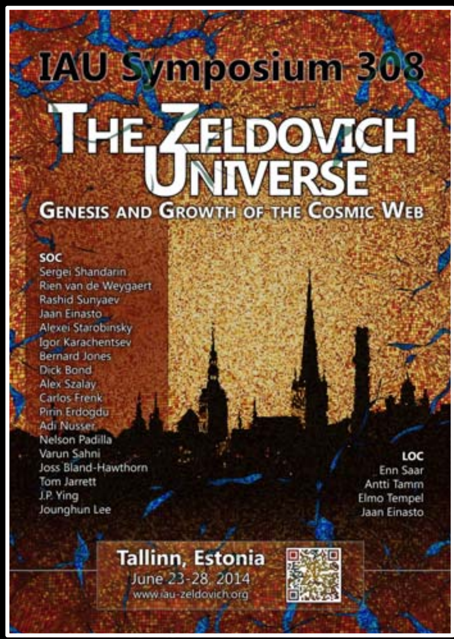
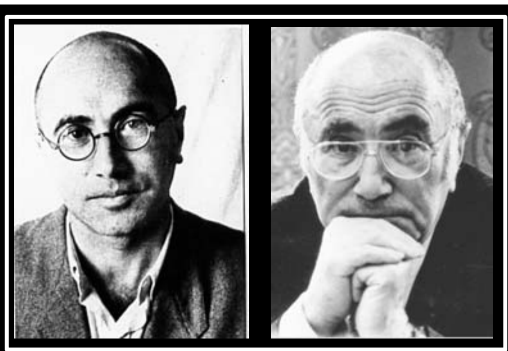
the Zel’dovich Formalism



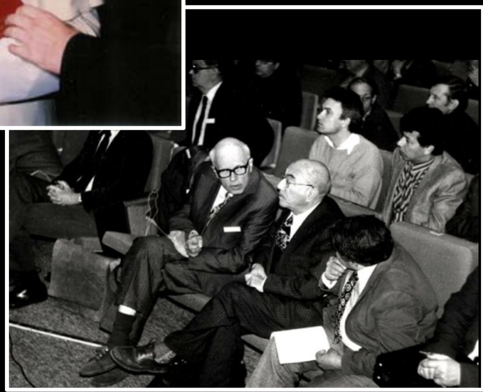


Yakov Borisovich Zel'dovich
Minsk, 1914- Moscow, 1987

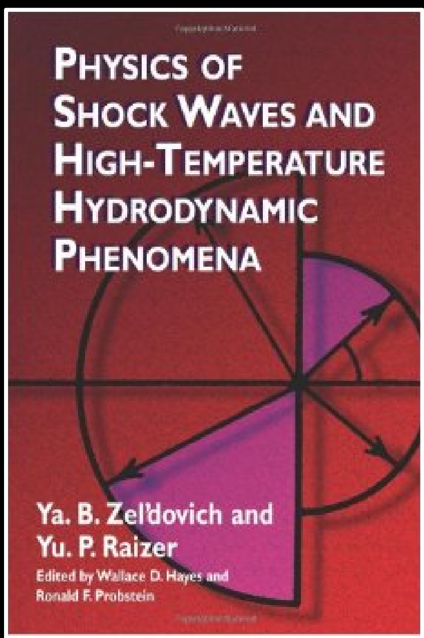
stamp Zeldovich,
Russia 2014

monument Zeldovich,
Minsk, Belorussia

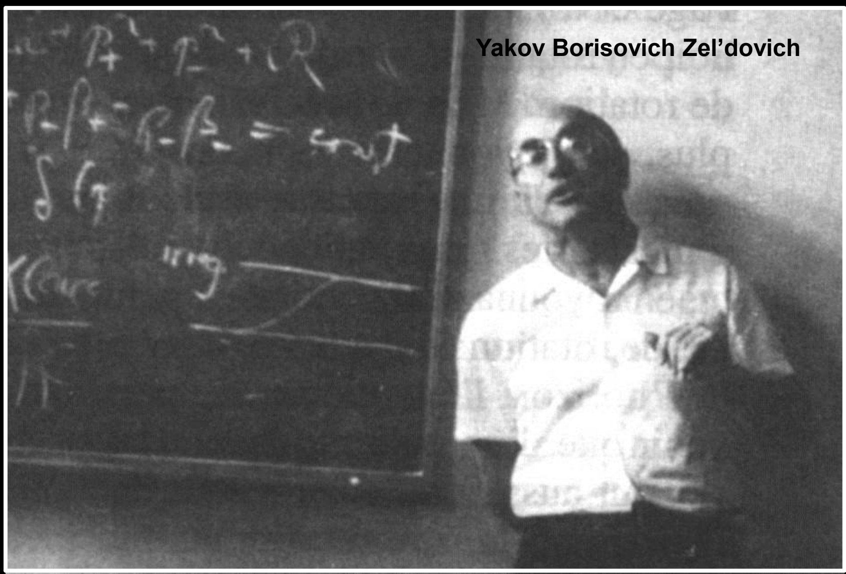
Zel'dovich & pope John Paul II



Zel'dovich & Andrei Sacharov



Zeldovich & Raizer
standard book on shock waves ...



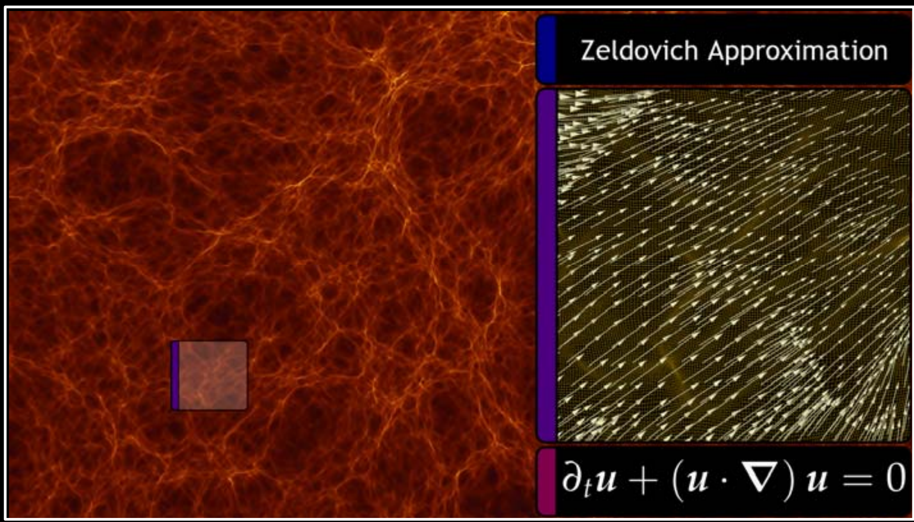
Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\,\phi_{lin}\left(\vec{q}\right)$$

Zel'dovich Approximation



Zel'dovich Formalism: Streaming & Caustics

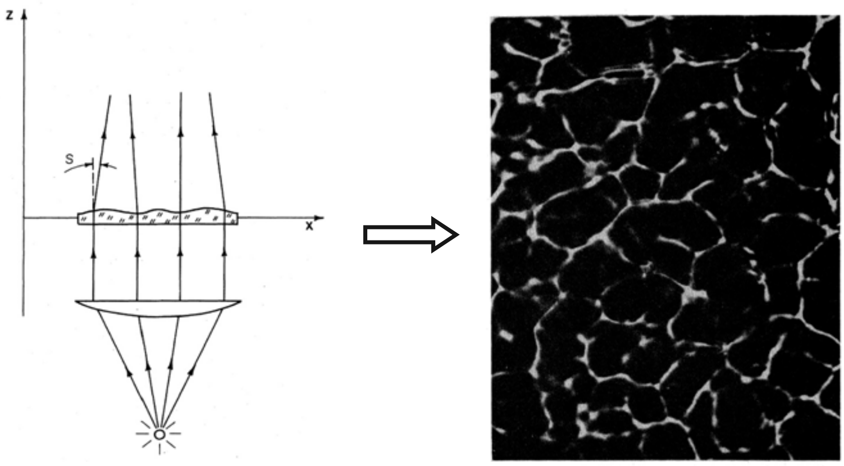
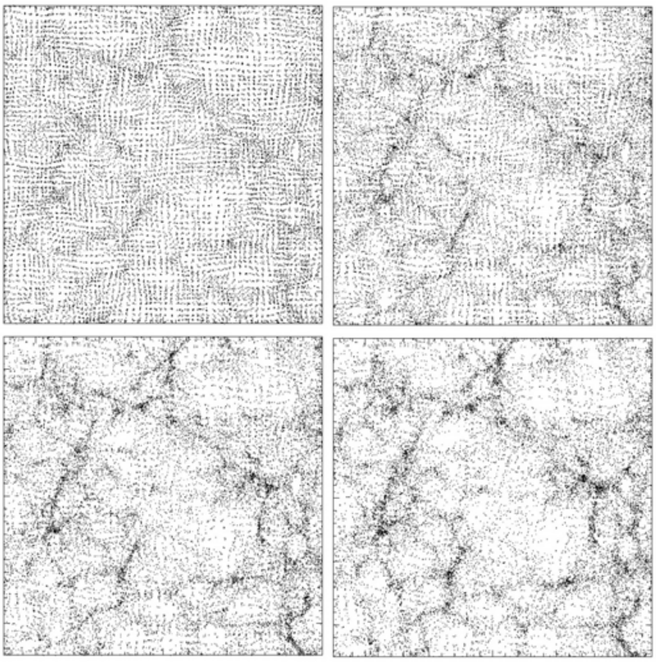


