Gaussian Random Fields

Lecture course LSS2014
University Groningen
Nov. 2014-Jan. 2015

Power Spectrum

Power Spectrum P(k)

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \hat{\delta}(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

$$(2\pi)^3 P(k_1)\, \delta_{\mathrm{D}}(\mathbf{k}_1 - \mathbf{k}_2) \equiv \langle \hat{f}(\mathbf{k}_1) \hat{f}^*(\mathbf{k}_2) \rangle$$

$$\Updownarrow$$

$$P(k) \propto \langle \hat{f}(\mathbf{k}) \hat{f}^*(\mathbf{k}) \rangle$$

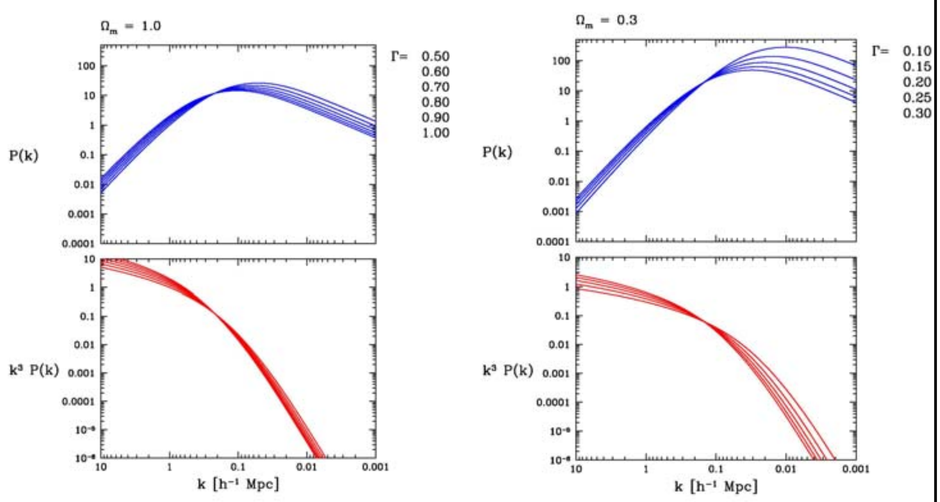
CDM Power Spectrum P(k)

$$P_{\mathrm{CDM}}(k) \propto \frac{k^n}{[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{1/2}} \times \frac{[\ln(1 + 2.34q)]^2}{(2.34q)^2}$$

$$q = k/\Gamma$$

$$\Gamma = \Omega_{m,\circ} h \exp\left\{-\Omega_b - \frac{\Omega_b}{\Omega_{m,\circ}}\right\}$$

Power Spectrum P(k)



