Cosmology,

lect. 3

Cosmological Principle
&
Friedmann-Lemaitre Equations

Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Cosmological Principle

General Relativity

A crucial aspect of any particular configuration is the geometry of spacetime: because Einstein's General Relativity is a metric theory, knowledge of the geometry is essential.

Einstein Field Equations are notoriously complex, essentially 10 equations. Solving them for general situations is almost impossible.

However, there are some special circumstances that do allow a full solution. The simplest one is also the one that describes our Universe. It is encapsulated in the

Cosmological Principle

On the basis of this principle, we can constrain the geometry of the Universe and hence find its dynamical evolution.

Cosmological Principle: the Universe Simple & Smooth

"God is an infinite sphere whose centre is everywhere and its circumference nowhere"
Empedocles, 5th cent BC

Cosmological Principle:

Describes the symmetries in global appearance of the Universe:

- **Homogeneous** → The Universe is the same everywhere:
  - physical quantities (density, T,p,…)
- **Isotropic** → The Universe looks the same in every direction
- **Universality** → Physical Laws same everywhere
- **Uniformly Expanding** → The Universe "grows" with same rate in
  - every direction
  - at every location

"all places in the Universe are alike"
Einstein, 1931

Geometry of the Universe

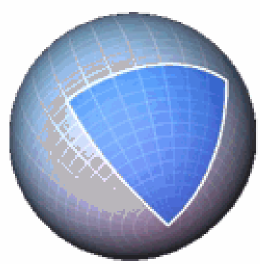
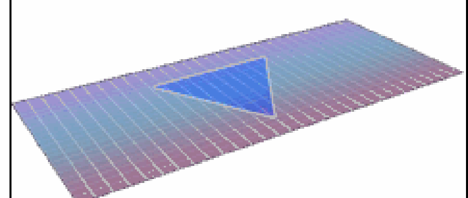
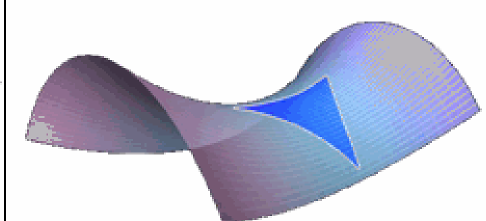
Fundamental Tenet

of (Non-Euclidian = Riemannian) Geometry

There exist no more than THREE uniform spaces:

1)	Euclidian (flat) Geometry	Euclides
2)	Hyperbolic Geometry	Gauß, Lobachevski, Bolyai
3)	Spherical Geometry	Riemann

uniform=
homogeneous & isotropic
(cosmological principle)

Property	Closed	Euclidean	Open
Spatial Curvature	Positive	Zero	Negative
Circle Circumference	$< 2\pi R$	$2\pi R$	$> 2\pi R$
Sphere Area	$< 4\pi R^2$	$4\pi R^2$	$> 4\pi R^2$
Sphere Volume	$< {}^4/_3\ \pi R^3$	${}^4/_3\ \pi R^3$	$> {}^4/_3\ \pi R^3$
Triangle Angle Sum	$> 180°$	$180°$	$< 180°$
Total Volume	Finite ($2\pi^2 R^3$)	Infinite	Infinite
Surface Analog	Sphere 	Plane 	Saddle 

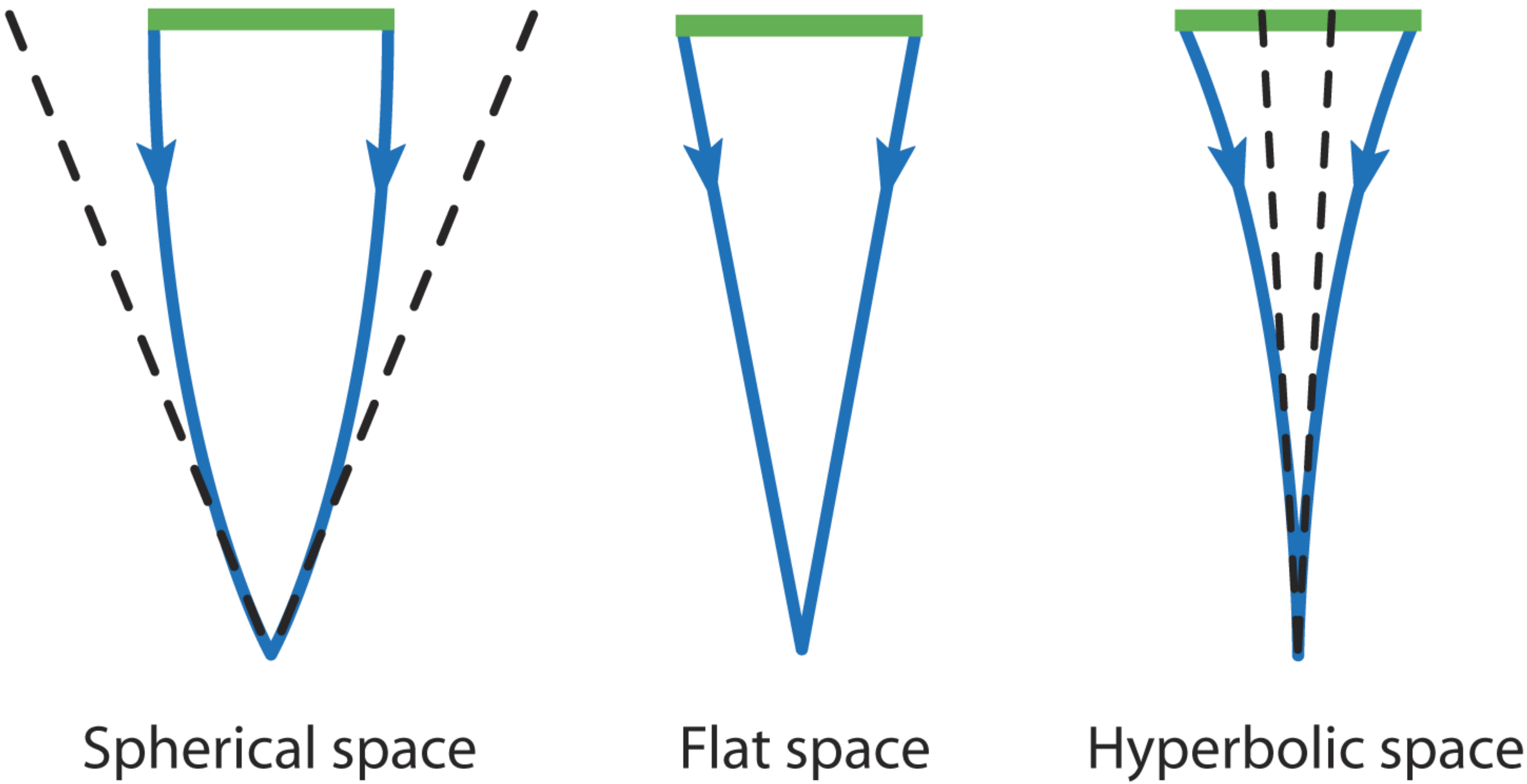
Robertson-Walker Metric

Distances in a uniformly curved spacetime is specified in terms of the Robertson-Walker metric. The spacetime distance of a point at coordinate (r,?,?) is:

