \alpha_i = \Delta_{ci} \\[2ex]
H_i r_i \, (\Delta_{ci} - \alpha_i)^{1/2} \, \dfrac{\sin \Phi_r}{1 - \cos \Phi_r} & \alpha_i < \Delta_{ci}
\end{cases}
$$

- and the components of the tidal tensor:

$$
\boxed{
\begin{aligned}
E_{rr}(r,t) &= \frac{8\pi G}{3} \left(\rho(r) - \langle \rho(r) \rangle \right) \\
&= \Omega H^2 \left(\delta(r) - \Delta(r) \right) \\
E_{\theta\theta}(r) = E_{\phi\phi}(r) &= -\tfrac{1}{2} E_{rr}(r)
\end{aligned}
}
\;*
$$

Notice: <u>overdensity</u> : $\delta < \Delta$: $E_{rr} < 0$: stretching along r.

<u>underdensity</u> : $\delta > \Delta$: $E_{rr} > 0$: compression in radial direction $\downarrow$ shell formati

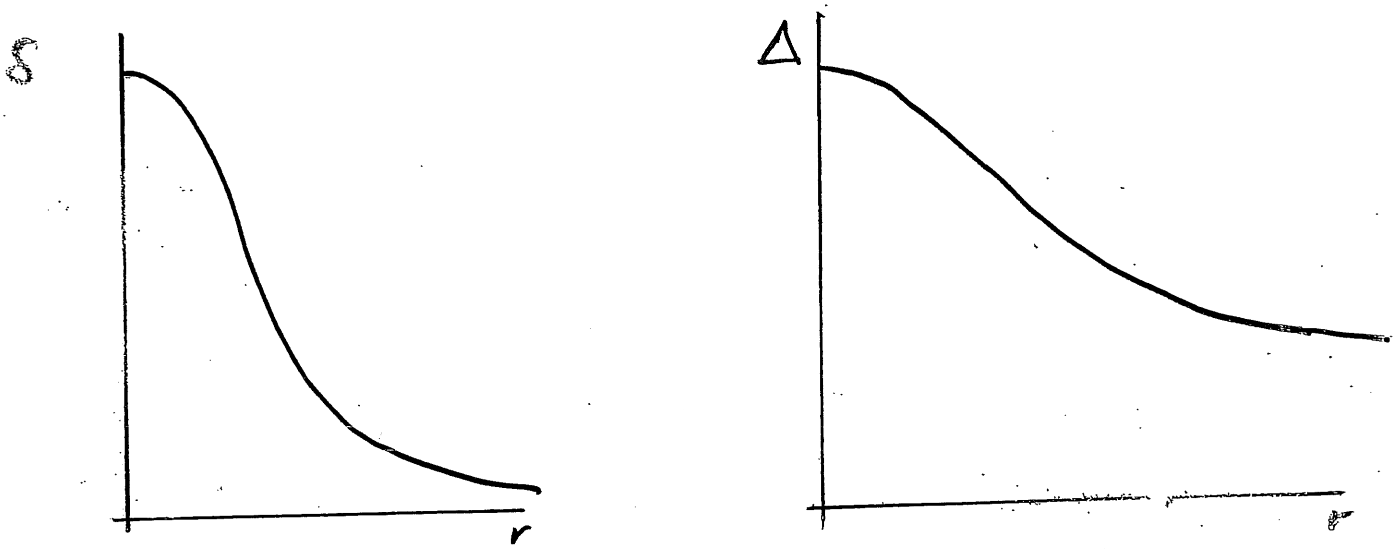
$$
* \quad E_{ij} \equiv \frac{\partial^2 \phi}{\partial r_i \, \partial r_j} - \frac{1}{3} \nabla^2 \phi \, \delta_{ij}
$$

Applications of the Spherical Model.

① Spherical Collapse of Tophat Overdensity

② Shell Crossing and Void Expansion.

(I) Spherical Collapse



For a spherical overdense shell, we know that:

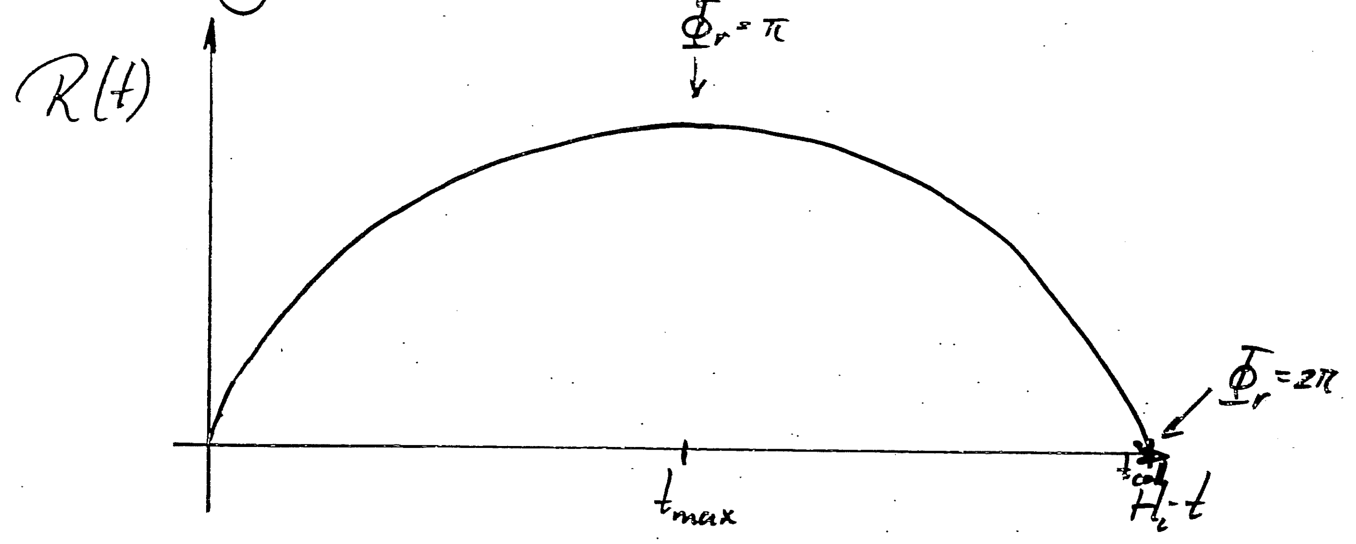
$$\begin{cases} R(t, r_i) = \frac{1}{2} \frac{1+\Delta_{ci}}{\Delta_{ci} - \alpha_i} \left(1 - \cos \Phi_r \right) \\ H_i\, t = \frac{1}{2} \frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}} \left(\Phi_r - \sin \Phi_r \right) \end{cases}$$

with

$$\Delta_{ci} = \Omega_i (1 + \Delta_i) - 1$$

$$\alpha_i = \begin{cases} 0 & \text{if } v_{pec,i} = 0 \\ -\frac{2}{3} f(\Omega_i) \Delta_i & \text{growing mode} \end{cases}$$

Qualitatively:



- Maximum expansion radius:

$$\Phi_r = \pi: \quad R(t_{max}) = \frac{1+\Delta_{ci}}{\Delta_{ci} - \alpha_i}$$

$$H_i t_{max} = \frac{\pi}{2} \frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}}$$

- Collapse $(R(t_{coll}) = 0)$:

$$\Phi_r = 2\pi: \quad R(t_{coll}) = 0$$

$$H_i t_{coll} = 2 H_i t_{max} = \pi \frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}}$$

For an Einstein-de Sitter Universe ($\Omega = 1$):

- $\Delta_{ci} = \Delta_i$

- $\alpha_i = \begin{cases} 0 & v_{pec,i} = 0 \quad \rightarrow \Delta_i - \alpha_i = \Delta_i \\ -\frac{2}{3}\Delta_i & \text{growing mode} \rightarrow \Delta_i - \alpha_i = \frac{5}{3}\Delta_i \end{cases}$

$$1 + \Delta(r,t) = \frac{9}{2} \frac{(\Phi_r - \sin\Phi_r)^2}{(1-\cos\Phi_r)^3}$$

$\Rightarrow$ **Maximum Expansion :**

(1) $$R_{max} = \frac{1+\Delta_i}{\Delta_i - \alpha_i} \approx \frac{1}{\Delta_i - \alpha_i} = \frac{1}{A\Delta_i}$$

($A = 1$ $v_{pec,i} = 0$ or
$A = \frac{5}{3}$ growing mode).