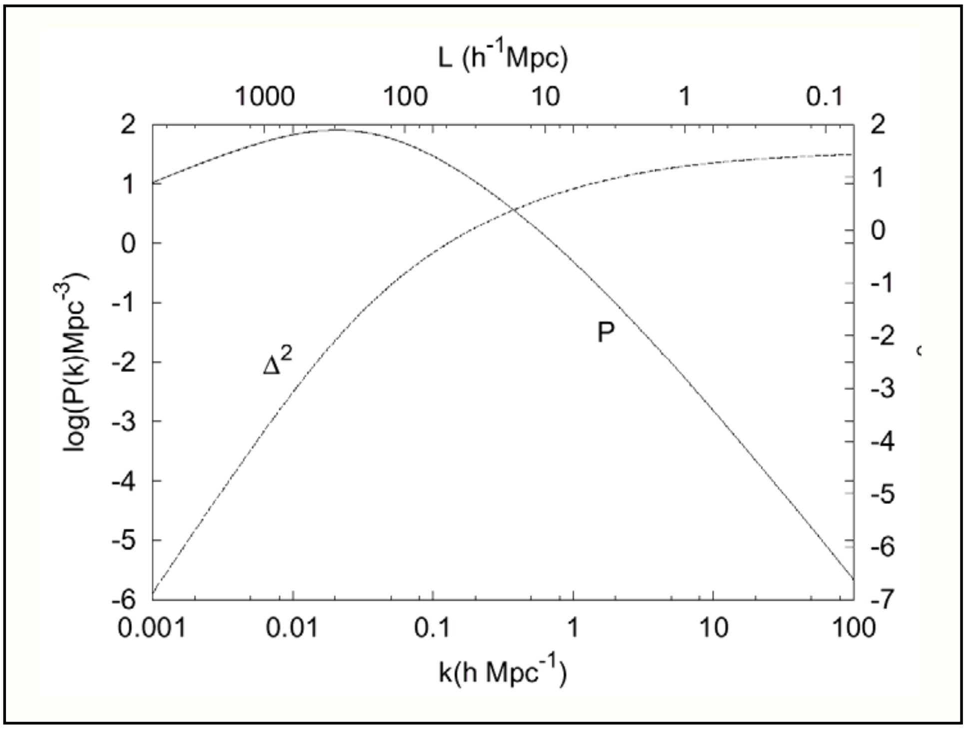
Power Spectrum - Correlation Function

$$P(k) = \int d^3 r \xi(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$$

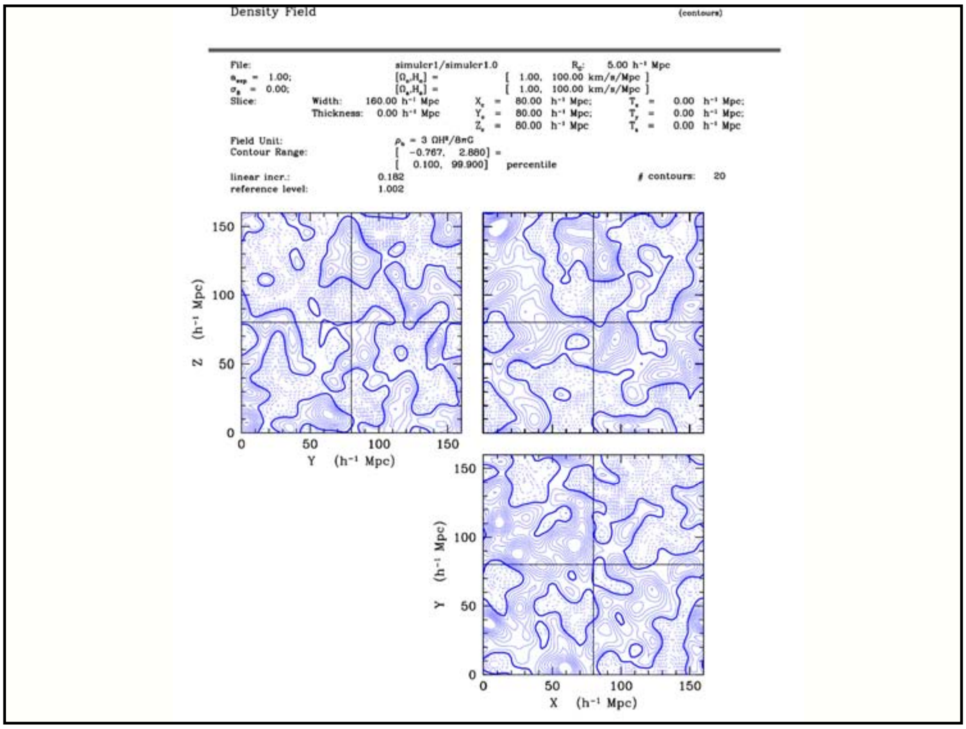
$$\xi(\vec{r}) = \int \frac{d^3 k}{(2\pi)^3} P(k) e^{-i\vec{k}\cdot\vec{r}}$$

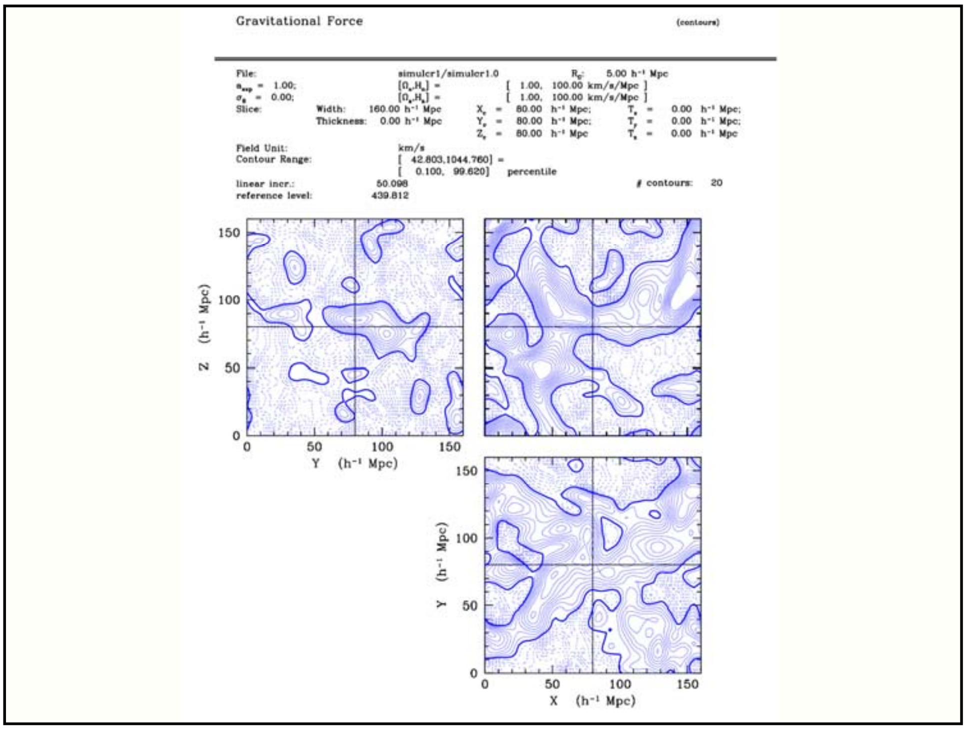
Isotropy: $$\xi(r) = 4\pi \int_0^\infty \frac{k^2 dk}{(2\pi)^3} P(k) \frac{\sin(kr)}{kr}$$