$$ds^2 = c^2 dt^2 - a(t)^2 \left\{ dr^2 + R_c{}^2 S_k{}^2 \left(\frac{r}{R_c} \right) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\}$$

where the function $S_k(r/R_c)$ specifies the effect of curvature on the distances between points in spacetime

$$S_k\left(\frac{r}{R_c}\right) = \begin{cases} \sin\left(\frac{r}{R_c}\right) & k = +1 \\ \frac{r}{R_c} & k = 0 \\ \sinh\left(\frac{r}{R_c}\right) & k = -1 \end{cases}$$


Spherical space
Flat space
Hyperbolic space

Friedmann-Robertson-Walker-Lemaitre
(FRLW)

Universe

Einstein Field Equation

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu,RW} \quad \Rightarrow \quad \Gamma^{\mu}{}_{\lambda\nu} \quad \Rightarrow \quad R_{\mu\nu} \ , \ R$$

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^{\mu} U^{\nu} - p g^{\mu\nu}$$

$$= diag\left(\rho c^2, p, p, p \right)$$

Einstein Field Equation

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$G^0{}_0 \quad \rightarrow \quad G^0{}_0 = 3\left(\dot{R}^2 + kc^2\right)/R^2 = \frac{8\pi G}{c^2}\rho c^2$$

$$G^1{}_1 \quad \rightarrow \quad G^1{}_1 = \left(2R\ddot{R} + \dot{R}^2 + kc^2\right)/R^2 = -\frac{8\pi G}{c^2} p$$

Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda}{3}R$$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{\Lambda}{3}R^2$$

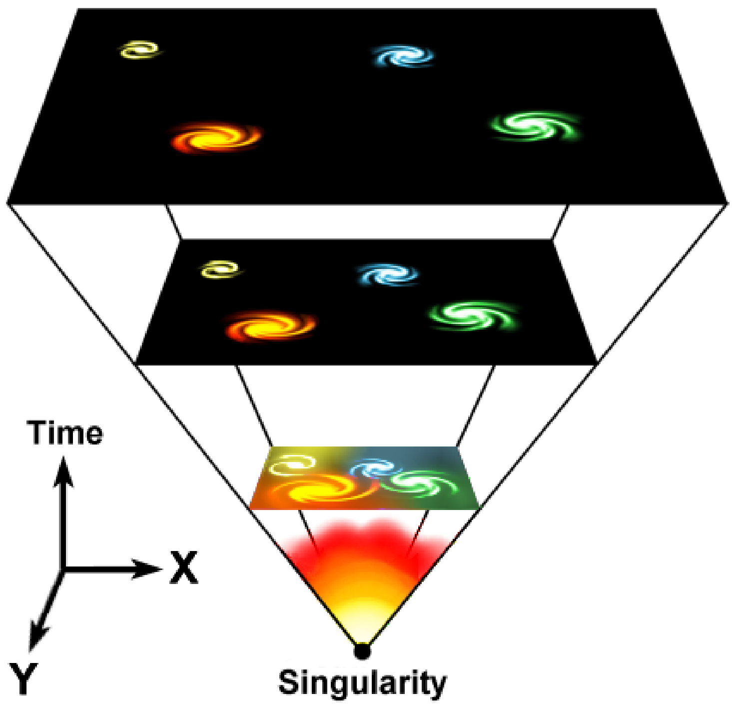
Cosmic Expansion Factor

Cosmic Expansion Factor

$$a(t) = \frac{R(t)}{R_0}$$

- Cosmic Expansion is a uniform expansion of space

$$\vec{r}(t) = a(t)\vec{x}$$



Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0{}^2} + \frac{\Lambda}{3}a^2$$

Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

density

pressure

cosmological constant

curvature term

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^{\,2}} + \frac{\Lambda}{3}a^2$$

Friedmann-Robertson-Walker-Lemaitre Universe

Because of General Relativity, the evolution of the Universe is fully determined by four factors:

- density $\rho(t)$
- pressure $p(t)$
- curvature $kc^2 / R_0^{\,2}$ $k = 0, +1, -1$

  R_0: present curvature radius
- cosmological constant Λ

- Density & Pressure:
  - in relativity, energy & momentum need to be seen as one physical quantity (four-vector)
  - pressure = momentum flux
- Curvature:
  - gravity is a manifestation of geometry spacetime
- Cosmological Constant:
  - free parameter in General Relativity
  - Einstein's "biggest blunder"
  - mysteriously, since 1998 we know it dominates the Universe

Friedmann-Robertson-Walker-Lemaitre Universe

Relativistic Cosmology

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0{}^2} + \frac{\Lambda}{3}a^2$$

Newtonian Cosmology

$$\ddot{a} = -\frac{4\pi G}{3}\rho a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + E$$

		↔		
$-kc^2 / R_0{}^2$	Curvature		E	Energy
Λ	Cosmological Constant			______
p	Pressure			______

Hubble Parameter

Hubble Expansion

- Cosmic Expansion is a uniform expansion of space
- Objects do not move themselves:
  they are like beacons tied to a uniformly expanding sheet:

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$\Bigg\} \quad H(t) = \frac{\dot{a}}{a}$$

Hubble Expansion

- Cosmic Expansion is a uniform expansion of space

- Objects do not move themselves:

  they are like beacons tied to a uniformly ex

Comoving Position

Hubble Parameter:

Hubble "constant":
$H_o \equiv H(t=t_o)$

$$\vec{r}(t) = a(t)\vec{x}$$

$$\dot{\vec{r}}(t) = \dot{a}(t)\vec{x} = \frac{\dot{a}}{a} a\vec{x} = H(t)\vec{r}$$

$$H(t) = \frac{\dot{a}}{a}$$

Hubble Parameter

- For a long time, the correct value of the Hubble constant H_o was a major unsettled issue:

H_o = 50 km s^{-1} Mpc^{-1} ⟷ H_o = 100 km s^{-1} Mpc^{-1}

- This meant distances and timescales in the Universe had to deal with uncertainties of a factor 2 !!!

