Tutorial I. Cosmology - First Steps, first thoughts

Question 1. Metric Spaces

To describe distances in a given space for a particular coordinate system, we need a distance recipy. The metric tensor is the translation for a coordinate system

$$ds^2 \equiv c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

with

$$g_{\mu\nu} \equiv \frac{\partial \vec{r}}{\partial x^\mu} \cdot \frac{\partial \vec{r}}{\partial x^\nu} \tag{2}$$

a) For a 3-dimensional space $\vec{r} = (x, y, z)$ derive the elements of the metric tensor $g_{\mu\nu}$, and write in matrix form, for:
- Euclidian coordinates (x,y,z)
- cylindrical coordinates (ρ, ϕ, z)
- spherical coordinates (r, θ, ϕ)

b) In addition, give the covariant metric tensor $g^{\mu\nu}$, which is the inverse of $g_{\mu\nu}$.

c) What is the metric tensor for Minkowski space in coordinate system $x^\mu = (ct, x, y, z)$.

To describe the curvature of space we need to specify the spatial variation of the geometry of space. This brings us to a key quantity in differential geometry, the **Christoffel symbol** $\Gamma^\alpha_{\beta\gamma}$ (also called the *affine connection*),

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left\{ \frac{\partial g_{\gamma\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\nu} \right\} . \tag{3}$$

d) Derive all Christoffel Symbol elements $\Gamma^\alpha_{\beta\gamma}$ for
- Euclidian coordinates (x,y,z)
- cylindrical coordinates (ρ, ϕ, z)
- spherical coordinates (r, θ, ϕ)

e) Subsequently, derive the equation of motion of a freely moving particle with mass m,

$$\frac{d^2 \xi^\beta}{dt^2} = 0 \tag{4}$$

in which ξ^μ is regular Cartesian coordinate system for an inertial system, in each of these coordinate systems,

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\lambda\nu} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

in which τ is the coordinate time, the proper time of the system,

$$ds^2 \equiv c^2 d\tau^2 \,, \tag{5}$$

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Question 2. Olbers' paradox

Let us compute how bright we expect the night sky to be in an infinite universe, populated by stars of luminosity L. Assume their average number density is n.

a) Consider a thin shell of stars, with radius r and thickness dr, centered on Earth. What is the radiation intensity $dJ(r)$ from this shell of stars ?

b) What do you notice with respect to dependence on distance r and shell thickness dr.

c) Compute the total intensity of starlight from all stars in the universe. What do you find ?

d) The inferred result needs some serious modification to be realistic. What are the main modifications.

e) Still, we find that the nightsky should be extremely bright. This is obviously not the case. What are three major contributing factors to this ?

Question 3. Galaxy Number Counts

One of the principal arguments for the Universe to be homogeneous concerns the number counts of galaxies, ie. the number $N(m)$ of galaxies brighter than apparent magnitude m that one would expect to have in a homogeneous Universe. The apparent magnitude m of the galaxy is the logarithm of the flux s of the galaxy,

$$m = cst. - 2.5 \log_{10} s \,. \tag{6}$$

Hubble in his seminal book "The realm of nebulae" used counts of galaxies to the limit of the 100-inch Mount Wilson telescope to show that on the largest scales the distribution of galaxies is homogeneous. He found that $N(m)$ scales as

$$N(m) \propto 10^{0.6m}, \tag{7}$$

which is what you expect for a homogeneous distribution of galaxies in a flat (Euclidian) space. This assignment seeks to prove that this is indeed what you expect in such a situation.

a) Given a galaxy of intrinsic luminosity L, what would be its magnitue m at a distance r (assuming Euclidian space) ?

b) If our survey can observe objects down to flux limit S, out to wbat distance r_S can a galaxy of luminosity L be seen ?

c) If the intrinsic number of galaxies with luminosity in the range $[L, L+dL]$ is $n(L)dL$, what would be the number of sources of intrinsic luminosity L, $N(\geq S, L)dL$ that you would count down to flux S, within a solid angle Ω on the sky (assuming they are uniformly distributed over the Universe) ?

d) Show - by integration over the luminosity function of sources - that the total number of sources with flux higher than S is given by

$$N(\geq S) = \frac{\Omega}{3(4\pi)^{3/2}} S^{-3/2} \int L^{3/2} n(L) dL \tag{8}$$

e) Given that we found $N(\geq S) \propto S^{-3/2}$, infer - using the definition of magnitude m - that

$$N(m) \propto 10^{0.6m}. \tag{9}$$

As this relation has been inferred by (tacitly) assuming a homogeneous Universe, we may confront the observed number counts with this theoretical relation to prove homogeneity.

Question 4. Newtonian Cosmology

In 1934 – i.e. way after Friedmann and Lemaitre derived their equations- Milne and McCrea showed that relations of the 'Friedmann' form can be derived using **non-relativistic Newtonian dynamics**.

a) Write down the field equation for the gravitational force in the non-relativistic limit.

b) Imagine you are a particle moving outside a spherically symmetric mass concentration of radius R with a total mass M and a density profile $\rho(r)$. What two essential simplifications can you invoke to derive your equation of motion ?

c) Write down the equation of motion (ie. the equation for your acceleration). In addition, derive the corresponding energy equation (conservation of energy).

d) We go one step further, and assume you are embedded within the spherically symmetric mass concentration. Imagine you are at a radius r, what will be your equation of motion ?

Subsequently, the situation becomes even more benevolent: we find ourselves in a homogeneous and isotropic medium.

e) Write down the equation of motion and the energy equation.

f) What three qualitative different situations can you distinguish on the basis of the energy E of a shell ?

g) Take a shell of initial radius $r_{1,i}$ and another shell of initial radius $r_{2,i}$, in how far does their evolution differ (or not) ? (assume that there are no non-radial motions). What does this imply for the evolution $r(t)$ for any shell in the mass distribution ?

h) What does the latter imply for the evolution of the density $\rho(t)$.

In principle, we are now all set to solve the equation of motion of the system, as a function of E. In fact, it is possible to derive the full solution for any spherically symmetric - not even homogeneous - mass distribution. This is the socalled *Spherical Model.* For this we refer to the follow-up MSc course of Cosmic Structure Formation.