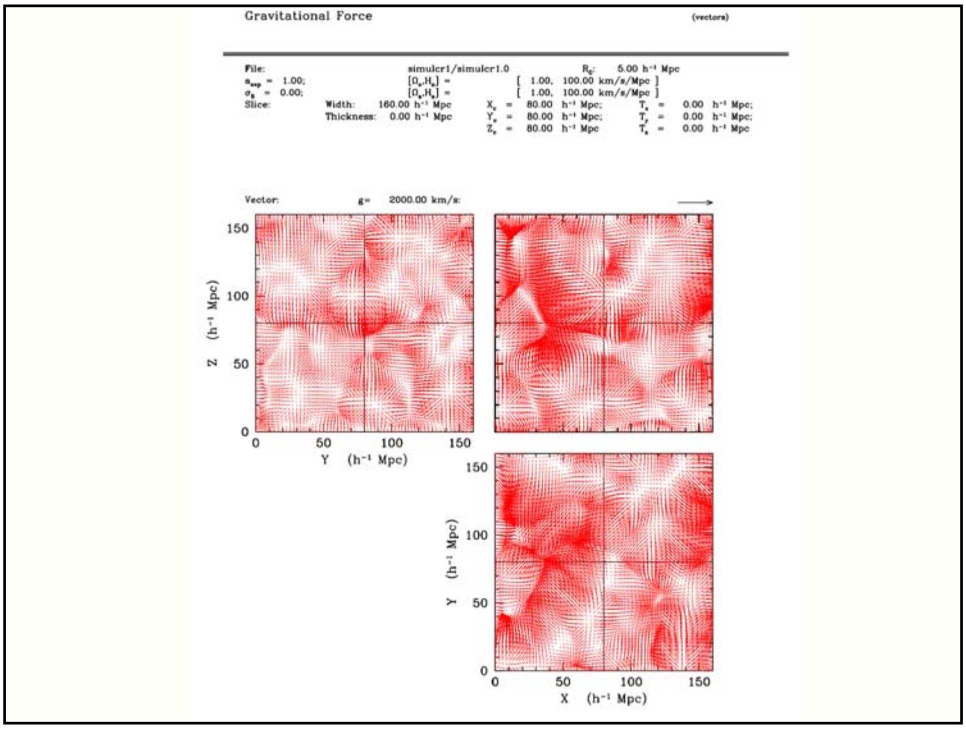
Delta-power $$\Delta^2(k) = \frac{1}{2\pi^2} P(k) k^2$$

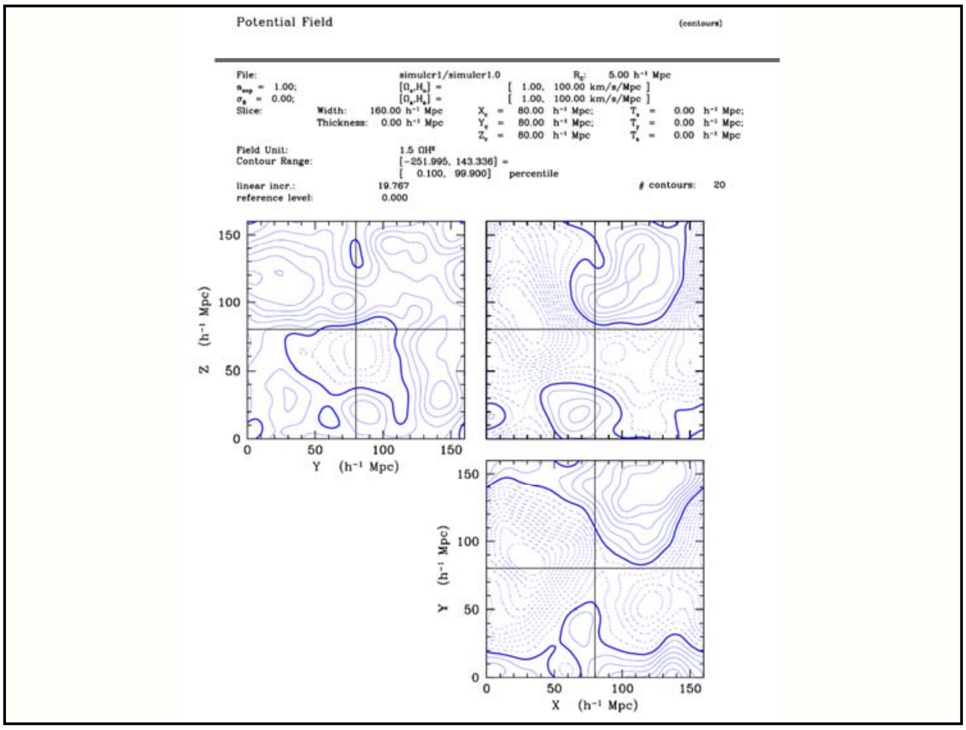

L (h-1Mpc)
1000
100
10
1
0.1
log(P(k)Mpc-3)
k(h Mpc-1)
0.001
0.01
0.1
1
10
100
P
Δ2
2
1
0
-1
-2
-3
-4
-5
-6
-7


Field Realizations


Density Field
(contours)
File: simulcr1/simulcr1.0
R_G: 5.00 h^{-1} Mpc
a_{exp} = 1.00;
σ_8 = 0.00;
$[\Omega_0, H_0]$ = [1.00, 100.00 km/s/Mpc]
$[\Omega_s, H_s]$ = [1.00, 100.00 km/s/Mpc]
Slice: Width: 160.00 h^{-1} Mpc Thickness: 0.00 h^{-1} Mpc
X_c = 80.00 h^{-1} Mpc; Y_c = 80.00 h^{-1} Mpc; Z_c = 80.00 h^{-1} Mpc
T_x = 0.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc; T_z = 0.00 h^{-1} Mpc
Field Unit: $\rho_u = 3\,\Omega H^2/8\pi G$
Contour Range: [-0.767, 2.880] = [0.100, 99.900] percentile
linear incr.: 0.182
reference level: 1.002
contours: 20
Z (h^{-1} Mpc)
Y (h^{-1} Mpc)
X (h^{-1} Mpc)
0
50
100
150


Gravitational Force
(contours)
File: simulcr1/simulcr1.0
R_G: 5.00 h^{-1} Mpc
a_{exp} = 1.00;
σ_8 = 0.00;
$[\Omega_0, H_0]$ = [1.00, 100.00 km/s/Mpc]
$[\Omega_s, H_s]$ = [1.00, 100.00 km/s/Mpc]
Slice: Width: 160.00 h^{-1} Mpc Thickness: 0.00 h^{-1} Mpc
X_c = 80.00 h^{-1} Mpc; Y_c = 80.00 h^{-1} Mpc; Z_c = 80.00 h^{-1} Mpc
T_x = 0.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc; T_z = 0.00 h^{-1} Mpc
Field Unit: km/s
Contour Range: [42.803,1044.760] = [0.100, 99.620] percentile
linear incr.: 50.098
reference level: 439.812
contours: 20
Z (h^{-1} Mpc)
Y (h^{-1} Mpc)
X (h^{-1} Mpc)
0
50
100
150