- Following major programs, such as Hubble Key Project, the Supernova key projects and the WMAP CMB measurements,

$$H_0 = 71.9^{+2.6}_{-2.7}\, km\, s^{-1} Mpc^{-1}$$

Hubble Constant: Tension

- As more accurate measurements of H_o become available,

  gradual rising tension

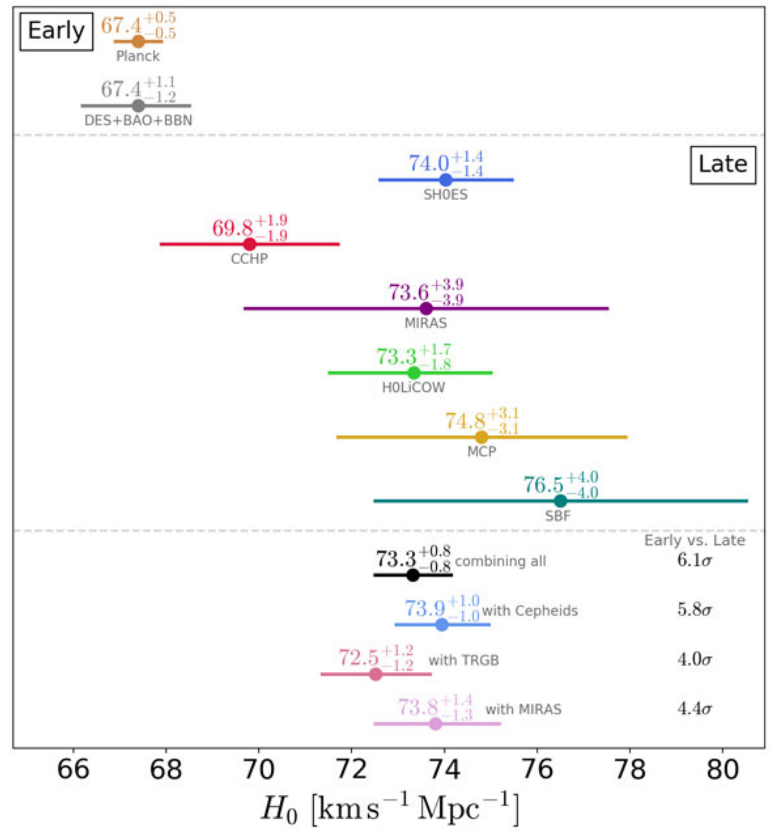
- CMB determination much lower than

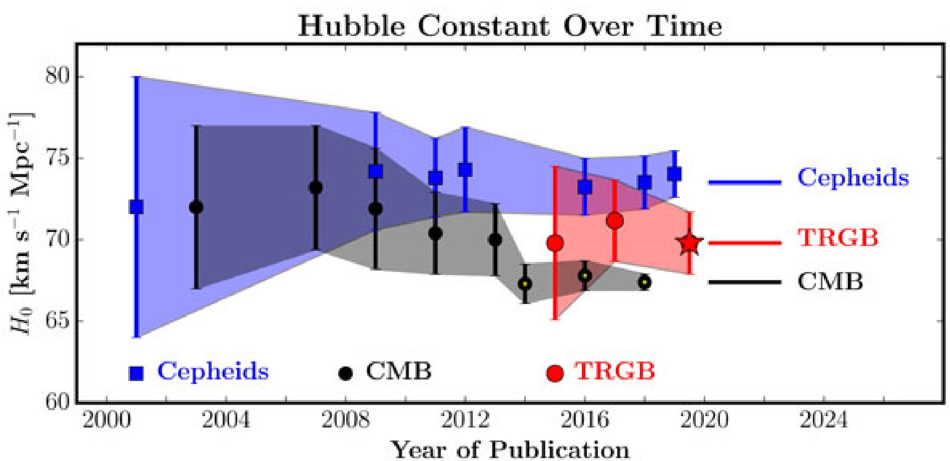
  "local values"

- Latest value:

  strong grav. lensing

  Ho = 82.4 +/- 8.3 km/s/Mpc





Hubble Time

$$t_H = \frac{1}{H}$$

$$H_0 = 100h \ km\, s^{-1} Mpc^{-1}$$

$$\Downarrow$$

$$t_0 = 9.78 h^{-1} \ Gyr$$

Cosmological Constant
&
FRW equations

Friedmann-Robertson-Walker-Lemaitre Universe

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a + \frac{\Lambda}{3}a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0{}^2} + \frac{\Lambda}{3}a^2$$

Dark Energy & Energy Density

$$\tilde{\rho} = \rho + \rho_\Lambda$$

$$\tilde{p} = p + p_\Lambda$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