Illustration of the formation of caustics due to streaming paths of light through deforming medium



Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

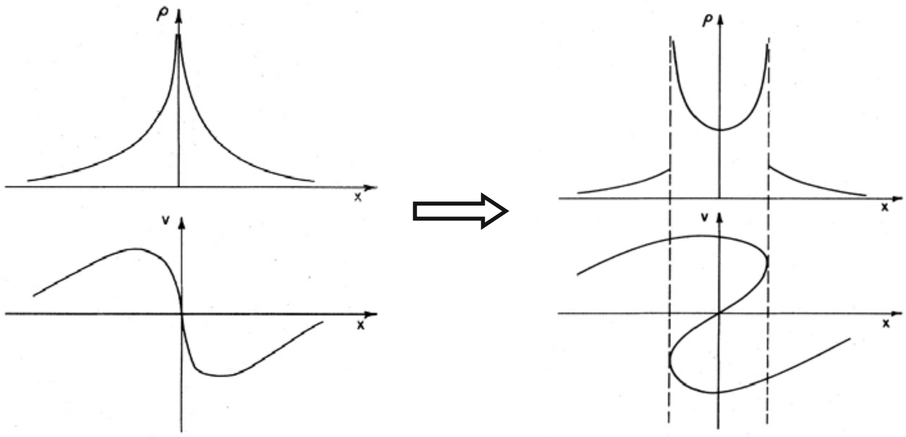
$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

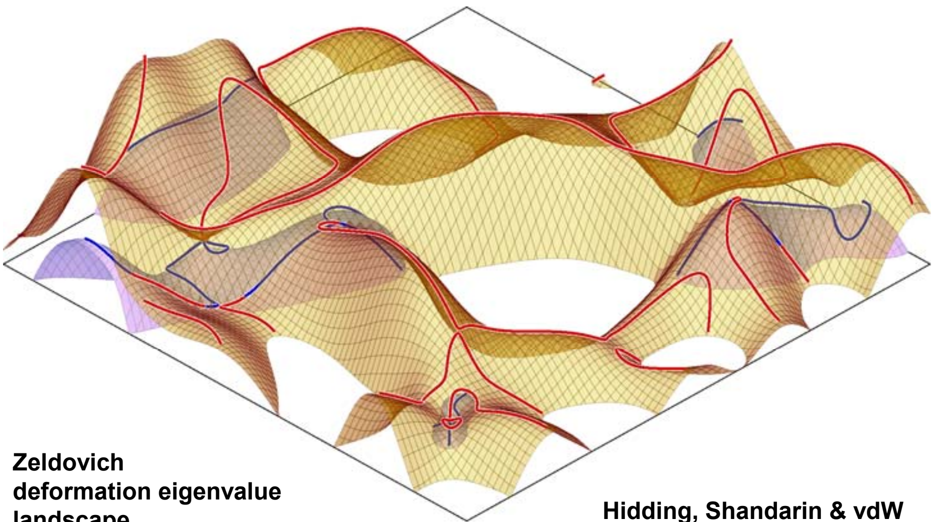
structure of the cosmic web determined by the spatial field of eigenvalues $\lambda_1, \lambda_2, \lambda_3$

Zel'dovich Formalism: Density Evolution



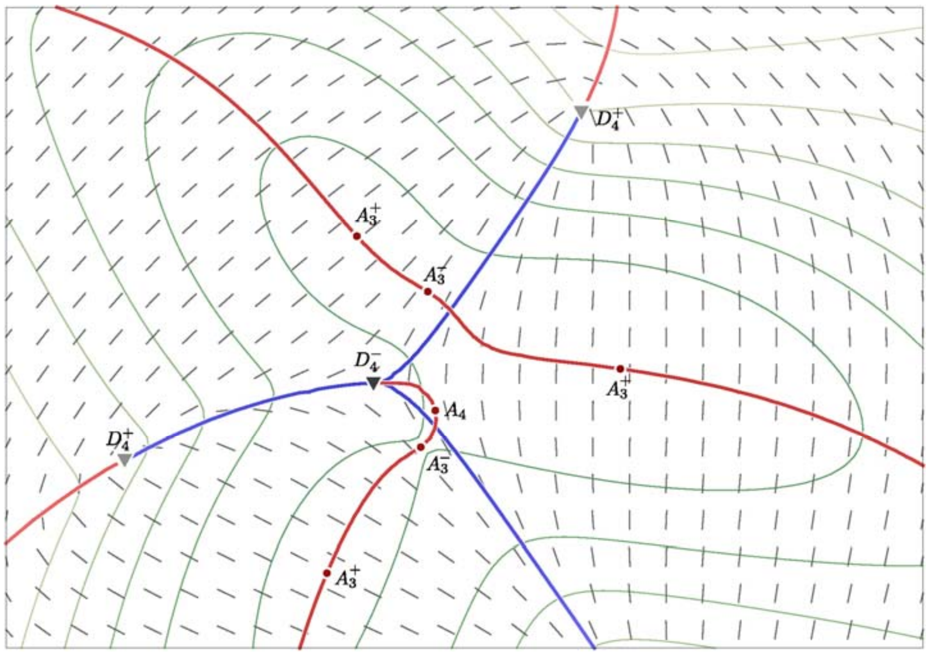
Density Profile through pancake, at moment of formation and shortly thereafter (multistream)

