STARTING CONDITIONS

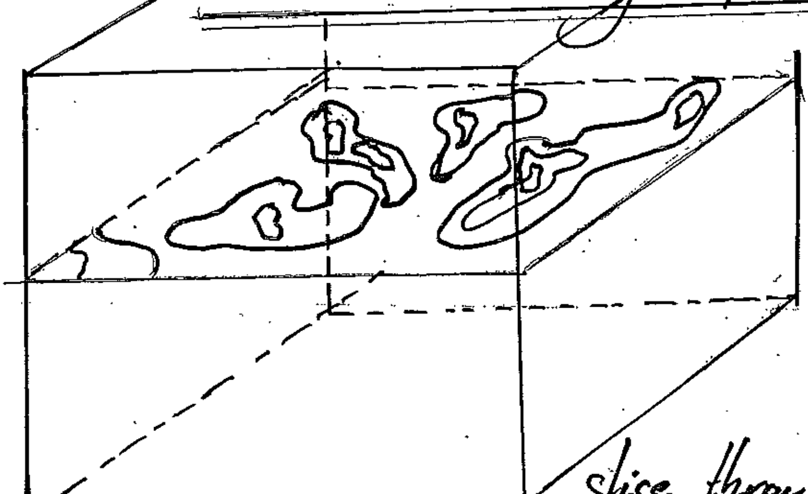
Stochastic Primordial Density Field.

OBSERVED UNIVERSE IS A REALIZATION OF UNDERLYING SPATIAL (!) STOCHASTIC PROCESS.

Note that as yet there is no first-principle cosmological theory describing the characteristics of the primordial density field, as yet it depends on the proposed structure formation scenario. Most studies work backward. They test a range of structure fomation scenarios by investigating the implied repereussions for observable quantities. In principle, if there is no agreement with predictions the scenario is deemed irrelevant.

How then do the characteristics of the primordial density noise field relate to the product of cosmic evolution?

COSMIC DENSITY FIELD: unstructured compilation of density fluctuations over a wide range of spatial scales:

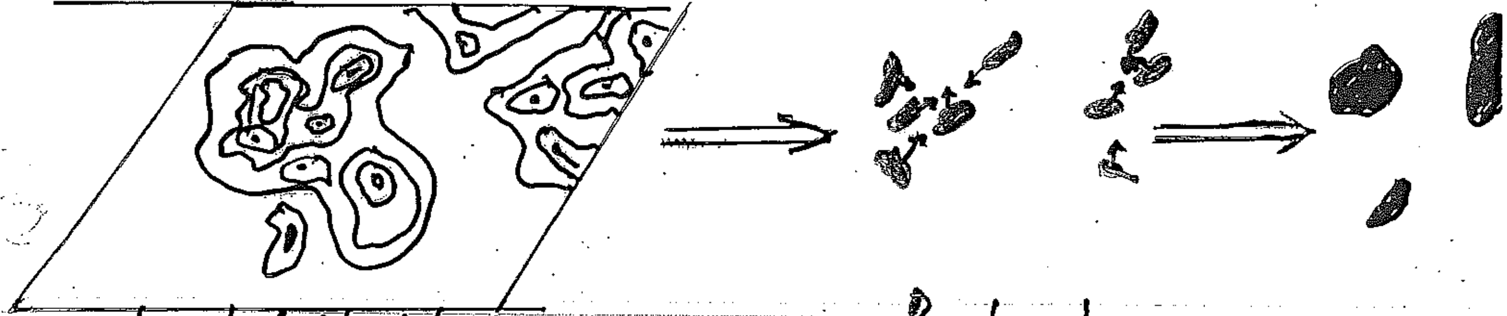


slice through 3-D density field.

The exact mode of structure evolution will depend on the character of the density field $\delta(\vec{x},t)$:

- fraction overdense ↔ underdense regions
  ↕
  in general, the distribution function $f(\delta)$, of density fluctuations (i.e. the "1-point" probability function).
- amount and amplitude of small-scale perturbations.
- coherence of the density fluctuation field.
- etc. ...

) Bottom-Up :

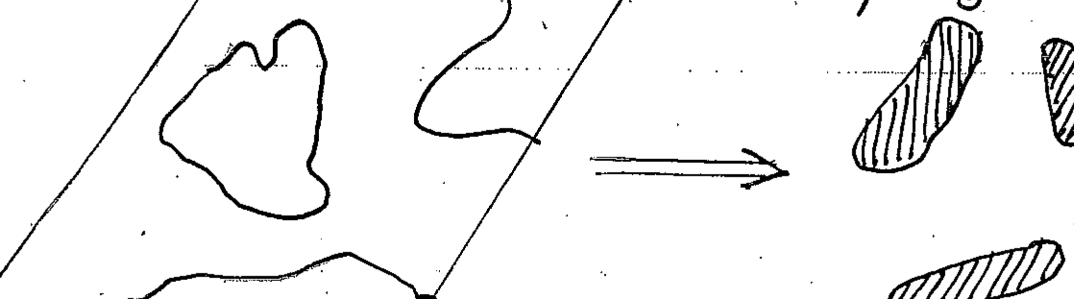


noisy density field: lot of small-scale power

hierarchical build-up:
1st small scale clumps → merging into ever bigger clumps.

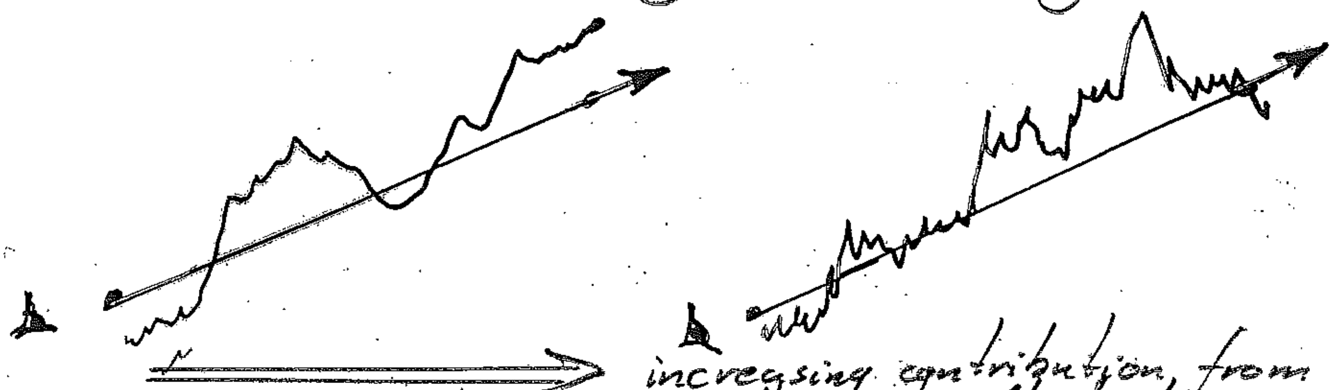
2) Top-Down :

Monologous collapse of large structures



- Thus, for the ensuing evolution, it is good to decompose the stochastic fluctuation field into its various spatial components:

- E.g., 1-D probe through 3-D density field:

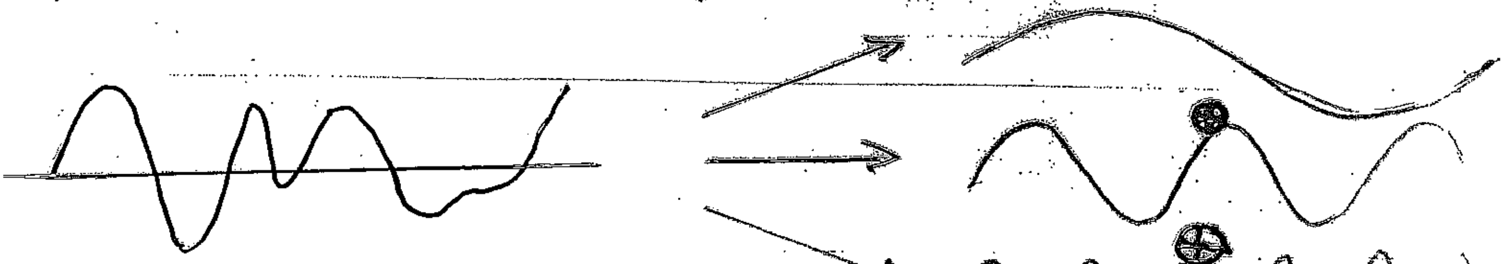


increasing contribution from small-scale fluctuations.

The global decomposition into the components on various scales is best done in terms of harmonic waves of wavelength ≡ scale perturbation, which is encapsulated in a Fourier decomposition:

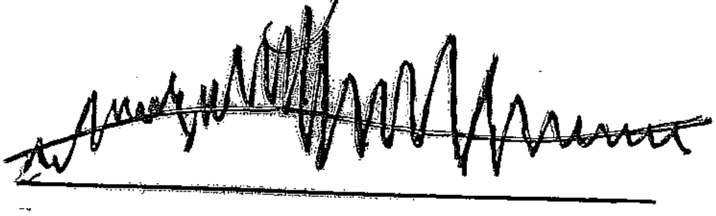
$$\begin{cases} \delta(\vec{x}) = \sum_{\vec{k}} \hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \vec{k} = \frac{2\pi}{\lambda_k}\,\hat{e}_k \qquad (\vec{k}: \text{wavevector}). \end{cases}$$

in which $\hat{\delta}(\vec{k}) \equiv$ amplitude wave $\vec{k}$



- Hierarchical scenario :

small-scale waves have higher amplitude than long waves :



small-scale noise creates most prominent fluctuation:

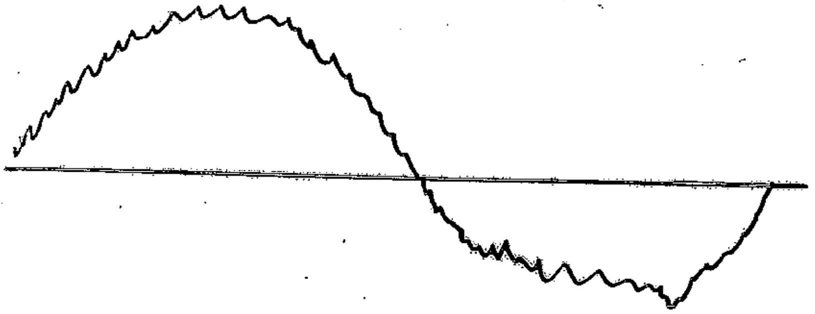
⇓

small scale fluctuations collapse first.

⇕

- Top-Down scenario :

Long waves have higher amplitude than small-scale waves :



large-scale wave collapses before small scale noise condenses into small-scale clumps.

To be able to make more quantitative statements on the structure evolution implied by the different possible primordial density fields, we will therefore have to investigate in more detail the decomposition embodied by the Fourier formulation:

Density Field:

$$* \begin{cases} \delta(\vec{x}) = \displaystyle\int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k}) = \displaystyle\int d\vec{x}\, \delta(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} \end{cases}$$

Fourier Conventions & Definitions.

For each of the relevant dynamical quantities (density δ; velocity $\vec{v}$; gravity $\vec{g}$; potential ϕ), we can define their respective Fourier components (amplitudes):

$$\begin{aligned}
\delta(\vec{x},t) &\iff \hat{\delta}(\vec{k},t). \\
\vec{v}(\vec{x},t) &\iff \hat{\vec{v}}(\vec{k},t) \leftarrow \text{Vector!} \\
\vec{g}(\vec{x},t) &\iff \hat{\vec{g}}(\vec{k},t) \\
\phi(\vec{x},t) &\iff \hat{\phi}(\vec{k},t).
\end{aligned}$$

- **Note:**
  - before proceeding we need to define carefully our Fourier conventions. (no any special significance, but once adopted important to follow consistently).
  - $\hat{f}(\vec{k}) \Rightarrow f(\vec{x})$: $e^{-i\vec{k}\cdot\vec{x}}$
    $f(\vec{x}) \Rightarrow \hat{f}(\vec{k})$: $e^{+i\vec{k}\cdot\vec{x}}$
  - $\frac{d\vec{k}}{(2\pi)^3}$: "Kaiser" convention: recall any measure "$d\vec{k}$" accompanied by $(2\pi)^3$
  - Arbitrary field $f(\vec{x},t) \iff \hat{f}(\vec{k},t)$:

$$f(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{f}(\vec{k},t)\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\hat{f}(\vec{k},t) = \int d\vec{x}\, f(\vec{x},t)\, e^{+i\vec{k}\cdot\vec{x}}$$

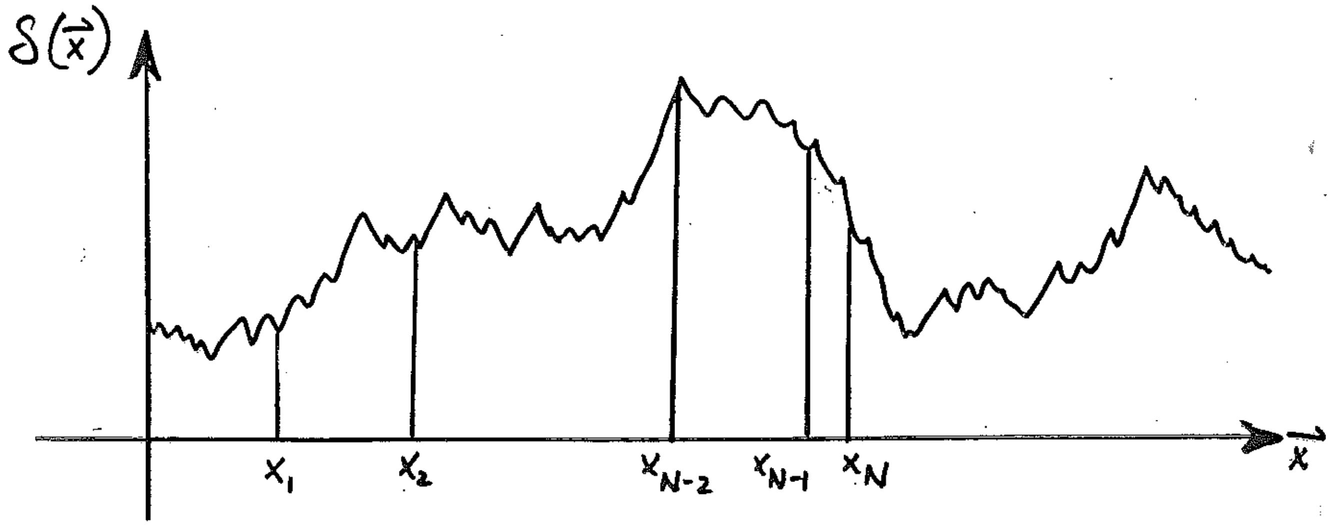
Preliminary, Fourier Conventions:

density
$$\left\{\begin{aligned} \delta(\vec{x}) &= \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k}) &= \int d\vec{x}\, \delta(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{aligned}\right.$$

elocity
$$\left\{\begin{aligned} \vec{v}(\vec{x}) &= \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\vec{v}}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\vec{v}}(\vec{k}) &= \int d\vec{x}\, \vec{v}(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{aligned}\right.$$

rav.
otential.
$$\left\{\begin{aligned} \phi(\vec{x}) &= \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{\phi}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\phi}(\vec{k}) &= \int d\vec{x}\, \phi(\vec{x})\, e^{+i\vec{k}\cdot\vec{x}} \end{aligned}\right.$$

Stochastic Density Fields.



