Tutorial II. Friedman-Robertson-Walker-Lemaitre

Question 1.
Hubble expansion and gravitationally bound objects

We have seen that galaxies are participating in the uniform Hubble expansion. Question is why we ourselves do not expand along. If this were so, we would not notice anything like expansion. Assume a Hubble parameter of $H_0 = 71\,\mathrm{km/s/Mpc}$. As a thought experiment compute

a) the expected Hubble expansion rate between your toes and the tip of your head.

b) the expected Hubble expansion rate between the core of the Earth and ourselves.

c) What is the reason behind the Hubble expansion being insignificant under these circumstances ? Suggestion: compute the gravitational binding energy/escape velocity at the surface of the Earth and compare to $v = Hr$.

d) Repeat the same exercise for Planet Earth wrt. the Sun and Dwarf Planet Pluto wrt. the Sun. Subsequently, consider the Sun and the Galaxy. Next, consider the Local Group (mainly M31 and the Galaxy). Then, consider the Local Group, or the Galaxy, wrt. the Local Supercluster dominated by the Virgo Cluster. Thus, what is your conclusion with respect to the scale at which the Hubble expansion becomes noticeable ? Note that you are expected to look up the relevant numbers yourself !

Question 2. Hubble expansion and Critical Density

The general Friedman-Robertson-Walker-Lemître equation for a Universe including a non-zero cosmological constant is

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a + \frac{\Lambda}{3} a \\ \dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2 \end{aligned} \tag{1}$$

a) The current estimate - on the basis of the Planck satellite measuements of the Cosmic Microwave Background - for the value of the Hubble constant H_0,

$$H_0 \equiv \frac{\dot{a}}{a}|_{t=t_0}\,, \tag{2}$$

is

$$H_0 = 67.7 \pm 0.4 \text{ km/s/Mpc}\,. \tag{3}$$

Given that 1Mpc $= 3.0857 \times 10^{19}$ km, compute the age of the Universe in seconds, and in years.

b) What is the corresponding value for the critical density ρ_{crit}, in units of g/cm^3,

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}\,. \tag{4}$$

What does this density correspond to in terms of Milky Way (halo) mass per Megaparsec (where you may assume a Milky Way halo mass of $10^{12}\,M_\odot$, with solar mass $M_\odot = 1.99 \times 10^{33}$g ?

Question 3. the Cosmic Energy Equation

The general Friedman-Robertson-Walker-Lemître equation for a Universe including a non-zero cosmological constant is

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a + \frac{\Lambda}{3} a \\ \dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2 \end{aligned} \quad (5)$$

This questions concerns the energy equation that follows from one of the other components of the Einstein field equation for a uniform Universe, and that therefore is equivalent to the Friedman-Robertson-Walker-Lemître universe.

a) Show, by using and combining the FRLW equations that

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0\,. \quad (6)$$

This is one of the key equations of cosmology, the *Energy Equation*.

b) Given that the internal energy U of the universe in a given volume V is $U = \rho c^2 V$, while the volume grows as the cubic of the expansion factor a(t), ie. $V \propto a^3$, show that the Energy Equation implies

$$dU = -p\,dV\,. \quad (7)$$

c) What does the previous result imply for the nature of the expansion of the Universe, in terms of its thermodynamic character ?

d) Take a pressureless medium of matter, ie. $p_m = 0$. What do you infer for the development of the matter density $\rho_m(t)$ as a function of expansion factor $a(t)$?

e) The equation of state $p(\rho, S)$ for radiation connects pressure and density of radiation,

$$p_{rad} = \frac{1}{3}\rho_{rad}c^2\,. \quad (8)$$

Infer the evolution of the radiation density $\rho rad(t)$ with expansion factor $a(t)$. Given that the total number of photons does not change with the expansion of the Universe (ie. there are no sources of creation of photons, ignoring the negligible contribution by stars), what does this imply for the energy of each individual photon ? Ie., how does the photon energy change as a function of cosmic expansion ? Why would this be ?