Singularities & Catastrophes



Zeldovich deformation eigenvalue landscape

Hidding, Shandarin & vdW 2014



Zel’dovich Formalism

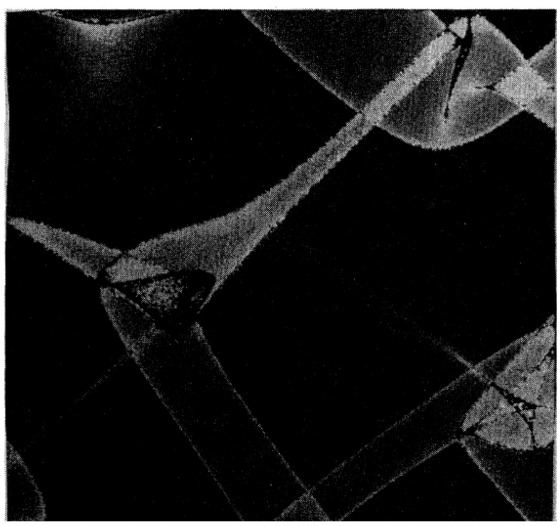
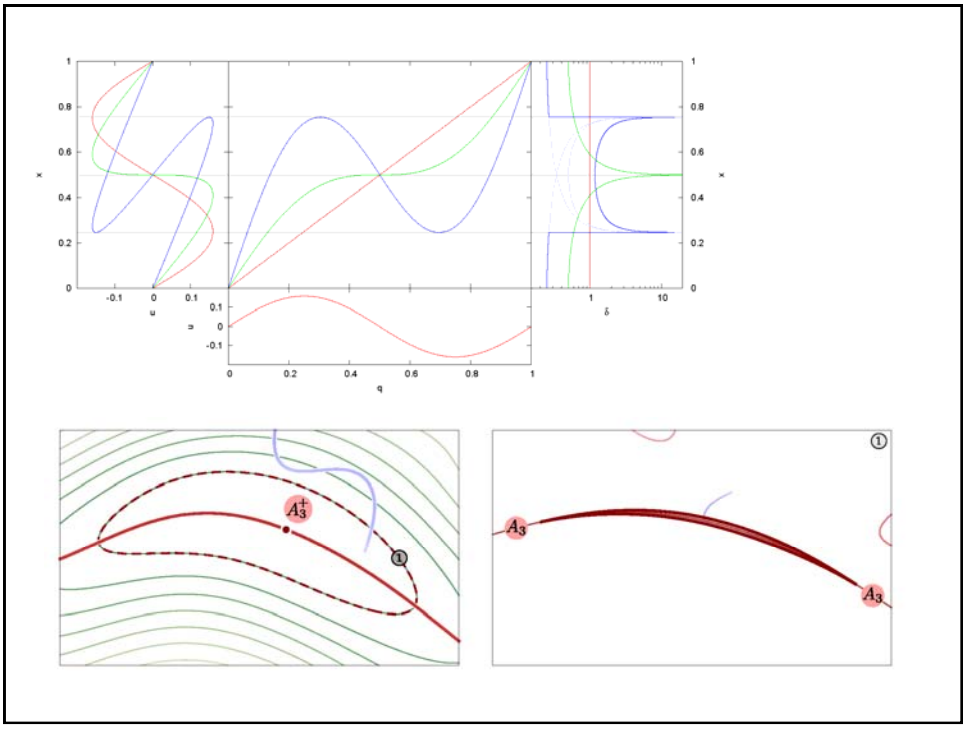
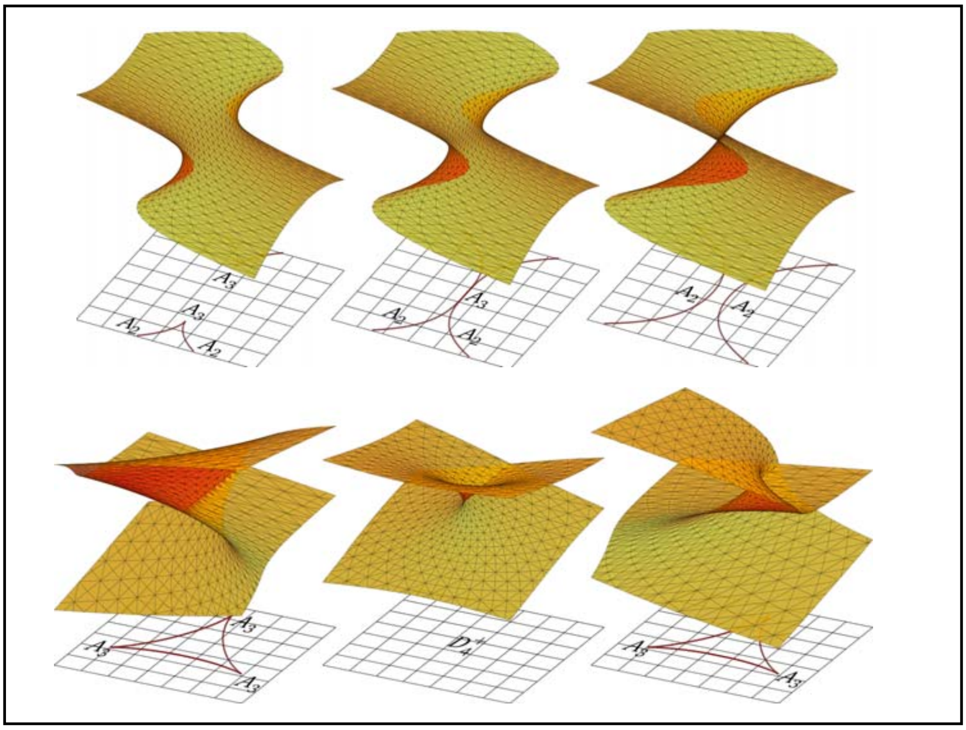


Illustration:

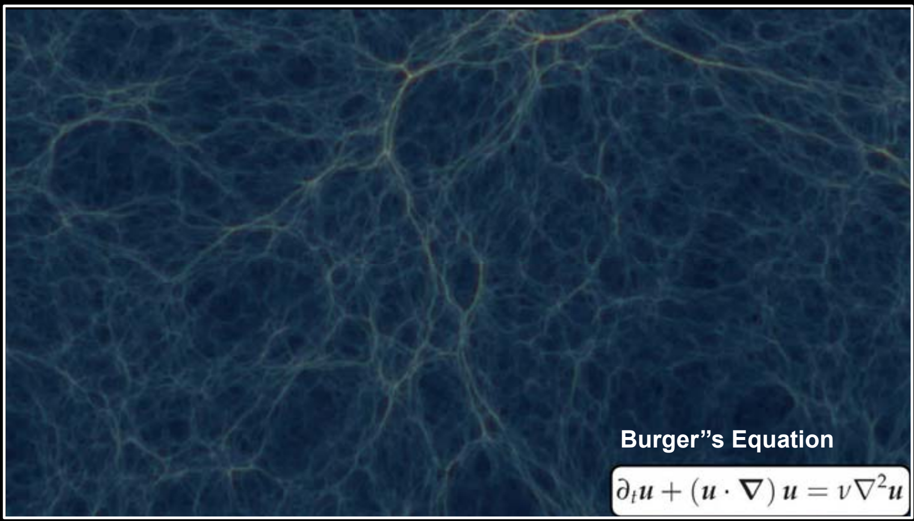
Caustic formation of weblike matter distribution according to the Zel’dovich formalism.