- A cosmological density field is a spatial "noise" process:
the value of the density $\delta(\vec{x})$ at a specific locatio $\vec{x}$ is a stochastic quantity:

a priori, we do not know the value $\delta(\vec{x})$, we may only ask its probability:

$$\underline{\mathcal{P}_1 = P_1(\delta(\vec{x}) = \delta')} = P(\delta)\,d\delta$$

- and, generically, what the density field realizatio is at N locations (or ... in a region of space):

<u>N-point probability function</u> :

$$\underline{\mathcal{P}_N = P[\delta(\vec{x}_1), \delta(\vec{x}_2), \ldots, \delta(\vec{x}_N)]\,d\delta_1\,d\delta_2 \ldots d\delta_N}$$

- <u>Notice:</u> in the case of <u>complete lack of any spatial coherence</u>,

The canonic density field pdf :

Gaussian Random Fields

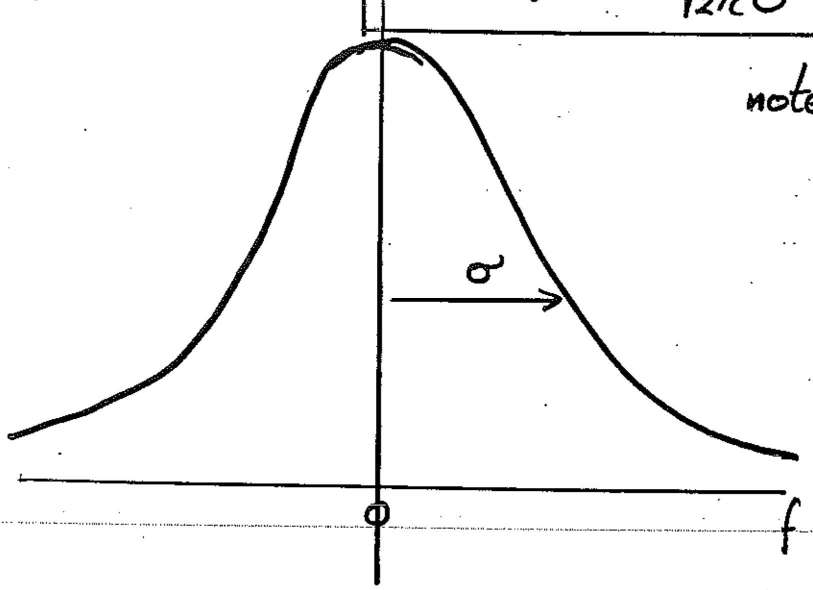
- **Note** : pdf = probability distribution function :
- Gaussian field :

$\mathcal{P}_N$ is a Gaussian distribution

1-point pdf:

$$\boxed{p(f)\,df = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{f^2}{2\sigma^2}\right)df}$$

note : "easy" notation:
$f = \delta$



- if you would measure in a Gaussian field the value δ at a zillion locations, you would find the above curve

$\Downarrow$

But, what about spatial coherence ?

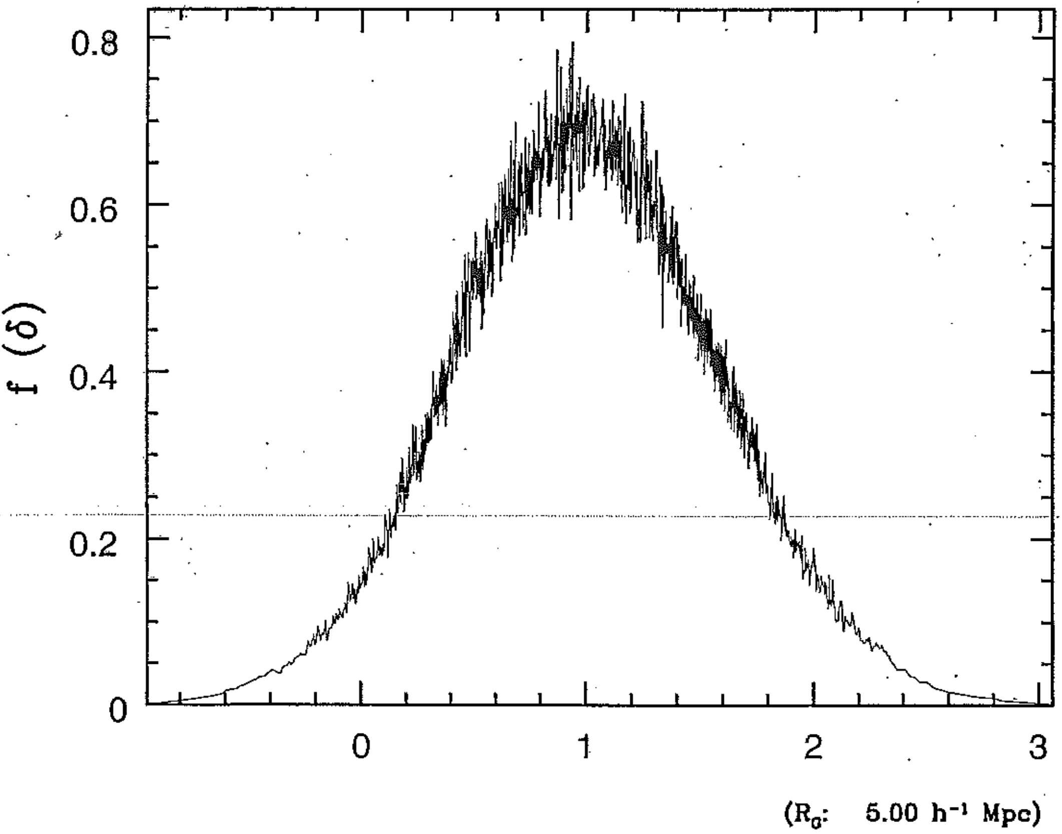
$\Downarrow$

[illegible] values at several locations

Probability Distribution Function:

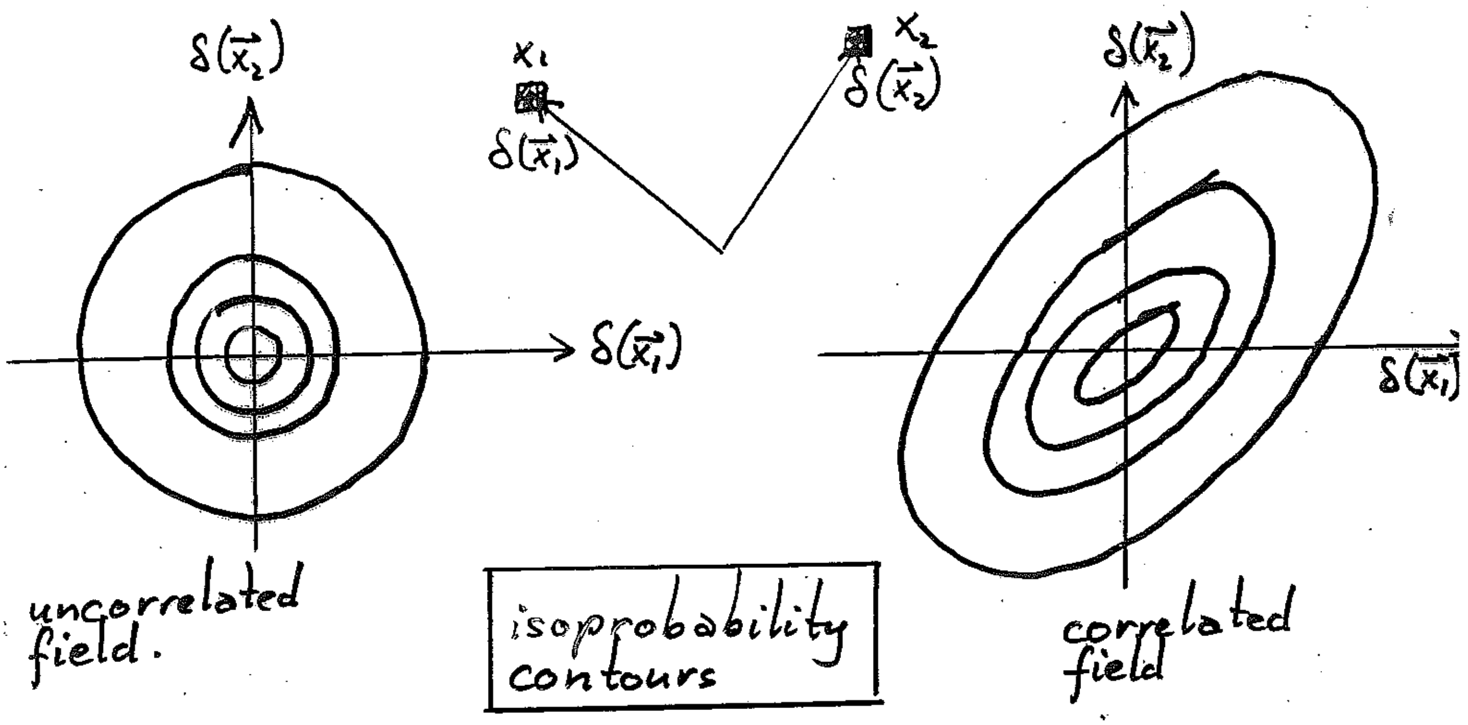
Density Field δ; R_G: 5.00 h^{-1} Mpc

File: simulcr1/simulcr1.0
a_{exp} = 1.00; $[\Omega_o, H_o]$ = [1.00, 100.00 km/s/Mpc]
σ_8 = 0.00; $[\Omega_a, H_a]$ = [1.00, 100.00 km/s/Mpc]



Average:	1.00
Median:	0.99
Sigma:	0.58
Skewness:	0.07
Curtosis:	0.05
Minimum:	−1.36
Maximum:	3.59
Low Value %:	50.61
High Value %:	49.39

Example : 2 point distribution function



Gaussian Random Field

$$\boxed{P(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}} \Rightarrow \xi(0) = \sigma^2$$

- Probability of field having value y_1 at $\vec{x}_1$ and y_2 at $\vec{x}_2$

$$\boxed{P(y_1, y_2)\, dy_1 dy_2 = \frac{e^{-(y_1\ y_2) M^{-1} \binom{y_1}{y_2}/2}}{(2\pi)(\det M)^{1/2}}\, dy_1\, dy_2}$$

in which:

$$M_{ij} = \begin{pmatrix} \xi(0) & \xi(\vec{x}_1, \vec{x}_2) \\ \xi(\vec{x}_1, \vec{x}_2) & \xi(0) \end{pmatrix}$$

covariance matrix:

$$M_{ij} = \begin{pmatrix} \xi(0) & \xi(\vec{x}_1,\vec{x}_2) \\ \xi(\vec{x}_1,\vec{x}_2) & \xi(0) \end{pmatrix}$$

$$\Rightarrow \quad \det M = \xi^2(0) - \xi^2(r) \quad ; \quad r = |\vec{x}_1 - \vec{x}_2|$$

$$\Rightarrow \quad M^{-1} = \frac{1}{\xi^2(0)-\xi^2(r)} \begin{pmatrix} \xi(0) & -\xi(r) \\ -\xi(r) & \xi(0) \end{pmatrix}$$

$$\Rightarrow \quad y^* M^{-1} y = \frac{1}{\xi^2(0)-\xi^2(r)} (y_1, y_2) \begin{pmatrix} \xi(0)y_1 - \xi(r)y_2 \\ -\xi(r)y_1 + \xi(0)y_2 \end{pmatrix}$$

$$= \frac{1}{\xi^2(0)-\xi^2(r)} \left(\xi(0)y_1^2 + \xi(0)y_2^2 - 2\xi(r)y_1y_2\right)$$

Which leads to $P(y_1, y_2)$:

$$\boxed{P(y_1,y_2) = \frac{1}{\sqrt{2\pi}\,(\xi^2(0)-\xi^2(r))^{1/2}} e^{-\frac{\xi(0)y_1^2+\xi(0)y_2^2-2\xi(r)y_1y_2}{2(\xi^2(0)-\xi^2(r))}}}$$

N-point probability distribution function:

Multivariate Gaussian:

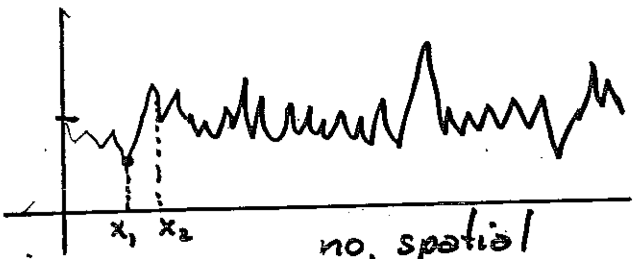
- $$\mathcal{P}_N = \frac{\exp\left[-\frac{1}{2}\sum_{i,j=1}^{N} f_i \left(M_{ij}^{-1}\right) f_j \right]}{\left[(2\pi)^N (\det M)\right]^{1/2}} \prod_{i=1}^{N} df_i$$

with: M: covariance matrix

$$\underset{\substack{\Uparrow\\ \text{statistical characterization}\\ \text{by 2-pt correlation function } \xi}}{M_{ij}} = \underset{\substack{\uparrow\\ \text{ensemble}\\ \text{average}}}{\left\langle f(\vec{x}_i) f(\vec{x}_j)\right\rangle} \equiv \underset{\substack{\uparrow\\ \text{2-pt}\\ \text{correlation}\\ \text{function}}}{\xi(\vec{x}_i - \vec{x}_j)} \underset{\substack{\uparrow\\ \text{assuming}\\ \text{homogeneous,}\\ \text{isotropic}\\ \text{stochastic process.}}}{=} \xi(|\vec{x}_i - \vec{x}_j|)$$

- ξ : 2-pt correlation function

- Quantification of mutual dependence field values at several locations:



no spatial coherence:

$\xi(x_1 - x_2) = 0$

both x_1 and x_2 in underdense region

$\xi(x_1 - x_2) \gg 0$

Random Phases:

* each mode is a complex number:

$$\hat{f}(\vec{k}) = \hat{f}_r(\vec{k}) + i\,\hat{f}_i(\vec{k}) = |\hat{f}(\vec{k})|\, e^{i\phi_k}$$

phase of wave

* $$\mathcal{P}(\hat{f}(\vec{k})) = \mathcal{P}(\hat{f}_r(\vec{k}), \hat{f}_i(\vec{k}))$$

$$= \mathcal{P}(\hat{f}_r(\vec{k}))\,\mathcal{P}(\hat{f}_i(\vec{k}))$$

$$= e^{-\frac{|\hat{f}(\vec{k})|^2}{2P(k)}}\ \frac{|\hat{f}(\vec{k})|\, d\,|\hat{f}(\vec{k})|}{2P(k)}\ \frac{d\phi_k}{2\pi}$$

Rayleigh distribution

* Strictly, a field is **Gaussian** only if
  - phases uniformly distributed in $[0, 2\pi]$
  - $|\hat{f}(\vec{k})|$ Rayleigh distributed
  - $|\hat{f}(\vec{k})|$ mutually independent.