Question 4. Solutions Matter-Dominated FRWL Universes

The general Friedman-Robertson-Walker-Lemître equation for a Universe including a non-zero cosmological constant is

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a \;+\; \frac{\Lambda}{3}\, a \\ \dot{a}^2 &= \frac{8\pi G}{3}\,\rho a^2 \;-\; \frac{kc^2}{R_0^2} \;+\; \frac{\Lambda}{3}\, a^2 \end{aligned} \tag{9}$$

We are going to investigate the simple situation of a matter-dominated Universe. Assuming matter can be assumed to be "cosmic dust" (true for dark matter, and baryonic matter on large scales) pressure can be ignored, ie. $p = 0$. In a matter-dominated Universe the cosmological constant is zero, $\Lambda = 0$.

a) On the basis of the full FRWL equations above, derive the following FRWL equations for a matter-dominated FRW Universe

$$\begin{aligned} \ddot{a} &= -\frac{1}{2}\,\Omega_0 H_0^2\, \frac{1}{a^2} \\ \dot{a}^2 &= \Omega_0 H_0^2\, \frac{1}{a} \;-\; H_0^2(\Omega_0 - 1)\,, \end{aligned} \tag{10}$$

in terms of the current Hubble parameter H_0 and density parameter Ω_0,

$$H_0 \;=\; \frac{\dot{a}}{a}\Big|_{t=t_0}\,; \qquad \Omega_0 = \frac{\rho(t_0)}{\rho_{crit}(t_0)} \;=\; \frac{8\pi G\rho(t_0)}{3H_0^2}\,. \tag{11}$$

thereby taking into account that the matter density $\rho = \rho_m$ evolves with expansion factor as

$$\rho(t) \;\propto 1/a^3\,, \tag{12}$$

which we derived in assignment (3).

b) Assume $k = 0$. What does this imply for Ω_0 ? Solve this equation for this situation, ie. derive the expansion factor $a(t)$. Note that this solution is known as the **Einstein-de Sitter Universe**.

Regretfully, for a general matter-dominated Universe you will not succeed in finding a direct solution. To be able to solve the FRWL equations we need to resort to a parameterized solution. Introduce the parameter Φ, the socalled **development angle**.

$$\frac{d}{d\Phi} \equiv \frac{1}{H_0\sqrt{\Omega_0 - 1}}\, a\, \frac{d}{dt} \qquad \Omega_0 > 1 \tag{13}$$

$$\frac{d}{d\Phi} \equiv \frac{1}{H_0\sqrt{1 - \Omega_0}}\, a\, \frac{d}{dt} \qquad \Omega_0 < 1 \tag{14}$$

c) Show that the FRWL equation (of motion) in terms of the development angle becomes

$$\frac{d^2 a}{d\Phi^2} = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} - a\,, \qquad \frac{dt}{d\Phi} = \frac{a}{H_0\sqrt{\Omega_0 - 1}} \qquad \Omega_0 > 1 \tag{15}$$

$$\frac{d^2 a}{d\Phi^2} = \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0} + a\,, \qquad \frac{dt}{d\Phi} = \frac{a}{H_0\sqrt{1 - \Omega_0}} \qquad \Omega_0 < 1$$

d) Solve these second-order differential equations by using the solution ansatz

$$a(\Phi) = c_1\, e^{b_1 \Phi} + c_2\, e^{b_2 \Phi} + \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} \tag{16}$$

After settling the values of b_1 and b_2, determine the values of c_1 and c_2 from the initial condition $a(t = 0) = a(\Phi = 0) = 0)$. Show that you obtain the following set of solutions:

- for a high-density $\Omega_0 > 1$ Universe

$$\begin{aligned} a(\Phi) &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}\,(1 - \cos\Phi) \\ H_0\, t(\Phi) &= \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\,(\Phi - \sin\Phi) \end{aligned} \tag{17}$$

- and for a low-density $\Omega_0 < 1$ Universe

$$\begin{aligned} a(\Phi) &= \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0}\,(\cosh\Phi - 1) \\ H_0\, t(\Phi) &= \frac{1}{2}\frac{\Omega_0}{(1 - \Omega_0)^{3/2}}\,(\sinh\Phi - \Phi) \end{aligned} \tag{18}$$

e) Make a graph of solutions $a(t)$ (i.e. expansion factor a vs. time t) for the three classes of solution: Einstein-de Sitter Universe, Open Universe ($\Omega_0 < 1$) and Closed Universe ($\Omega_0 > 1$)