From: Buchert 1989


x
u
q
δ
A_3^+
A_3
A_3


A_3
A_2
A_3
A_2
D_4^+

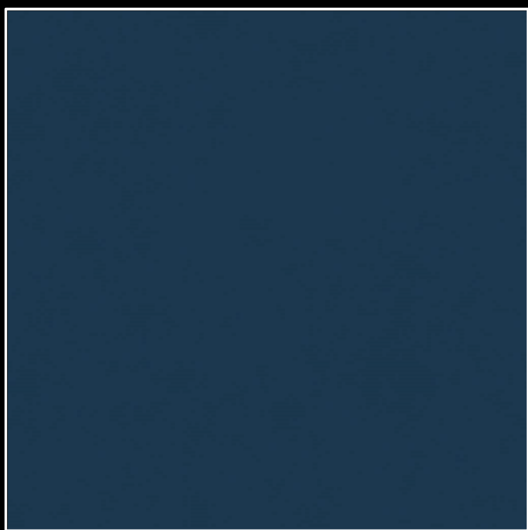
Adhesion Approximation



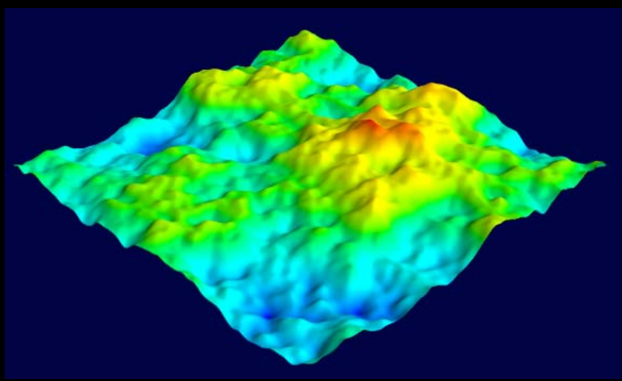
Adhesion Approximation

Gurbatov, Saichev & Shandarin 1987

Hidding 2012



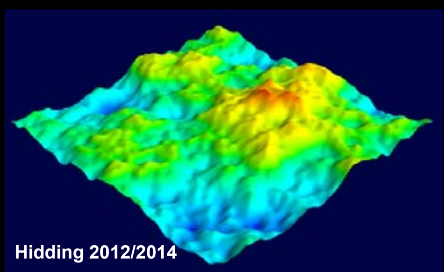
Velocity & Gravity Potential

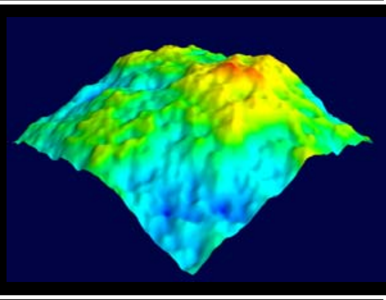


$$\vec{u}(\vec{q}) = \vec{\nabla}\Phi(\vec{q})$$

Burger's Equation: Hopf Solution

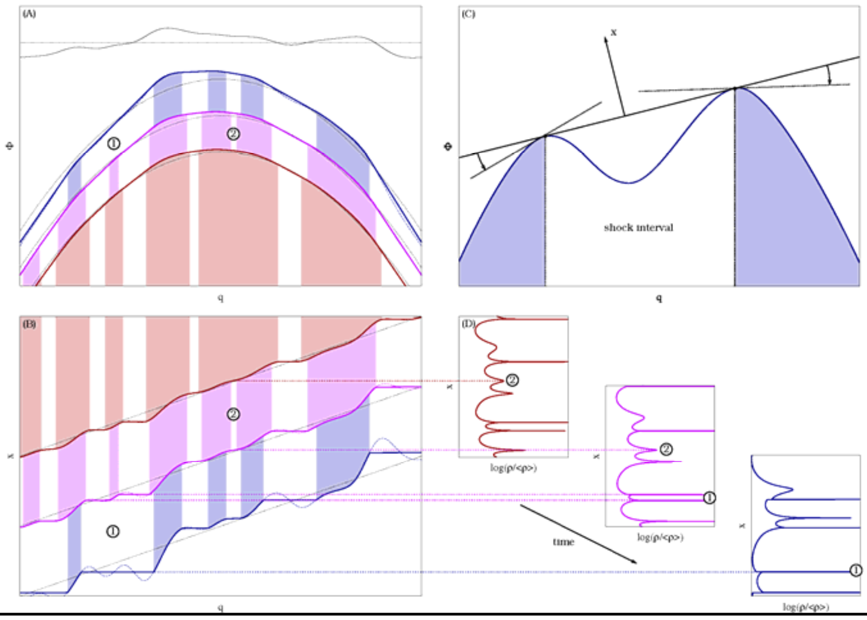
$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = \nu\,\nabla^2 \vec{u}$$

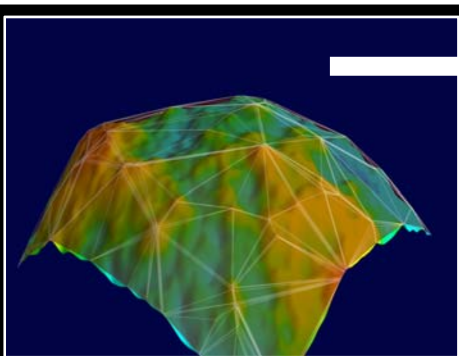


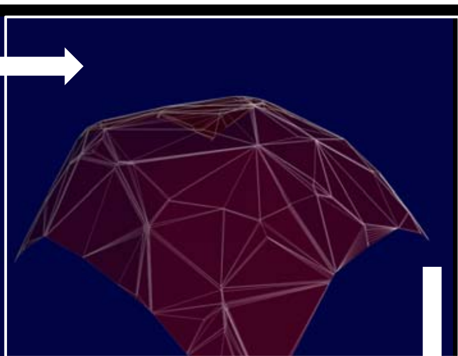


$$\Phi(\vec{x},t) + \frac{x^2}{2} = \max_q \left[\left(t\Phi_0\left(q\right) - \frac{q^2}{2}\right) + \vec{x}\cdot\vec{q}\right]$$

Burger's Equation: Hopf Solution



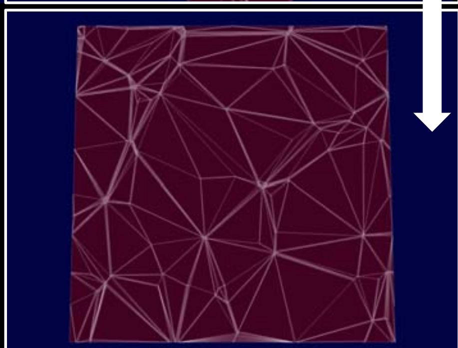




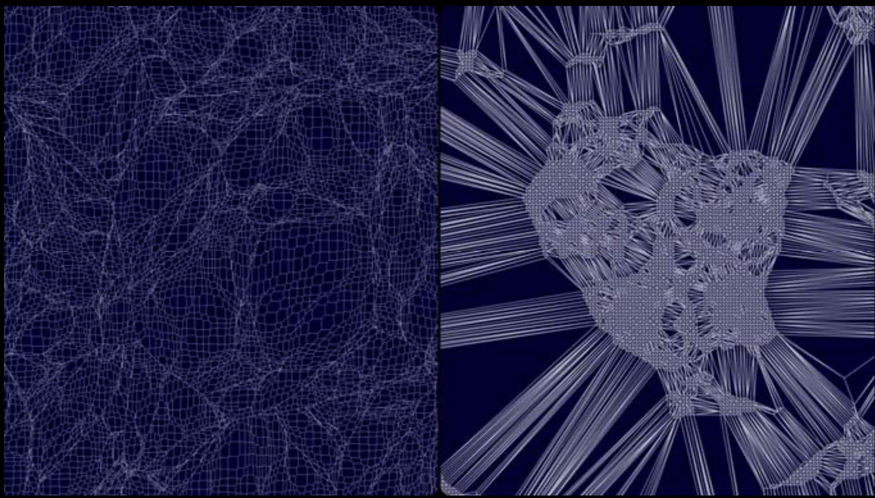
Hidding 2012/2014

Convex Hull
quadratically lifted potential field

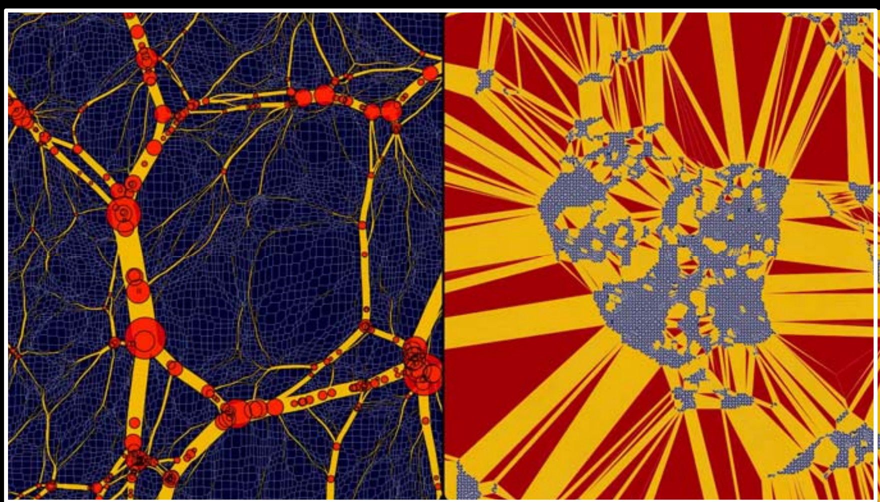
Delaunay tessellation
generated by maxima potential field



Eulerian – Lagrangian Voronoi - Delaunay



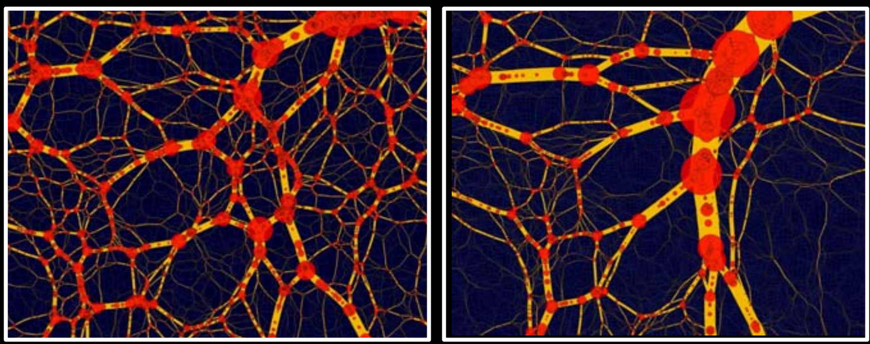
Eulerian – Lagrangian Voronoi - Delaunay