(2) $$H_i t_{max} = \frac{\pi}{2} \frac{1+\Delta_i}{(\Delta_i - \alpha_i)^{3/2}} \approx \frac{\pi}{2} A^{-3/2} \frac{1}{\Delta_i^{3/2}}$$

(3) $$\boxed{1 + \Delta(t_{max}) = \frac{\langle \rho(R_{max}) \rangle}{\bar{\rho}(t_{max})} = \left(\frac{3\pi}{4}\right)^2 \approx 5.55}$$

$\Rightarrow$ Any (tophat) density fluctuation, independent of value Δ_i, turns around into collapse when $\underline{\underline{1 + \Delta \approx 5.6}}$

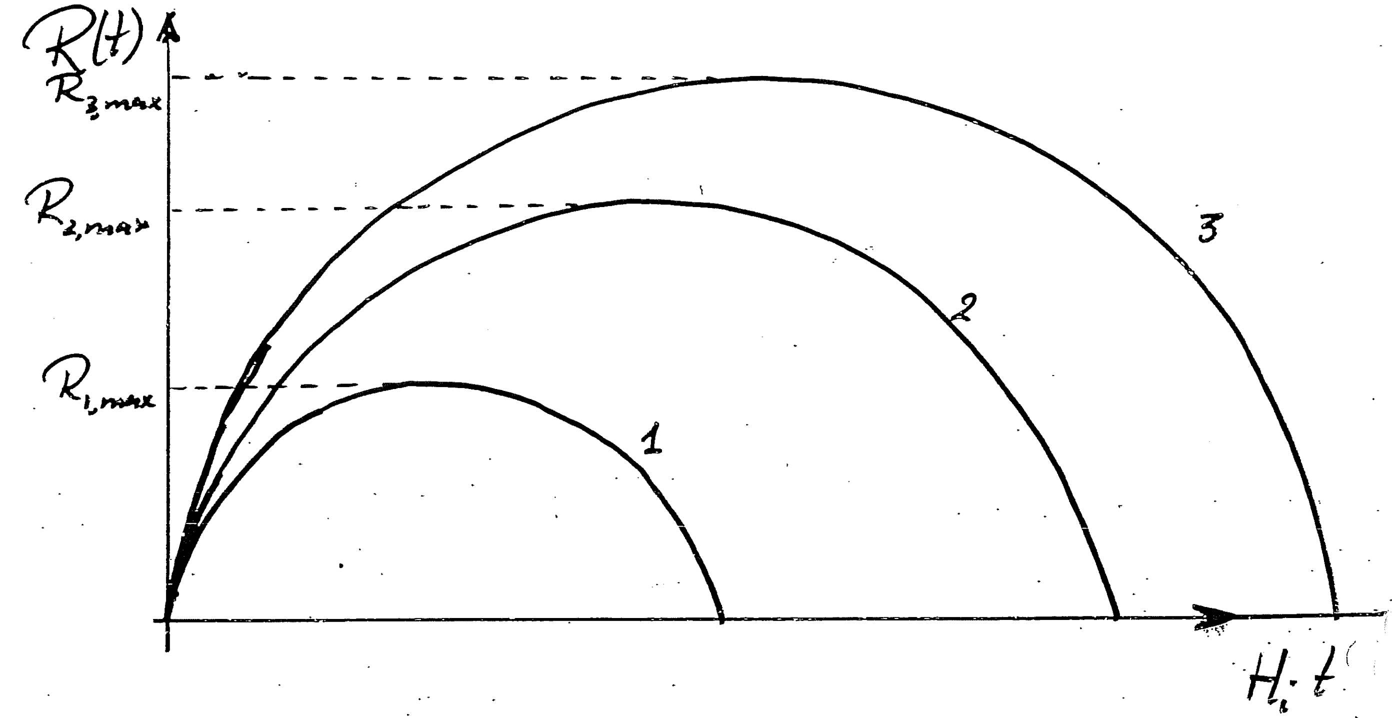
Collapse :

(1) $$R_{coll} = 0.$$

(2) $$H_i t_{coll} \approx \pi A^{-3/2} \frac{1}{\Delta_i^{3/2}}$$

(3) $$1 + \Delta(t_{coll}) = \infty$$

Comparison 3 tophat collapses:



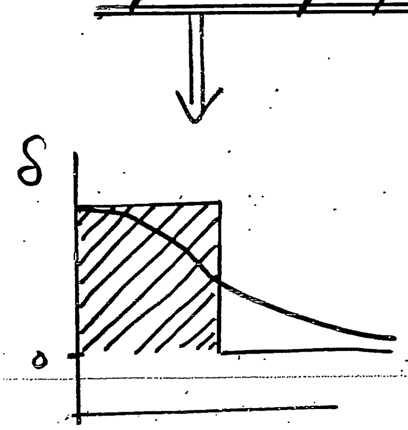
$$\Delta_{1,i} > \Delta_{2,i} > \Delta_{3,i}$$

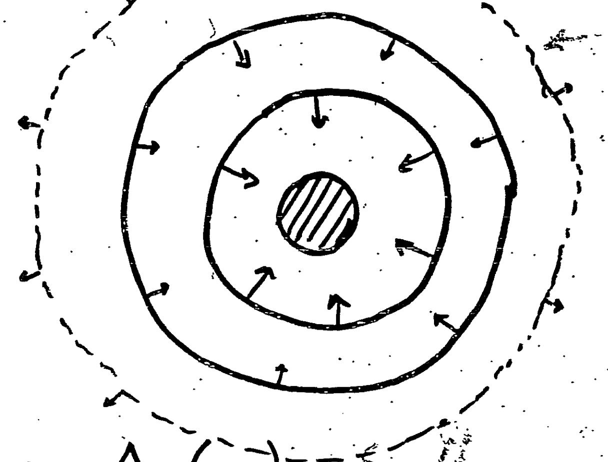
$$\Rightarrow \quad R_{1,max} < R_{2,max} < R_{3,max}$$

$$t_{1,max} < t_{2,max} < t_{3,max}$$

$$\Rightarrow \quad \underline{t_{1,coll} < t_{2,coll} < t_{3,coll}}$$

Notice above not only valid for density perturbation with tophat profile, but also valid for individual shells



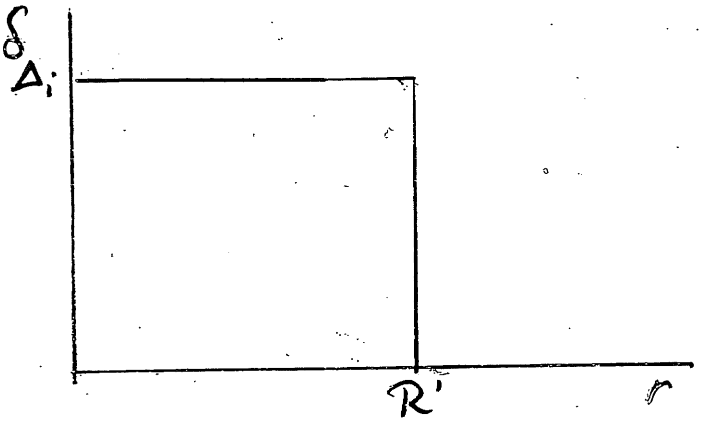


$\Rightarrow$ Outer shells, with lower $\Delta_i(r_i)$ collapse later, i.e. fall onto already collapsed core.