$$p_\Lambda = -\frac{\Lambda c^2}{8\pi G}$$

Friedmann-Robertson-Walker-Lemaitre Universe

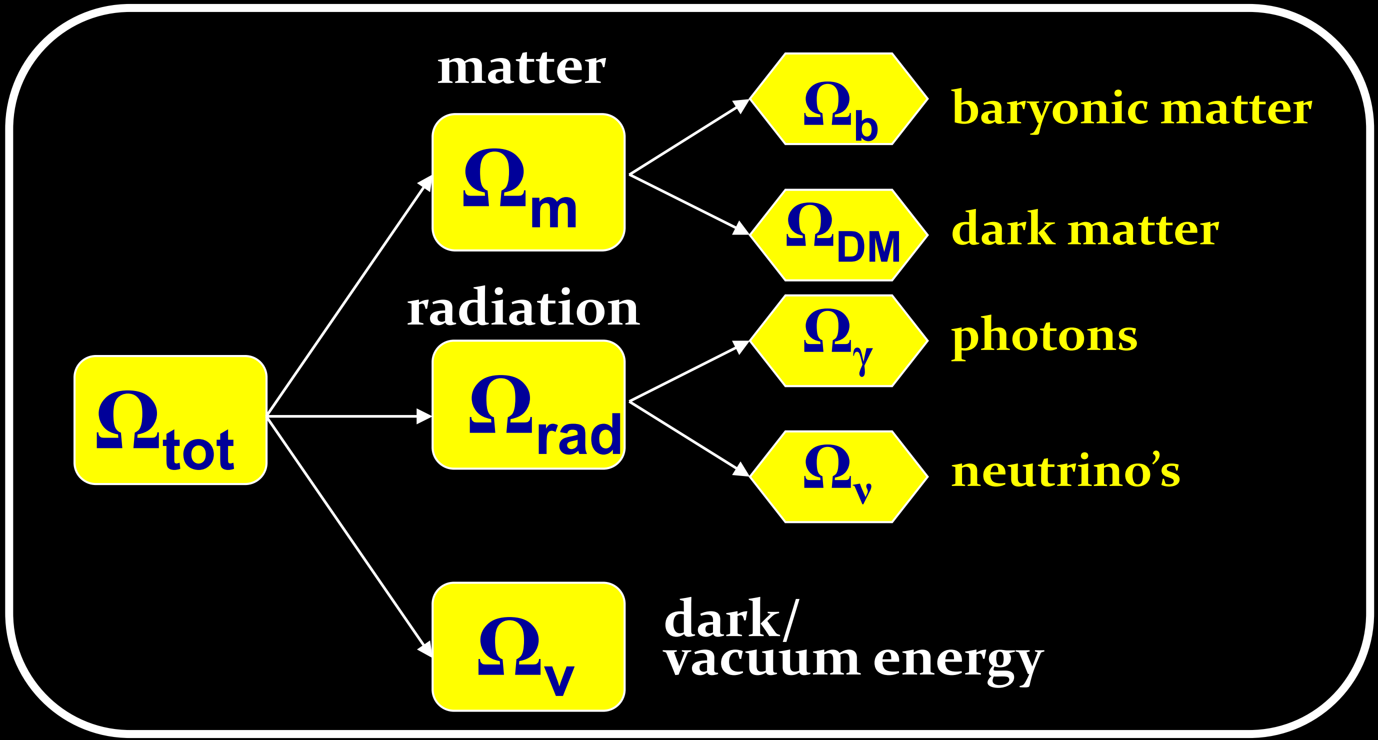
$$\ddot{a} = -\frac{4\pi G}{3}\left(\tilde{\rho} + \frac{3\tilde{p}}{c^2}\right)a$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\tilde{\rho} - \frac{kc^2 / R_0{}^2}{a^2}$$

Cosmic Constituents

Cosmic Constituents

The total energy content of Universe made up by various constituents, principal ones:



In addition, contributions by

- gravitational waves
- magnetic fields,
- cosmic rays ...

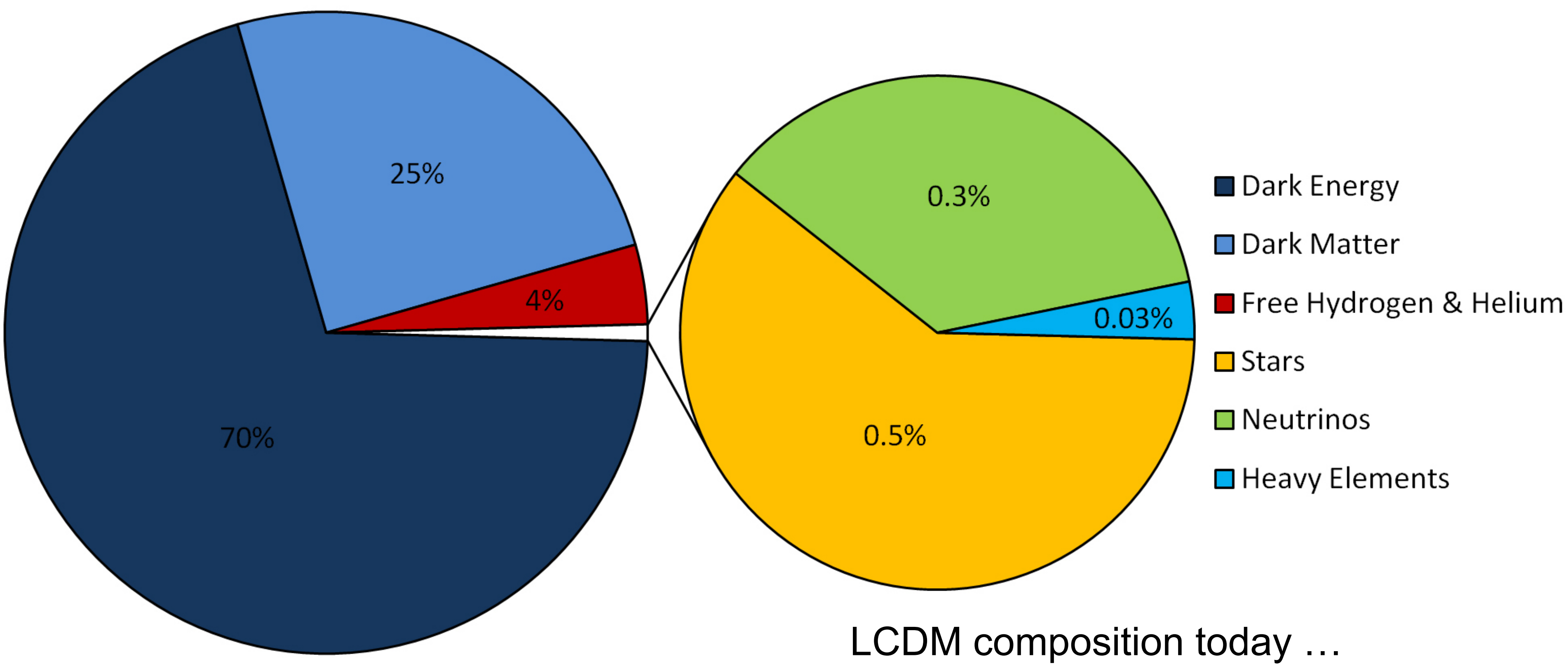
Poor constraints on their contribution: henceforth we will not take them into account !