The Random Field Power Spectrum

2-pt, (auto)correlation function quantifies mutual dependence field at different locations:

Look at correlation function $\xi(|\vec{x}_1 - \vec{x}_2|)$:

$$\left.\begin{aligned} &- \xi(\vec{x}_1 - \vec{x}_2) = \langle f(\vec{x}_1)\, f(\vec{x}_2) \rangle \\ &- f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, \hat{f}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \end{aligned}\right\}$$

$$\Rightarrow \quad \xi(\vec{x}_1 - \vec{x}_2) = \left\langle \int \frac{d\vec{k}_1}{(2\pi)^3} \hat{f}(\vec{k}_1) e^{-i\vec{k}_1\cdot\vec{x}_1} \int \frac{d\vec{k}_2}{(2\pi)^3} \hat{f}(\vec{k}_2)^* e^{i\vec{k}_2\cdot\vec{x}_2} \right\rangle$$

$$= \iint \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \rangle e^{-i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2}$$

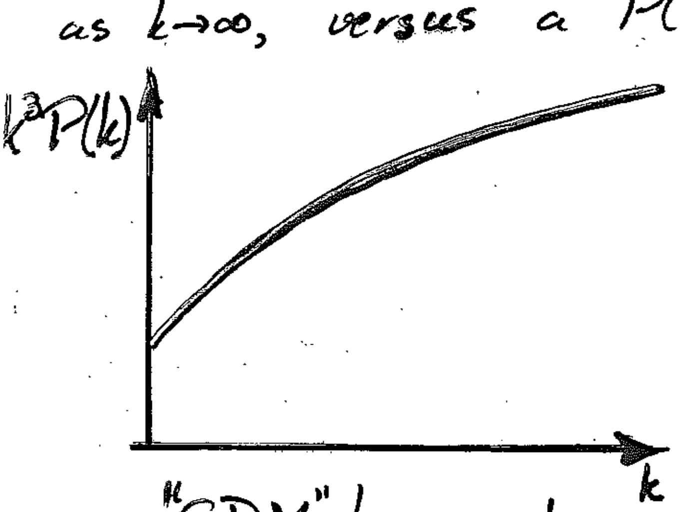
$$\Rightarrow \quad \boxed{\xi(\vec{x}_1 - \vec{x}_2) = \int \frac{d\vec{k}}{(2\pi)^3}\, \mathcal{P}(\vec{k})\, e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}$$

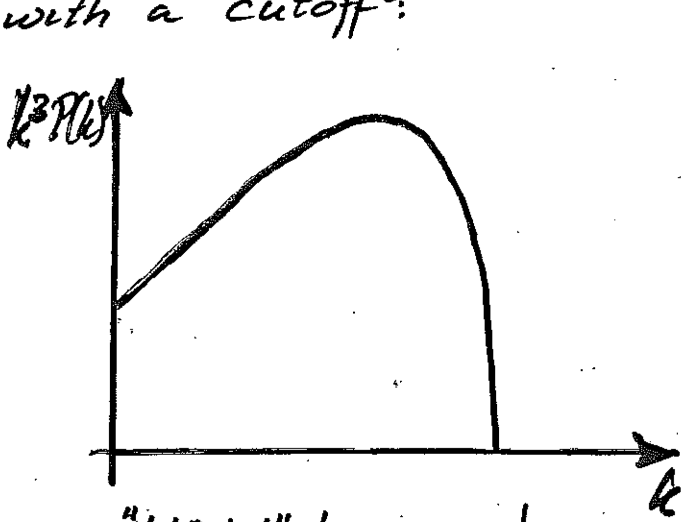
with Power Spectrum $\mathcal{P}(k)$:

$$\boxed{(2\pi)^3\, \mathcal{P}(\vec{k}_1)\, \delta_D(\vec{k}_1 - \vec{k}_2) = \langle \hat{f}(\vec{k}_1)\, \hat{f}^*(\vec{k}_2) \rangle}$$

Density Fluctuation Spectrum $P(k)$: Evolutionary Implications

For comparison it is illuminating to assess the spatial structure in a field with a $P(k)$ continuously increasing as $k \to \infty$, versus a $P(k)$ with a cutoff:



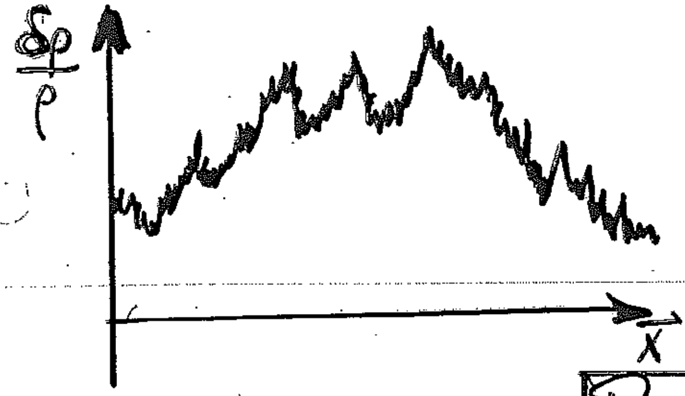


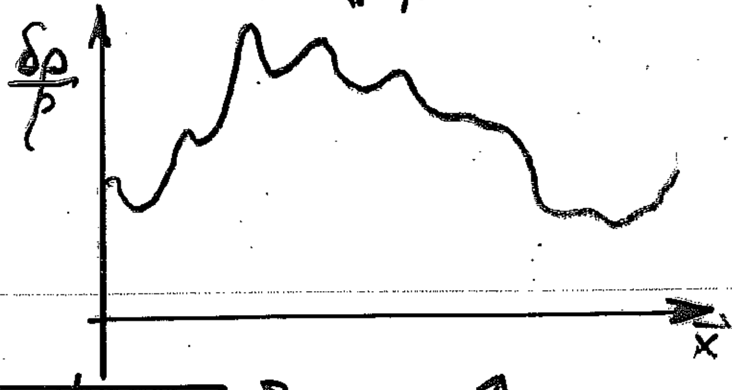
"CDM" type spectrum

$\Downarrow$ dominant small-scale fluctuations

"HDM" type spectrum, with cutoff.

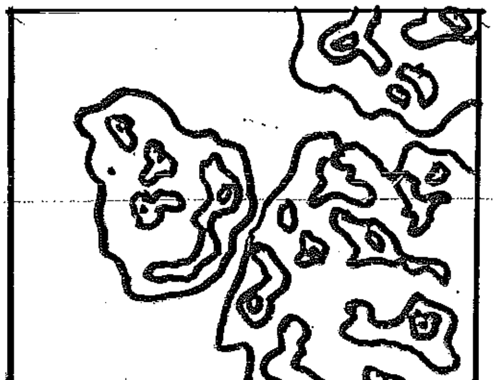
$\Downarrow$ small-scale fluctuations absent

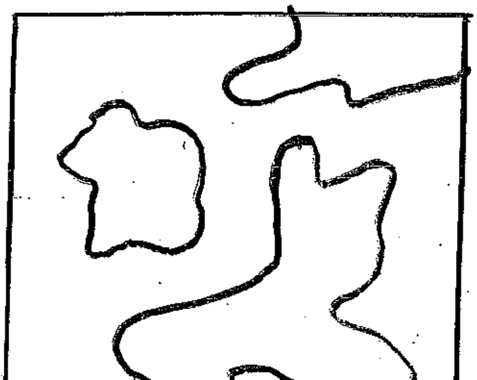




Realizations Density Field:

"linear" profiles.





Power Spectrum, Odds and Ends

* correct definition:

$$\boxed{(2\pi)^3 P(k_1)\, \delta_D(\vec{k}_1 - \vec{k}_2) = \langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \rangle}$$

* Often you find the following, not so accurate definition:

$$P(k) = \langle |\hat{f}(\vec{k})|^2 \rangle$$

notice though it reveals true meaning $P(k)$:

average contribution diverse wave numbers $\vec{k}$ to fluctuation field.

* from :

$$\mathcal{P}[f] \propto \exp - \left[\int \frac{d\vec{k}}{(2\pi)^3} \frac{|\hat{f}(\vec{k})|^2}{2P(k)} \right]$$

- Power spectrum specifies density field characteristics <u>completely</u> if Gaussian.

- Specifies relative contribution to density field from waves at different wavenumbers $\vec{k}$

$$\Downarrow$$

Determines character of density field evolution

$$\Downarrow$$

P(k) Holy Grail of COSMIC structure formatio

Justification for Gaussianity Assumption

① Physical Justification

Inflation

If fluctuations were generated during primordial inflationary phase, from quantum fluctuations that got inflated into macroscopic ripples, then field Gaussian.

② Mathematical Justification

Central Limit Theorem

$$\delta(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\,\hat{\delta}(\vec{k})\,e^{-i\vec{k}\cdot\vec{x}}$$

$\uparrow$

density field superposition of large number of <u>independent vectors</u>.

$\downarrow$

according to central limit theorem this implies directly: $\boxed{\text{Gaussianity}}$

Relations between $\hat{\delta}(\vec{k})$, $\hat{\vec{v}}(\vec{k})$ and $\hat{\phi}(\vec{k})$

In linear regime, equations of motion in Fourier space

(1) $$\frac{d\hat{\delta}(\vec{k})}{dt} - \frac{1}{a}\, i\vec{k}\cdot\hat{\vec{v}}(\vec{k}) = 0$$ continuity equation

(2) $$\frac{\hat{\phi}(\vec{k})}{a^2} = -4\pi G\bar{\rho}\,\frac{\hat{\delta}(\vec{k})}{k^2}$$ Poisson equation

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(1) $$\Rightarrow \quad \hat{\vec{v}}(\vec{k}) = -a\,\frac{i\vec{k}}{k^2}\,\frac{d\hat{\delta}(\vec{k})}{dt} = -Haf\,\frac{i\vec{k}}{k^2}\,\hat{\delta}(\vec{k})$$

$$\vec{v}(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\,\hat{\vec{v}}(\vec{k})\,e^{-i\vec{k}\cdot\vec{x}}$$

$$\boxed{\vec{v}(\vec{x},t) = \underbrace{\int \frac{d\vec{k}}{(2\pi)^3}\, -Haf\,\frac{i\vec{k}}{k^2}\,\underbrace{\hat{\delta}(\vec{k})}_{\text{Gaussian variable}}\,e^{-i\vec{k}\cdot\vec{x}}}_{\text{linear combination of Gaussian... : remains a Gaussian}}}$$

$\longrightarrow$ $\vec{v}$ also Gaussian variable

$\Rightarrow$ Gaussian $\vec{v}$ completely specified by its spectrum $P_v(k)$:

$$P_v(k) \propto \langle|\vec{v}(\vec{k})|^2\rangle = (Haf)^2\,\frac{\langle|\hat{\delta}(\vec{k})|^2\rangle}{k^2}$$

$$\boxed{P_v(k) = (Haf)^2\,\frac{P(k)}{k^2}}$$

(2): $\Rightarrow \quad \hat{\phi}(\vec{k}) = -\frac{3}{2}\Omega H^2 a^2 \, \frac{1}{k^2}\,\hat{\delta}(\vec{k})$.

$$\boxed{\phi(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3} \; -\frac{3}{2}\Omega H^2 a^2 \, \frac{1}{k^2}\,\hat{\delta}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}}$$

$\longrightarrow$ ϕ also Gaussian variable

$\Rightarrow$ with power spectrum $P_\phi(k)$:

$$\boxed{P_\phi(k) = \left(\frac{3}{2}\Omega H^2 a^2\right)^2 \; \frac{P(k)}{k^4}}$$