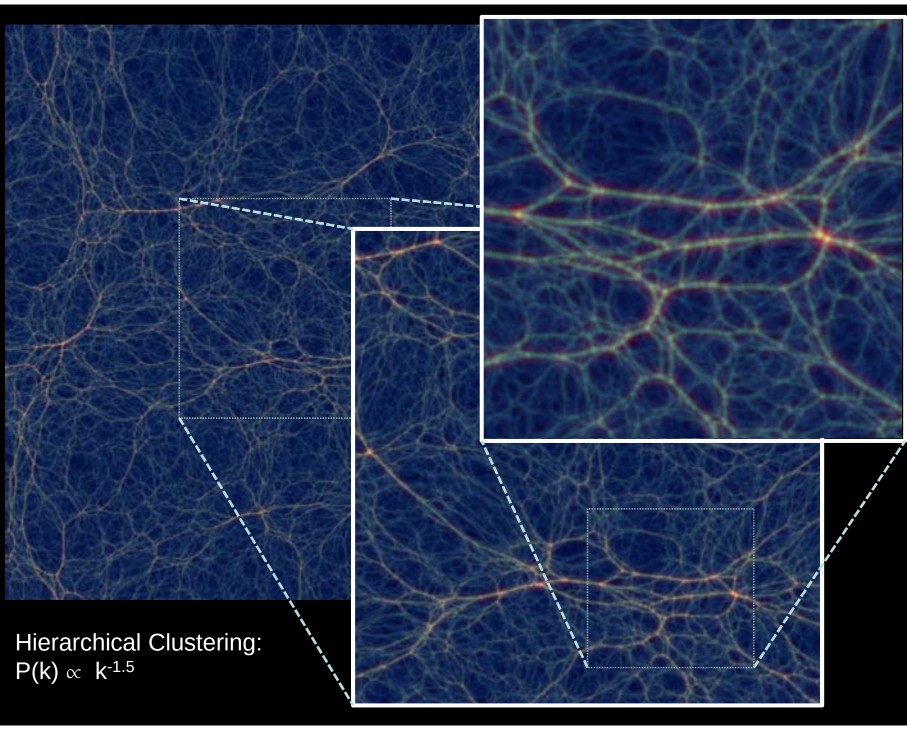


Cosmological Sensitivity

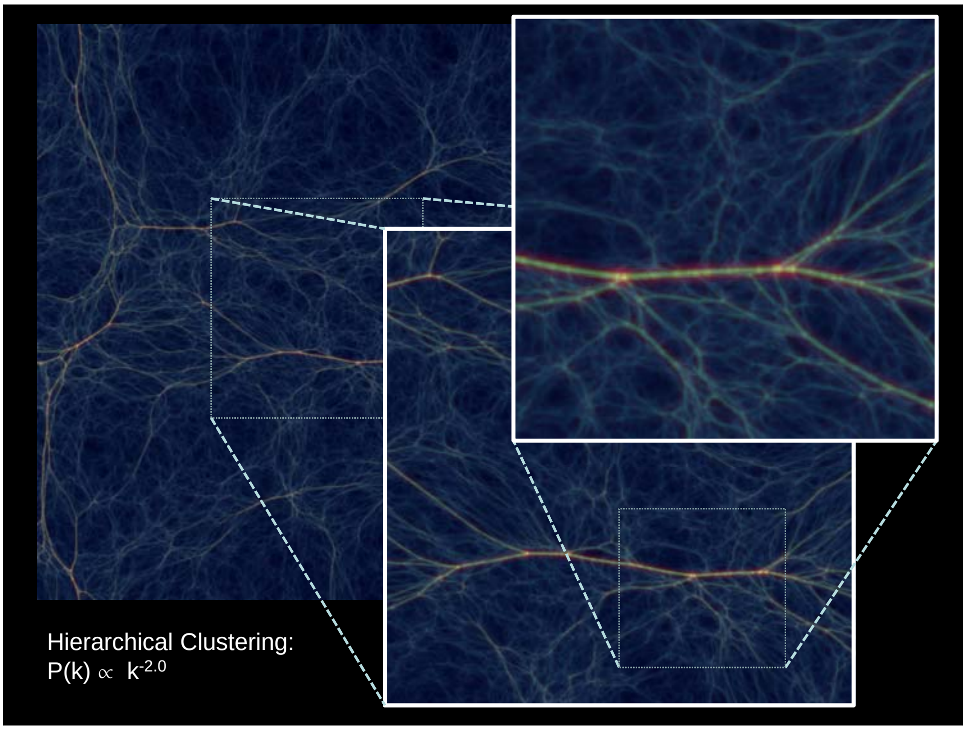
the morphology of the weblike network is
highly sensitive to the underlying cosmology

$P(k) \propto k^{-1.5}$ **Hidding 2012/2014** $P(k) \propto k^{-2.0}$





Hierarchical Clustering:
$P(k) \propto k^{-1.5}$



the Spherical Model

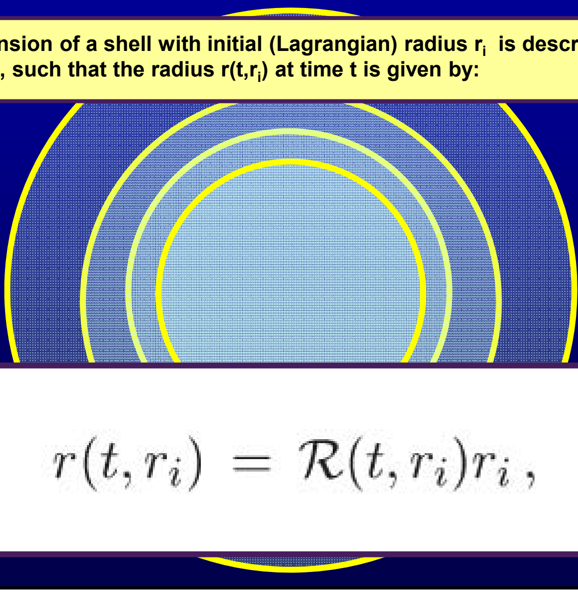
Spherical Model

The spherical model (Gunn & Gott 1972) describes the evolution of a spherical mass distribution. It forms THE reference point for all further evaluations of structure formation.

- Because of Birkhoff's theorem we may see the evolution of each individual mass shell as due only to the integrated mass distribution within its radius.
- As long as two mass shells are not crossing – e.g. due to the faster infall of an outer shell into an overdensity -- the motion of a shell – with radius r -- is simply that of an individual spherical shell attracted by a point mass M(r), with M(r) the integrated mass within radius r.
- Perhaps not surprisingly, the equations of motion for the mass shells are the same as that of Friedmann-Robertson-Walker universes for an equivalent density parameter Ω(r).
- These equations of motion for each mass shell can be solved analytically for any decently behaving mass profile (i.e. the mass profile should be sufficiently centrally concentrated to prevent shell crossing).
- The spherical model is equally valid for overdensities as well as for underdensities.

Spherical Model

Contraction/Expansion of a shell with initial (Lagrangian) radius r_i is described by a scale factor $R(t,r_i)$, such that the radius $r(t,r_i)$ at time t is given by:



$$r(t, r_i) = \mathcal{R}(t, r_i) r_i \,,$$

Spherical Model

The motion is fully determined by the average mass density Δ(r,t) within a radius r,

$$\Delta(r,t) = \frac{3}{r^3}\int_0^r \left[\frac{\rho(y,t)}{\rho_u(t)} - 1\right] y^2\, dy = \frac{3}{r^3}\int_0^r \delta(y,t)\, y^2\, dy\,,$$

$$1 + \Delta_{ci} = \Omega_i\,[1 + \Delta(t_i, r_i)]$$

$$\alpha_i = \left(\frac{v_i}{H_i r_i}\right)^2 - 1\,.$$

and by the the peculiar velocity $v_{pec,i}$ of the shell. For this we usually take the peculiar velocity predicted by linear theory for the growing mode.

$$v_{pec,i} = -\frac{H_i r_i}{3} f(\Omega_i)\Delta(r_i, t_i)\,,$$

$$\alpha_i = -\frac{2}{3} f(\Omega_i)\Delta(r_i, t_i)\,.$$

It is convenient to describe the density perturbation with respect to a EdS Universe, in terms of Δ_i and the velocity perturbation with respect to the Hubble expansion in terms of parameter α_i .