LCDM Cosmology

- Concordance cosmology

  - model that fits the majority of cosmological observations

  - universe dominated by Dark Matter and Dark Energy



LCDM composition today …

Cosmic Energy Inventory

1	dark sector				0.954 ± 0.003
1.1	dark energy			0.72 ± 0.03	
1.2	dark matter			0.23 ± 0.03	
1.3	primeval gravitational waves			$\lesssim 10^{-10}$	
2	primeval thermal remnants				0.0010 ± 0.0005
2.1	electromagnetic radiation			$10^{-4.3 \pm 0.0}$	
2.2	neutrinos			$10^{-2.9 \pm 0.1}$	
2.3	prestellar nuclear binding energy			$-10^{-4.1 \pm 0.0}$	
3	baryon rest mass				0.045 ± 0.003
3.1	warm intergalactic plasma			0.040 ± 0.003	
3.1a	virialized regions of galaxies	0.024 ± 0.005			
3.1b	intergalactic	0.016 ± 0.005			
3.2	intracluster plasma			0.0018 ± 0.0007	
3.3	main sequence stars		spheroids and bulges	0.0015 ± 0.0004	
3.4			disks and irregulars	0.00055 ± 0.00014	
3.5	white dwarfs			0.00036 ± 0.00008	
3.6	neutron stars			0.00005 ± 0.00002	
3.7	black holes			0.00007 ± 0.00002	
3.8	substellar objects			0.00014 ± 0.00007	
3.9	HI + HeI			0.00062 ± 0.00010	
3.10	molecular gas			0.00016 ± 0.00006	
3.11	planets			10^{-6}	
3.12	condensed matter			$10^{-5.6 \pm 0.3}$	
3.13	sequestered in massive black holes			$10^{-5.4}(1 + \epsilon_n)$	
4	primeval gravitational binding energy				$-10^{-6.1 \pm 0.1}$
4.1	virialized halos of galaxies			$-10^{-7.2}$	
4.2	clusters			$-10^{-6.9}$	
4.3	large-scale structure			$-10^{-6.2}$	

Fukugita & Peebles 2004

Critical Density & Omega

FRW Dynamics

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0{}^2}$$

Critical Density:

- For a Universe with Ω=o
- Given a particular expansion rate H(t)
- Density corresponding to a flat Universe (k=o)

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

FRW Dynamics

In a FRW Universe,
densities are in the order of the critical density,

$$\rho_{crit} = \frac{3H_0^{\ 2}}{8\pi G} = 1.8791 h^2 \times 10^{-29}\, g\, cm^{-3}$$

$$\begin{aligned} \rho_0 &= 1.8791 \times 10^{-29}\, \Omega h^2 \;\; g\, cm^{-3} \\ &= 2.78 \times 10^{11}\, \Omega h^2 \quad\quad M_{\odot} Mpc^{-3} \end{aligned}$$

FRW Dynamics

In a matter-dominated Universe,
the evolution and fate of the Universe entirely determined
by the (energy) density in units of critical density:

$$\Omega \equiv \frac{\rho}{\rho_{crit}} = \frac{8\pi G \rho}{3H^2}$$

Arguably, ⍰ is the most important parameter of cosmology !!!

Present-day
Cosmic Density:

$$\rho_0 = 1.8791 \times 10^{-29}\, \Omega h^2 \;\; g\, cm^{-3}$$
$$= 2.78 \times 10^{11}\, \Omega h^2 \quad M_{\odot} Mpc^{-3}$$

FRWL Dynamics
&
Cosmological Density

FRW Dynamics

- The individual contributions to the energy density of the Universe can be figured into the ⍰ parameter:

- radiation

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$$

- matter

$$\Omega_m = \Omega_{dm} + \Omega_b$$

- dark energy/ cosmological constant

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

Critical Density

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

$\Omega < 1$	$k = -1$	*Hyperbolic*	*Open Universe*
$\Omega = 1$	$k = 0$	*Flat*	*Critical Universe*
$\Omega > 1$	$k = +1$	*Spherical*	*Close Universe*

FRW Universe: Curvature

There is a 1-1 relation between the total energy content of the Universe and its curvature. From FRW equations:

$$k = \frac{H^2 R^2}{c^2}(\Omega - 1)$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

$\Omega < 1$	$k = -1$	*Hyperbolic*	*Open Universe*
$\Omega = 1$	$k = 0$	*Flat*	*Critical Universe*
$\Omega > 1$	$k = +1$	*Spherical*	*Close Universe*

Radiation, Matter & Dark Energy

The individual contributions to the energy density of the Universe can be figured into the ⍰ parameter:

- radiation

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_{crit}} = \frac{\sigma T^4 / c^2}{\rho_{crit}} = \frac{8\pi G \sigma T^4}{3H^2 c^2}$$

- matter

$$\Omega_m = \Omega_{dm} + \Omega_b$$

- dark energy/ cosmological constant

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

$$\Omega = \Omega_{rad} + \Omega_m + \Omega_\Lambda$$

Cosmic Constituents:

Evolving Energy Density

FRW Energy Equation

To infer the evolving energy density ?(t) of each cosmic component, we refer to the cosmic energy equation. This equation can be directly inferred from the FRW equations

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

The equation forms a direct expression of the adiabatic expansion of the Universe, ie.

$U = \rho c^2 V$ Internal energy

$V \propto a^3$ Expanding volume

$$dU = -p\,dV$$

FRW Energy Equation

To infer ⍰(t) from the energy equation, we need to know the pressure p(t) for that particular medium/ingredient of the Universe.

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

To infer p(t), we need to know the nature of the medium, which provides us with the equation of state,

$$p = p(\rho, S)$$

Cosmic Constituents:
Evolution of Energy Density

- **Matter:**

$$\rho_m(t) \propto a(t)^{-3}$$

□ **Radiation:**

$$\rho_{rad}(t) \propto a(t)^{-4}$$

□ **Dark Energy:**

$$\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$$

$$\Downarrow \quad w = -1$$

$$\rho_\Lambda(t) = cst.$$

Dark Energy:

Equation of State

Einstein Field Equation

curvature side

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

energy-momentum side

Equation of State

$$T^{\mu\nu}{}_{vac} \equiv \frac{\Lambda c^4}{8\pi G} g^{\mu\nu}$$

restframe ⇒

$$T^{\mu\nu}{}_{vac} \equiv \frac{\Lambda c^4}{8\pi G} \eta^{\mu\nu}$$

$$\eta^{00} = 1, \quad \eta^{ii} = -1$$

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^{\mu} U^{\nu} - p g^{\mu\nu}$$

restframe:

$$\left. \begin{aligned} T^{00}{}_{vac} &= \rho_{vac} c^2 \\ T^{ii}{}_{vac} &= p \end{aligned} \right\} \Rightarrow \quad \rho_{vac} c^2 = \frac{\Lambda c^4}{8\pi G}, \quad p = -\frac{\Lambda c^4}{8\pi G}$$

Equation of State

$$\rho_{vac}c^2 = \frac{\Lambda c^4}{8\pi G}$$

$$p = -\frac{\Lambda c^4}{8\pi G}$$

$$p_{vac} = -\rho_{vac}c^2$$

Dynamics

Relativistic Poisson Equation:

$$\nabla^2 \phi = 4\pi G \left(\rho + \frac{3p}{c^2} \right)$$

$$\rho_{vac} + \frac{3p_{vac}}{c^2} = -2\rho_{vac} < 0; \qquad \rho_{vac} = \frac{\Lambda}{8\pi G}$$

⇓

$$\nabla^2 \phi < 0$$

Repulsion !!!

