# Cosmology,

## lect. 4

# Dynamics of FRWL Universes

# Dynamics

# FRW Universe

# Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0{}^2} + \frac{\Lambda}{3}a^2$$

# Cosmic Constituents:

# Evolving Energy Density

# FRW Energy Equation

**To infer the evolving energy density ?(t) of each cosmic component, we refer to the cosmic energy equation. This equation can be directly inferred from the FRW equations**

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

**The equation forms a direct expression of the adiabatic expansion of the Universe, ie.**

$U = \rho c^2 V$ **Internal energy**

$V \propto a^3$ **Expanding volume**

$$dU = -p dV$$

# FRW Energy Equation

To infer $\rho(t)$ from the energy equation, we need to know the pressure p(t) for that particular medium/ingredient of the Universe.

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

To infer p(t), we need to know the nature of the medium, which provides us with the equation of state,

$$p = p(\rho, S)$$

# Cosmic Constituents:
## Evolution of Energy Density

- **Matter:** $\rho_m(t) \propto a(t)^{-3}$
- **Radiation:** $\rho_{rad}(t) \propto a(t)^{-4}$
- **Dark Energy:** $\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$

$$\Downarrow \quad w = -1$$

$$\rho_\Lambda(t) = cst.$$

# FRWL Dynamics
# &
# Cosmological Density

# FRW Dynamics

- The individual contributions to the energy density of the Universe can be figured into the ? parameter:

- radiation

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$$

- matter

$$\Omega_m = \Omega_{dm} + \Omega_b$$

- dark energy/
  cosmological constant

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

# Critical Density

**There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:**

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

| | | | |
|---|---|---|---|
| $\Omega < 1$ | $k = -1$ | *Hyperbolic* | *Open Universe* |
| $\Omega = 1$ | $k = 0$ | *Flat* | *Critical Universe* |
| $\Omega > 1$ | $k = +1$ | *Spherical* | *Close Universe* |

# FRW Universe: Curvature

**There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:**

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

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# Radiation, Matter & Dark Energy

**The individual contributions to the energy density of the Universe can be figured into the ⍰ parameter:**

- **radiation**

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$$

- **matter**

$$\Omega_m = \Omega_{dm} + \Omega_b$$

- **dark energy/ cosmological constant**

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

# General Solution Expanding FRW Universe

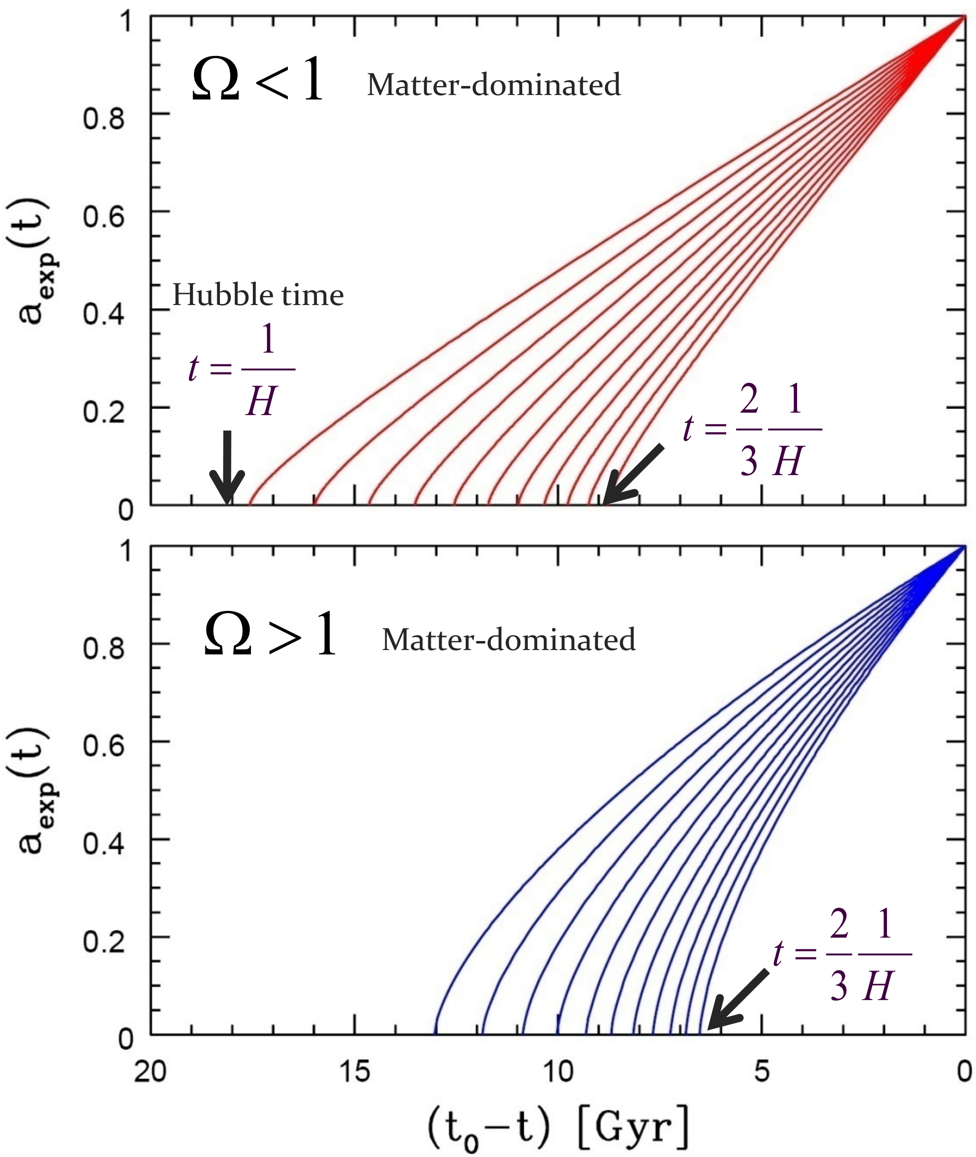
**From the FRW equations:**

$$\frac{H(t)^2}{H_0{}^2} = \frac{\Omega_{rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

$a(t)$ Expansion history Universe

$$H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)}}$$

# Age of the Universe




Age of a FRW universe at
Expansion factor a(t)

$$H\,t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad}}{a^2} + \frac{\Omega_m}{a} + \Omega_\Lambda a^2 + \left(1 - \Omega\ \right)}}$$

# Specific Solutions
# FRW Universe

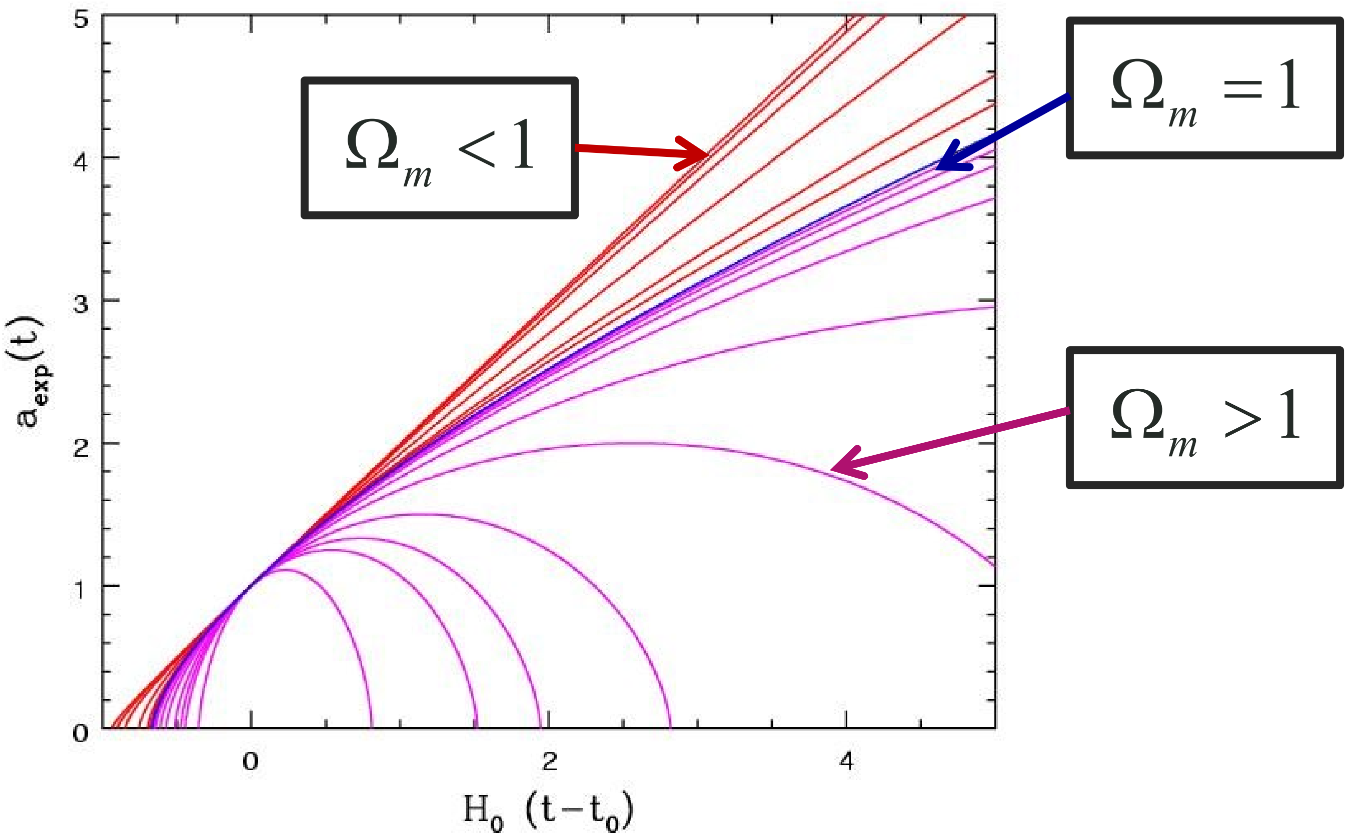
**While general solutions to the FRW equations is only possible by numerical integration, analytical solutions may be found for particular classes of cosmologies:**

- **Single-component Universes:**
  - empty Universe
  - flat Universes, with only radiation, matter or dark energy
- **Matter-dominated Universes**
- **Matter+Dark Energy flat Universe**