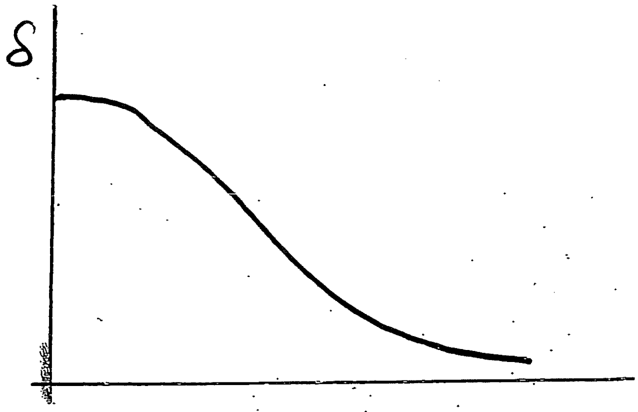
Comoving Radius Infalling Shells:

$r(t, r_i) = R(t, r_i)\, r_i$ — physical radius.

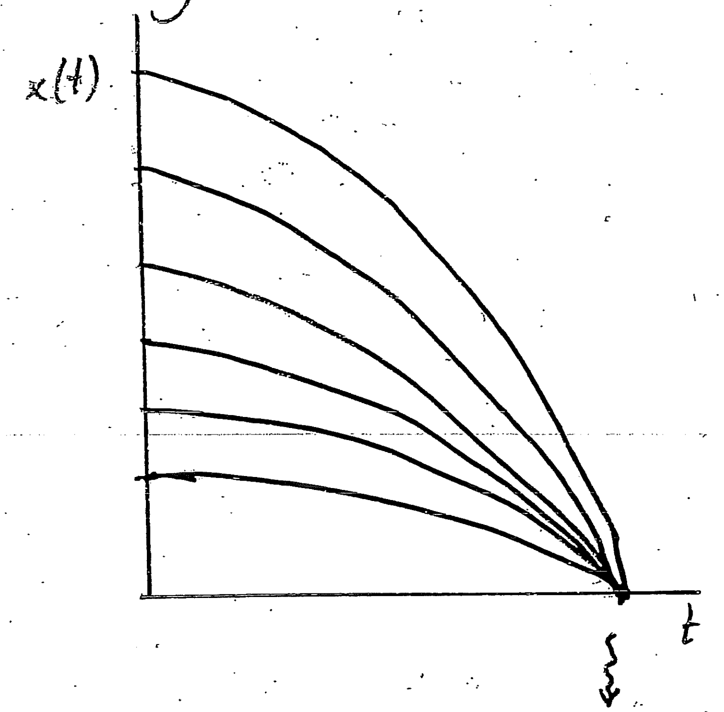
$x(t, r_i) = R(t, r_i)\, \dfrac{r_i}{a(t)}$ — comoving radius.



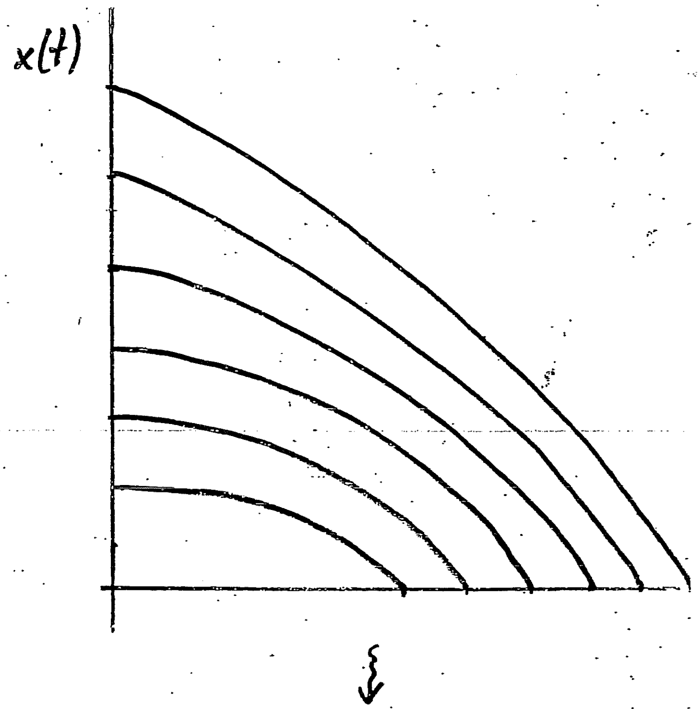
Tophat perturbation
(homogeneous sphere)



peak with 'general' density profile



$t_{coll} \propto \dfrac{1}{\Delta_i}$



outer shells fall in later.

• <u>Initial density excess collapsed objects:</u>

• An object of density excess Δ_i collapses at expansion factor $a(t_{coll})$:

$$H_i\, t_{coll} = \pi \frac{1+\Delta_i}{(\Delta_i - \alpha_i)^{3/2}} = \frac{2}{3}\frac{t_{coll}}{t_i} = \frac{2}{3}\left(\frac{a_{coll}}{a_i}\right)^{3/2}$$

$$\Rightarrow \begin{cases} \dfrac{a_{coll}}{a_i} = 2^{2/3}\left(\dfrac{3\pi}{4}\right)^{2/3}\dfrac{(1+\Delta_i)^{2/3}}{(\Delta_i - \alpha_i)} \approx 2^{2/3}\left(\dfrac{3\pi}{4}\right)^{2/3}\dfrac{1}{A\Delta_i} \\[2ex] \dfrac{a_0}{a_i}\Delta_i = \Delta_{0,lin} \quad \text{(linearly extrapolated density excess)} \end{cases}$$

$$\Rightarrow \boxed{a_{coll} = 2^{2/3}\left(\frac{3\pi}{4}\right)^{2/3} A^{-1}\,\frac{1}{\Delta_{0,lin}}}$$

$$a_{coll} = \begin{cases} \dfrac{2.81}{\Delta_{0,lin}} & (\delta_{pec,i} = 0) \\[2ex] \boxed{\dfrac{1.69}{\Delta_{0,lin}}} & \text{(growing mode)} \end{cases}$$

(and corresponding $a_{turn} = 1.77/\Delta$ and $\boxed{a_{turn} = 1.06/\Delta}$

⇒ At expansion factor $a(t)$, the objects that collapse, had linearly extrapolated $\Delta_{0,lin}$:

$$\boxed{\Delta_{0,lin} = \frac{1.69}{a(t)}} \quad \text{growing mode}$$

Collapse :

- $r_{coll} = 0.$: $\underline{\underline{\Phi_r = 2\pi.}}$

- $H_i t_{coll} = \pi \left(\frac{3}{5}\right)^{3/2} \frac{1}{\Delta_i^{3/2}}$

$$\Rightarrow \boxed{a_{coll} = 2^{2/3} \left(\frac{3\pi}{4}\right)^{2/3} \frac{3}{5\Delta_{0,lin}} \approx \frac{1.69}{\Delta_{0,lin}}}$$

$(\Delta_{0,lin} = 1.69)$

Virialization

Full collapse (to one point) will not happen:

spherical model will break down
discreteness + random velocities particles become substantial:

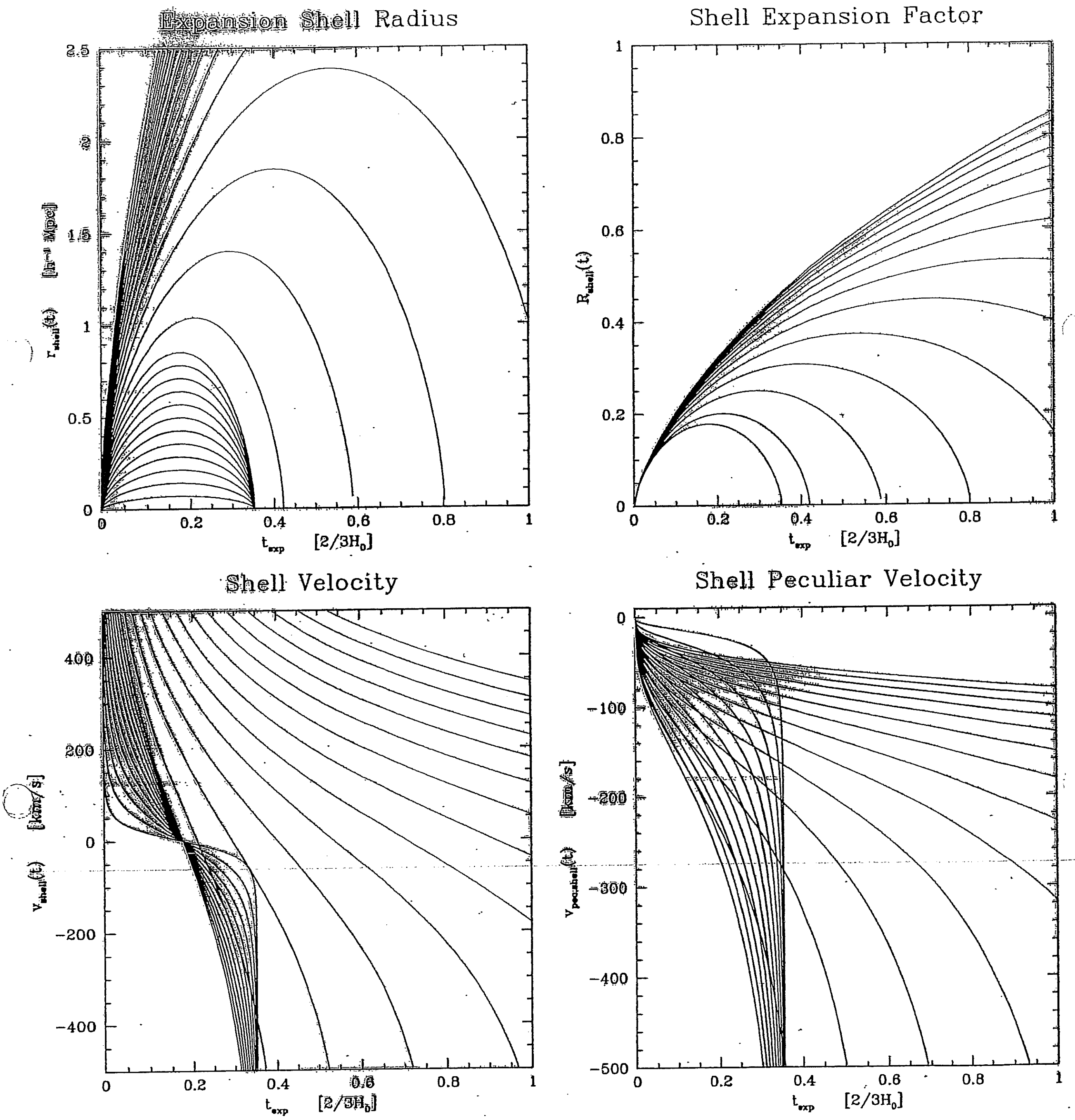
- distribution of energy through "violent relaxation" (Lynden-Bell):

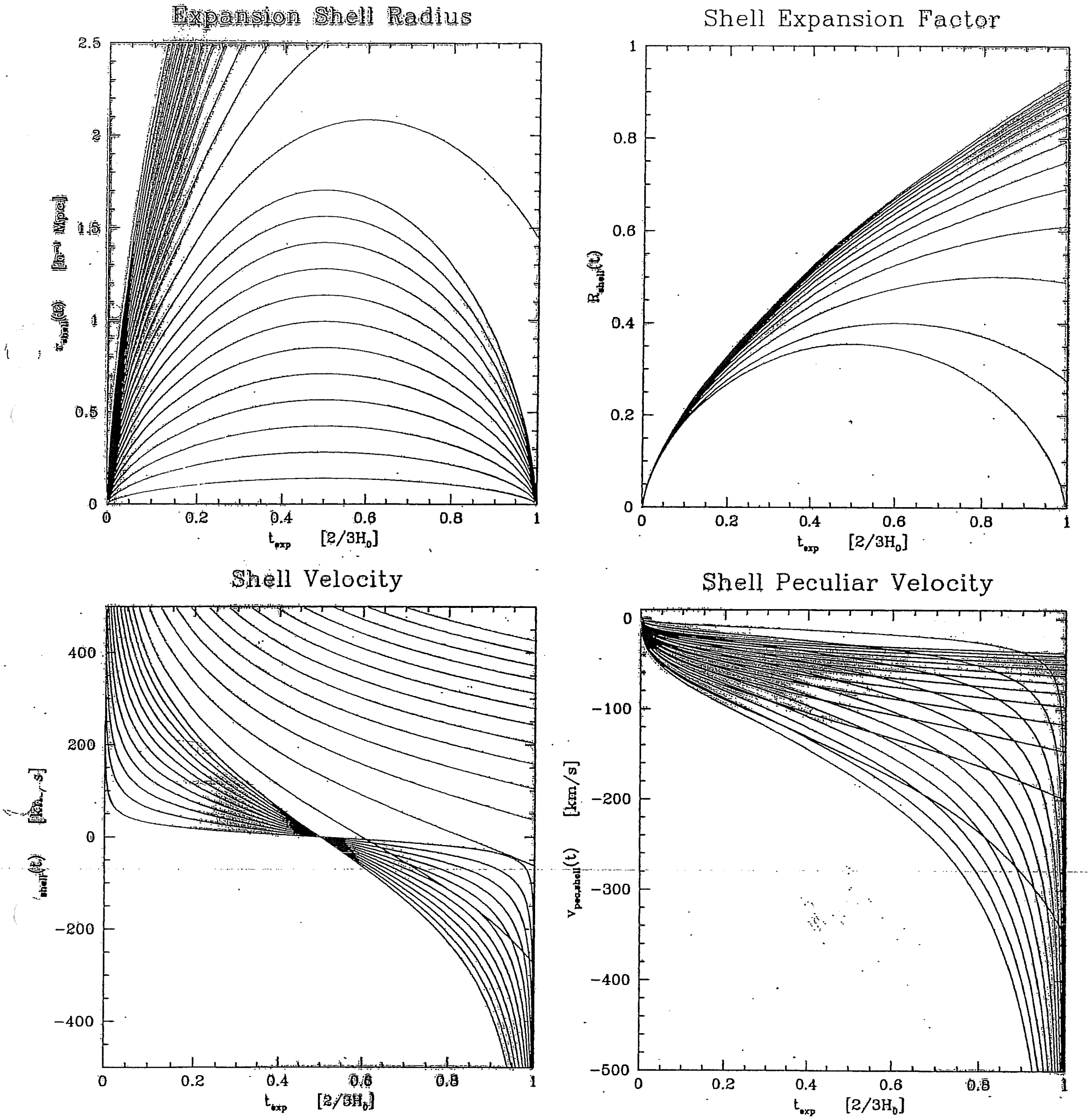
fast changes of potential ϕ on time-scales $\lesssim t_{dyn} \sim \sqrt{G\rho}^{\,-1}$: relaxation mechanism operates far more efficient than 2-body relax.