- Summarizing :

$$\boxed{P(k) \;\propto\; k^2 P_v(k) \;\propto\; k^4 P_\phi(k)}$$

- This implies :
  - density perturbation spectrum more sensitive to small-scale density field, ($\lesssim$ 10-25 h^{-1} Mpc,
  - velocity perturbation spectrum to medium-scale (~ 10 - 200 h^{-1} Mpc): velocity fields
  - and potential spectrum $P_\phi(k)$ to

```
Density Field                                                          (contours)

File:                          simulcr1/simulcr1.0          R_G:   5.00 h^-1 Mpc
a_exp =  1.00;                 [Ω_o,H_o] =         [ 1.00,  100.00 km/s/Mpc ]
σ_8  =  0.00;                  [Ω_a,H_o] =         [ 1.00,  100.00 km/s/Mpc ]
Slice:        Width:     160.00 h^-1 Mpc     X_c  =  80.00  h^-1 Mpc;     T_x  =   0.00  h^-1 Mpc;
              Thickness:   0.00 h^-1 Mpc     Y_c  =  80.00  h^-1 Mpc;     T_y  =   0.00  h^-1 Mpc;
                                             Z_c  =  80.00  h^-1 Mpc      T_z  =   0.00  h^-1 Mpc

Field Unit:                    ρ_b = 3 ΩH²/8πG
Contour Range:                 [  -0.767,   2.880] =
                               [   0.100,  99.900]    percentile
linear incr.:                  0.182                                   # contours:   20
reference level:               1.002
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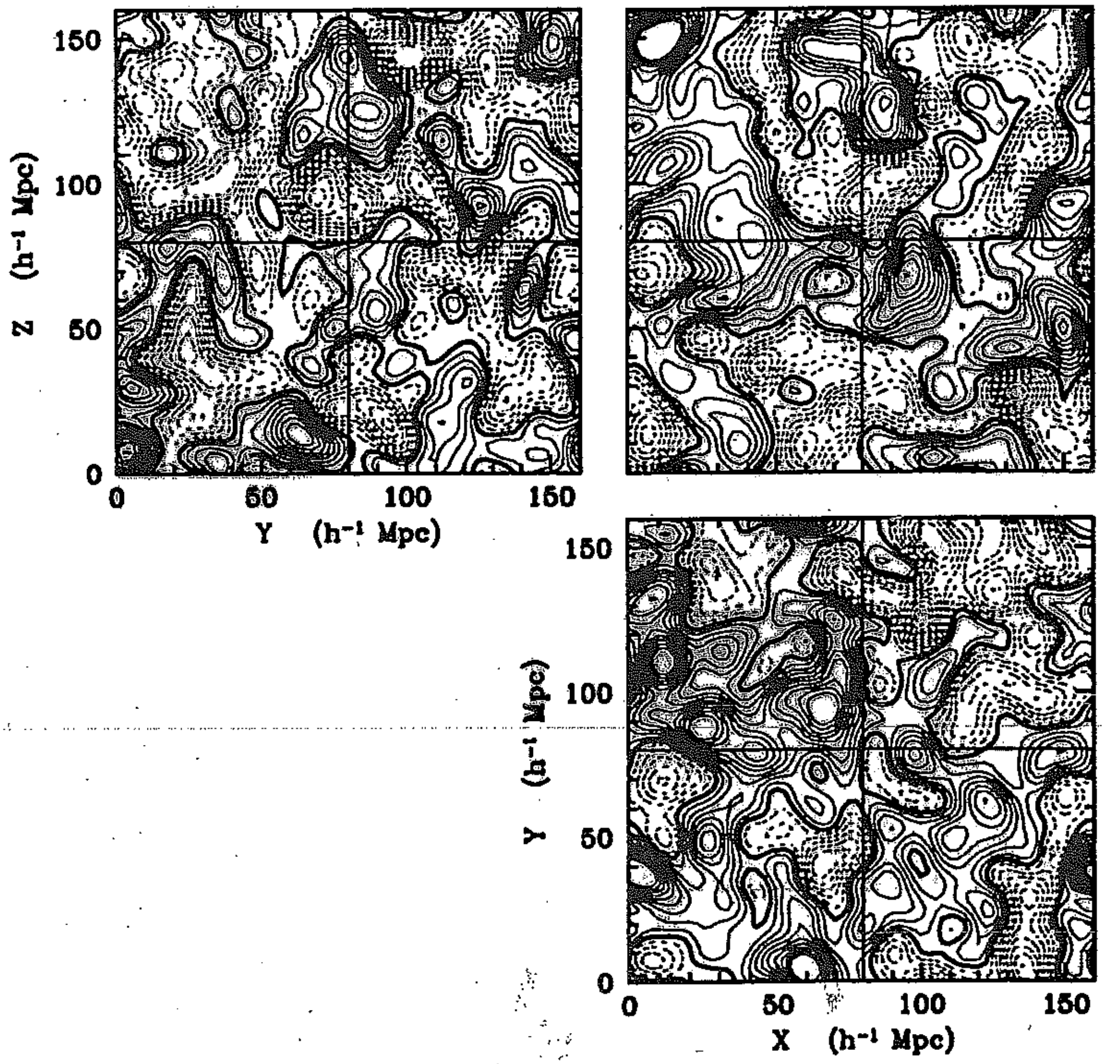


Gravitational Force

(contours)

File: simulcr1/simulcr1.0 R_s: 5.00 h^{-1} Mpc

a_{exp} = 1.00; $[\Omega_o,H_o]$ = [1.00, 100.00 km/s/Mpc]

σ_8 = 0.00; $[\Omega_s,H_s]$ = [1.00, 100.00 km/s/Mpc]

Slice: Width: 160.00 h^{-1} Mpc X_c = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;

Thickness: 0.00 h^{-1} Mpc Y_c = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;

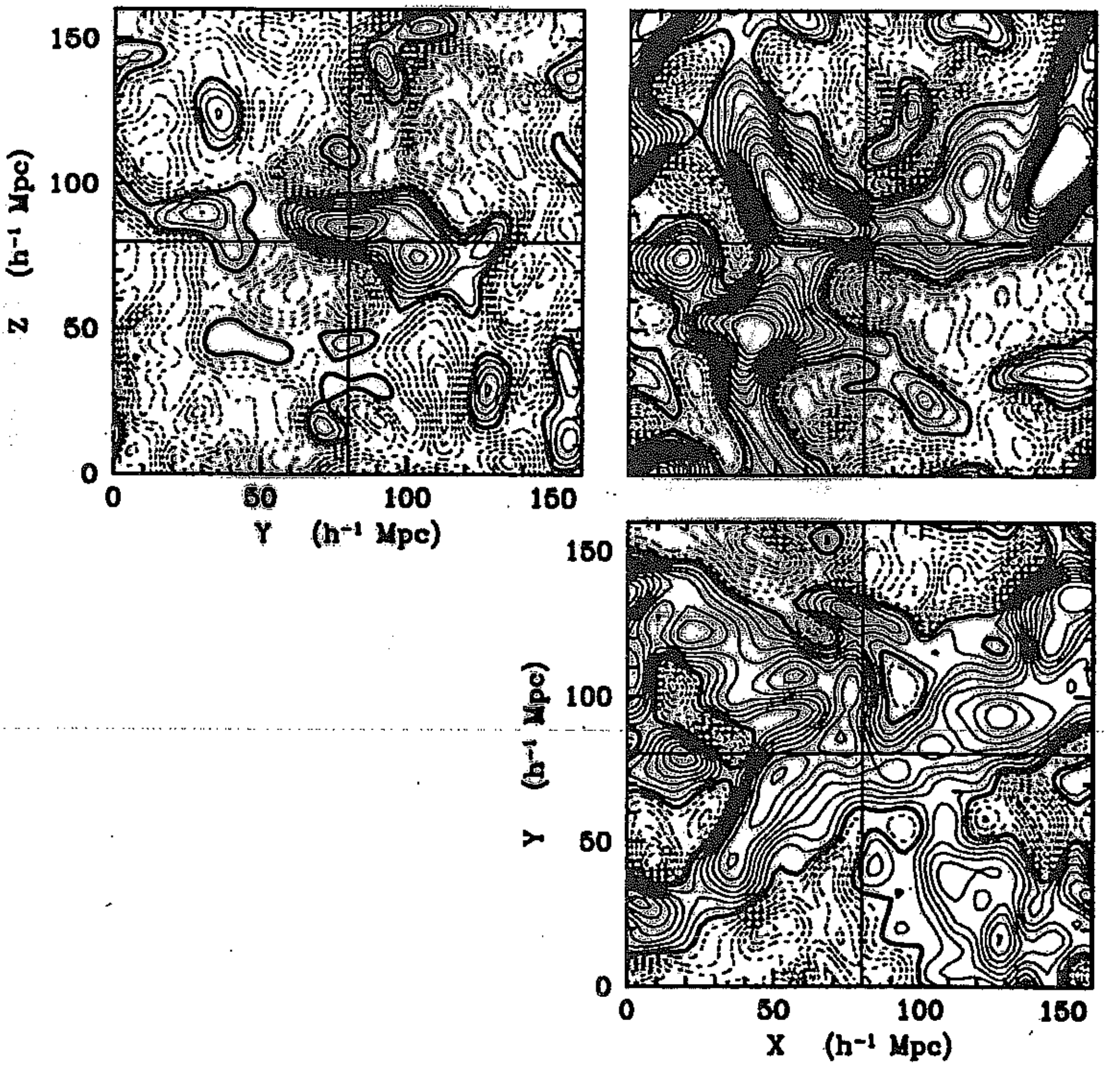
Z_c = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc

Field Unit: km/s

Contour Range: [42.803,1044.760] =
[0.100, 99.820] percentile

linear incr.: 50.098 **# contours:** 20

reference level: 439.812



Potential Field

(contours)

File: simulcr1/simulcr1.0 R_s: 5.00 h^{-1} Mpc

a_{exp} = 1.00; $[\Omega_o, H_o]$ = [1.00, 100.00 km/s/Mpc]

σ_g = 0.00; $[\Omega_a, H_a]$ = [1.00, 100.00 km/s/Mpc]

Slice: Width: 160.00 h^{-1} Mpc X_o = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;

Thickness: 0.00 h^{-1} Mpc Y_o = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;

Z_o = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc

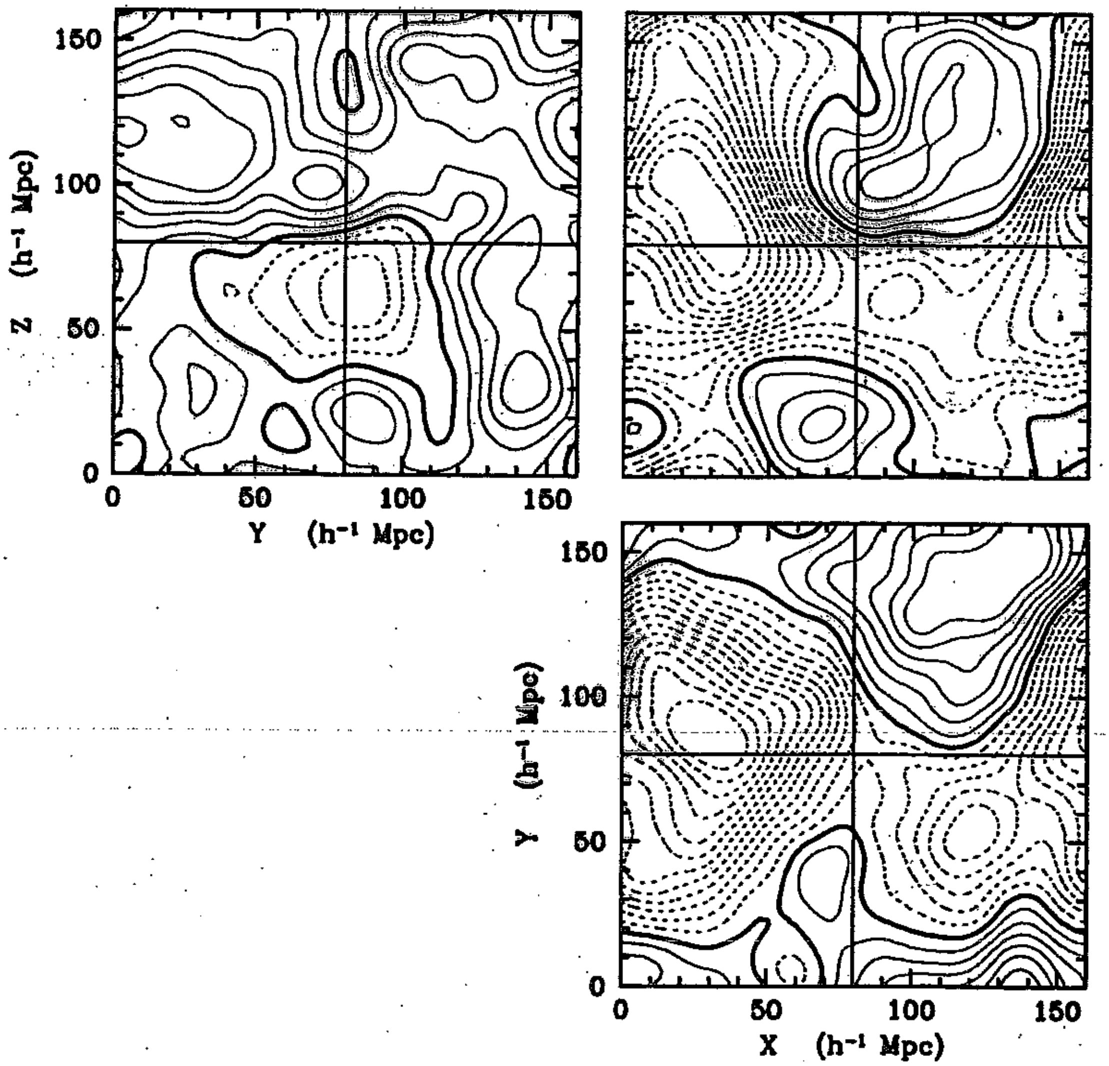
Field Unit: 1.5 ΩH^2

Contour Range: [−251.995, 143.336] =

[0.100, 99.900] percentile

linear incr.: 19.767 **# contours:** 20

reference level: 0.000



Gravitational Force

(vectors)

File: simulcr1/simulcr1.0 R_g: 5.00 h^{-1} Mpc

a_{exp} = 1.00; [Ω_o,H_o] = [1.00, 100.00 km/s/Mpc]

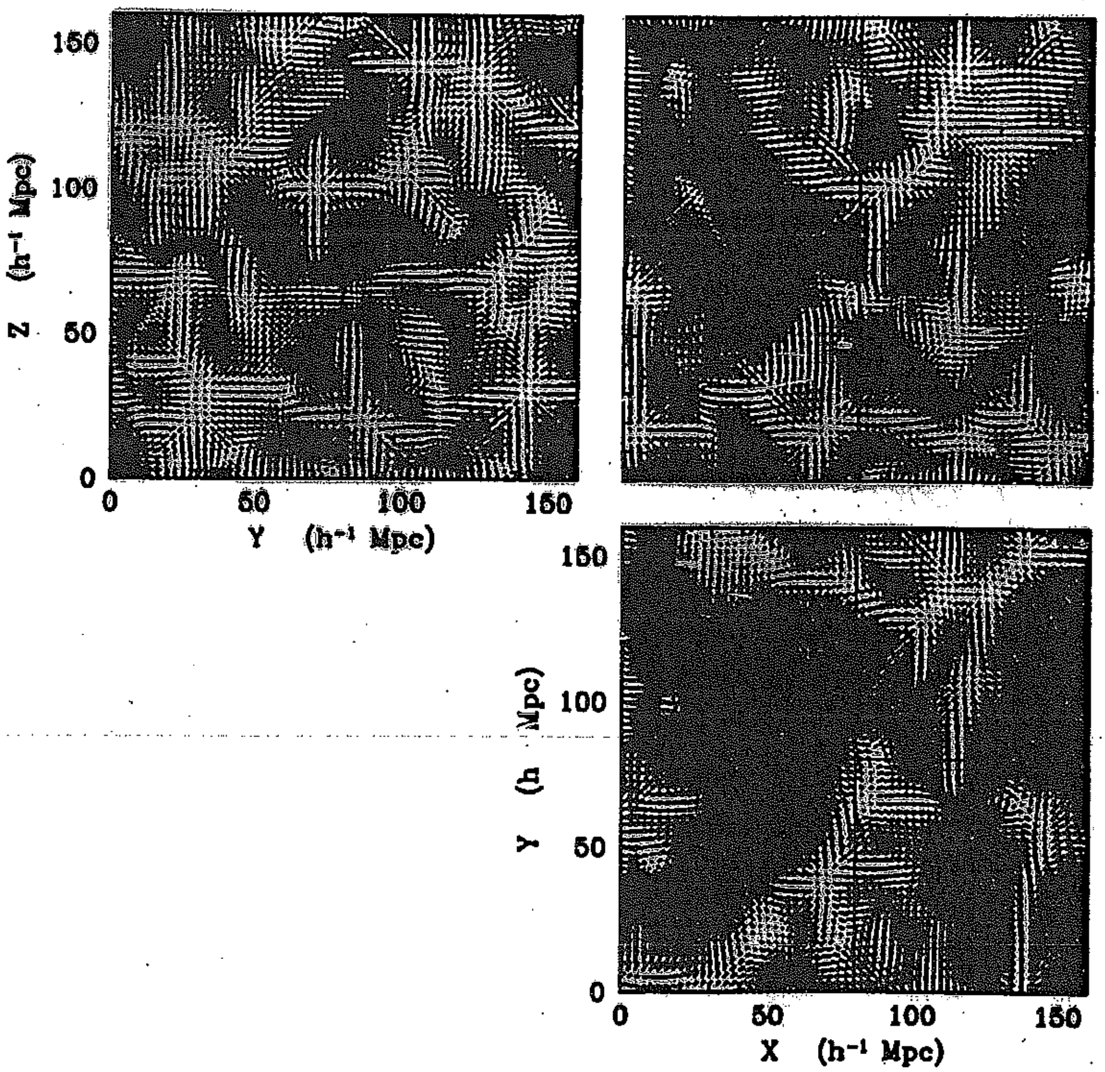
σ_g = 0.00; [Ω_o,H_o] = [1.00, 100.00 km/s/Mpc]

Slice: Width: 160.00 h^{-1} Mpc X_c = 80.00 h^{-1} Mpc; T_x = 0.00 h^{-1} Mpc;

Thickness: 0.00 h^{-1} Mpc Y_c = 80.00 h^{-1} Mpc; T_y = 0.00 h^{-1} Mpc;

Z_c = 80.00 h^{-1} Mpc T_z = 0.00 h^{-1} Mpc

Vector: g= 2000.00 km/s:



Power Spectra: Cosmic Physics

- Inflation predicts a

$$\boxed{\text{Harrison-Zel'dovich Spectrum.}}$$

$$\Longrightarrow \boxed{P_{prim}(k) = Ak.}$$

- Harrison-Zel'dovich Spectrum:
  "constant curvature spectrum"

$$\sigma_\phi^2 = \int \frac{dk}{(2\pi)^3} P_\phi(k)$$

$$= \int \frac{k^2 dk}{2\pi^2} \frac{P(k)}{k^4} \Longleftarrow P(k) \propto k^n$$

$$\propto A \int \frac{d \log k}{2\pi^2} \frac{k^n}{k}$$

$$\propto A \int \frac{d \log k}{2\pi^2} k^{n-1}$$