# Matter-Dominated Universes

- **Assume radiation contribution is negligible:**
- **Zero cosmological constant:**
- **Matter-dominated, including curvature**

$$\Omega_{rad,0} \approx 5\times10^{-5}$$
$$\Omega_{\Lambda} = 0$$



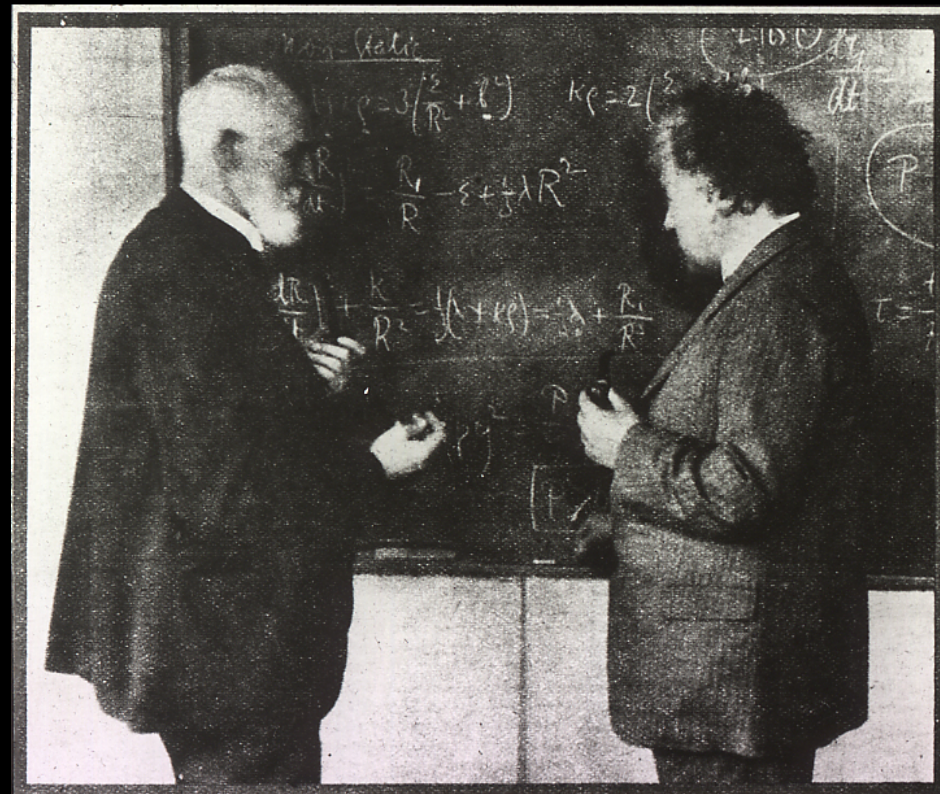
# Einstein-de Sitter Universe

$$\left.\begin{aligned}\Omega_m &= 1\\ \Omega_\Lambda &= 0\end{aligned}\right\} \quad k = 0$$

FRW: $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

Age
EdS Universe: $$t_0 = \frac{2}{3}\frac{1}{H_0}$$



Albert Einstein and Willem de Sitter discussing the Universe. In 1932 they published a paper together on the Einstein-de Sitter universe, which is a model with flat geometry containing matter as the only significant substance.

# Free Expanding "Milne" Universe

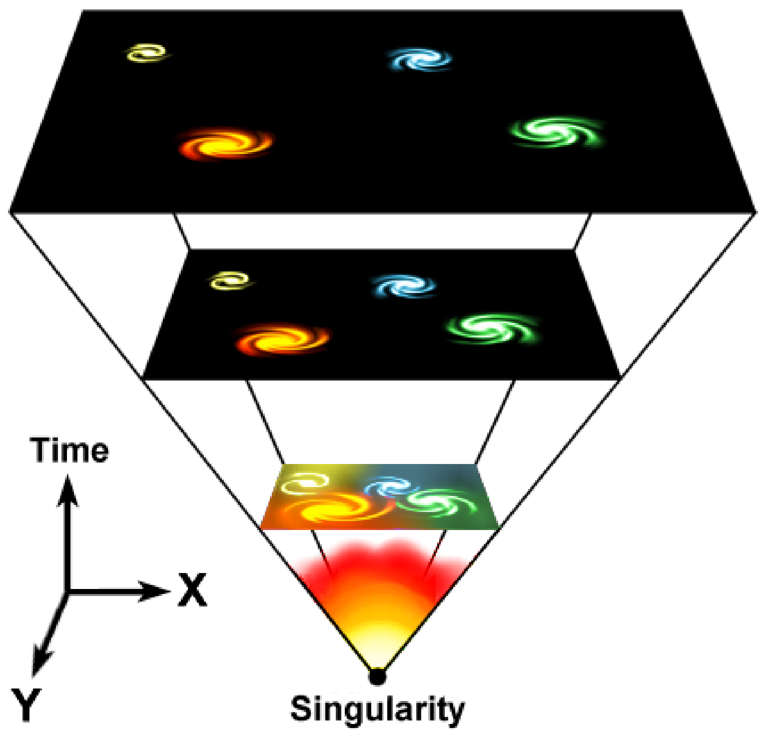
$$\left.\begin{array}{l}\Omega_m = 0 \\ \Omega_\Lambda = 0\end{array}\right\} k = -1$$

Empty space is curved

FRW: $\dot{a}^2 = -\frac{kc^2}{R_0^{\,2}} = cst.$

$$a(t) = \left(\frac{t}{t_0}\right)$$

Age
Empty Universe: $t_0 = \frac{1}{H_0}$



# Expansion
# Radiation-dominated Universe

$$\left.\begin{aligned}\Omega_{rad} &= 1\\ \Omega_m &= 0\\ \Omega_\Lambda &= 0\end{aligned}\right\} k = 0$$

In the very early Universe, the energy density is completely dominated by radiation. The dynamics of the very early Universe is therefore fully determined by the evolution of the radiation energy density:

FRW: $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G \rho_0}{3}\frac{1}{a^2}$$ ⟵ $$\rho_{rad}(a) \propto \frac{1}{a^4}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

Age Radiation Universe: $$t_0 = \frac{1}{2}\frac{1}{H_0}$$

# De Sitter Expansion

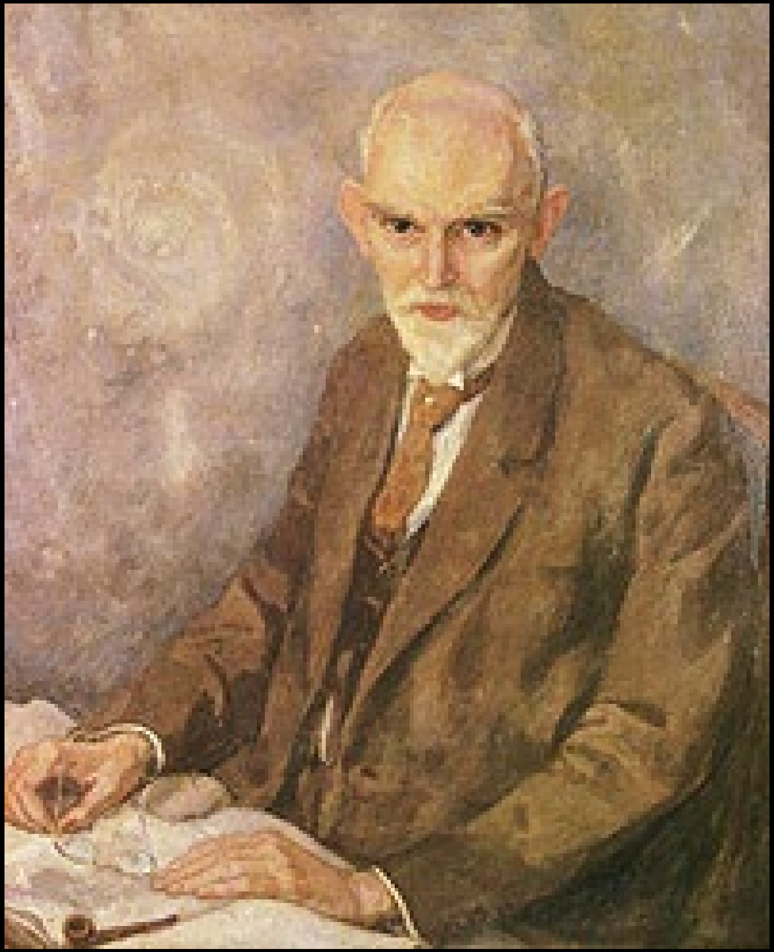
$$\left.\begin{aligned}\Omega_m &= 0\\ \Omega_\Lambda &= 1\end{aligned}\right\} \quad k = 0$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^{\,2}} \quad \Rightarrow \quad H_0 = \sqrt{\frac{\Lambda}{3}}$$

FRW: $\dot{a}^2 = \frac{\Lambda}{3} a^2 \quad \Rightarrow \quad \dot{a} = H_0 a$

$$a(t) = e^{H_0 (t - t_0)}$$

Age
De Sitter Universe: infinitely old



Willem de Sitter (1872-1934; Sneek-Leiden)
director Leiden Observatory
alma mater: Groningen University