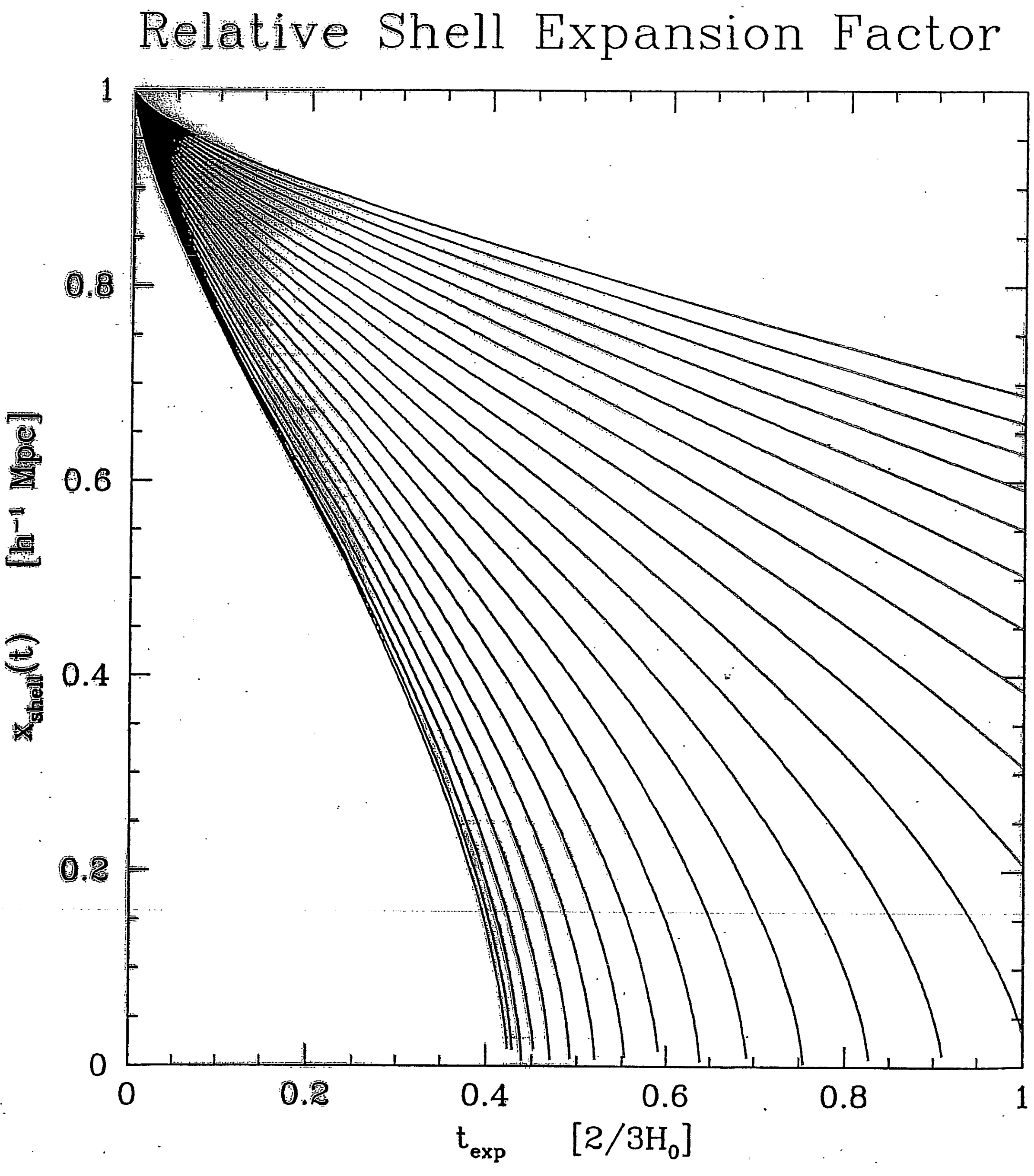
$$\Downarrow$$

Virialized System

radius : r_{vir}
velocity dispersion : σ_v
density : ρ_{vir}


Expansion Shell Radius
$r_{shell}(t)$ [h^{-1} Mpc]
t_{exp} [2/3H$_0$]
Shell Expansion Factor
$R_{shell}(t)$
t_{exp} [2/3H$_0$]
Shell Velocity
$v_{shell}(t)$ [km/s]
t_{exp} [2/3H$_0$]
Shell Peculiar Velocity
$v_{pec,shell}(t)$ [km/s]
t_{exp} [2/3H$_0$]


Expansion Shell Radius
r_shell(t) [h^-1 Mpc]
t_exp [2/3H_0]
Shell Expansion Factor
R_shell(t)
t_exp [2/3H_0]
Shell Velocity
v_shell(t) [km/s]
t_exp [2/3H_0]
Shell Peculiar Velocity
v_pec,shell(t) [km/s]
t_exp [2/3H_0]


Relative Shell Expansion Factor
1
0.8
0.6
0.4
0.2
0
$x_{shell}(t)$ [h^{-1} Mpc]
0
0.2
0.4
0.6
0.8
1
t_{exp} [$2/3H_0$]

$\Phi_r = \pi$:

- $$H_i t_{max} = \frac{\pi}{2}\left(\frac{3}{5}\right)^{3/2} \frac{1}{\Delta_i^{3/2}} = \frac{2}{3}\left(\frac{a_{max}}{a_i}\right)^{3/2}$$

$$\Rightarrow \boxed{a_{max} = \left(\frac{3\pi}{4}\right)^{2/3} \frac{3}{5\Delta_{o,lin}} \simeq \frac{1.06}{\Delta_{o,lin}}}$$

- $$1 + \Delta(t_{max}) = \left(\frac{3\pi}{4}\right)^2 \approx 5.55$$

$$\Rightarrow \boxed{\frac{\rho_{max}}{\rho_u} \approx 5.55} \qquad (\Delta_{o,lin} = 1.06)$$

- **Energy** $E = U + T$

U : Potential Energy

T : Kinetic Energy

$$U(R) = -\int_0^R \frac{G M(r)\, dM}{r}$$

$$\left.\begin{aligned} M(r) &= \frac{4\pi}{3}\rho_r(t)\, r^3 \\ dM &= 4\pi \rho_r(t)\, r^2 dr \end{aligned}\right\} \quad U = -\frac{3}{5}\frac{M^2(R)}{R}$$

$t = t_{max}$: $E = U \qquad (T = 0)$.

$$\boxed{E_{sphere} = -\frac{3}{5}\frac{G M^2(R_{max})}{R_{max}}}$$

Virial Theorem:

$$|U| = 2T$$

* **virial radius r_{vir}:**

$$U(r_{vir}) = -\frac{3}{5}\frac{GM^2(r_{vir})}{r_{vir}}$$

$$E = U(r_{vir}) + T(r_{vir}) = U(r_{max}) = -\frac{3}{5}\frac{M^2}{R_{max}}$$

$$\Longrightarrow T(r_{vir}) = \frac{3}{5}GM^2\left(\frac{1}{r_{vir}} - \frac{1}{r_{max}}\right)$$

$$\overset{VT}{=} \frac{1}{2}|U(r_{vir})| = \frac{1}{2}\frac{3}{5}GM^2\left(\frac{1}{r_{vir}}\right) = \frac{3}{5}GM^2\frac{1}{r_{max}}$$

$$\boxed{r_{vir} = \frac{1}{2}r_{max} = \frac{1}{2}\times\frac{3}{5}\frac{1}{\Delta_{0,lin}}}$$