$\Rightarrow \boxed{\text{if } n=1}$ contributions to potential perturbations "scale-free", all equal per logarithmic bin.

Power spectrum:

$$\xi(\vec{x}) \overset{FT}{\Longleftrightarrow} P(k)$$

- (auto) correlation function $\xi(\vec{x})$ and power spectrum $P(k)$ are each others Fourier transform.

$$\boxed{\begin{aligned} \xi(\vec{x}) &= \int \frac{d\vec{k}}{(2\pi)^3} P(k)\, e^{-i\vec{k}\cdot\vec{x}} \\ P(\vec{k}) &= \int d\vec{x}\, \xi(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} \end{aligned}}$$

- $\xi(\vec{x}) = \langle f(\vec{x}_1 + \vec{x})\, f(\vec{x}_1) \rangle$

$\Downarrow$

Thus, density fluctuations σ^2:

$$\sigma^2 = \langle f(\vec{x}_1) f(\vec{x}_1) \rangle = \xi(\vec{x}=0)$$

$$\Downarrow$$

$$\boxed{\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k)} \Longrightarrow$$

$P(k)$ quantifies the contribution to the total perturbation σ^2 by spatial frequencies $\vec{k}$

Physical spectra:

- Inflation predicts Harrison-Zel'dovich spectrum:

$$P_{prim}(k) = Ak$$

- Subsequently, physical processes affect growth of different waves k differently. Expressed in transfer function $T(k)$:

$$T^2(k,a) = \frac{P(k,a)}{P(k,a_i)} ;$$

$$\boxed{P(k) = A k T^2(k)}$$

- For example:

① Cold Dark Matter:

$$P_{CDM}(k) = \frac{Ak}{(1 + 1.7q + 9.0q^{3/2} + 1.0q^2)^2} \quad ; \quad q = \frac{k}{(\Omega_{CDM} h^2 M_{p}}$$

② Hot Dark Matter:

$$P_\nu(k) = e^{\left[-0.16(kR_\nu) - \frac{(kR_\nu)^2}{2}\right]} \frac{1}{1 + 1.6q + (4.0q)^{3/2} + (0.92q)^2}$$

$$\left(q = \frac{k}{(\Omega_\nu h^2 Mpc^{-1})} \; ; \; R_\nu = \frac{2.6}{\Omega_\nu h^2} \; Mpc \right)$$

- Other examples $P(k) = Ak^{n}$

- ## Power-law Power Spectra

$$\boxed{P(k) \propto A k^{n}}$$

$$n(k) = \frac{d\log P}{d\log k} :$$

① $\underline{\underline{n > -3}}$

hierarchical clustering.

② $\underline{\underline{n \leq 1}}$

If $n>1$ then ultraviolet divergences: small-scale perturbations diverge.

- ## Post-horizon Power Spectrum

After a perturbation with wavenumber $\vec{k}$ enters the horizon:

- it can start growing under force of gravity.
- physical processes start to effect growth of primordial perturbations and modify it.
  - baryons } pressure :
  - photons } Jeans damping & oscillation.
  - Silk damping : photon free-streamin

- dark matter : free streaming

$$\Downarrow$$

- free streaming length: $\ell \propto \frac{1}{n_s \sigma_s}$

$$\Downarrow$$

n_s : density DM species
σ_s : cross-section

- all waves with:

$$k \gtrsim k_s \approx \frac{1}{\ell} \propto n_s \sigma_s$$

- damped.

$$\Downarrow$$

$\Longrightarrow$ signatures in spectrum of

$$k_s \simeq n_s \sigma_s \propto \Omega_s h^2$$

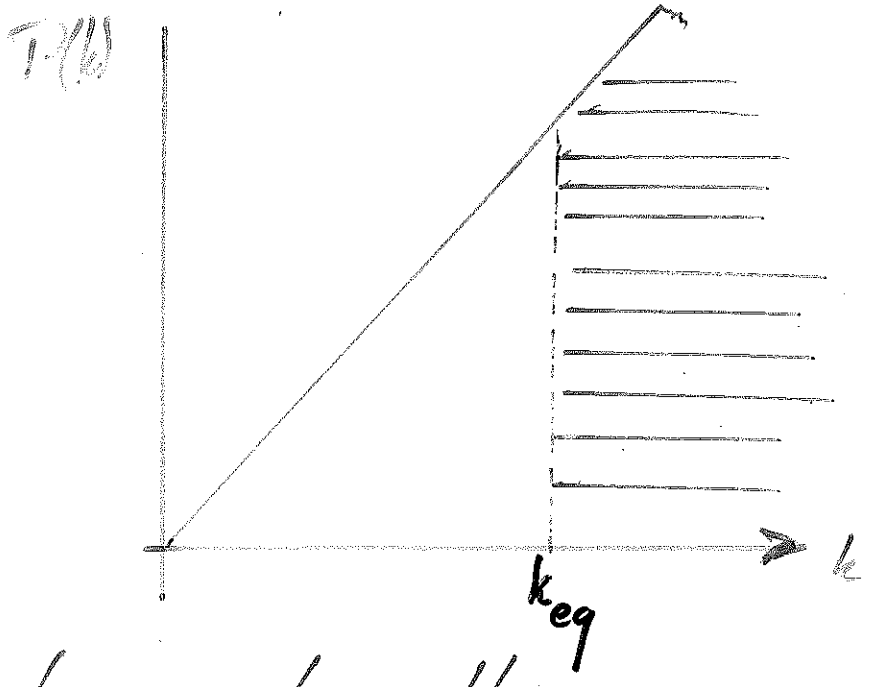
$$\Downarrow$$

$$\frac{k}{\Omega_s h^2} \approx \frac{1}{\Omega_s h} h^{-1} k \equiv \frac{k}{\Gamma_s} \qquad k : [h^{-1} \mathrm{Mpc}]$$

$$q \equiv \frac{k}{\Gamma_s}$$

- **Recall:**
  - $g_{00} = g^{c}_{00} + h_{00} \quad \propto \frac{\Phi}{c^2}$
  - $h_{00} \propto \frac{\Phi}{c^2} \implies$ equal contributions to potential give equal contributions to curvature $h_{\mu\nu}$.

- Harrison-Zel'dovich



- fluctuations enter with same amplitude the horizon, then **(after $t > t_{eq}$)** start growing with $D(t) \propto a(t)$ **($\Omega \approx 1$)**:

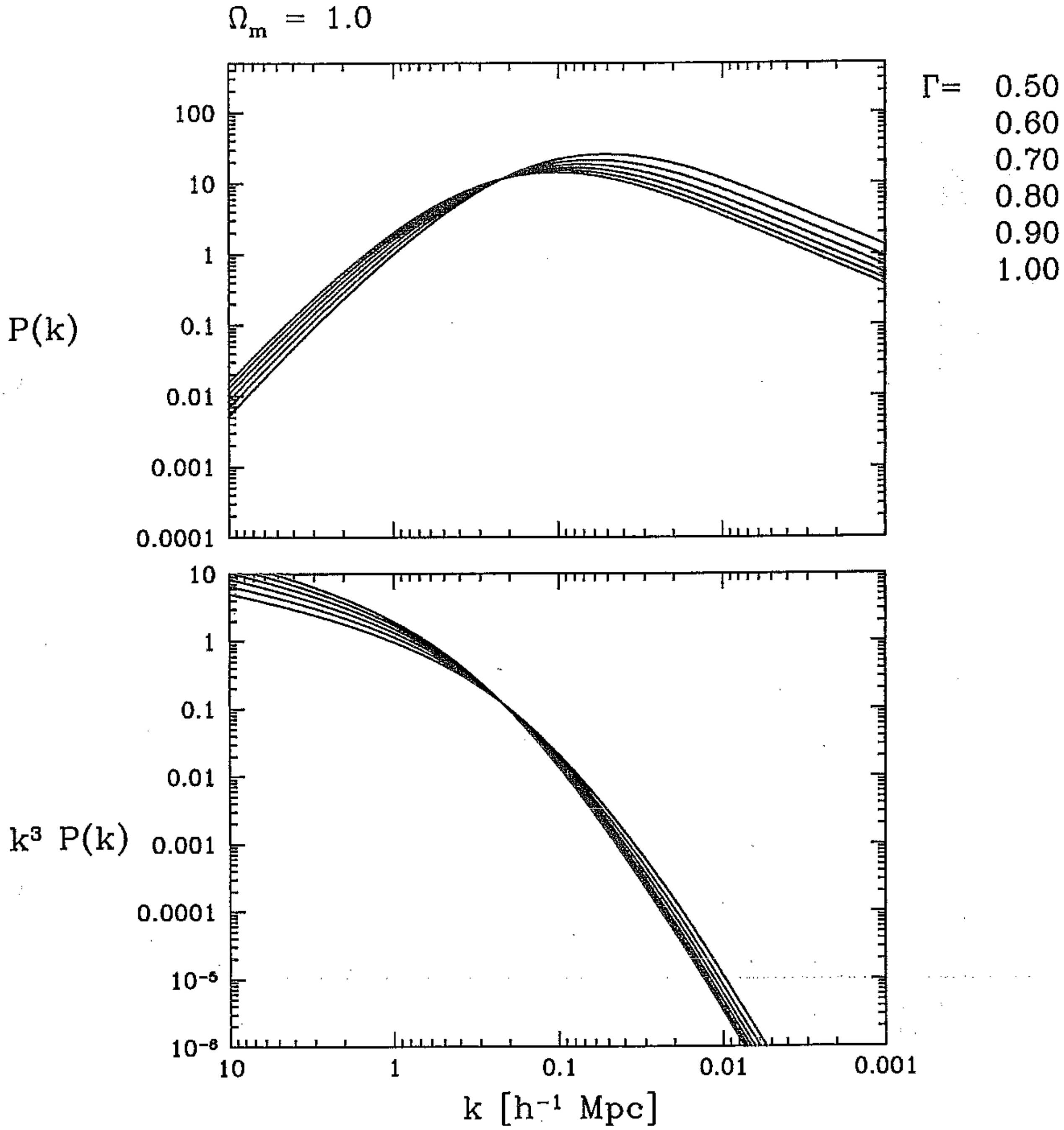
$$\hat{\delta}(k, t) \propto \frac{D(t)}{D(t_{hor})} \hat{\delta}(k, t_{hor})$$

$$\Downarrow$$

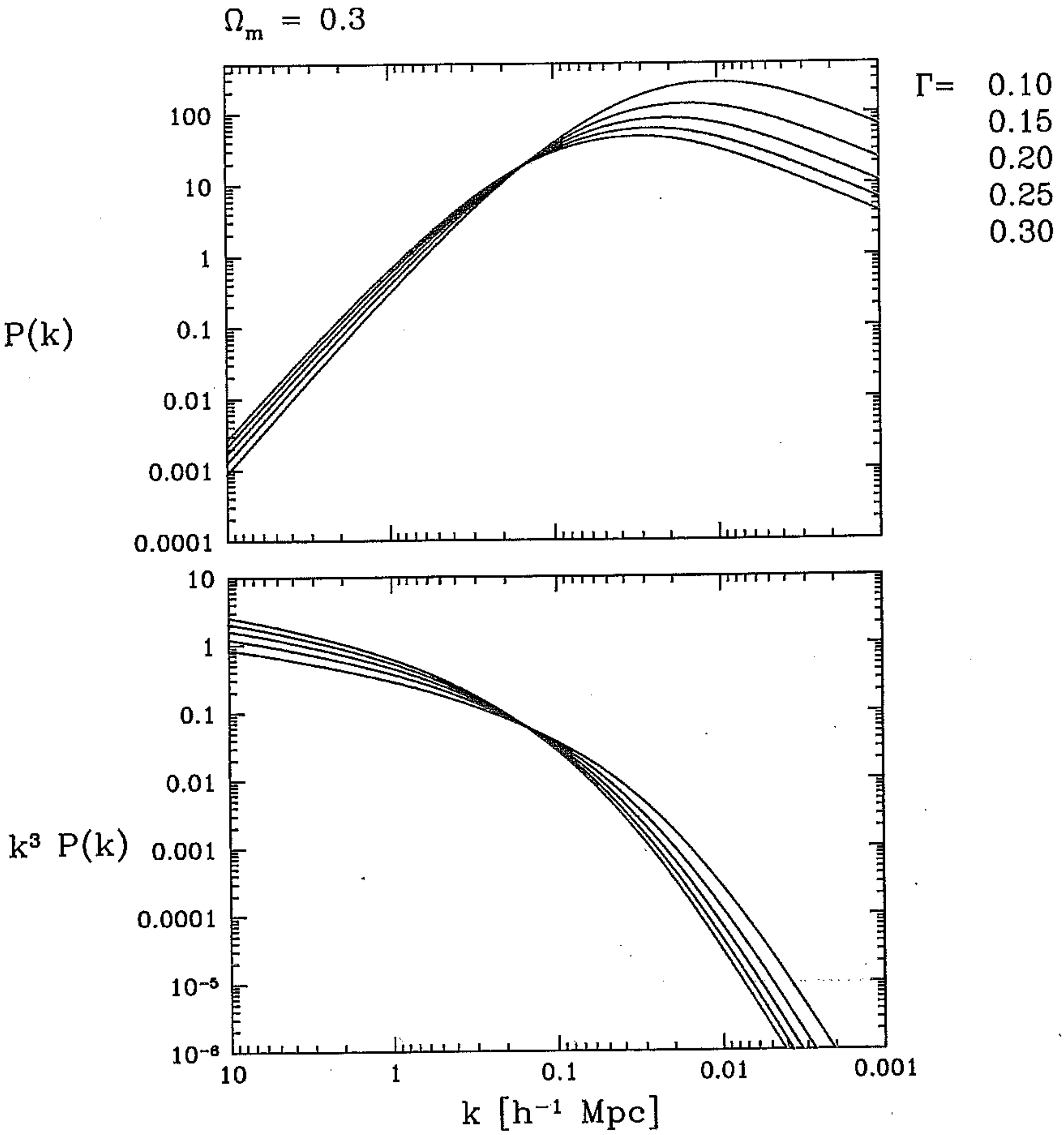
$\hat{\delta}(k$

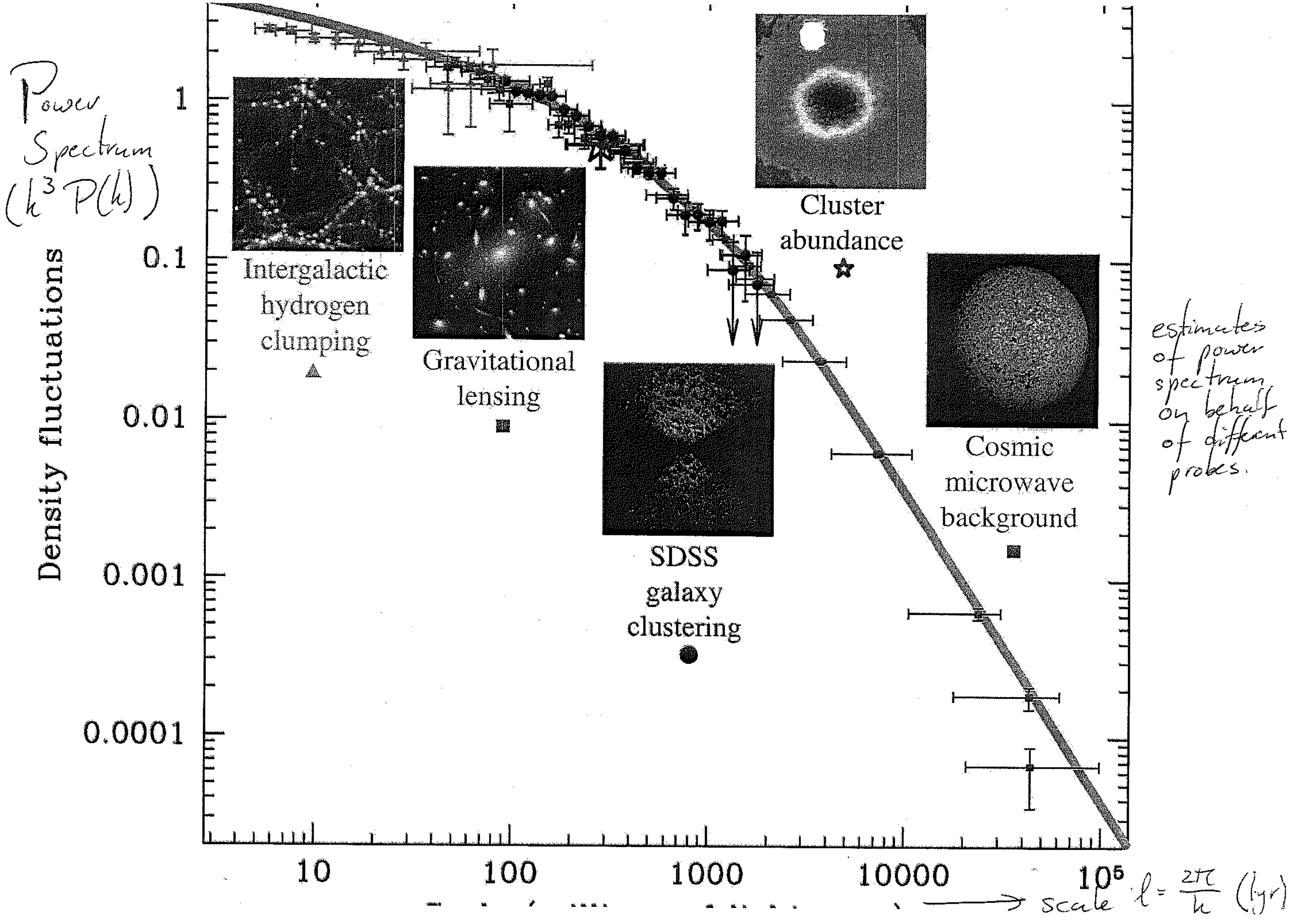