Dark Energy & Cosmic Acceleration

Nature Dark Energy:

(Parameterized) Equation of State

$$p(\rho) = w\rho c^2$$

Cosmic Acceleration:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a$$

Gravitational Repulsion:

$$p = w\rho c^2 \quad \Leftrightarrow \quad w < -\frac{1}{3} \quad \Rightarrow \quad \ddot{a} > 0$$

Dark Energy & Cosmic Acceleration

DE equation of State

$$p(\rho) = w\rho c^2$$

$$\rho_w(a) = \rho_w(a_0)\, a^{-3(1+w)}$$

Cosmological Constant:

$$\Lambda: \qquad w = -1 \qquad \rho_w = cst.$$

-1/3 > w > -1:

$$\rho_w \propto a^{-3(1+w)} \qquad 1 + w > 0$$

decreases with time

Phantom Energy:

$$\rho_w \propto a^{-3(1+w)} \qquad 1 + w < 0$$

increases with time

Dynamic Dark Energy

DE equation of State

$$p(\rho) = w\rho c^2$$

Dynamically evolving dark energy, parameterization:

$$w(a) = w_0 + (1-a)w_a \approx w_\phi(a)$$

$$\rho_w(a) = \rho_w(a_0)\ \exp\left\{-3\int_1^a \frac{1+w_\phi(a')}{a'}da'\right\}$$

General Flat FRW Universe

$$k = 0$$

$$\rho_v(t) \propto a(t)^{-3(1+w)} \quad \Leftarrow \quad p = w\rho_v c^2$$

FRW:

$$a(t) \propto t^{\frac{2}{3+3w}}$$

Acceleration Parameter

FRW Dynamics: Cosmic Acceleration

Cosmic acceleration quantified by means of dimensionless deceleration parameter q(t):

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

$$q = \frac{\Omega_m}{2} + \Omega_{rad} - \Omega_\Lambda$$

$$q \approx \frac{\Omega_m}{2} - \Omega_\Lambda$$

Examples:

$\Omega_m = 1; \quad \Omega_\Lambda = 0;$
$q = 0.5$

$\Omega_m = 0.3; \quad \Omega_\Lambda = 0.7;$
$q = -0.65$

Dynamical Evolution

FRWL Universe

General Solution
Expanding FRW Universe

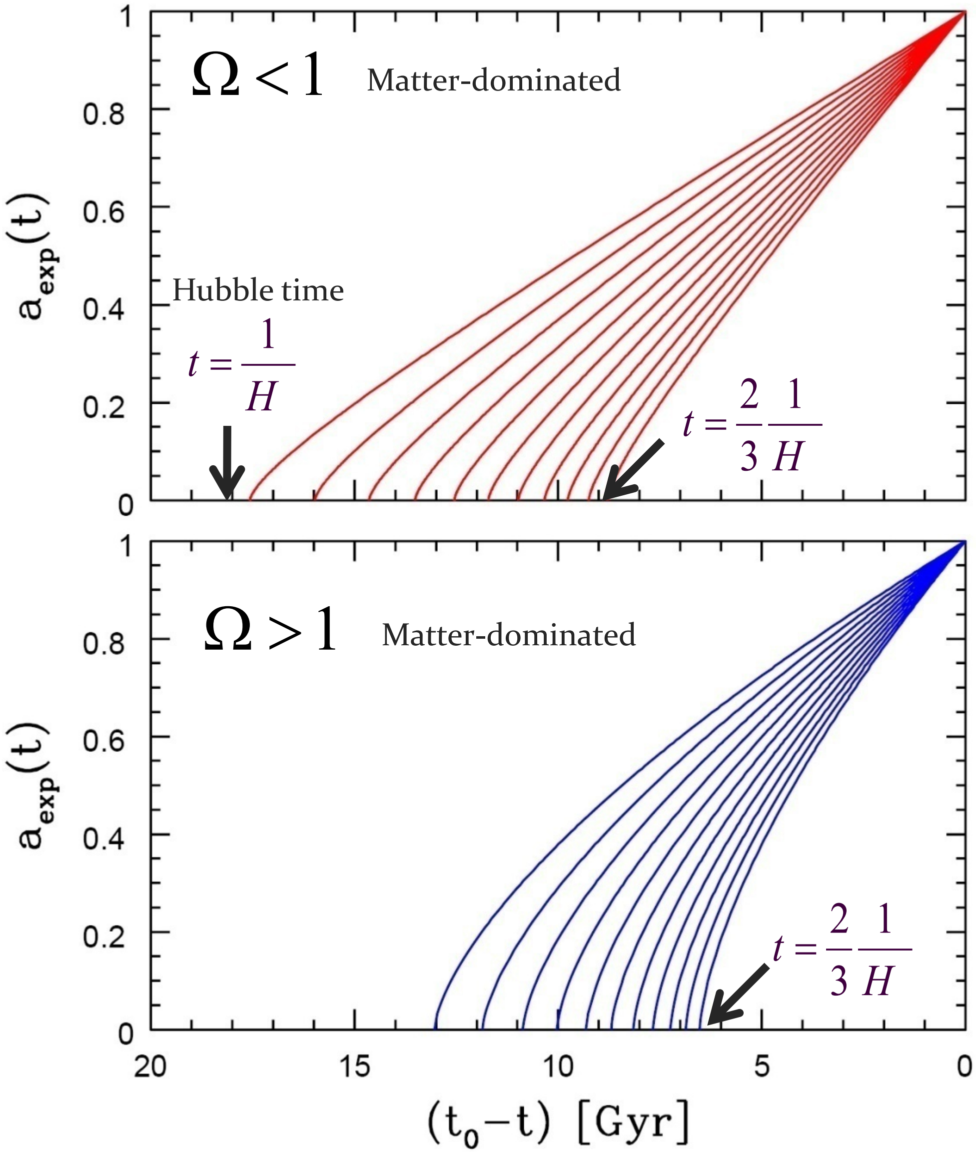
From the FRW equations:

$$\frac{H(t)^2}{H_0^{\,2}} = \frac{\Omega_{rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

$a(t)$ Expansion history Universe

$$H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)}}$$

Age of the Universe



Age of a FRW universe at
Expansion factor a(t)

$$H\,t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{rad}}{a^2} + \frac{\Omega_m}{a} + \Omega_\Lambda a^2 + (1-\Omega)}}$$