* **virial velocity "σ_v"**

$$T_{vir} = \frac{1}{2}M\sigma_r^2 = \frac{3}{5}M^2\frac{1}{r_{max}}$$

$$\boxed{\sigma_v = \sqrt{\frac{6GM}{5r_{max}}}} = \sqrt{2GM\Delta_{0,lin}}$$

* **virialization epoch**

$$r(t) = \frac{1}{2}r_{max}: \qquad \Phi_r = \frac{3\pi}{2}$$

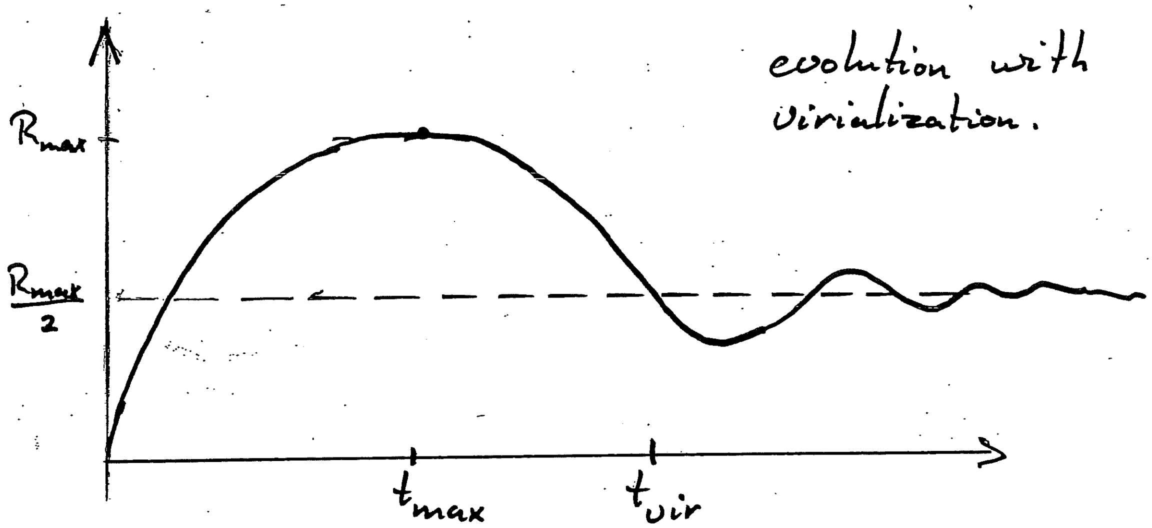
$$\frac{\rho_{vir}}{\rho_u} = 1 + \Delta(t_{vir}) = \frac{9}{2}\left(\frac{3\pi}{2} + 1\right)^2$$

$$= \frac{(9\pi + 6)^2}{8} \approx 147$$

$$\Downarrow$$

$$\boxed{\delta_{vir} = \frac{\rho_{vir}}{\rho_u} \approx 147}$$

$$\boxed{a_{vir} = \left(\frac{3}{2}\right)^{2/3}\left(\frac{3\pi}{4}\right)^{2/3} \frac{3}{5\Delta_{0,lin}} \simeq \frac{1.40}{\Delta_{0,lin}}}$$



evolution with virialization.

$$\rho_{vir} \gtrsim 147\ \rho_u = 147 \times \rho_{0,u}\,(1 + z_{vir})^3$$

$$\boxed{\rho_{vir} \gtrsim 147\ \frac{3\Omega_0 H_0^2}{8\pi G}\left(1 + z_{vir}\right)^3}$$

$$\rho_{vir} \gtrsim 147 \frac{3\Omega_0 H_0^2}{8\pi G}(1+z_{vir})^3$$

$\Longrightarrow$ if you know ρ_{vir}, you can tell when the object got virialized at the earliest:

VT:

$$\sigma_v^2 = \frac{6GM}{10\, r_{vir}}$$

$$\sigma_v^6 = \left(\frac{3}{5}\right)^3 \frac{G^3 M^3}{r_{vir}^3}$$

$$= \left(\frac{3}{5}\right)^3 \frac{G^3 M^3}{\left(\frac{3}{4\pi}\right)\frac{1}{\rho_{vir}} M}$$

$$\approx \left(\frac{4\pi}{3}\right) \rho_{vir}\, G^3 M^2 \left(\frac{3}{5}\right)^3$$

for object with mass M and velocity dispersion σ_v:

$$\rho_{vir} \approx \left(\frac{5}{3}\right)^3 \frac{\sigma_v^6}{\left(\frac{4\pi}{3}\right) G^3 M^2}$$

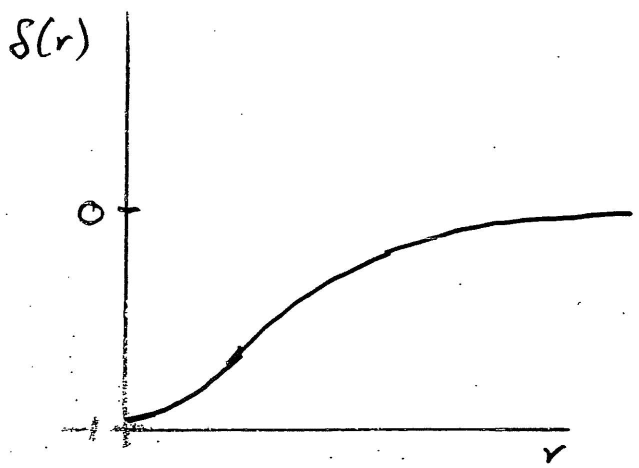
$$\Longrightarrow \boxed{(1+z_{vir}) \lesssim 0.47 \left(\frac{\sigma_v}{100\ \mathrm{km/s}}\right)^2 \left(\frac{M}{10^{12} M_\odot}\right)^{-2/3} (\Omega_0 h^2)^{-1/3}}$$

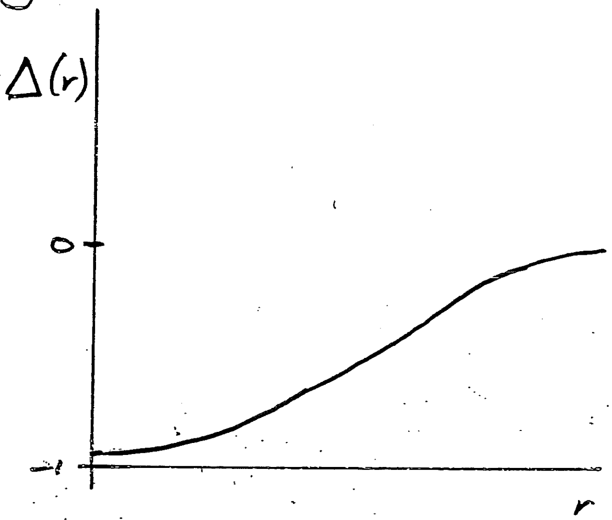
<u>Galaxies</u>: $M \sim 10^{12} M_\odot$, $\sigma_v \sim 300$ km/s $\Big\}$ $z_{vir} \lesssim 10$

<u>Clusters</u>: $\sigma_v \sim 1000$ km/s, $M \sim 10^{15} M_\odot$ $\Big\}$ $z_{vir} \lesssim 1$