Gravitational Force
(vectors)
File: simulcr1/simulcr1.0
R_G: 5.00 h^{-1} Mpc
a_{exp} = 1.00;
σ_8 = 0.00;
$[\Omega_o,H_o]$ = [1.00, 100.00 km/s/Mpc]
$[\Omega_s,H_s]$ = [1.00, 100.00 km/s/Mpc]
Slice: Width: 160.00 h^{-1} Mpc
Thickness: 0.00 h^{-1} Mpc
X_c = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;
Y_c = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;
Z_c = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc
Vector: g= 2000.00 km/s:
Z (h^{-1} Mpc)
Y (h^{-1} Mpc)
X (h^{-1} Mpc)
0
50
100
150


Potential Field
(contours)
File: simulcr1/simulcr1.0
R_G: 5.00 h^{-1} Mpc
a_{exp} = 1.00;
σ_8 = 0.00;
$[\Omega_o,H_o]$ = [1.00, 100.00 km/s/Mpc]
$[\Omega_s,H_s]$ = [1.00, 100.00 km/s/Mpc]
Slice: Width: 160.00 h^{-1} Mpc
Thickness: 0.00 h^{-1} Mpc
X_c = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;
Y_c = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;
Z_c = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc
Field Unit: 1.5 ΩH^2
Contour Range: [-251.995, 143.336] =
[0.100, 99.900] percentile
linear incr.: 19.767
reference level: 0.000
contours: 20
Z (h^{-1} Mpc)
Y (h^{-1} Mpc)
X (h^{-1} Mpc)
0
50
100
150

Power Spectrum

Phases & Pattern Information

Random Field Phases

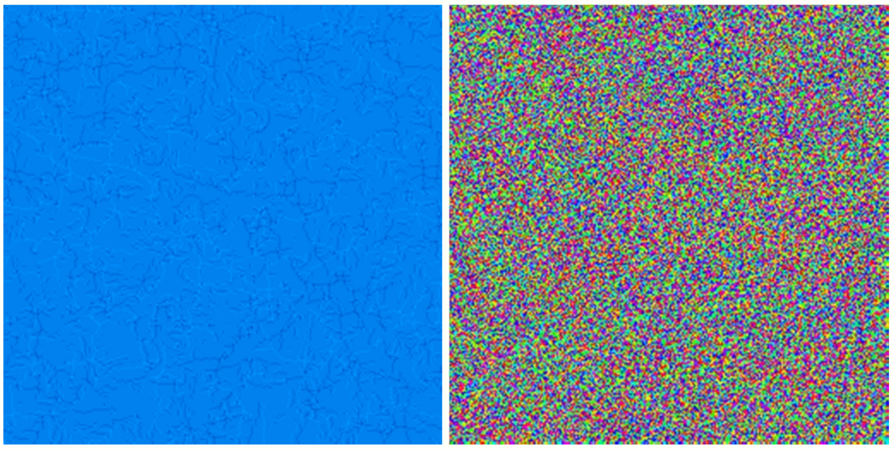
$$f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{f}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\hat{f}(\vec{k}) = \hat{f}_r(\vec{k}) + i\,\hat{f}_i(\vec{k}) = \left|\hat{f}(\vec{k})\right| e^{i\theta(k)}$$

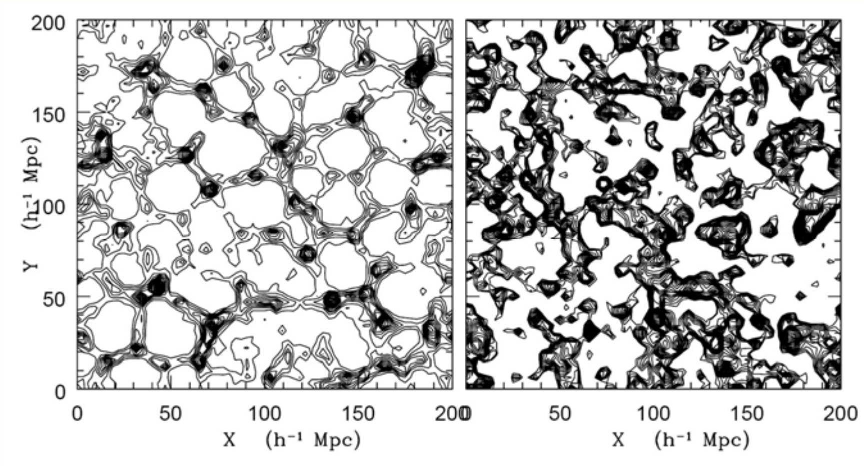
When a field is a Random Gaussian Field, its phases θ(k) are uniformly distributed over the interval [0,2π]:

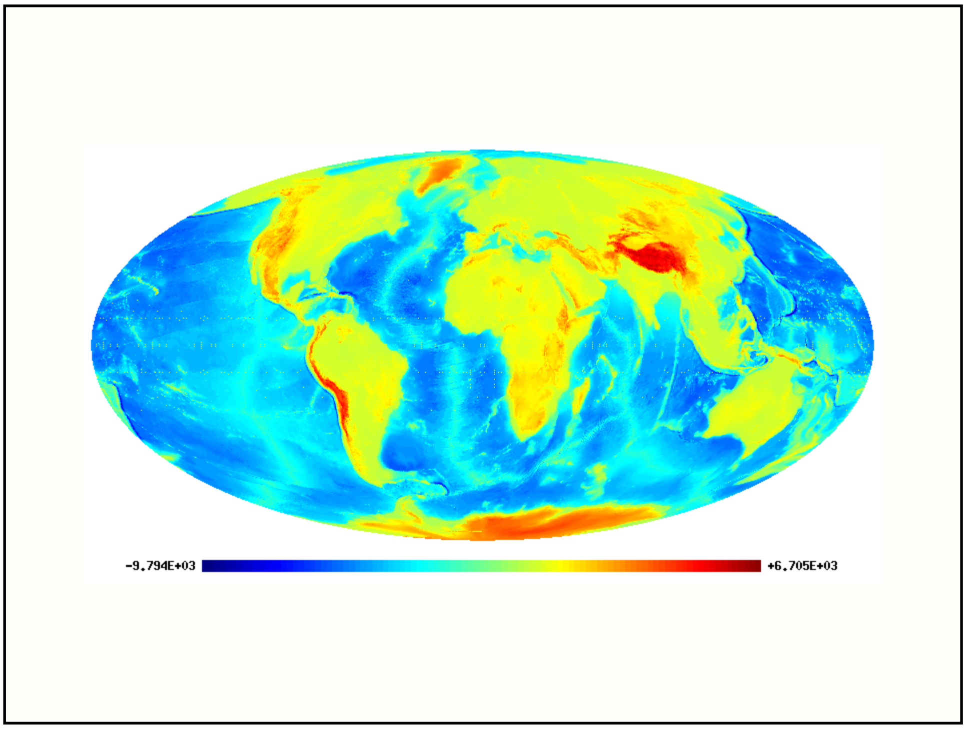
$$\theta(k) \in U[0, 2\pi]$$

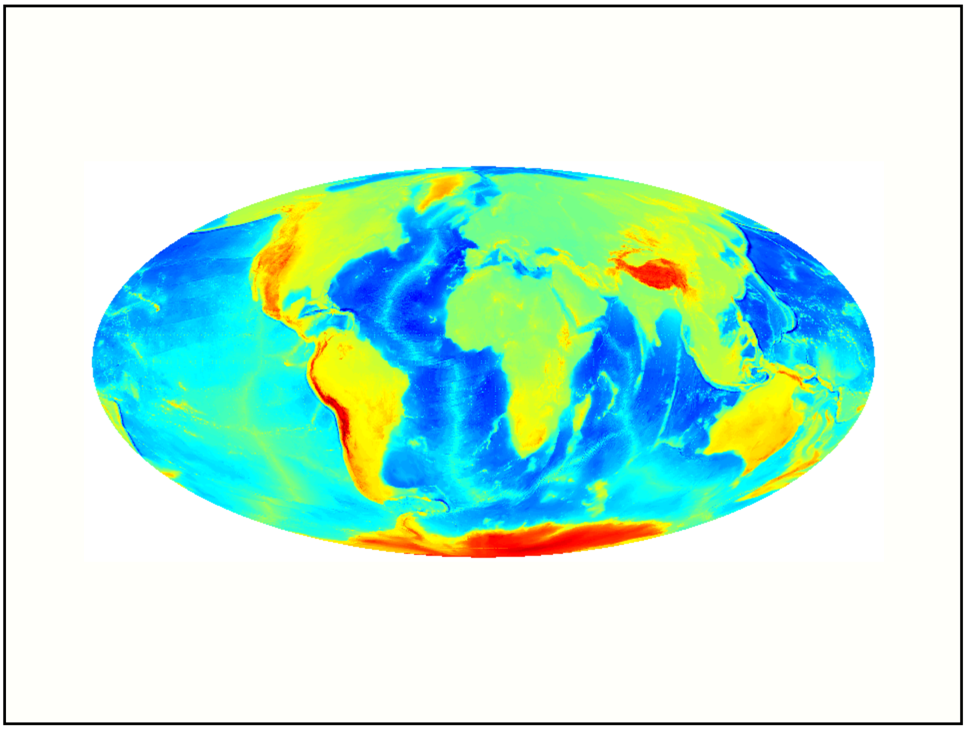
As a result of nonlinear gravitational evolution, we see the phases acquire a distinct non-uniform distribution.