# General Flat FRW Universe

$$k = 0$$

$$\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$$

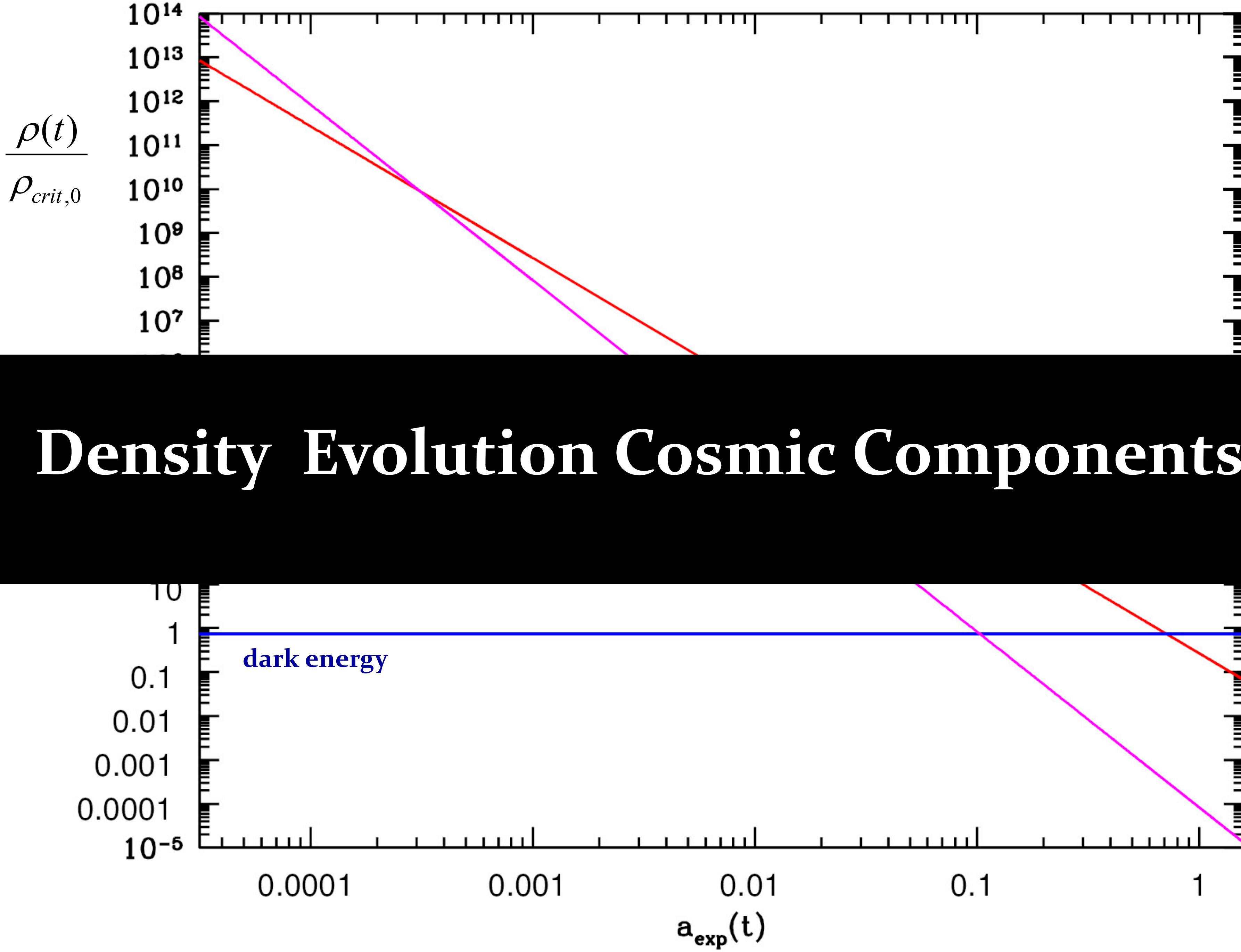
FRW:

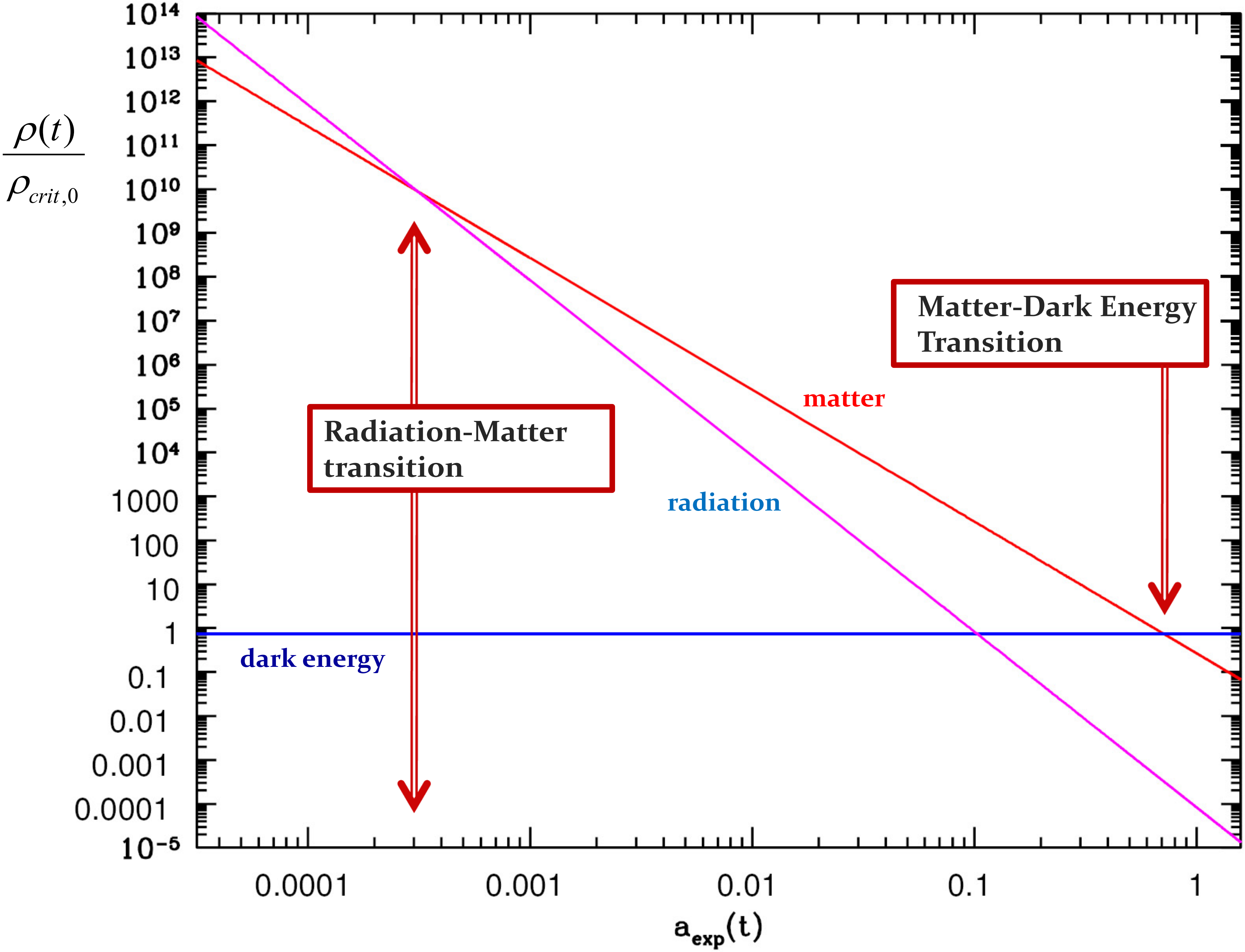
$$a(t) \propto t^{\frac{2}{3+3w}}$$

# Evolving

# Cosmic Composition

# Density Evolution Cosmic Components

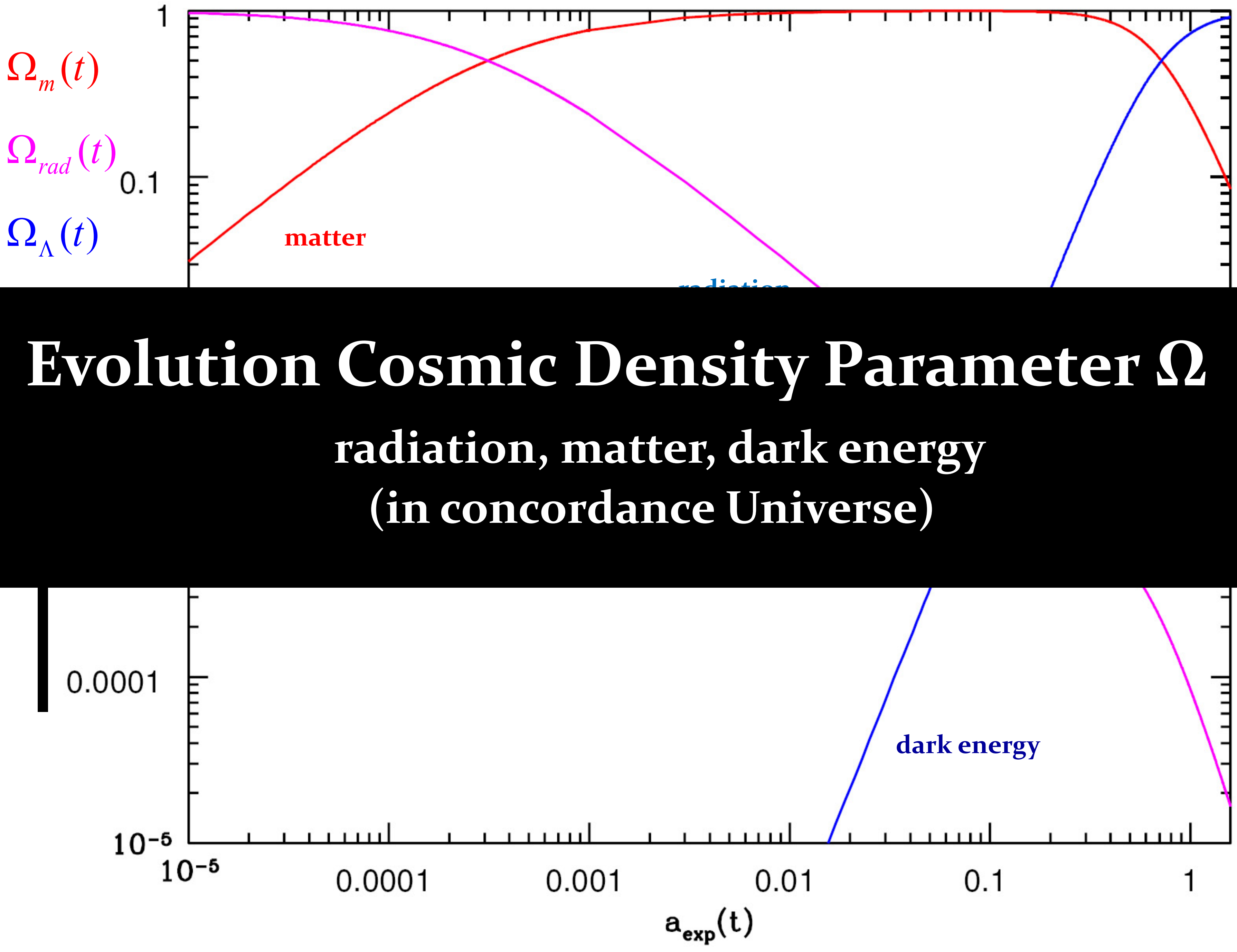


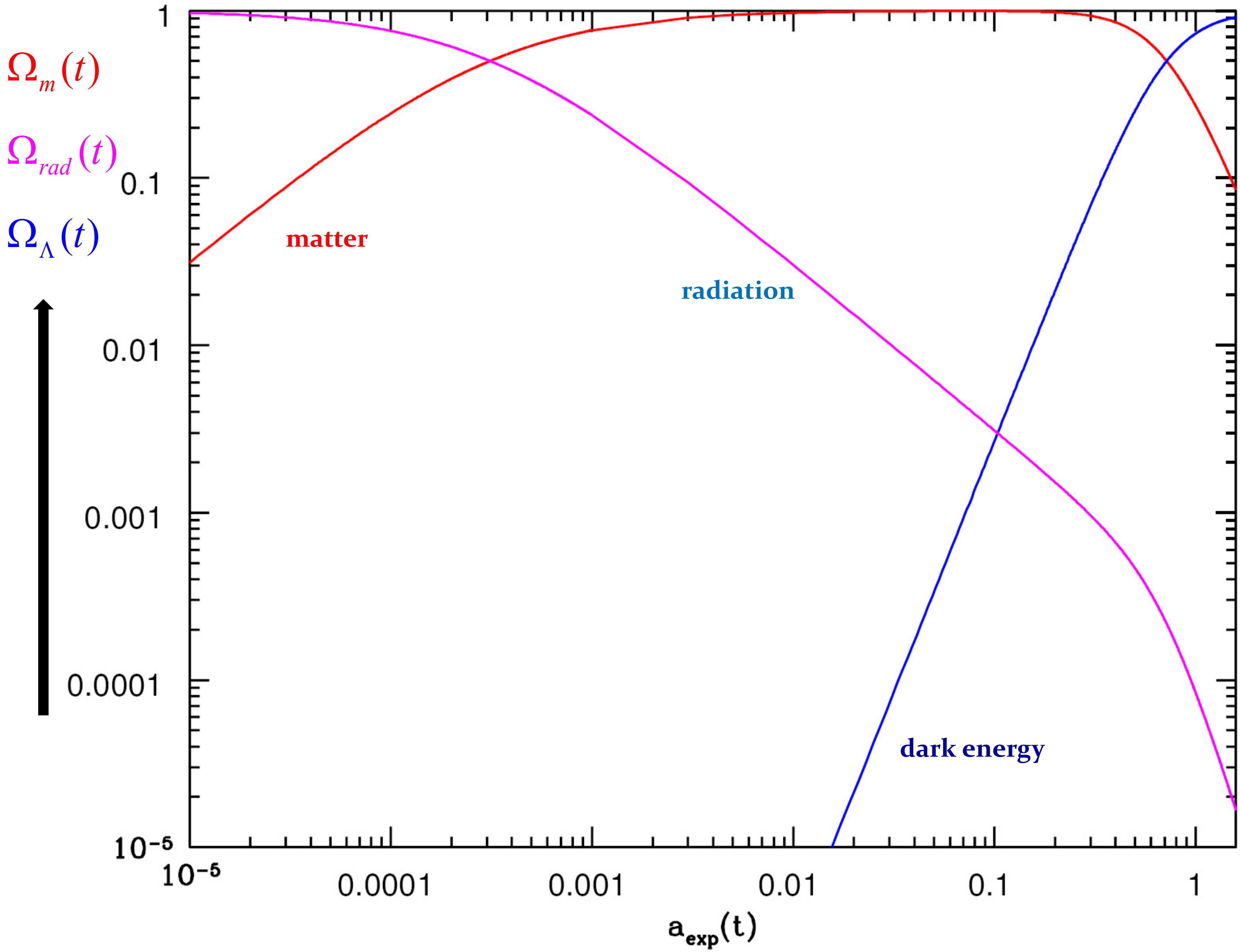


$\frac{\rho(t)}{\rho_{crit,0}}$
$10^{14}$
$10^{13}$
$10^{12}$
$10^{11}$
$10^{10}$
$10^{9}$
$10^{8}$
$10^{7}$
$10^{6}$
$10^{5}$
$10^{4}$
1000
100
10
1
0.1
0.01
0.001
0.0001
$10^{-5}$
0.0001
0.001
0.01
0.1
1
$a_{exp}(t)$
Matter-Dark Energy Transition
matter
Radiation-Matter transition
radiation
dark energy

# Evolution Cosmic Density Parameter Ω

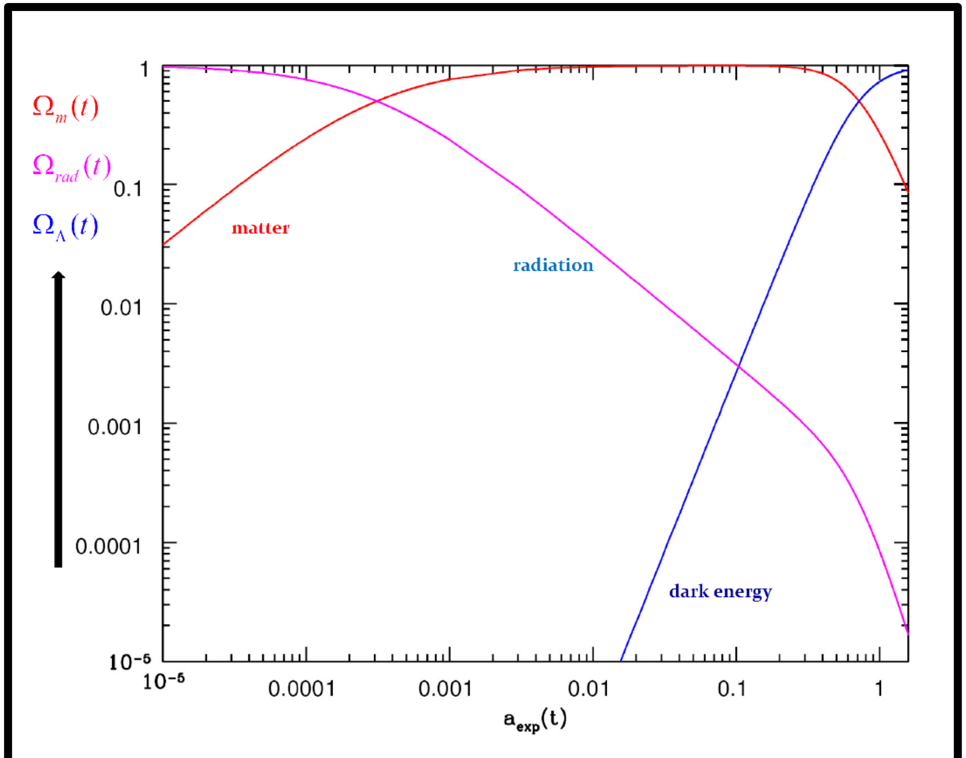
## radiation, matter, dark energy
## (in concordance Universe)

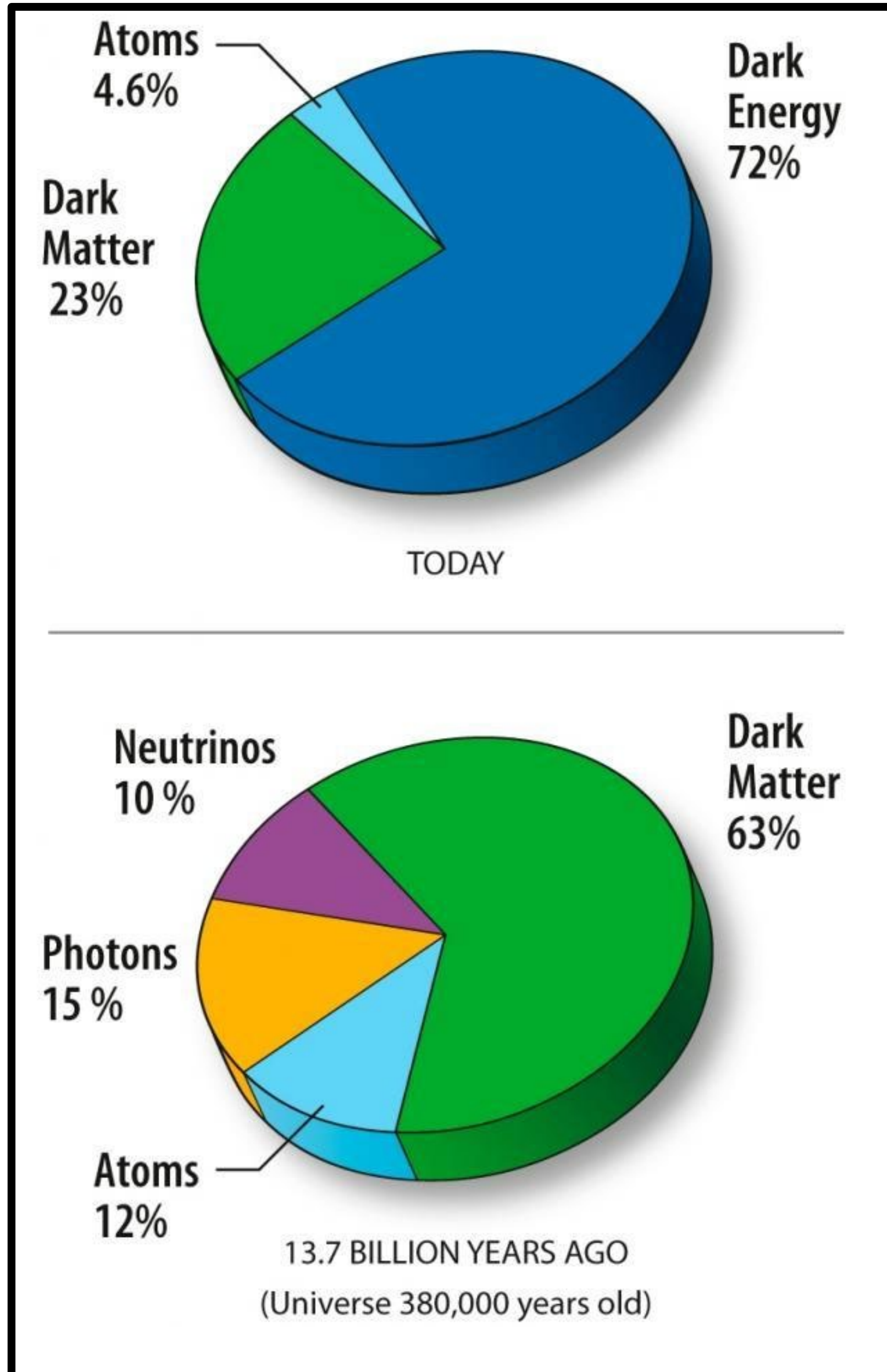




$\Omega_m(t)$
$\Omega_{rad}(t)$
$\Omega_\Lambda(t)$
1
0.1
0.01
0.001
0.0001
$10^{-5}$
matter
radiation
dark energy
$10^{-5}$
0.0001
0.001
0.01
0.1
1
$a_{exp}(t)$

# Evolving Composition FRWL Universe



# Cosmological Transitions

# Dynamical Transitions

Because radiation, matter, dark energy (and curvature) of the Universe evolve differently as the Universe expands, at different epochs the energy density of the Universe is alternately dominated by these different ingredients.