II) Shell Crossing & Void Expansion

- **Spherical Underdensity** (with some profile $\delta(r)$



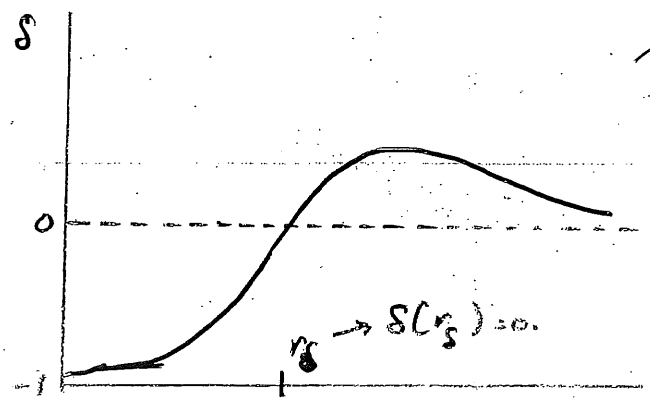


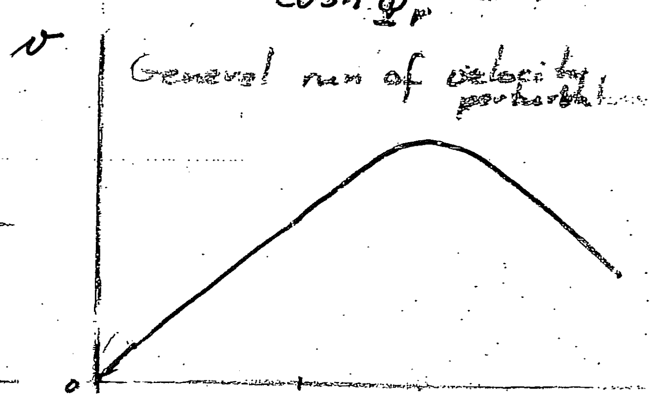
- **Motion of (open) shells:**

$$\begin{cases} R(t,r_i) = \dfrac{1}{2}\,\dfrac{1+\Delta_{ci}}{\alpha_i-\Delta_{ci}}\,(\cosh\Phi_r - 1) \\[2ex] H_i t = \dfrac{1}{2}\,\dfrac{1+\Delta_{ci}}{(\alpha_i-\Delta_{ci})^{3/2}}\,(\sinh\Phi_r - \Phi_r) \end{cases}$$

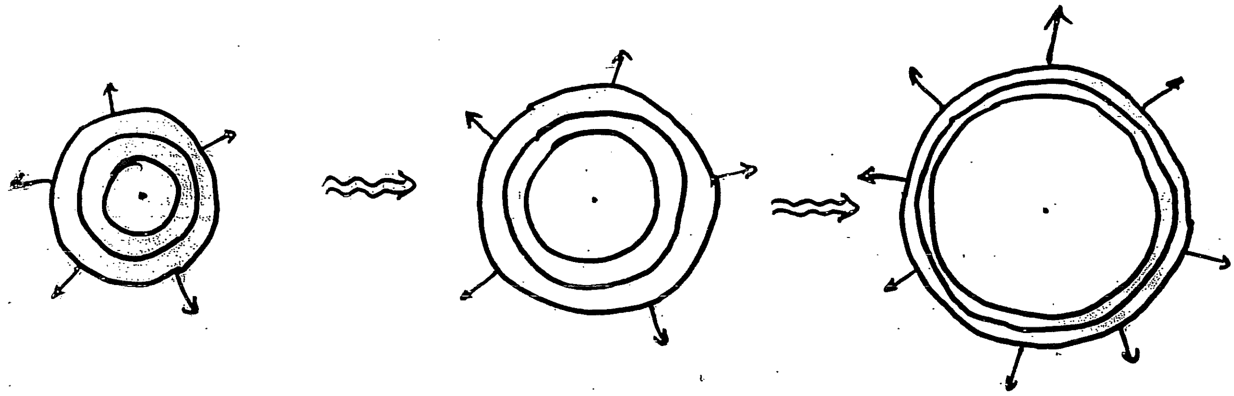
Expansion velocity:

$$v(r,t) = H_i\, r_i\, (\alpha_i - \Delta_{ci})^{1/2}\,\frac{\sinh\Phi_r}{\cosh\Phi_r - 1}$$

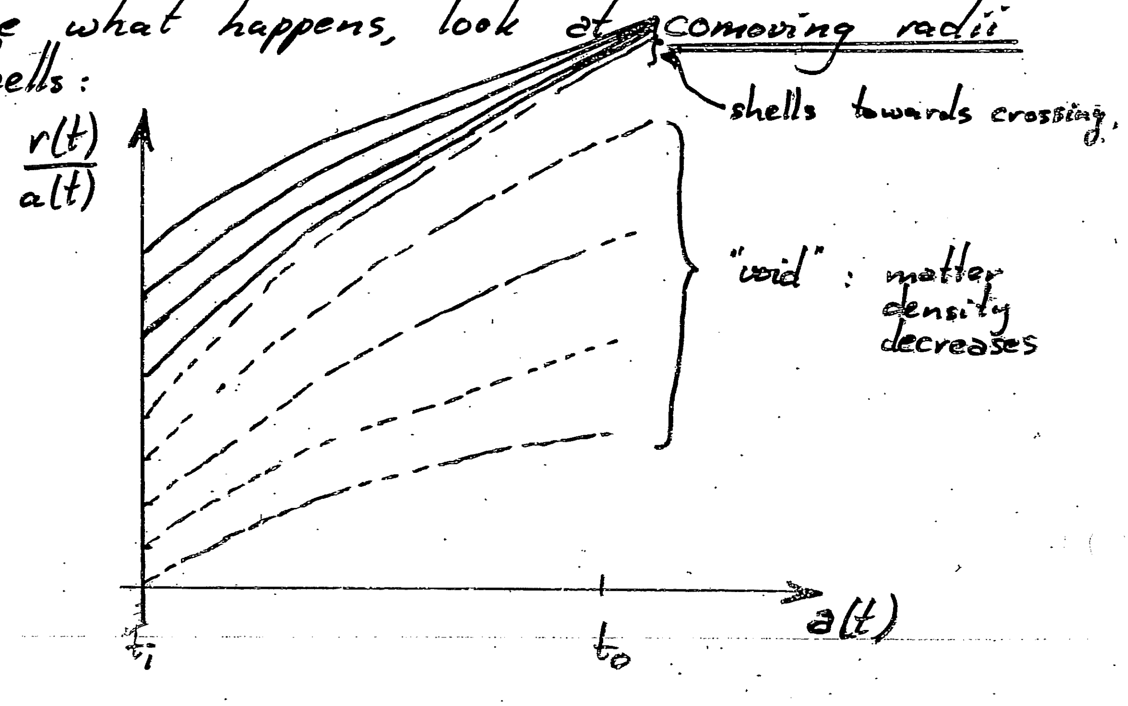




If inner shell expands faster than outer shell, this can lead to <u>'shell crossing'</u> :



To see what happens, look at <u>comoving radii</u> of shells :



<u>Which shells cross and when?</u>

- Shell radius $\quad r(t) = r(t, r_i, \epsilon_i, \Phi_r)$.

  (with $\epsilon_i = -\Delta_i$: initial density deficit
  $\qquad r_i$: initial radius)

- Then, shell crossing if :

$$\zeta = \left(\frac{\partial r}{\ } + \frac{\partial r}{\ } \frac{\partial \epsilon_i}{\ } + \frac{\partial r}{\ } \frac{\partial \Phi_r}{\ } \right) \zeta$$

$$\Rightarrow \quad \boxed{\delta r = \frac{r(t)}{r_i}\,\delta r_i \left\{ 1 - \frac{\partial \ln G_i}{\partial \ln r_i} \left(1 - \frac{3}{2}\, \frac{\sinh \bar{\Phi}_r (\sinh \bar{\Phi}_r - \bar{\Phi}_r)}{(\cosh \bar{\Phi}_r - 1)^2} \right) \right\}}$$