$$P_{\mathrm{CDM}}(k) \propto \frac{k^n}{[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{1/2}} \times \frac{[\ln(1 + 2.34q)]^2}{(2.34q)^2} \tag{13}$$

$$\begin{aligned} q &= k/\Gamma \\ \Gamma &= \Omega_{m,\circ} h \exp\left\{-\Omega_b - \frac{\Omega_b}{\Omega_{m,\circ}}\right\} \end{aligned}$$



Ω_m = 1.0
Γ= 0.50
0.60
0.70
0.80
0.90
1.00
P(k)
100
10
1
0.1
0.01
0.001
0.0001
k^3 P(k)
10
1
0.1
0.01
0.001
0.0001
10^{-5}
10^{-6}
10
1
0.1
0.01
0.001
k [h^{-1} Mpc]

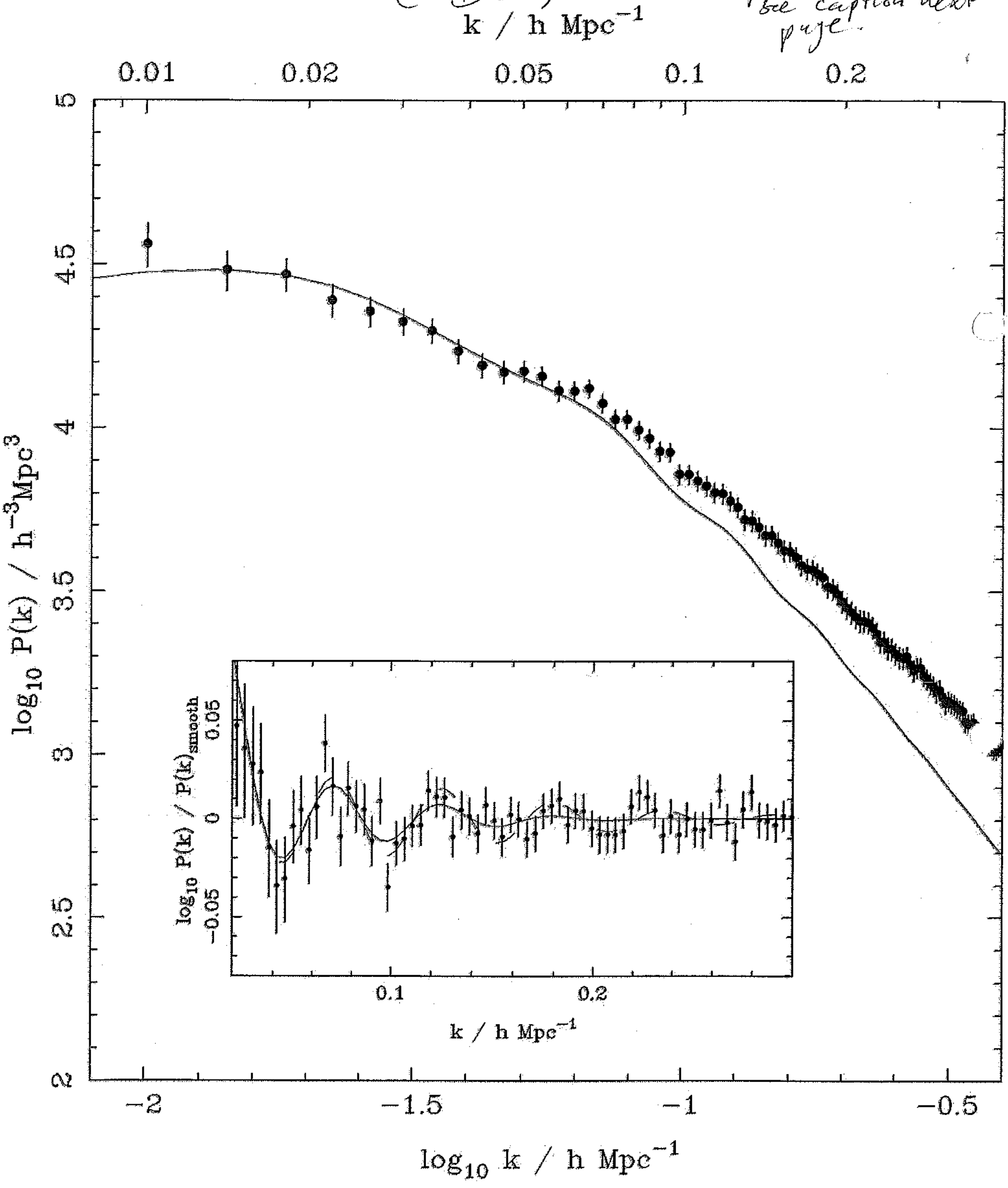

Ω_m = 0.3
Γ= 0.10
0.15
0.20
0.25
0.30
P(k)
100
10
1
0.1
0.01
0.001
0.0001
k³ P(k)
10
1
0.1
0.01
0.001
0.0001
10⁻⁵
10⁻⁶
10
1
0.1
0.01
0.001
k [h⁻¹ Mpc]



estimates of power spectrum on behalf of different probes.

Power Spectrum Sloan Digital Sky Survey (SDSS)

for explanation see caption next page.



from Tegmark et al.

Power Spectrum SDSS

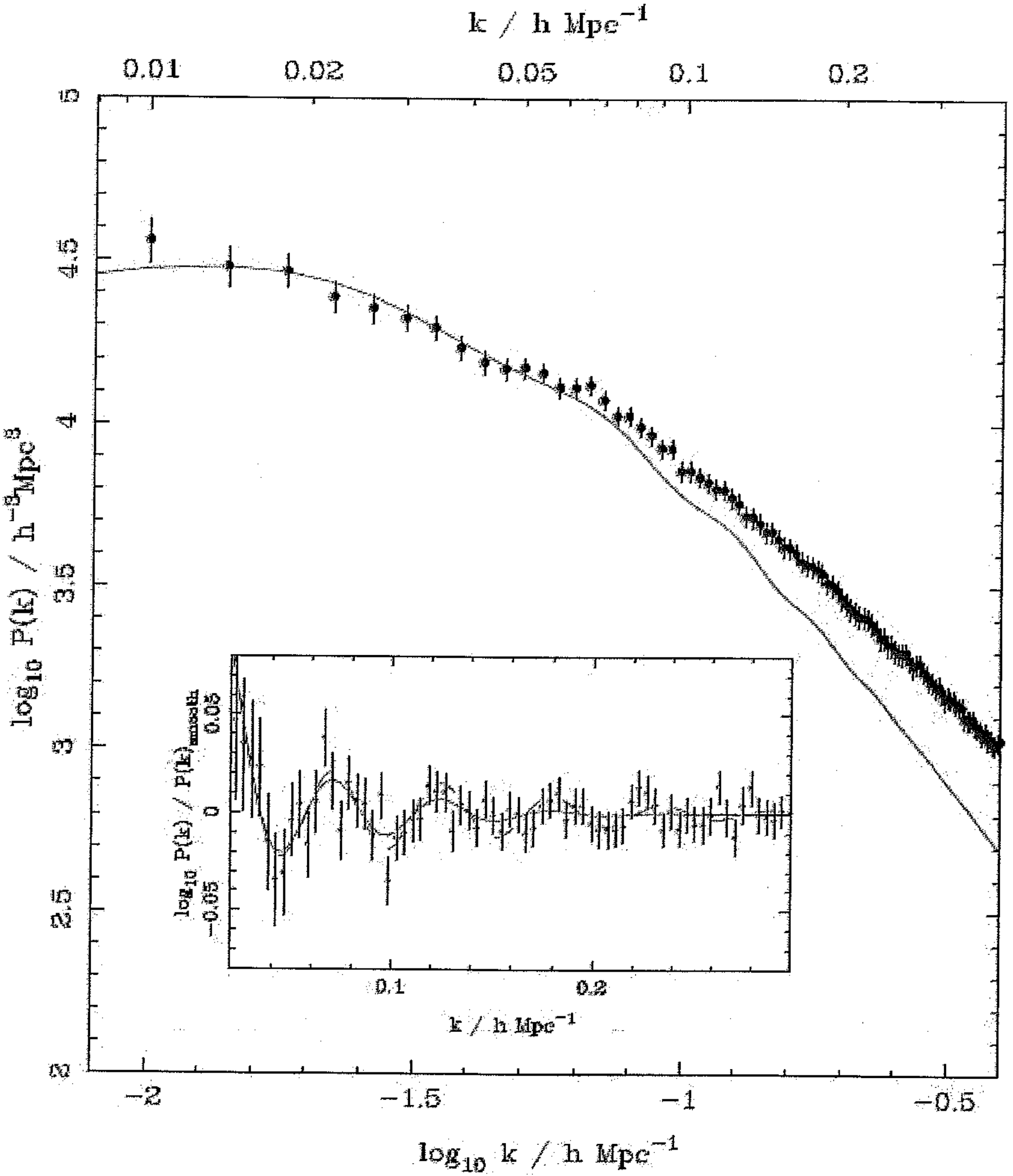
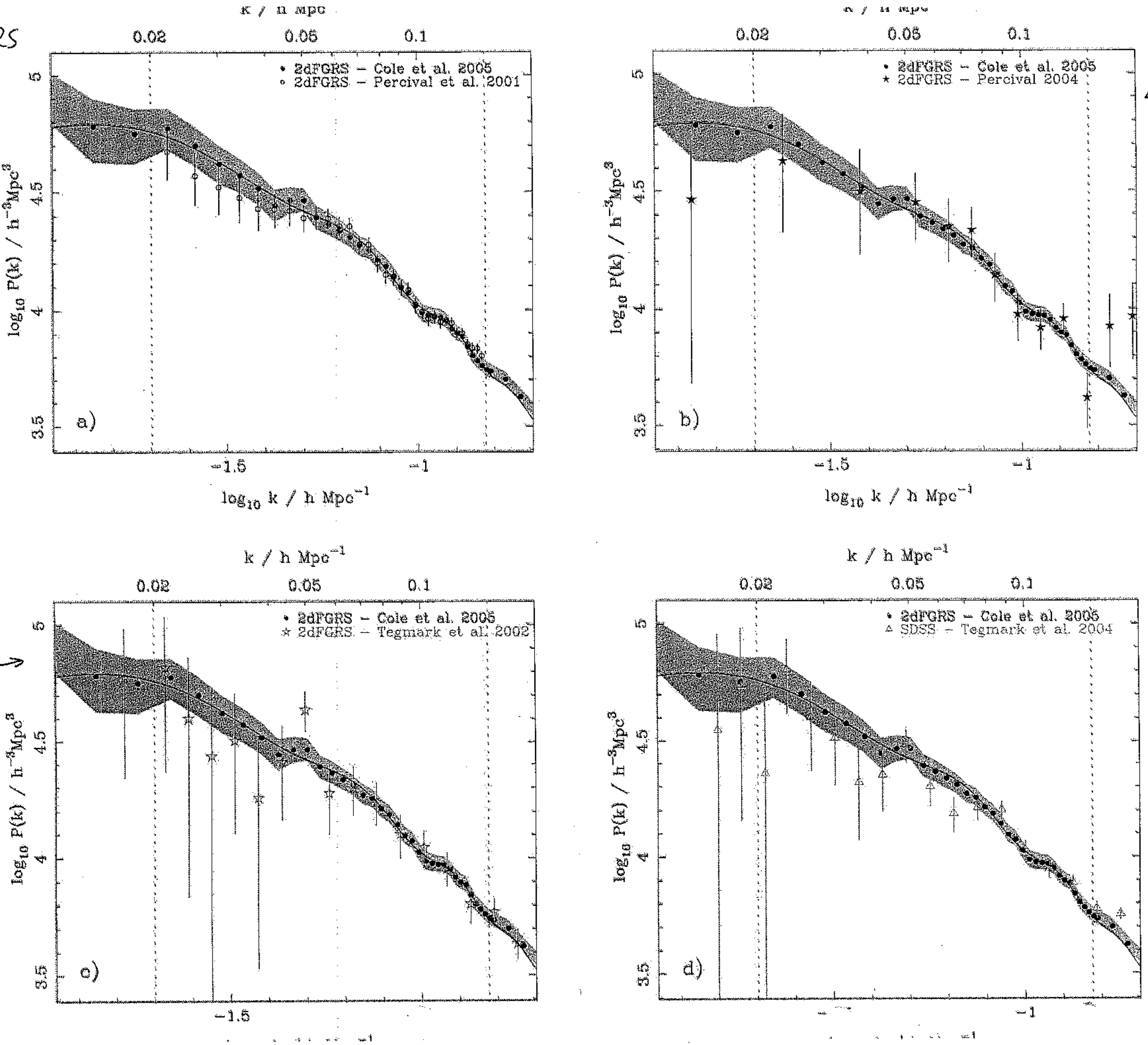


FIG. 12.— The redshift-space power spectrum recovered from the combined SDSS main galaxy and LRG sample, optimally weighted fo
oth density changes and luminosity dependent bias (solid circles with 1-σ errors). A flat Λ cosmological distance model was assumed wit
$\Omega_M = 0.24$. Error bars are derived from the diagonal elements of the covariance matrix calculated from 2000 log-normal catalogues create
or this cosmological distance model, but with a power spectrum amplitude and shape matched to that observed (see text for details)
The data are correlated, and the width of the correlations is presented in Fig. 10 (the correlation between data points drops to < 0.33 fo
$\Delta k > 0.01\,h\,\mathrm{Mpc}^{-1}$). The correlations are smaller than the oscillatory features observed in the recovered power spectrum. For compariso
we plot the model power spectrum (solid line) calculated using the fitting formulae of Eisenstein & Hu (1998); Eisenstein et al. (2006), fo
he best fit parameters calculated by fitting the WMAP 3-year temperature and polarisation data, $h = 0.73$, $\Omega_M = 0.24$, $n_s = 0.96$ an
$\Omega_b/\Omega_M = 0.174$ (Spergel et al. 2006). The model power spectrum has been convolved with the appropriate window function to match th
measured data, and the normalisation has been matched to that of the large-scale ($0.01 < k < 0.06\,h\,\mathrm{Mpc}^{-1}$) data. The deviation from
this low Ω_M linear power spectrum is clearly visible at $k \gtrsim 0.06\,h\,\mathrm{Mpc}^{-1}$, and will be discussed further in Section 6. The solid circles wit
σ errors in the inset show the power spectrum ratioed to a smooth model (calculated using a cubic spline fit as described in Percival et al
2006) compared to the baryon oscillations in the (WMAP 3-year parameter) model (solid line), and shows good agreement. The calculatio
of the matter density from these oscillations will be considered in a separate paper (Percival et al. 2006). The dashed line shows the sam
model without the correction for the damping effect of small-scale structure growth of Eisenstein et al. (2006). It is worth noting that thi
model is not a fit to the data, but a prediction from the CMB experiment.

2dFGRS



2dFGRS

Power Spectrum

2dFGRS,
various different studies.

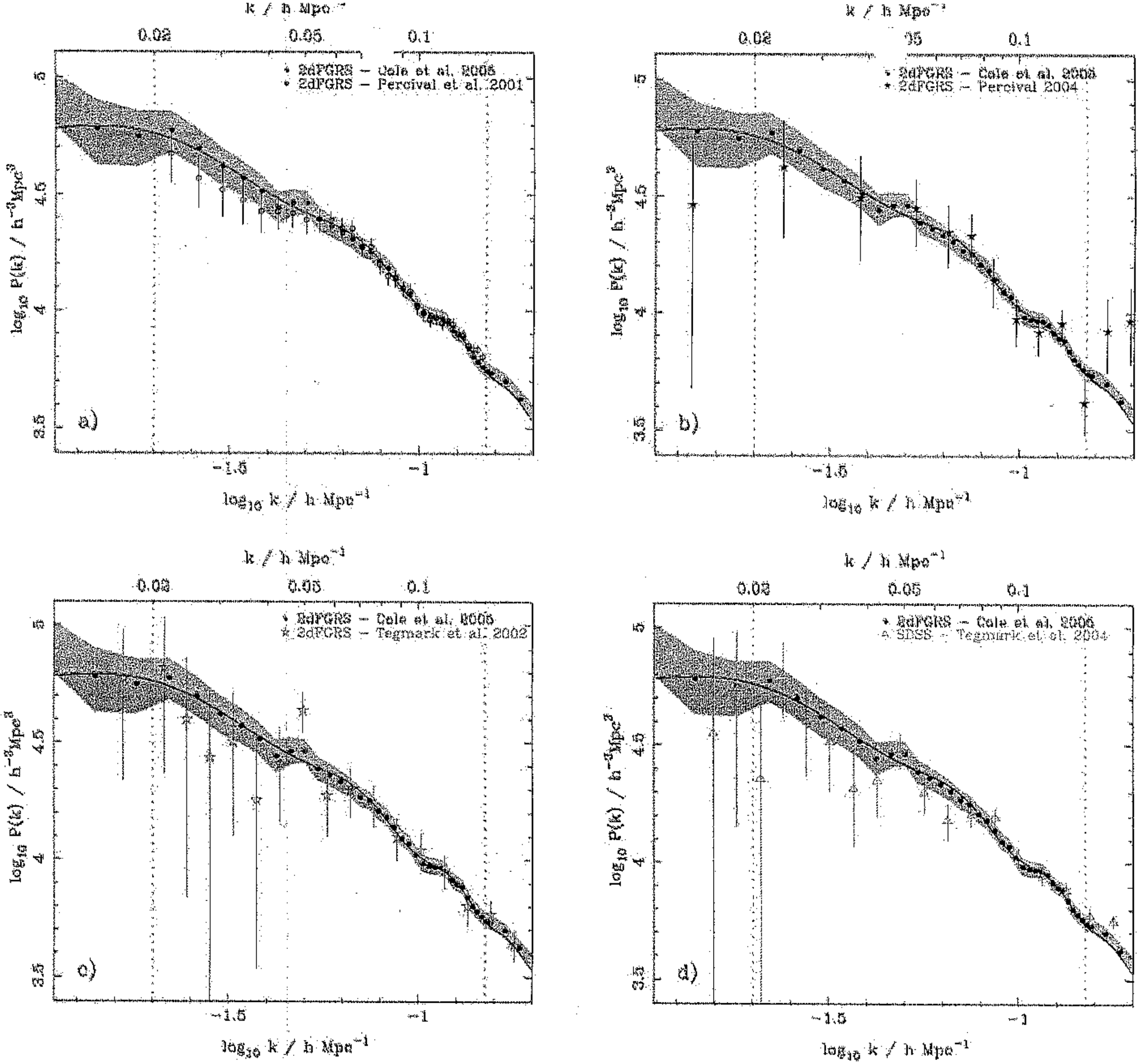
solid line: theoretical curve

dFGRS.

SDSS.

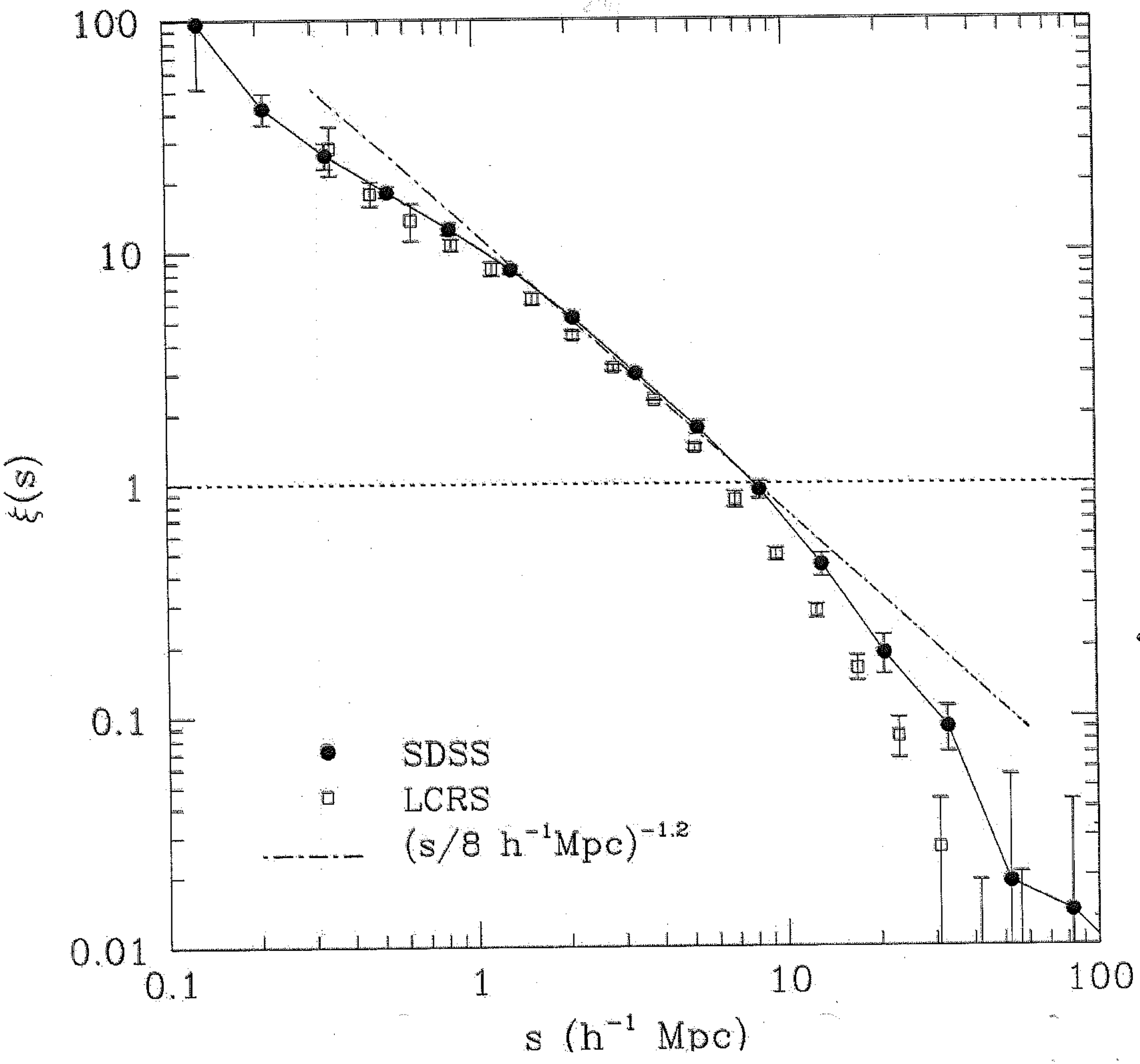
(see caption next page)

rom
Percival
et al.



Power spectra
2dFGRS &
SDSS
↓