As the Universe is dominated by either radiation, matter, curvature or dark energy, the cosmic expansion a(t) proceeds differently.

We therefore recognize the following epochs:

- radiation-dominated era
- matter-dominated era
- curvature-dominated expansion
- dark energy dominated epoch

The different cosmic expansions at these eras have a huge effect on relevant physical processes

# Dynamical Transitions

- Radiation Density Evolution

$$\rho_{rad}(t) = \frac{1}{a^4}\rho_{rad,0}$$

- Matter Density Evolution

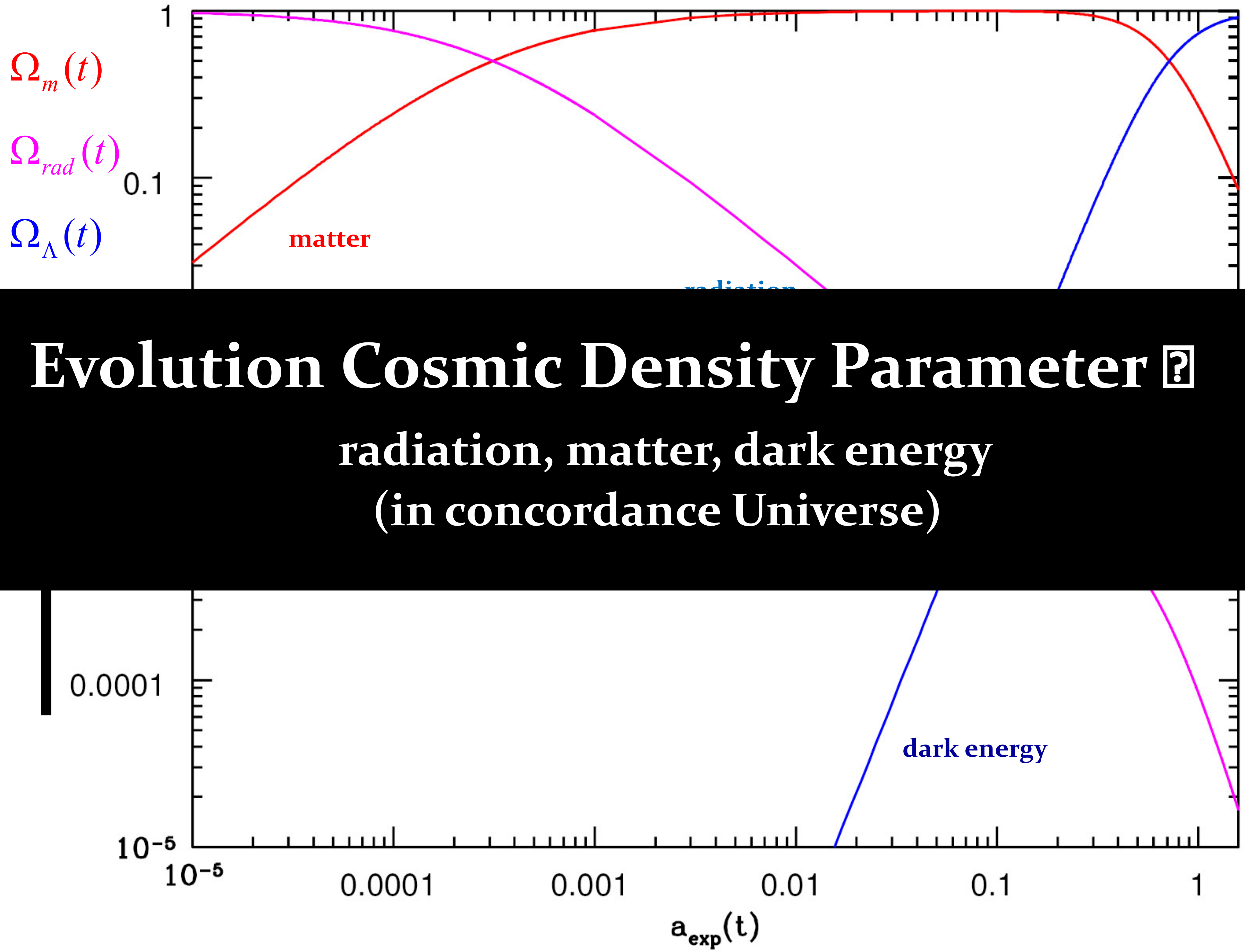
$$\rho_m(t) = \frac{1}{a^3}\rho_{m,0}$$

- Curvature Evolution

$$\frac{kc^2}{R(t)^2} = \frac{1}{a^2}\frac{kc^2}{R_0{}^2} = \frac{1}{a^2}\left(1-\Omega_0\right)$$

- Dark Energy (Cosmological Constant) Evolution

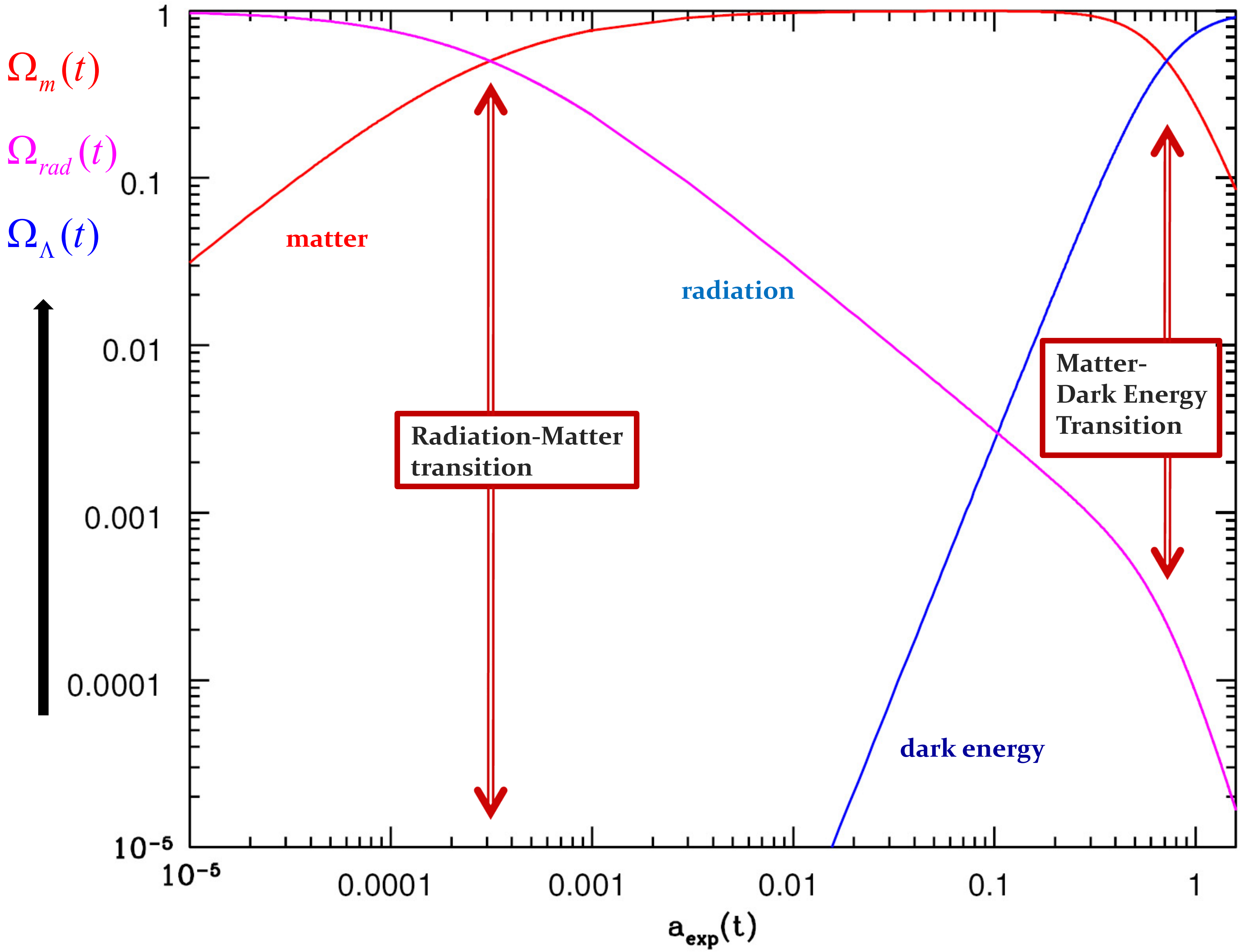
$$\rho_\Lambda(t) = cst. = \rho_{\Lambda 0}$$




# Evolution Cosmic Density Parameter

**radiation, matter, dark energy**

**(in concordance Universe)**



$\Omega_m(t)$
$\Omega_{rad}(t)$
$\Omega_\Lambda(t)$
matter
radiation
dark energy
Radiation-Matter transition
Matter-Dark Energy Transition
1
0.1
0.01
0.001
0.0001
$10^{-5}$
$10^{-5}$
0.0001
0.001
0.01
0.1
1
$a_{exp}(t)$