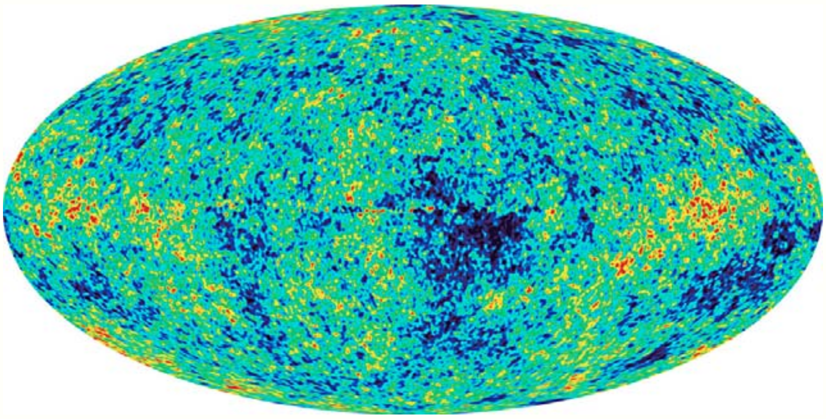


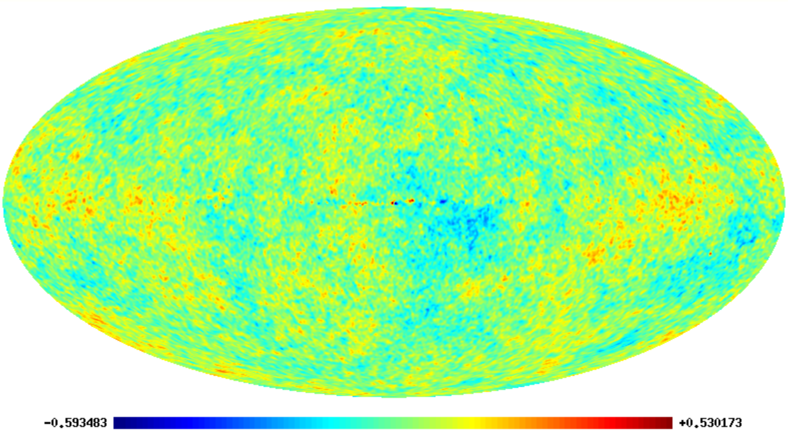
Power Spectrum: Pattern Information & Phases