Figure 16. The redshift-space power spectrum calculated in this paper (solid circles with 1σ errors shown by the shaded region) compared with other measurements of the 2dFGRS power-spectrum shape by (a) Percival et al. (2001), (b) Percival (2005), and (c) Tegmark et al. (2002). For the data with window functions, the effect of the window has been approximately corrected by multiplying by the net effect of the window on a model power spectrum with $\Omega_m h = 0.168$, $\Omega_b/\Omega_m = 0.0$, $h = 0.72$ & $n_s = 1$. A zero-baryon model was chosen in order to avoid adding features into the power spectrum. All of the data are renormalized to match the new measurements. Panel (d) shows the uncorrelated SDSS real-space $P(k)$ estimate of Tegmark et al. (2004), calculated using their 'modelling method' with no FOG compression (their Table 3). These data have been corrected for the SDSS window as described above for the 2dFGRS data. The solid line shows a model linear power spectrum with $\Omega_m h = 0.168$, $\Omega_b/\Omega_m = 0.17$, $h = 0.72$, $n_s = 1$ and normalization matched to the 2dFGRS power



correlation function
$\xi(s)$

in redshift space
(ie. not corrected for peculiar motions)

● Sloan DSS
□ Las Campanas Redshift Survey.

from Zehavi 2001
et al.

Identifying Protostructures in a Primordial Density Field.

Filtering Stochastic Density Fields.

Power Spectrum $P(k)$

$$\boxed{\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k)}$$

$$\Big\Downarrow \text{ isotropic random field.}$$

- $\sigma^2 = \int \frac{k^2 dk}{2\pi^2} P(k) = \int \frac{d\log k}{2\pi^2} k^3 P(k)$

- $\boxed{\underline{\underline{k^3 P(k)}} \quad \text{is power per logarithmic frequency band } d \log k}$

To appreciate the meaning of the power spectrum in terms of physical structures, it is better to discuss the fluctuations in terms of a <u>physical scale R</u> or <u>physical mass scale M.</u>

- Recall that σ^2 per se does not provide information on contributions by various scales
- While σ^2 may even diverge, if the integral is infinite :
- To discuss density fluctuations,

Fluctuation Mass Scales

- In order to identify (embryonic) structures in a primordial cosmological density field we look at the field filtered on scale R:

$$\boxed{\delta_f(\vec{r}) = \int d\vec{x}\, \delta(\vec{x})\, W_f(\vec{x},\vec{r};R)}$$

(see accompanying page on 'filtering random field'.

- The corresponding Mass Scale $\overline{M}_f$

$$\boxed{\overline{M}_f = \int d\vec{x}\, \overline{\rho}\, W_f(\vec{x},R)}$$

For example, for a regular Tophat Filter W_{TH}, we deal with a straight "mean" mass $\overline{M}_f$:

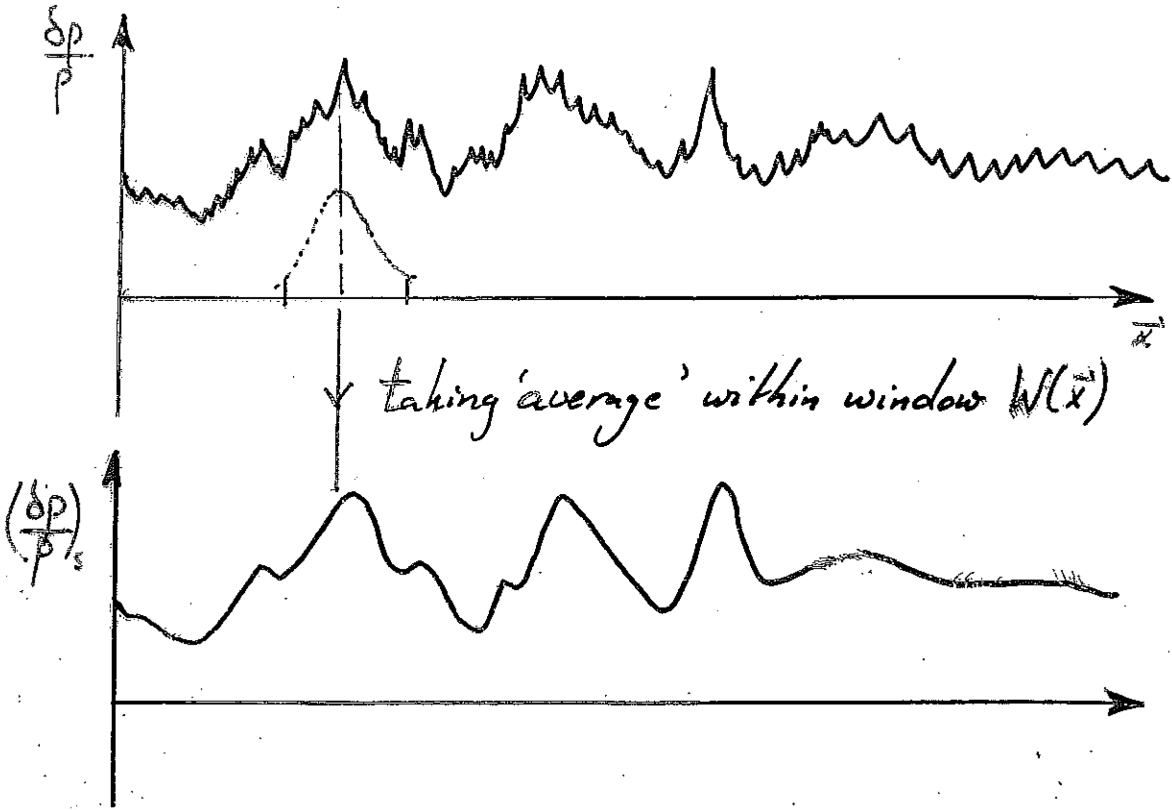
$$\boxed{\overline{M}_f = \frac{4\pi}{3}\overline{\rho}R^3}$$

- Field fluctuations on mass scale M_f are then:

$$\sigma_M^2 = \frac{\langle (M_f - \overline{M}_f)^2 \rangle}{\overline{M}_f^2} = \frac{\langle \{\int d\vec{x}\, \delta(\vec{x})\, W_f(\vec{x},R)\}^2 \rangle}{\{\int d\vec{x}\, W_f(\vec{x},R)\}^2}$$

so that for a normalized filter W_f: $\int d\vec{x}\, W_f = 1$,

Filtering of random field



- $$\boxed{f_G(\vec{x}) = \int d\vec{y}\, f(\vec{y})\, W_G(\vec{y}, \vec{x})}$$

$W_G(\vec{y}, \vec{x})$: filter function

- How does this translate into Fourier space:

$$f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{f}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$$

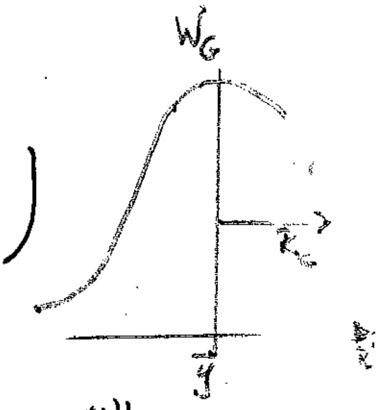
$$W_G(\vec{y}, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{W}(\vec{k})\, e^{-i\vec{k}\cdot(\vec{y}-\vec{x})}$$

$$\Rightarrow \boxed{f_G(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} f(\vec{k})\, \hat{W}^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}}$$