# Radiation-Matter Transition

- Radiation Density Evolution

$$\rho_{rad}(t) = \frac{1}{a^4}\rho_{rad,0}$$

- Matter Density Evolution

$$\rho_m(t) = \frac{1}{a^3}\rho_{m,0}$$

⍰ Radiation energy density decreases more rapidly than matter density: this implies radiation to have had a higher energy density before a particular cosmic time:

$$a_{rm} = \frac{\Omega_{rad,0}}{\Omega_{m,0}}$$

$$\frac{\rho_{m,0}}{a^3} = \frac{\rho_{rad,0}}{a^4}$$

$a < a_{rm}$ Radiation dominance

$a > a_{rm}$ Matter dominance

# Matter-Dark Energy Transition

- Matter Density Evolution

$$\rho_m(t) = \frac{1}{a^3}\rho_{m,0}$$

- Dark Energy Density Evolution

$$\rho_\Lambda(t) = cst. = \rho_{\Lambda 0}$$

 While matter density decreases due to the expansion of the Universe, the cosmological constant represents a small, yet constant, energy density. As a result, it will represent a higher density after

$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}} \Leftarrow \frac{\rho_{m,0}}{a^3} = \rho_{\Lambda,0}$$

$a < a_{m\Lambda}$ Matter dominance

$a > a_{m\Lambda}$ Dark energy dominance

# Matter-Dark Energy Transition

$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}}$$

$$\left.\begin{aligned}\Omega_{\Lambda,0} &= 0.27 \\ \Omega_{m,0} &= 0.73\end{aligned}\right\} \quad \begin{aligned} a_{m\Lambda} &= 0.72 \\ a_{m\Lambda}{}^{\dagger} &= 0.57\end{aligned}$$

Flat Universe

$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}}$$

Note: a more appropriate characteristic transition is that at which the deceleration turns into acceleration:

$$a_{m\Lambda}{}^{\dagger} = \sqrt[3]{\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}} = \sqrt[3]{\frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}}$$

# Evolution
# Cosmological Density Parameter

Limiting ourselves to a flat Universe
(and discarding the contribution by and evolution of curvature):

to appreciate the dominance of radiation, matter and dark energy in the subsequent cosmological eras, it is most illuminating to look at the evolution of the cosmological density parameter of these cosmological components:

$$\Omega_{rad}(t) \Longleftrightarrow \Omega_m(t) \Longleftrightarrow \Omega_\Lambda(t)$$

e.g.

$$\Omega_m(t) = \frac{\Omega_{m,0}\, a^4}{\Omega_{rad,0} + \Omega_{m,0}\, a + \Omega_{\Lambda,0}\, a^4}$$

# Evolution
## Cosmological Density Parameter

From the FRW equations, one can infer that the evolution of ? goes like (for simplicity, assume matter-dominated Universe),

$$\left(\frac{1}{\Omega}-1\right)=a(t)\left(\frac{1}{\Omega_0}-1\right) \iff \Omega(z)=\frac{\Omega_0(1+z)}{1+\Omega_0 z}$$

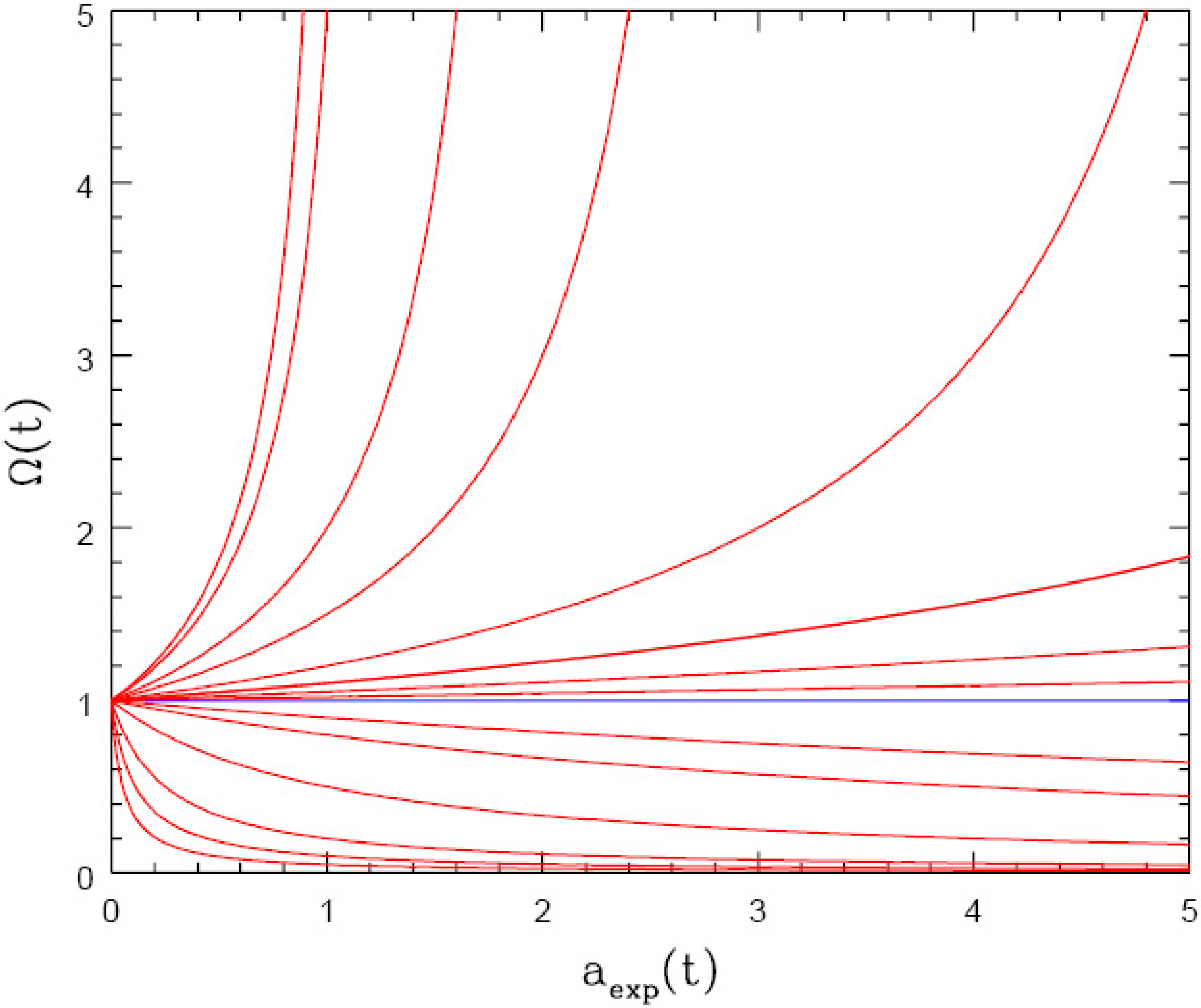
These equations directly show that

$$a \downarrow 0 \implies \Omega \to 1$$

$$k=\frac{H^2R^2}{c^2}(\Omega-1)$$

implying that the early Universe was very nearly flat …

# Flatness Evolution




$$\left(\frac{1}{\Omega}-1\right)=a(t)\left(\frac{1}{\Omega_0}-1\right)$$

- At radiation-matter equiv.

  $$\left|1-\Omega_{rm}\right|\leq 2\times 10^{-4}$$

- Big Bang nucleosynthesis

  $a_{nuc}\approx 3.6 \times 10^{-8}$

  $$\left|1-\Omega_{nucl}\right|\leq 3\times 10^{-14}$$

- Planck time

  $$\left|1-\Omega_{P}\right|\leq 1\times 10^{-60}$$

# Concordance Universe

# Concordance Universe Parameters

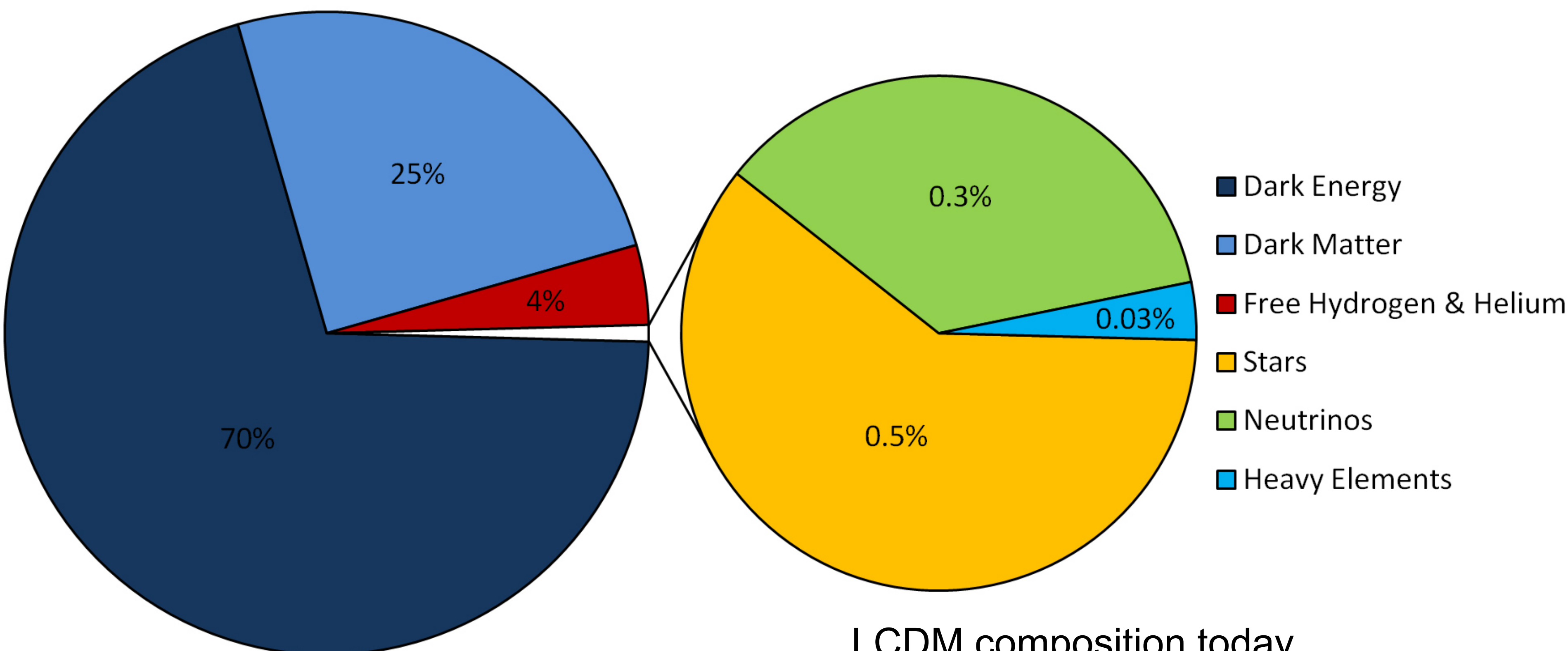
| | | | |
|---|---|---|---|
| **Hubble Parameter** | | $H_0 = 71.9 \pm 2.6 \; km \; s^{-1} \; Mpc^{-1}$ | |
| **Age of the Universe** | | $t_0 = 13.7 \pm 0.12 \; Gyr$ | |
| **Temperature CMB** | | $T_0 = 2.725 \pm 0.001 K$ | |
| **Matter** | **Baryonic Matter**<br>**Dark Matter** | $\Omega_m = 0.27$ | $\Omega_b = 0.0456 \pm 0.0015$<br>$\Omega_{dm} = 0.228 \pm 0.013$ |
| **Radiation** | **Photons (CMB)**<br>**Neutrinos (Cosmic)** | $\Omega_{rad} = 8.4 \times 10^{-5}$ | $\Omega_\gamma = 5 \times 10^{-5}$<br>$\Omega_\nu = 3.4 \times 10^{-5}$ |
| **Dark Energy** | | $\Omega_\Lambda = 0.726 \pm 0.015$ | |
| **Total** | | $\Omega_{tot} = 1.0050 \pm 0.0061$ | |