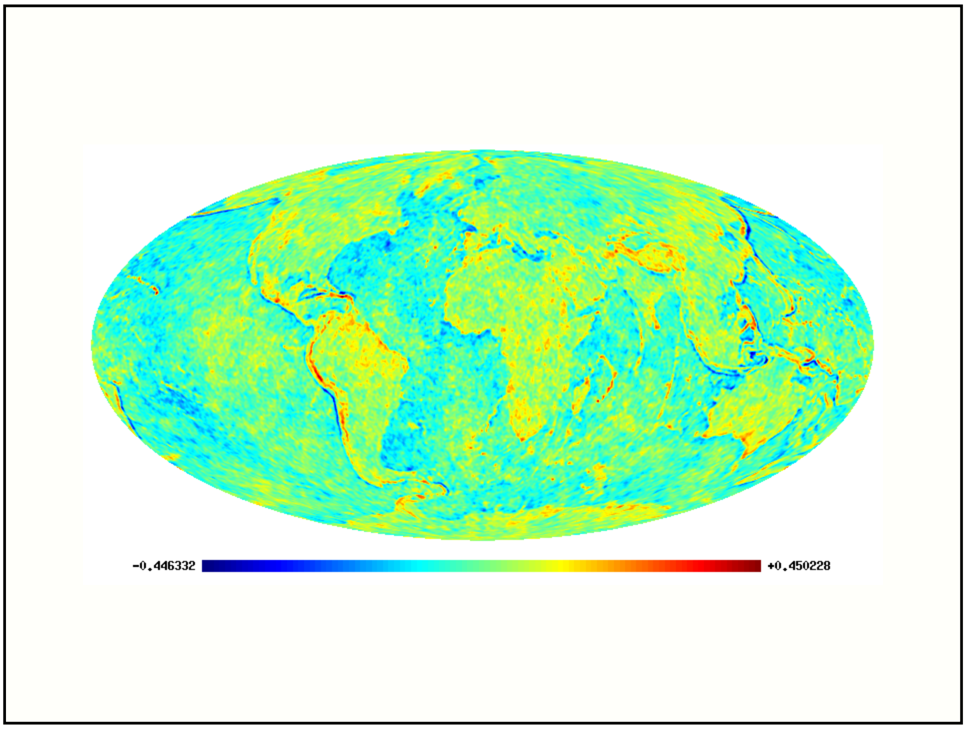


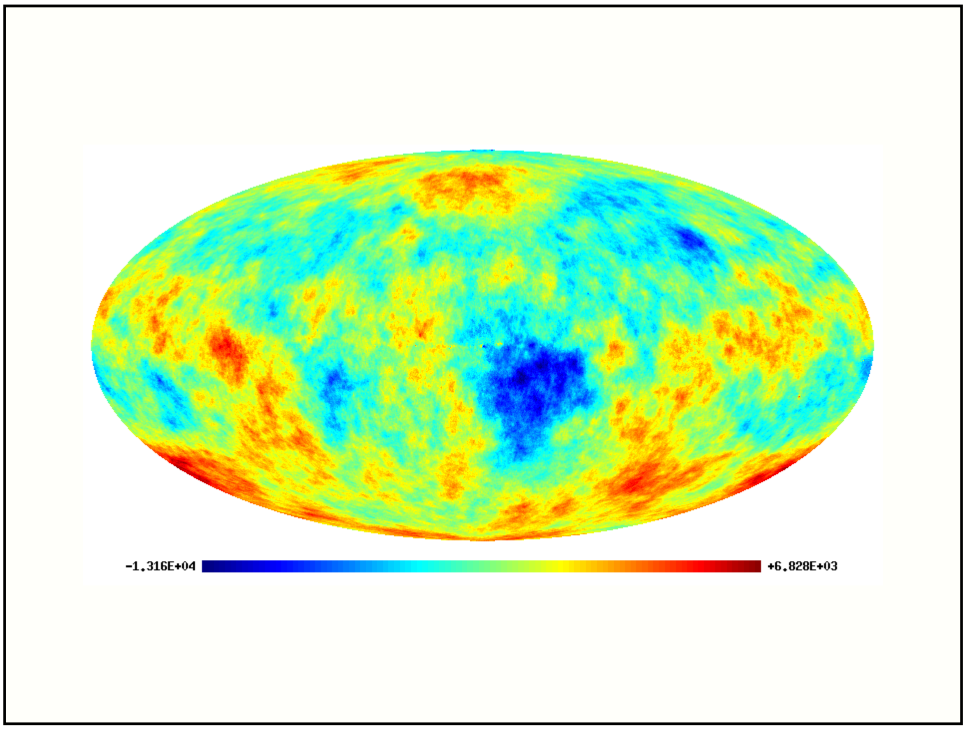

-9.794E+03
+6.705E+03

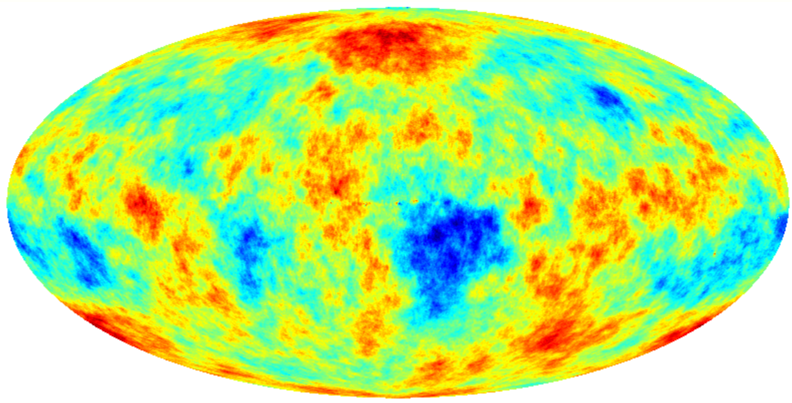




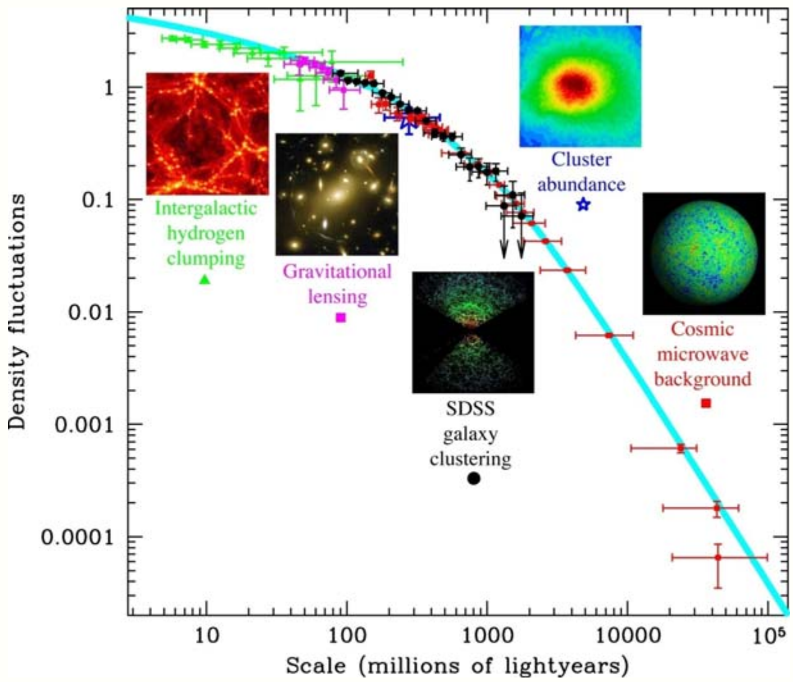

-0.593483
+0.530173


-0.446332
+0.450228


-1.316E+04
+6.828E+03



Power Spectrum Measurement



CDM Power Spectrum P(k)

$$P_{\rm CDM}(k) \propto \frac{k^n}{[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{1/2}} \times \frac{[\ln(1 + 2.34q)]^2}{(2.34q)^2}$$

$$q = k/\Gamma$$

$$\Gamma = \Omega_{m,\circ} h \exp\left\{-\Omega_b - \frac{\Omega_b}{\Omega_{m,\circ}}\right\}$$

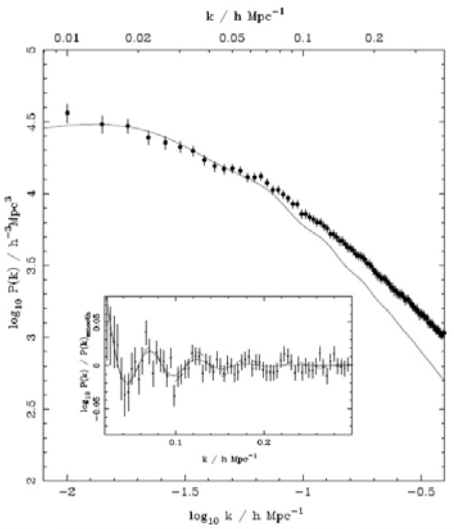


FIG. 12.— The redshift-space power spectrum recovered from the combined SDSS main galaxy and LRG sample, optimally weighted for both density changes and luminosity dependent bias (solid circles with 1-σ errors). A flat Λ cosmological distance model was assumed with $\Omega_M = 0.24$. Error bars are derived from the diagonal elements of the covariance matrix calculated from 2000 log-normal catalogues created for this cosmological distance model, but with a power spectrum amplitude and shape matched to that observed (see text for details). The data are correlated, and the width of the correlations is presented in Fig. 10 (the correlation between data points drops to < 0.33 for $\Delta k > 0.01\,h\,\mathrm{Mpc}^{-1}$). The correlations are smaller than the oscillatory features observed in the recovered power spectrum. For comparison we plot the model power spectrum (solid line) calculated using the fitting formulae of Eisenstein & Hu (1998); Eisenstein et al. (2006), for the best fit parameters calculated by fitting the WMAP 3-year temperature and polarisation data, $h = 0.73$, $\Omega_M = 0.24$, $n_s = 0.96$ and $\Omega_b/\Omega_M = 0.174$ (Spergel et al. 2006). The model power spectrum has been convolved with the appropriate window function to match the measured data, and the normalisation has been matched to that of the large-scale ($0.01 < k < 0.06\,h\,\mathrm{Mpc}^{-1}$) data. The deviation from this low Ω_M linear power spectrum is clearly visible at $k \gtrsim 0.06\,h\,\mathrm{Mpc}^{-1}$, and will be discussed further in Section 6. The solid circles with 1σ errors in the inset show the power spectrum ratioed to a smooth model (calculated using a cubic spline fit as described in Percival et al. 2006) compared to the baryon oscillations in the (WMAP 3-year parameter) model (solid line), and shows good agreement. The calculation of the matter density from these oscillations will be considered in a separate paper (Percival et al. 2006). The dashed line shows the same model without the correction for the damping effect of small-scale structure growth of Eisenstein et al. (2006). It is worth noting that this model is not a fit to the data, but a prediction from the CMB experiment.