Two most important filters:

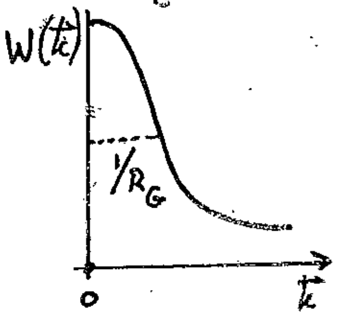
- **Gaussian filter:**

$$W_G(\vec{y}, \vec{x}) = \frac{1}{(2\pi R_G^2)^{3/2}} \exp\left(- \frac{|\vec{y}-\vec{x}|^2}{2R_G^2}\right)$$



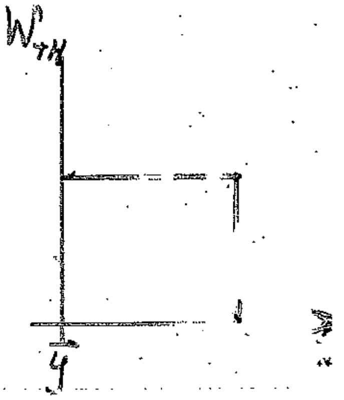
R_G: filter scale

$$\Rightarrow \quad \boxed{\hat{W}(\vec{k}) = e^{-\frac{k^2 R_G^2}{2}}}$$



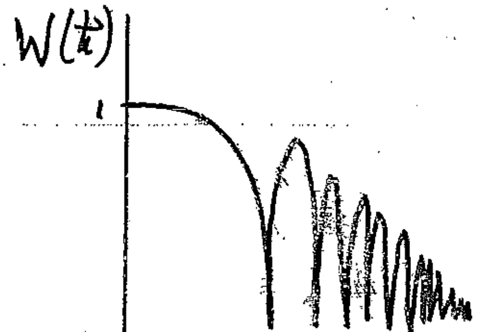
- **Tophat filter:**

$$W_{TH}(\vec{y}, \vec{x}) = \begin{cases} \frac{1}{\frac{4\pi}{3} R_{TH}^3} & |\vec{y}-\vec{x}| \leq R_{TH} \\ 0 & \text{else} \end{cases}$$



R_{TH}: filter scale.

$$W(\vec{k}, R_{TH}) = \frac{3\left[\sin(kR_{TH}) - (kR_{TH})\cos(kR_{TH})\right]}{(kR_{TH})^3}$$



fluctuations, mass scale M, cont'd:

$$\boxed{\sigma_M^2 = \int \frac{dk}{2\pi^2}\, k^2 P(k)\, \hat{W}^2(kR_f)}$$

- To understand the ramifications of a particular power spectrum for the scale-dependence of mass fluctuations, evaluate a

① Power-law power spectrum: $P(k) = Ak^n$

② Gaussian filter $W_G(kR)$

(the Gaussian filter behaves distinctly better than the tophat filter):

$$W_G(kR_G) \to 0 \qquad k < \frac{1}{R_G}$$

- $$\sigma_M^2 = \int \frac{dk}{2\pi^2}\, k^2 P(k)\, e^{-k^2 R_G^2}$$

$$\Longrightarrow \quad \sigma_M^2 = A \int \frac{dk}{2\pi^2}\, k^{(2+n)}\, e^{-k^2 R_G^2}$$

$$\Longrightarrow \quad \boxed{\sigma_M^2 = \frac{A}{2\pi^2}\,\frac{1}{n+3}\,\Gamma\left(\frac{n+3}{2}\right) R_G^{-(n+3)}}$$

- $M_G = 4.37 \times 10^{12}\, \Omega_0 h^{-1} R_G^3\, M_\odot$

$R_G \propto M_G^{1/3}$

$$\Longrightarrow \quad \sigma^2 \quad M^{-\frac{n+3}{3}}$$

fluctuations, mass scale M, cont'd:

$$\boxed{\sigma_M \propto M^{-\frac{n+3}{6}}} \equiv M^{-\alpha}$$

The exponent $\alpha = \frac{3+n}{6}$: Mass index

Poisson distribution :

random distribution $\overset{N}{\vee}$ mass particles in volume V:

$$\sigma_M = \frac{\delta N}{N} \simeq N^{-1/2} \propto M^{-1/2} \Longrightarrow n=0$$

Spectral index $n=0$: white noise

- Hierarchical clustering : $\boxed{n > -3}$
  - As long as $n > -3$:

$$\sigma_{M_1} < \sigma_{M_2} \quad \text{for} \quad M_1 > M_2$$

Fluctuations LARGER for SMALLER scales: structure on smaller scales will grow faster, condense out earlier than on large scales:

Hierarchical Clustering

- In principle this may be viewed on all scales for any spectrum $P(k)$:

$$\boxed{n(k) \equiv \frac{d \log P}{d \log k}} \Longrightarrow n(k) > -3$$

Hierarchical Clustering

fluctuations, mass scales M, cont'd:

- Mass scales & filter scales :

	Ω_0	h	R_G (h^{-1} Mpc)
"Galaxy" : $M \approx 10^{12} M_\odot$	1.0	0.5	0.97
	0.3	0.5	1.45
	0.1	0.5	2.09
"Cluster" : $M \approx 10^{14} M_\odot$	1.0	0.5	4.5
	0.3	0.5	6.7
	0.1	0.5	9.7

Mass scale M_f and Filter scale R_f

$$\boxed{M_f = \int d\vec{x}\; W_f(\vec{x}, R_f)\, \bar{\rho}}$$

$$\Downarrow$$

① Tophat Filter:

$$\boxed{W_{TH} = \begin{cases} 0 & x > R_{TH} \\ \dfrac{1}{\frac{4\pi}{3} R_{TH}^3} & x \leq R_{TH} \end{cases}}$$

$$\Longrightarrow \quad M_{TH} = \frac{4\pi}{3} \bar{\rho} R_{TH}^3$$

$$\bar{\rho} = \Omega_0 \rho_{crit} = \Omega_0 \frac{3H^2}{8\pi G} = \Omega_0 h^2\, 1.8791 \cdot 10^{-29}\, g/cm^3$$

$$= 2.78 \times 10^{11}\, \Omega_0 h^2\; M_\odot / Mpc^3$$

$$\longrightarrow \quad \boxed{M_{TH} = 1.16 \times 10^{12}\, \Omega_0 h^2 R_{TH}^3\; M_\odot} \; (R_{TH} : h^{-1} Mpc)$$

② Gaussian Filter:

$$\boxed{W_G = \frac{1}{(2\pi R_G^2)^{3/2}}\, e^{-\frac{x^2}{2 R_G^2}}}$$

$$\Longrightarrow \quad M_G = (2\pi)^{3/2} \bar{\rho} R_G^3$$

$$\boxed{M_G = 4.37 \times 10^{12}\, \Omega_0 h^2 R_G^3\; M_\odot} \quad (R_G : h^{-1} Mpc)$$