# LCDM Cosmology

- Concordance cosmology

  - model that fits the majority of cosmological observations

  - universe dominated by Dark Matter and Dark Energy




LCDM composition today …

# Concordance Expansion

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left\{ \left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^{3}} \right\}$$

transition epoch:

matter-dominate to
□ dominated

$a_{m□}$□0.75

$$a_{m\Lambda} = \sqrt[3]{\frac{\Omega_{m0}}{1-\Omega_{m0}}}$$

# Concordance Expansion

We can recognize two extreme regimes:

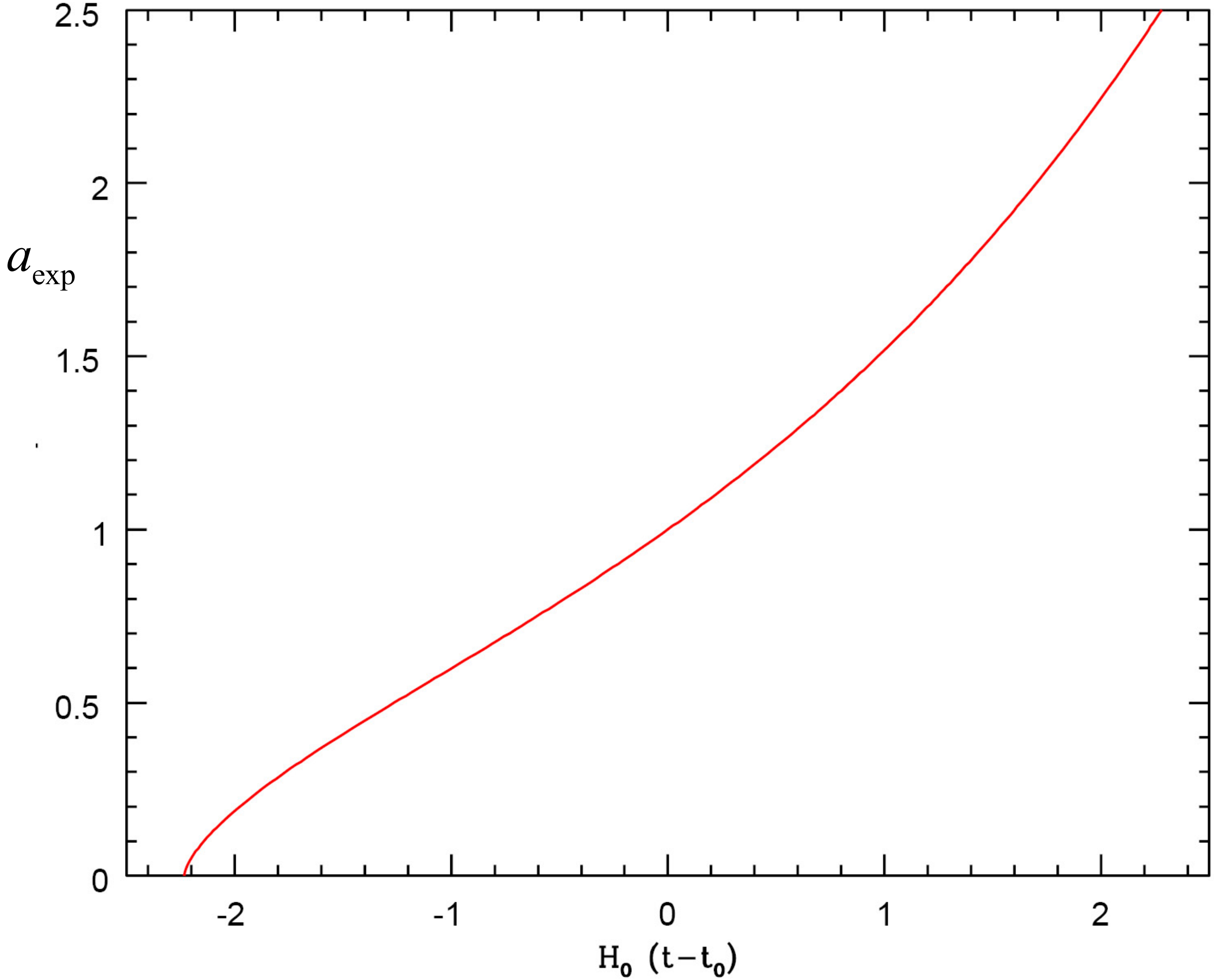
- $a \ll a_{m\Lambda}$ very early times
  matter dominates the expansion, and $\Omega_m \approx 1$: Einstein-de Sitter expansion,

$$a(t) \approx \left( \frac{3}{2} \sqrt{\Omega_{m,0}} H_0 t \right)^{2/3}$$

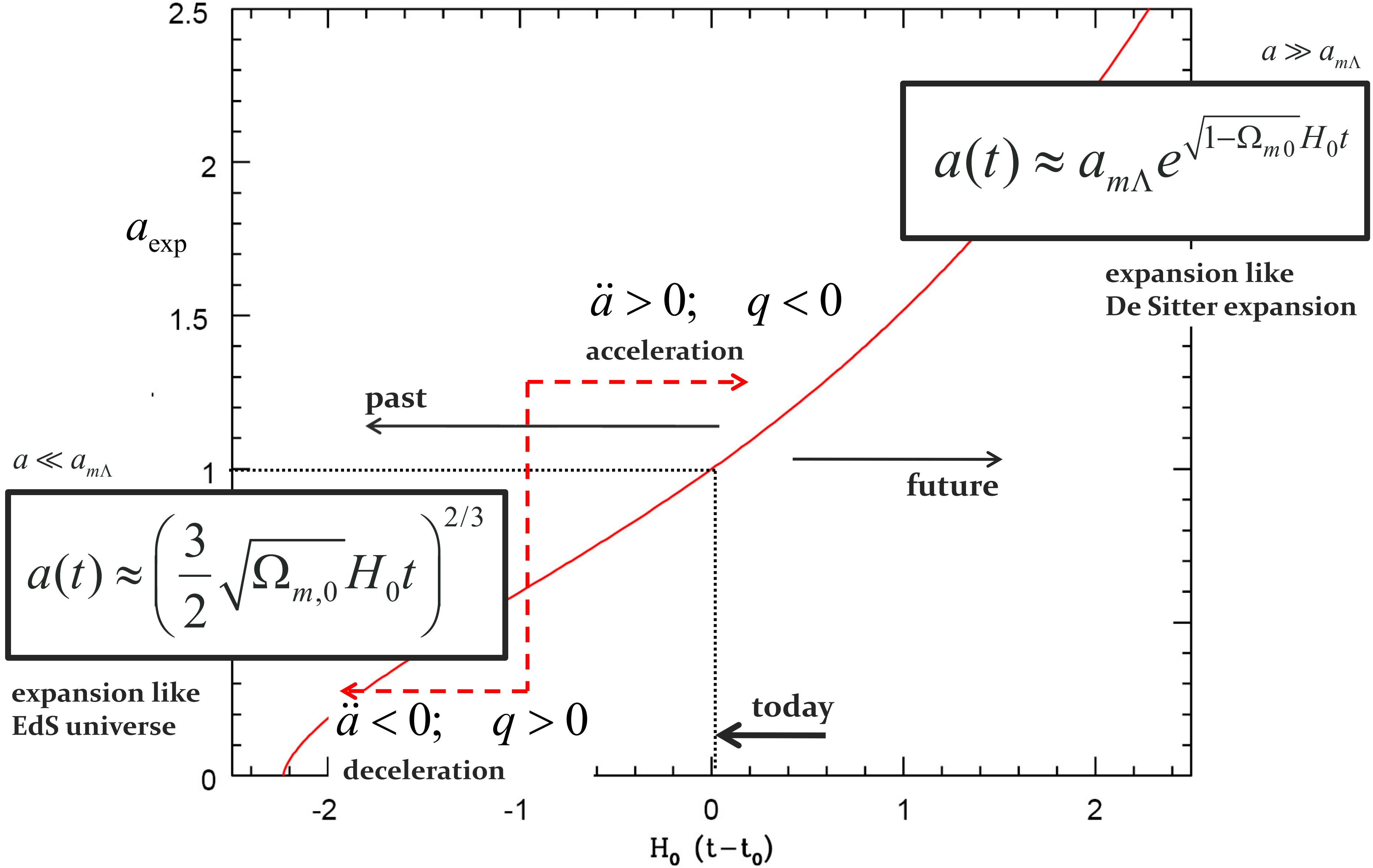
- $a \gg a_{m\Lambda}$ very late times
  matter has diluted to oblivion, and $\Omega_m \approx 0$: de Sitter expansion driven by dark energy

$$a(t) \approx a_{m\Lambda} e^{\sqrt{1-\Omega_{m0}} H_0 t}$$

# Concordance Expansion



# Concordance Expansion



# Matter-Dark Energy Transition

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Note: a more appropriate characteristic transition is that at which the deceleration turns into acceleration:

$$a_{m\Lambda}{}^{\dagger} = \sqrt[3]{\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}} = \sqrt[3]{\frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}}$$