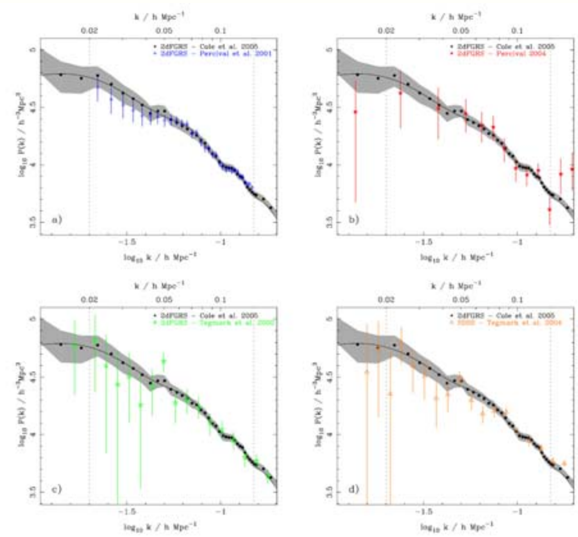
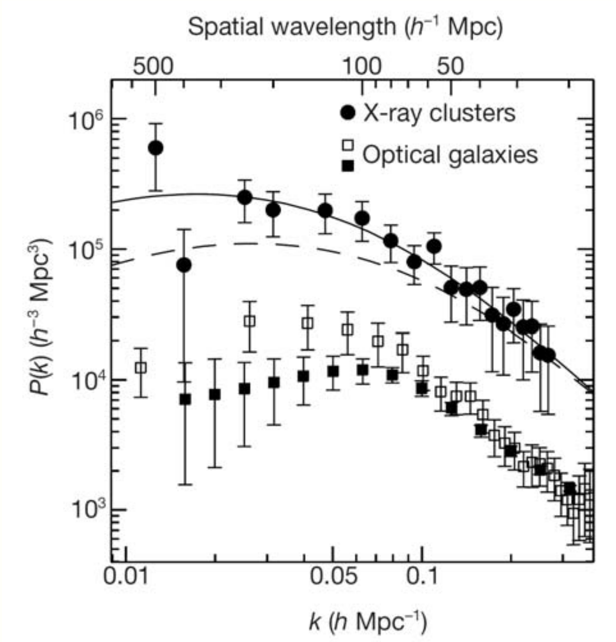


Figure 16. The redshift-space power spectrum calculated in this paper (solid circles with 1σ errors shown by the shaded region) compared with other measurements of the 2dFGRS power-spectrum shape by (a) Percival et al. (2001), (b) Percival (2005), and (c) Tegmark et al. (2002). For the data with window functions, the effect of the window has been approximately corrected by multiplying by the net effect of the window on a model power spectrum with $\Omega_m h = 0.168$, $\Omega_b/\Omega_m = 0.0$, $h = 0.72$ & $n_s = 1$. A zero-baryon model was chosen in order to avoid adding features into the power spectrum. All of the data are renormalized to match the new measurements. Panel (d) shows the uncorrelated SDSS real-space $P(k)$ estimate of Tegmark et al. (2004), calculated using their 'modelling method' with no FOG compression (their Table 3). These data have been corrected for the SDSS window as described above for the 2dFGRS data. The solid line shows a model linear power spectrum with $\Omega_m h = 0.168$, $\Omega_b/\Omega_m = 0.17$, $h = 0.72$, $n_s = 1$ and normalization matched to the 2dFGRS power spectrum.



Ergodic Theorem

Statistical Cosmological Principle

Cosmological Principle:

Universe is Isotropic and Homogeneous

Homogeneous & Isotropic Random Field $\psi(\vec{x})$:

Homogenous $p[\psi(\vec{x}+\vec{a})] = p[\psi(\vec{x})]$

Isotropic $p[\psi(\vec{x}-\vec{y})] = p[\psi(|\vec{x}-\vec{y}|)]$

Within Universe one particular realization $\psi(\vec{x})$:

Observations: only spatial distribution in that one particular $\psi(\vec{x})$

Theory: $p[\psi(\vec{x})]$

Ergodic Theorem

Ensemble Averages $\Longleftrightarrow$ **Spatial Averages over one realization of random field**

- Basis for statistical analysis cosmological large scale structure
- In statistical mechanics Ergodic Hypothesis usually refers to time evolution of system, in cosmological applications to spatial distribution at one fixed time

Ergodic Theorem

Validity Ergodic Theorem:

- Proven for Gaussian random fields with continuous power spectrum
- Requirement:

spatial correlations decay sufficiently rapidly with separation

such that

many statistically independent volumes in one realization

⇩

All information present in complete distribution function $p[\psi(\vec{x})]$ available from single sample $\psi(\vec{x})$ over all space

Fair Sample Hypothesis

- Statistical Cosmological Principle

+

- Weak cosmological principle
  (small fluctuations initially and today over Hubble scale)

+

- Ergodic Hypothesis

fair sample hypothesis
(Peebles 1980)