Spherical Model

The solutions for the scale factor of overdense/underdense shells can be written in the same parameterized form, by means of shell angle Θ, as we know from the solutions for FRW universes,

$$\mathcal{R}(\Theta_r) = \begin{cases} \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})}\;(\cosh\Theta_r - 1) & \Delta_{ci} < \alpha_i, \\ \\ \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)}\;(1 - \cos\Theta_r) & \Delta_{ci} > \alpha_i\,, \end{cases}$$

with time dependence specified by

$$t(\Theta_r) = \begin{cases} \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}}\;(\sinh\Theta_r - \Theta_r) & \Delta_{ci} < \alpha_i \\ \\ \frac{1}{2}\,\frac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}}\;(\Theta_r - \sin\Theta_r) & \Delta_{ci} > \alpha_i \end{cases}$$

Spherical Model

The corresponding peculiar velocity of the shell

$$v_{pec}(r,t) = v(r,t) - H_u(t)r(t)\,,$$

can be inferred from

$$v_{pec}(r,t) = H_u(t)r(t)\left\{\frac{g(\Theta_r)}{g(\Theta_u)} - 1\right\}$$

with

$$g(\Theta) = \begin{cases} \dfrac{\sinh\Theta\,(\sinh\Theta - \Theta)}{(\cosh\Theta - 1)^2} & \text{open}\,, \\ \dfrac{2}{3} & \text{critical}\,, \\ \dfrac{\sin\Theta\,(\Theta - \sin\Theta)}{(1-\cos\Theta)^2} & \text{closed} \end{cases}$$

Evolution

Spherical Tophat Halo

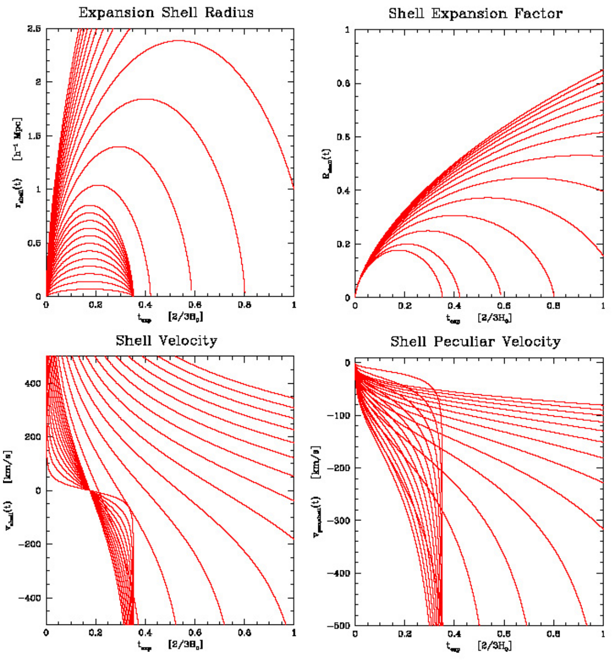
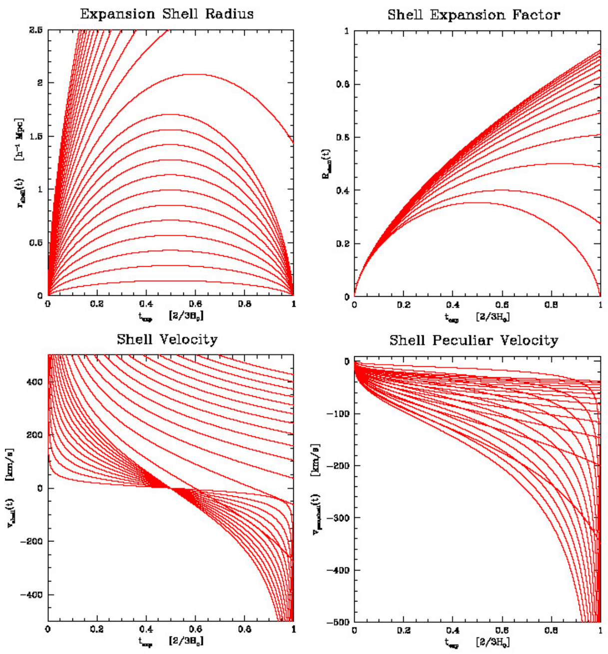


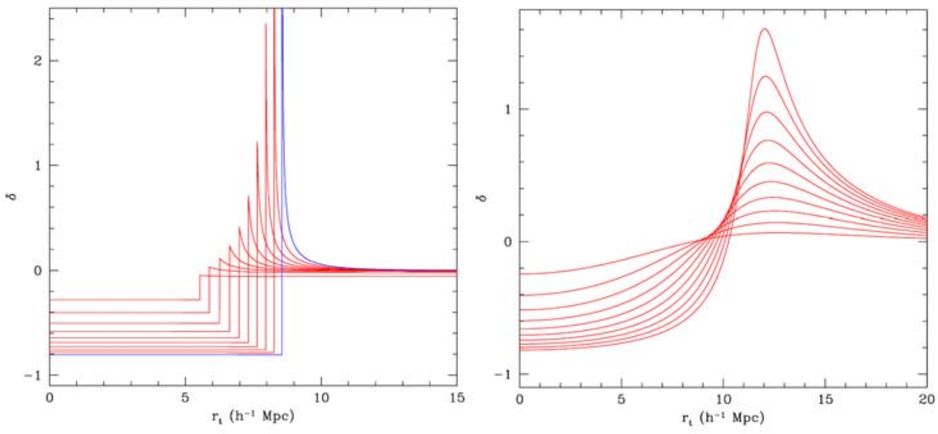
Figure 7. Spherical Peak 2

Evolution

Spherical Tophat Halo



Spherical Void Evolution



Spherical Model

Having determined the evolution of the radius and velocity of each spherical shell of the density perturbation, we may then proceed to derive the corresponding evolution of the density profile of the shell. Here we limit ourselves to the integrated density profile Δ(r,t),

$$1 + \Delta(r,t) \; = \; \frac{1 + \Delta_i(r_i)}{\mathcal{R}^3} \, \frac{a(t)^3}{a_i^3} \; ,$$

whose solution can be specified in terms of a density function f(θ),

$$1 + \Delta(r,t) = f(\Theta_r)/f(\Theta_u),$$

Spherical Model

whose solution can be specified in terms of a density function f(θ),

$$f(\Theta) \; = \; \begin{cases} \dfrac{(\sinh\Theta - \Theta)^2}{(\cosh\Theta - 1)^3} & \text{open}\,, \\ 2/9 & \text{critical}\,, \\ \dfrac{(\Theta - \sin\Theta)^2}{(1 - \cos\Theta)^3} & \text{closed}\,, \end{cases}$$

At maximum expansion of an overdense shell, Θ=π, defining the turnaround radius of the matter concentration, we thus find that the integrated overdensity of the shell is

$$1 + \Delta(r, t_{ta}) = (3\pi/4)^2$$

~ 5.6

Spherical Model

In the "imaginary" situation in which the overdensity would have continued to evolve linearly, it would have reached an overdensity dictated by the linear growth factor D(t) for the corresponding background Universe. For the situation of an Einstein-de Sitter Universe, with

$$D \propto (t/t_0)^{2/3}$$

a mass overdensity reaches its turnaround at a linear overdensity

$$\Delta_{lin}(z_{\rm ta}) = \delta_{\rm ta} = (3/5)(3\pi/4)^{2/3} \approx 1.062.$$

The consequences of this finding are truely wonderful: the cosmologist may resort to the primordial density field, search for the peaks in this Gaussian field, and assuming they are spherical (which they are not at all), and identify the ones that reach turnaround at some redshift z. Even more useful is the equivalent case for final collapse.

Spherical Model

Collapse, ie. $\Delta=\infty$, happens when the density fluctuation would have reached a linear overdensity of

$$\Delta_{lin}(z_{\rm c}) = \delta_{\rm c} = \left(\frac{3}{5}\right)\left(\frac{3\pi}{2}\right)^{2/3} \approx 1.686 .$$

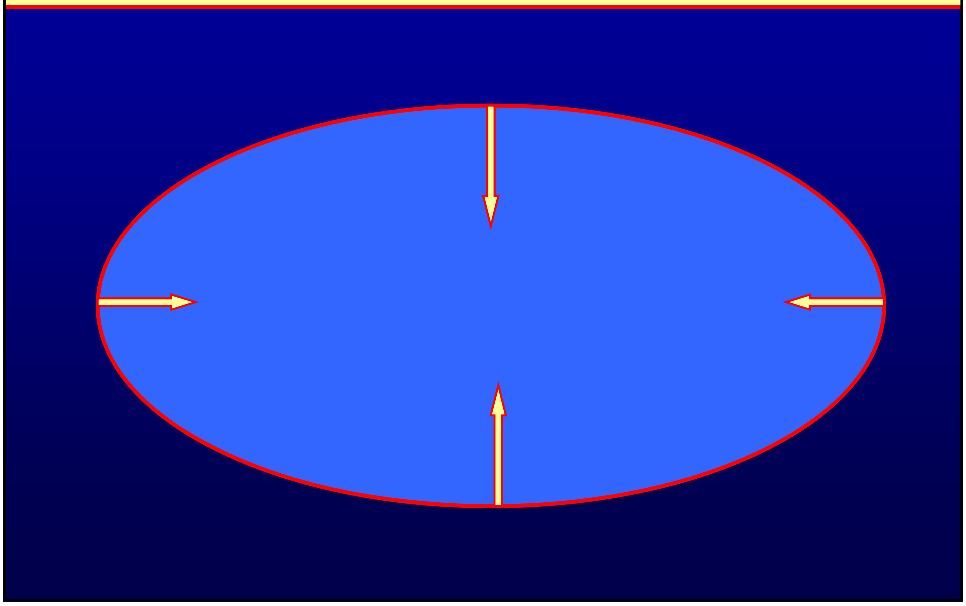
The fact that this is a universal value, valid for any (spherical) density peak, makes it into one of the most crucial numbers in the theory of structure formation. We may thus find the collapse redshifts z_{coll} for any primordial density peak,