Specific Solutions
FRW Universe

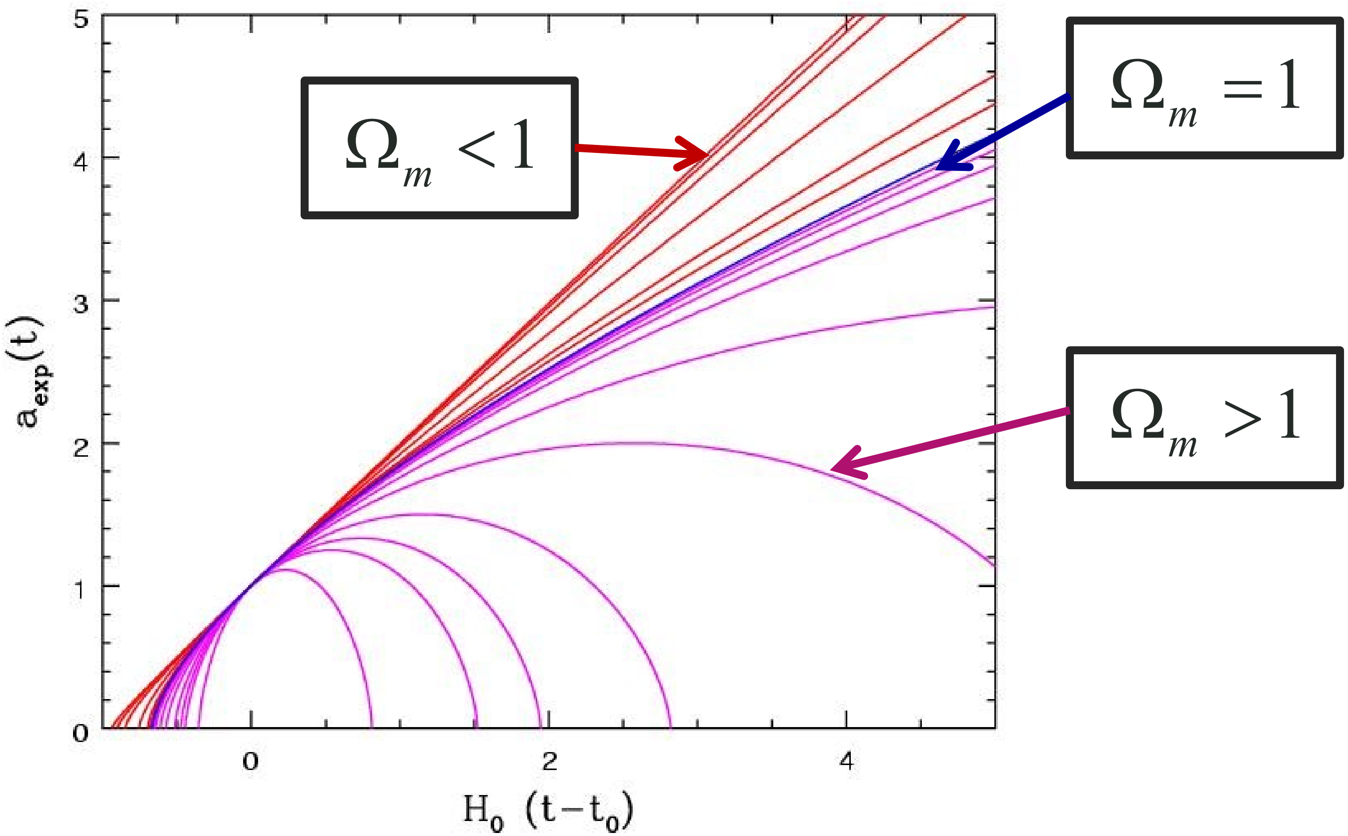
While general solutions to the FRW equations is only possible by numerical integration, analytical solutions may be found for particular classes of cosmologies:

- **Single-component Universes:**
  - empty Universe
  - flat Universes, with only radiation, matter or dark energy
- **Matter-dominated Universes**
- **Matter+Dark Energy flat Universe**

Matter-Dominated Universes

- Assume radiation contribution is negligible:
- Zero cosmological constant:
- Matter-dominated, including curvature

$$\Omega_{rad,0} \approx 5\times10^{-5}$$
$$\Omega_{\Lambda} = 0$$



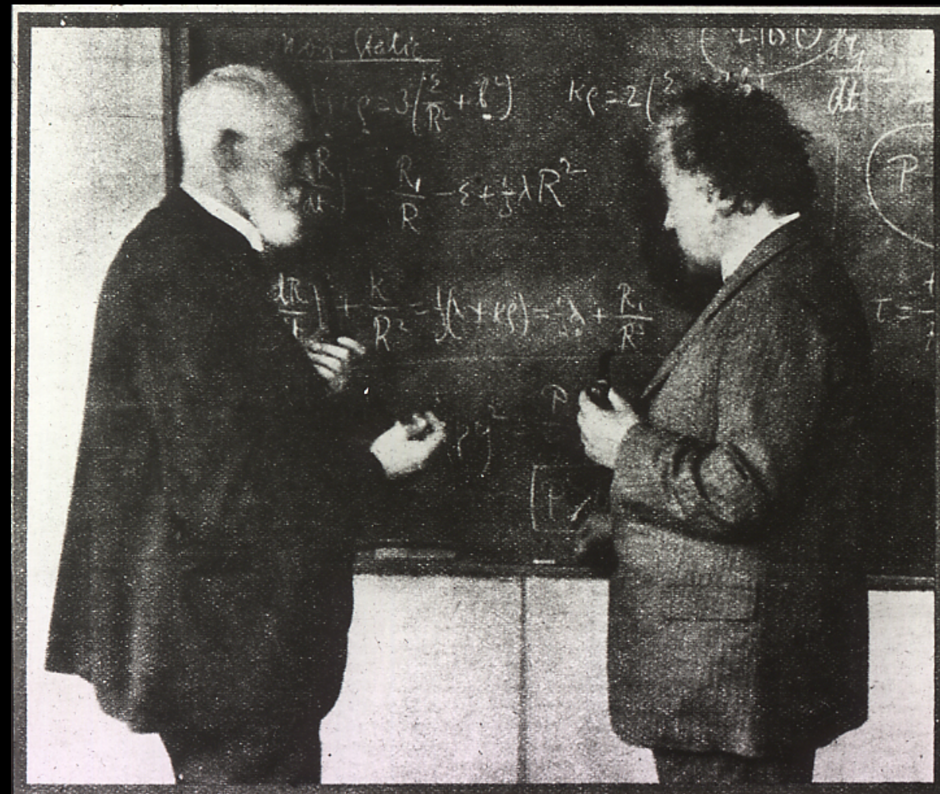
Einstein-de Sitter Universe

$$\left.\begin{array}{l}\Omega_m = 1 \\ \Omega_\Lambda = 0\end{array}\right\} \quad k = 0$$

FRW: $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G \rho_0}{3}\frac{1}{a}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

Age EdS Universe: $$t_0 = \frac{2}{3}\frac{1}{H_0}$$



Albert Einstein and Willem de Sitter discussing the Universe. In 1932 they published a paper together on the Einstein-de Sitter universe, which is a model with flat geometry containing matter as the only significant substance.