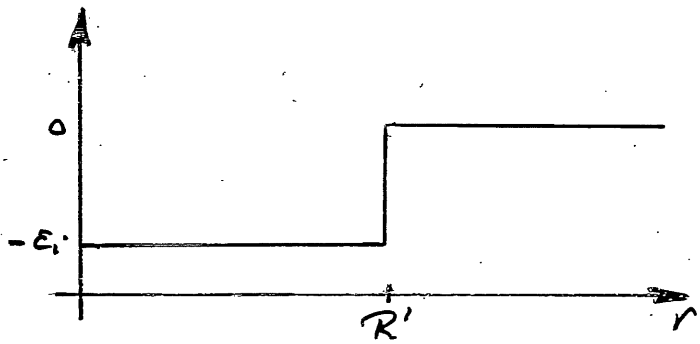
with:

$$G_i(\varepsilon_i, \Omega_i) = \left\{ \Omega_i + \tfrac{2}{3} f(\Omega_i) \right\} \varepsilon_i + (1 - \Omega_i)$$

$$G_i(\varepsilon_i, \Omega_i) \equiv A(\Omega_i)\,\varepsilon_i + B(\Omega_i)$$

$$\Rightarrow \quad \boxed{\frac{\partial \ln G_i}{\partial \ln r_i} = \frac{A(\Omega_i)\,\varepsilon_i}{A(\Omega_i)\,\varepsilon_i + B(\Omega_i)}\; \frac{\partial \ln \varepsilon_i}{\partial \ln r_i}}$$

- <u>Concentrate on Tophat Void:</u>



- $\Delta M(r_1) = \Delta M(r_2) \quad \Rightarrow \quad \varepsilon_i\, r_i^3 = \text{cst}$ outside void $(r > R')$

$$\Rightarrow \quad \boxed{\frac{\partial \ln \varepsilon_i}{\partial \ln r_i} = -3} \quad \Rightarrow \quad \boxed{\frac{\partial \ln G_i}{\partial \ln r_i} = -3\, \frac{A(\Omega_i)\,\varepsilon_i}{A(\Omega_i)\,\varepsilon_i + B(\Omega_i)}}$$

<u>Shell Crossing</u> $(\delta r = 0)$: $\quad (\longrightarrow \bar{\Phi}_{sc})$

$$\frac{\sinh \bar{\Phi}_{sc} (\sinh \bar{\Phi}_{sc} - \bar{\Phi}_{sc})}{(\cosh \bar{\Phi}_{sc} - 1)^2} = \frac{8\, A(\Omega_i)\,\varepsilon_i + 2\, B(\Omega_i)}{9\, A(\Omega_i)\,\varepsilon_i}$$

Einstein-de Sitter Universe

- $\Omega_i = \Omega = 1 \quad \Rightarrow \quad B(\Omega_i) = 1 - \Omega_i = 0.$

$$G(\varepsilon_i, \Omega_i) = A\varepsilon_i + B = A\varepsilon_i$$

- $$\frac{\sinh\Phi_{sc}(\sinh\Phi_{sc} - \Phi_{sc})}{(\cosh\Phi_{sc} - 1)^2} = \frac{8}{9} \quad \Rightarrow \quad \boxed{\Phi_{sc} = 3.527918}$$

- Note: Φ_{sc} independent of mode velocity perturbation.

Epoch of shell-crossing:

- $$H_i t = \frac{1}{2}\frac{1}{G_i^{3/2}}(\sinh\Phi_r - \Phi_r)$$

- $$H_i t \overset{\Omega=1.}{=} \frac{2}{3}\frac{t}{t_i} = \frac{2}{3}\left(\frac{a}{a_i}\right)^{3/2}$$

- $$G_i = A\,\varepsilon_i \quad \Longrightarrow \quad G_i\frac{a_0}{a_i} = A\frac{a_0}{a_i}\varepsilon_i = A\tilde{\varepsilon}_0$$

($\hat{\varepsilon}_0$: linearly extrapolated perturbation).

$$\Rightarrow \quad \boxed{a_{sc} = \left(\frac{3}{4}\right)^{2/3}(\sinh\Phi_{sc} - \Phi_{sc})^{2/3}\frac{1}{A}\frac{1}{\tilde{\varepsilon}_0}}$$

$$\boxed{a_{sc} = \frac{4.6765}{A(\Omega_i)}\frac{1}{\tilde{\varepsilon}_0} = \begin{cases} \dfrac{2.8059}{\tilde{\varepsilon}_0} & \text{growing mode} \\[2ex] \dfrac{4.6765}{\tilde{\varepsilon}_0} & v_i = 0 \end{cases}}$$

- ## Mass Perturbation at Shell Crossing:

$$\Delta_{lin,sc} \equiv -\left(\frac{\delta M}{M}\right)_{sc} = \varepsilon_i \frac{a(t_{sc})}{a(t_i)}$$

linearly extrapolated density deficit.

$$\Rightarrow \quad \Delta_{lin,sc} = \left(\frac{3}{4}\right)^{2/3} \frac{\varepsilon_i}{G_i(\varepsilon_i, \Omega_i)} \left(\sinh \Phi_{sc} - \Phi_{sc}\right)^{2/3}$$

$\downarrow \Omega_i = 1.$

$$\boxed{\Delta_{lin,sc} = \left(\frac{3}{4}\right)^{2/3} \frac{1}{A} \left(\sinh \Phi_{sc} - \Phi_{sc}\right)^{2/3} = \frac{4.6765}{A}}$$

$$\boxed{\Delta_{lin,sc} = \begin{cases} 2.8059 \\ 4.6765 \end{cases}}$$

growing mode

$v_{pec,i} = 0$

- ## Radius of void at shell-crossing:

Edge of void at t_i: $r(t_i) = r_b$

t_{sc}: $r(t_{sc}) = r_{sc}$

$$\boxed{\frac{r_{sc}}{r_b} = \mathcal{R} = \frac{1}{2} \frac{1}{G_i} \left(\cosh \Phi_{sc} - 1\right)}$$

combine with:

$$\frac{4}{3} \frac{t_{sc}}{t_i} \frac{1}{\sinh \Phi_{sc} - \Phi_{sc}} = \frac{1}{G_i^{3/2}}$$

$$\Longrightarrow \quad \boxed{\frac{r_{sc}}{r_b} = \frac{1}{2}\left(\frac{4}{3}\right)^{2/3} \frac{\cosh \Phi_{sc} - 1}{(\sinh \Phi_{sc} - \Phi_{sc})^{2/3}} \left(\frac{t_{sc}}{t_i}\right)^{2/3}}$$

$$= 1.7151 \left(\frac{t_{sc}}{t_i}\right)^{2/3}$$

Note: independent of mode perturbation.

- in comoving coordinates:

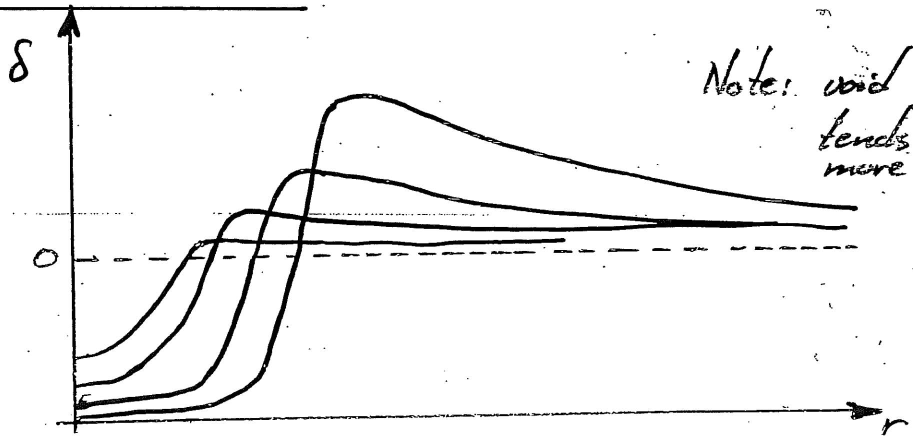
$$r_{sc} = a_{sc} R_{sc}$$

$$r_i = a_i R_i.$$

$$\Longrightarrow \quad \boxed{\frac{R_{sc}}{R_i} = 1.7151}$$

- Void undergoes shell crossing when it has expanded by a factor 1.7151 in comoving space.

Generic Void Evolution:



Note: void profile tends more and more to tophat