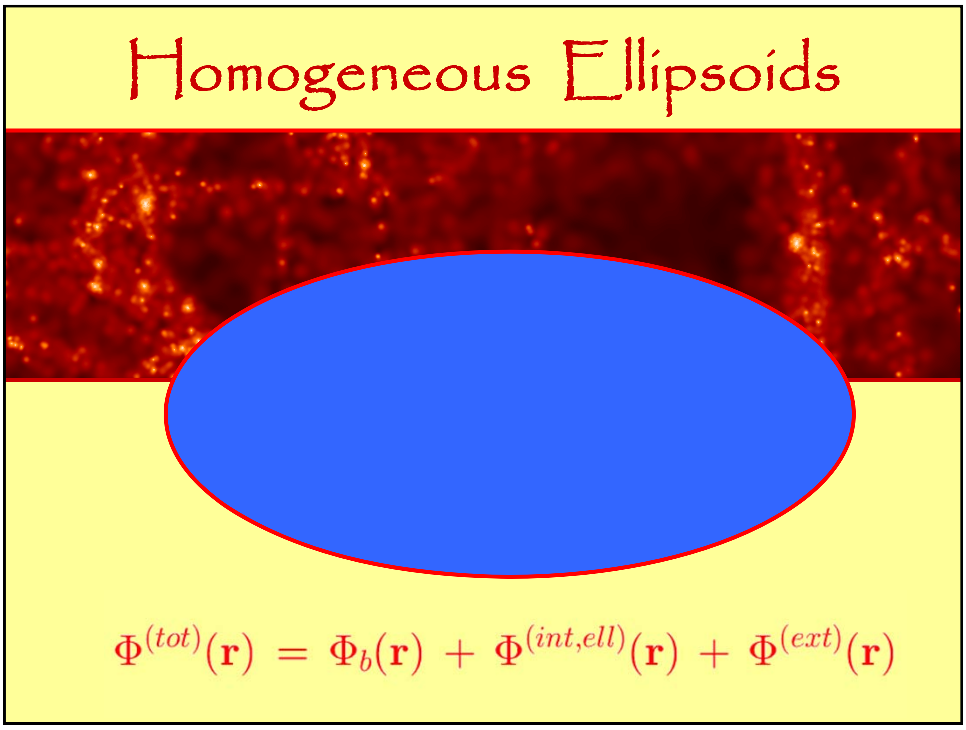
$$D(z_{\rm coll})\,\Delta_{lin,0} = \delta_{\rm c} \ .$$

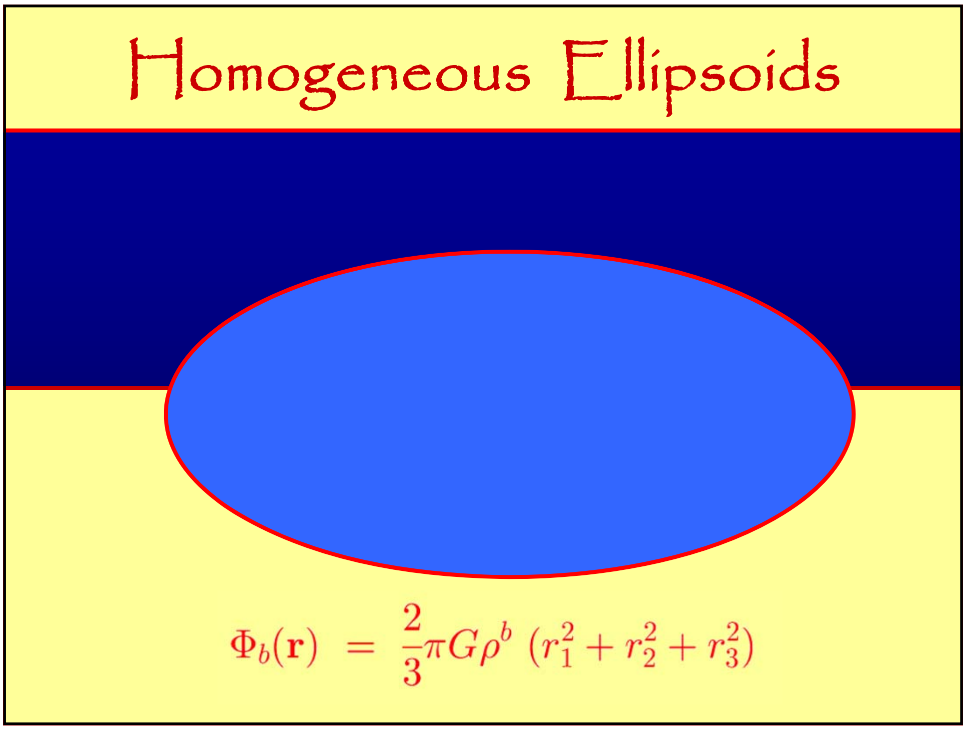
$$1 + z_{coll} = \frac{\Delta_{lin,0}}{1.686}.$$

the Ellipsoidal Model

Homogeneous Ellipsoids




Homogeneous Ellipsoids
$$\Phi^{(tot)}(\mathbf{r}) \;=\; \Phi_b(\mathbf{r}) \;+\; \Phi^{(int,ell)}(\mathbf{r}) \;+\; \Phi^{(ext)}(\mathbf{r})$$

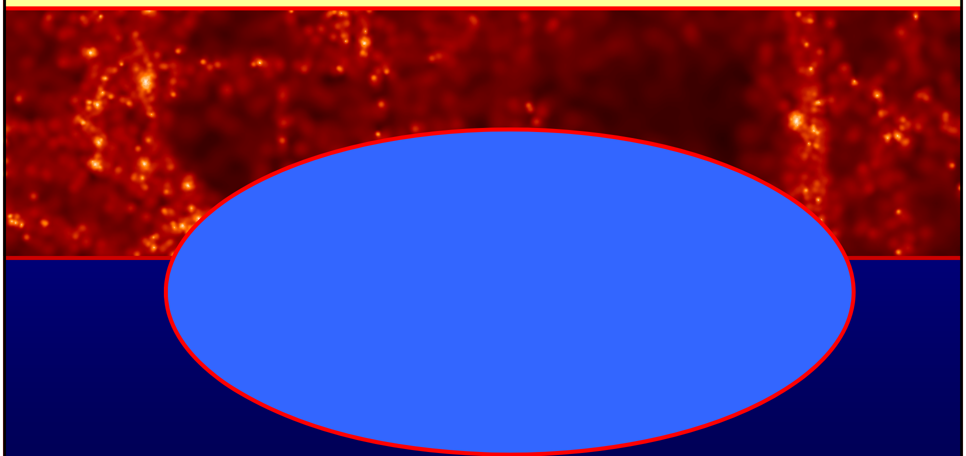

Homogeneous Ellipsoids
$$\Phi_b(\mathbf{r}) \;=\; \frac{2}{3}\pi G \rho^b \,(r_1^2 + r_2^2 + r_3^2)$$

Homogeneous Ellipsoids

$$\begin{aligned}\Phi^{(int,ell)}(\mathbf{r}) &= \frac{1}{2}\sum_{m,n}\Phi^{(int,ell)}_{mn} r_m r_n \\ &= \frac{2}{3}\pi G(\rho^{ell}-\rho^{b})\,(r_1^2+r_2^2+r_3^2) + \frac{1}{2}\sum_{m,n} T^{(int)}_{mn} r_m r_n\end{aligned}$$

$$T^{(int)}_{mn} \equiv \frac{\partial^2 \Phi^{(int,ell)}}{\partial r_m \partial r_n} - \frac{1}{3}\nabla^2 \Phi^{(int,ell)}\;\delta_{mn}$$