Free Expanding "Milne" Universe

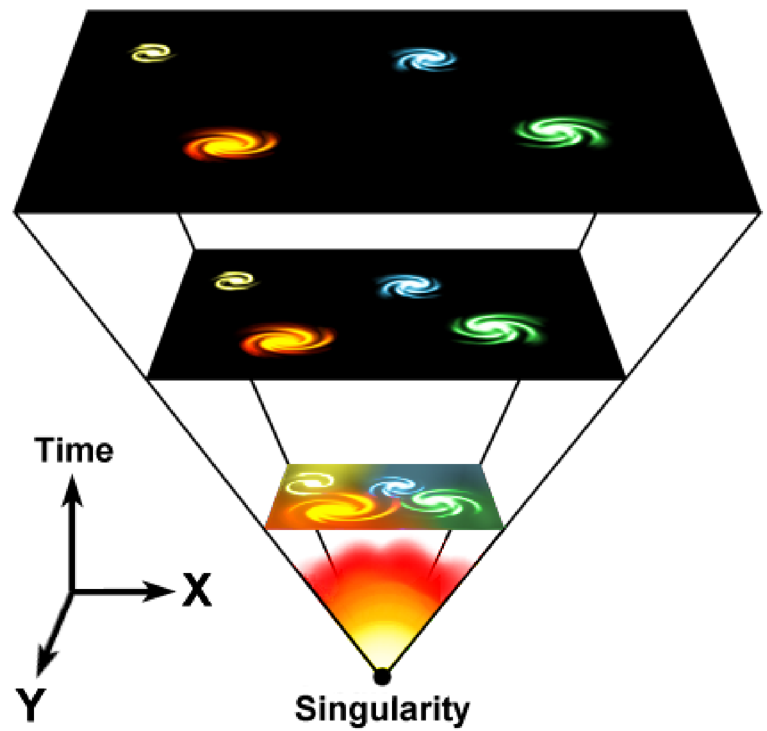
$$\left.\begin{array}{l}\Omega_m = 0 \\ \Omega_\Lambda = 0\end{array}\right\} k = -1$$

Empty space is curved

FRW: $\dot{a}^2 = -\frac{kc^2}{R_0^{\ 2}} = cst.$

$$a(t) = \left(\frac{t}{t_0}\right)$$

Age Empty Universe: $t_0 = \frac{1}{H_0}$



Expansion
Radiation-dominated Universe

$$\left.\begin{array}{l}\Omega_{rad} = 1 \\ \Omega_m = 0 \\ \Omega_\Lambda = 0\end{array}\right\} \quad k = 0$$

In the very early Universe, the energy density is completely dominated by radiation. The dynamics of the very early Universe is therefore fully determined by the evolution of the radiation energy density:

FRW: $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G \rho_0}{3}\frac{1}{a^2}$$

$$\rho_{rad}(a) \propto \frac{1}{a^4}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

Age Radiation Universe:

$$t_0 = \frac{1}{2}\frac{1}{H_0}$$

De Sitter Expansion

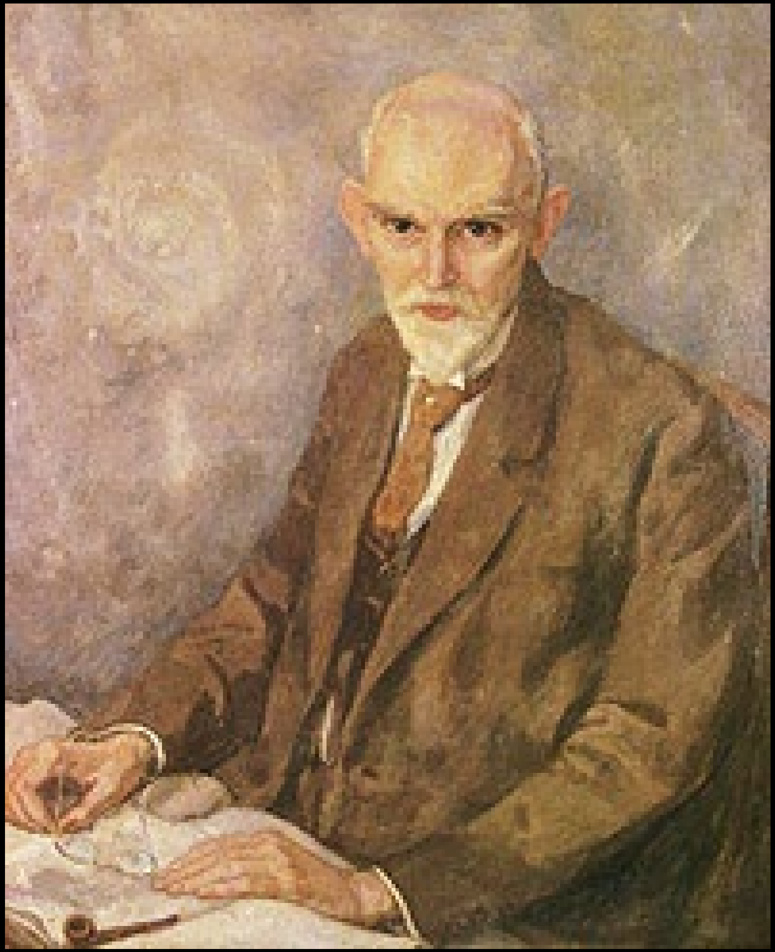
$$\left.\begin{array}{l}\Omega_m = 0 \\ \Omega_\Lambda = 1\end{array}\right\} \quad k = 0$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^{\,2}} \quad \Rightarrow \quad H_0 = \sqrt{\frac{\Lambda}{3}}$$

FRW: $\dot{a}^2 = \frac{\Lambda}{3} a^2 \quad \Rightarrow \quad \dot{a} = H_0 a$

$$a(t) = e^{H_0 (t - t_0)}$$

Age
De Sitter Universe: infinitely old



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