# Key Epochs Concordance Universe

| | | | |
|---|---|---|---|
| Radiation-Matter Equality | | $a_{eq} = 2.8\times10^{-4}$ | $t_{eq} = 4.7\times10^{4}\ yr$ |
| Recombination/ Decoupling | | $a_{rec} \approx 1/1091$<br>$z_{rec} = 1090.88 \pm 0.72$ | $t_{rec} = 3.77 \pm 0.03\times10^{5}\ yrs$ |
| Reionization | Optical Depth<br>Redshift | $\tau_{reion} = 0.084 \pm 0.016$<br>$z_{reion} = 10.9 \pm 1.4$ | $t_{reion} = 432^{+90}_{-67}\times10^{6}\ yrs$ |
| Matter-Dark Energy Transition | Acceleration<br>Energy | $a_{m\Lambda}{}^{\dagger} \approx 0.60;\ z_{m\Lambda}{}^{\dagger} \approx 0.67$<br>$a_{m\Lambda} \approx 0.75;\ z_{m\Lambda} \approx 0.33$ | $t_{m\Lambda} = 9.8\ Gyr$ |
| Today | | $a_{0} = 1$ | $t_{eq} = 13.72 \pm 0.12\ Gyr$ |

**General FRWL**
**cxpansion histories:**

**cosmic “phase diagram”**

# Cosmological Evolution Modes

It is interesting to inspect the possible expansion histories for generic FRWL cosmologies with matter & cosmological constant.

- The expansion histories entirely determined by 2 parameters:

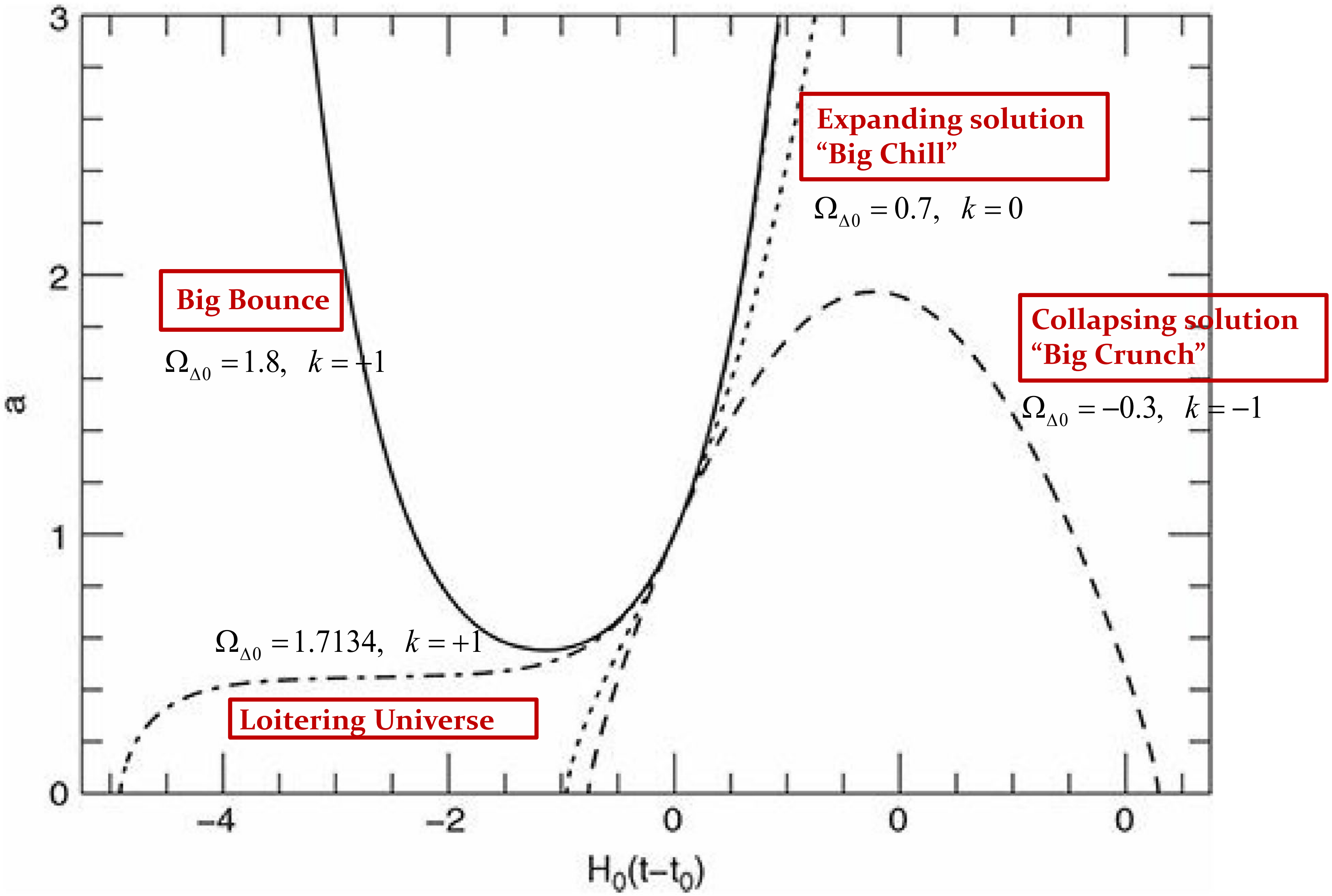
  matter density $\Omega_{m,0}$

  cosmological constant $\Omega_{\Lambda,0}$

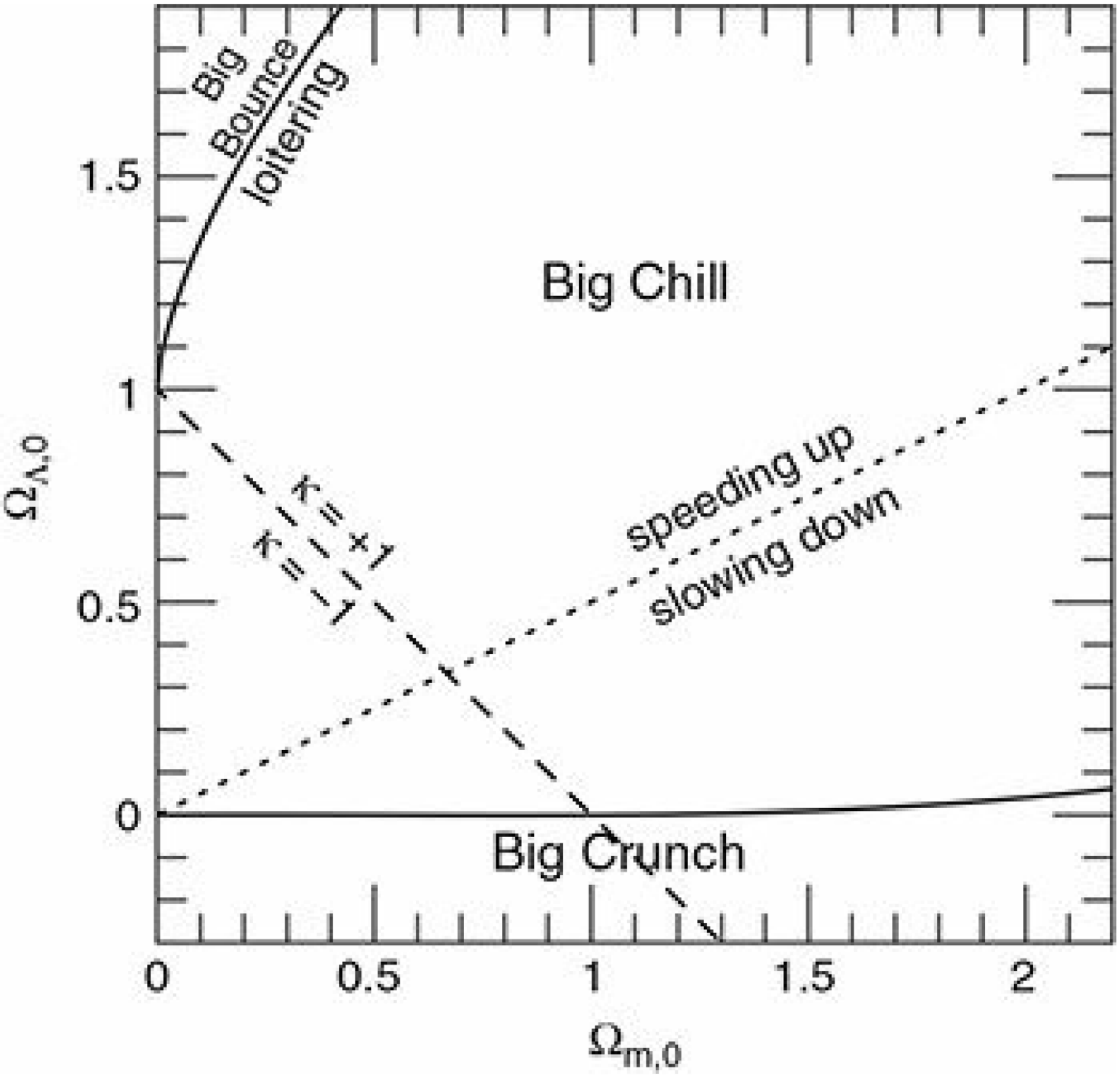
- 4 (qualitatively) different and possible modes of cosmic evolution:

  1) Bouncing universe
  2) Collapsing universe “Big Crunch”
  3) Loitering universe
  4) Expansion (only) universe

# Cosmological Evolution Modes



# Cosmological Evolution Modes




$$\Omega_{\Lambda 0} \leftrightarrow \Omega_{m0}$$

**Expansion Modes:**

**different combinations $\Omega_{m0}$ and $\Omega_{\Lambda 0}$**

**In the diagram you can identify regions of**

- **curvature**

$$k = \frac{H_0^2 R_0^2}{c^2}(\Omega_{m0} + \Omega_{\Lambda 0} - 1)$$

- **acceleration**

$$q = \frac{\Omega_{m0}}{2} - \Omega_{\Lambda 0}$$