Homogeneous Ellipsoids



$$\Phi^{(ext)}(\mathbf{r}) = \frac{1}{2}\sum_{m,n} T^{(ext)}_{mn} r_m r_n \quad \Longleftarrow \quad T^{(ext)}_{mn}(t) \equiv \frac{\partial^2 \Phi^{(ext)}}{\partial r_m \partial r_n}$$

Homogeneous Ellipsoids

$$\Phi^{(int,ell)}(\mathbf{r}) = \pi G \left(\rho^{ell} - \rho^{b}\right) \sum_{m} \alpha_m r_m^2$$

$$T^{(int)}_{mn} = 2\pi G(\rho^{ell} - \rho^{b}) \left(\alpha_m - \frac{2}{3}\right) \delta_{mn}$$

$$\alpha_m = c_1 c_2 c_3 \int_0^\infty (c_m^2 + \lambda)^{-1} \prod_{n=1}^{3} \frac{1}{\sqrt{c_n^2 + \lambda}} \mathrm{d}\lambda$$

Homogeneous Ellipsoids

$$\frac{d^2 r_m}{dt^2} = -\frac{4\pi}{3} G \rho^b r_m(t) - \sum_n \Phi^{(int,ell)}_{mn} r_n(t) - \sum_n T^{(ext)}_{mn} r_n(t)$$

Homogeneous Ellipsoids

$$r_m(t) = \sum_k R_{mk}(t) r_{k,i}$$

$$\frac{d^2 R_{mk}}{dt^2} = -\frac{4\pi}{3}\pi G R_{mk} - \sum_n \Phi_{mn}^{(int,ell)} R_{nk} - \sum_n T_{mn} R_{nk}$$

$$\frac{d^2 R_{mk}}{dt^2} = -2\pi G \left[\alpha_m \rho^{ell} + (\frac{2}{3} - \alpha_m)\ \rho^b\right] R_{mk} - T_{mm}^{(ext)} R_{mk}$$

Homogeneous Ellipsoids

$$R_{mn}(t_i) = R_m(t_i)\delta_{mn}$$

$$\frac{d^2 R_{mk}}{dt^2} = -2\pi G \left[\alpha_m \rho^{ell} + (\frac{2}{3} - \alpha_m)\ \rho^b\right] R_{mk} - T_{mm}^{(ext)} R_{mk}$$

$$\Downarrow$$

$$\frac{d^2 R_m}{dt^2} = -2\pi G \left[\alpha_m \rho^{ell} + (\frac{2}{3} - \alpha_m)\ \rho^b\right] R_m - T_{mm}^{(ext)} R_m$$

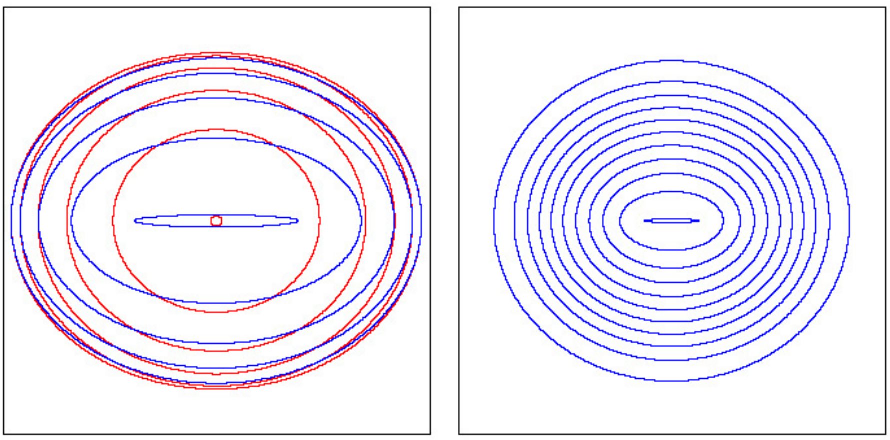
Homogeneous Ellipsoids

$$
\begin{aligned}
v_{pec,m}(t_i) &= \frac{2f(\Omega_i)}{3H_i\Omega_i} g_{pec,m}(t_i) \\
&= -\frac{1}{2} H_i f(\Omega_i) \left[\alpha_{m,i}\delta_i + \frac{4T^{(ext)}_{mm,\circ}}{3\Omega_0 H_0^2} D_i \right] r_{m,i}
\end{aligned}
$$

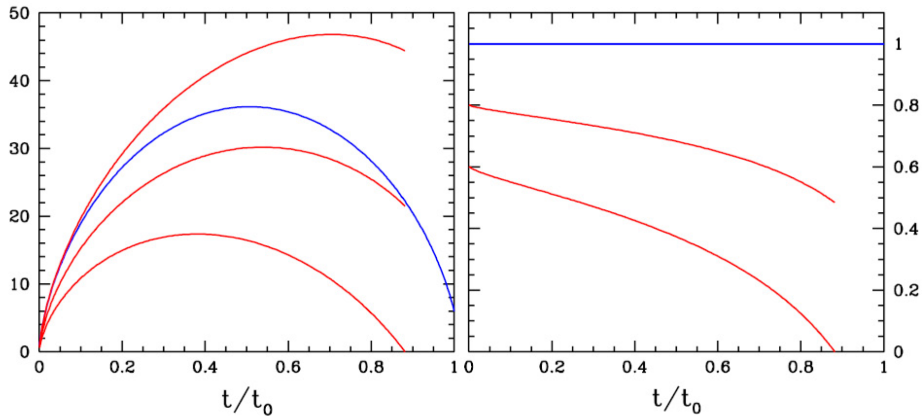
$$
\frac{d^2 R_m}{dt^2} = -2\pi G \left[\alpha_m \rho^{ell} + \left(\frac{2}{3} - \alpha_m\right) \rho^b \right] R_m - T^{(ext)}_{mm} R_m
$$

$$
c_m(t) = R_m(t) c_{m,i} \Longrightarrow (c_1, c_2, c_3)
$$

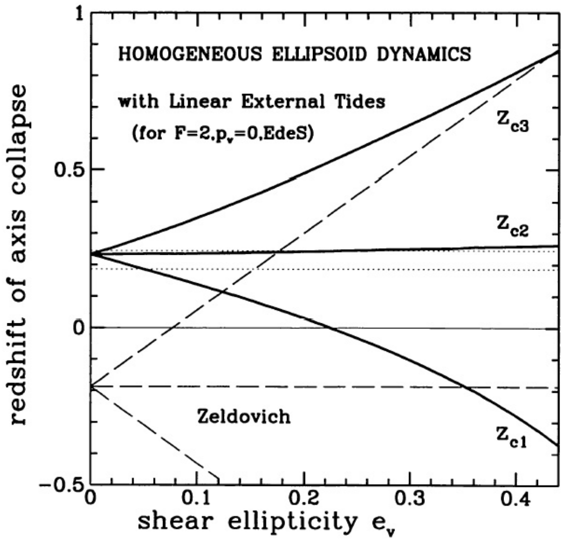
Homogeneous Ellipsoids



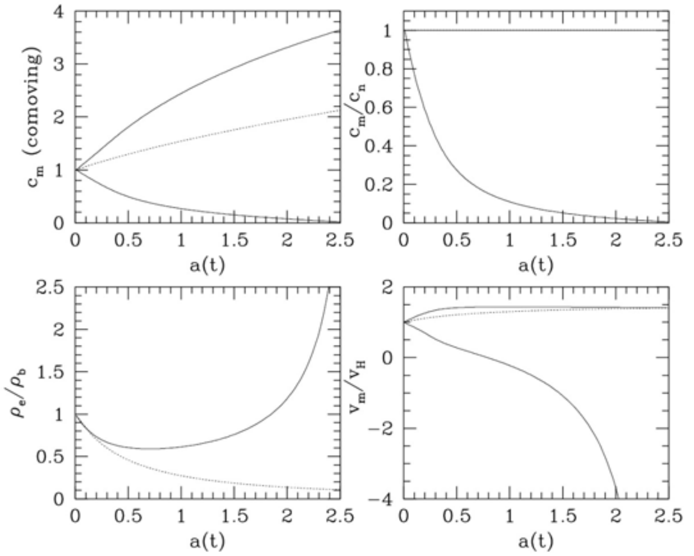
Homogeneous Ellipsoids



Homogeneous Ellipsoids



Homogeneous Ellipsoids



Homogeneous Ellipsoids