# Tutorial IV
# Observational Cosmology

Exercise 1:

Coordinate Distance and Mattig's Formulae.

Earlier during the lectures, when discussing Robertson-Walker geometries, we encountered the issue of how to translate our theoretical models into observationally relevant properties.

The main issue in translating the geometry of space into observational realities is the relation between the "theoretical" coordinate distance $r$ (the comoving coordinate location of an object comoving with the expansion of the Universe, usually taken as the hypothetical location at the spacetime hypersurface at present time) and the redshift $z$ of an object. The relations $r(z)$ are called *Mattig's formula*. In general it is not possible to find analytical expressions for the expansion history, but for a matter-dominated Universe this is perfectly feasible.

To keep it simple, we are first going to observe in matter-dominated Universe. At the end of the sections on the Robertson-Walker metric, we derived the general relation between coordinate distance $r$ and redshift $z$,

$$r = \frac{c}{H_0} \int_0^z \frac{dy}{H(y)/H_0} \tag{1}$$

- Show that for a matter-dominated Universe you obtain the following relation for the coordinate distance $r(z)$:

$$r = \frac{c}{H_0} \int_0^z \frac{dy}{(1+y)\sqrt{1+\Omega_0 y}} \tag{2}$$

- Calculate the coordinate distance $r(z)$ for an object in an Einstein-de Sitter Universe ($\Omega_0 = 1$). That is, express $r$ in terms of redshift $z$.

- Calculate the coordinate distance $r(z)$ for an object in an empty matter-dominated Universe ($\Omega_0 = 0$).

To be able to assess observational probes we also need to have expressions for the curvature measure $R_0 S_k(r/R_0)$, with

$$S_k(r/R_0) = \begin{cases} \sinh(r/R_0) & k = -1 \\ \\ r/R_0 & k = 0 \\ \\ \sin(r/R_0) & k = +1 \end{cases} \tag{3}$$

- Calculate $R_0 S_k(r/R_0)$ for an Einstein-de Sitter Universe, and show that it is equal to

$$R_0 S_k(r/R_0) \;=\; \frac{2c}{H_0}\left\{1 - \frac{1}{\sqrt{1+z}}\right\} \tag{4}$$

- Calculate $R_0 S_k(r/R_0)$ for an empty $\Omega_0 = 0$ Universe, and show that

$$R_0 S_k(r/R_0) \;=\; \frac{c}{2H_0}\, z\, \frac{2+z}{1+z} \tag{5}$$

The general expressions for Mattig's formulae in a matter-dominated Universe are:

$$R_0\, S_k(r/R_0) \;=\; \frac{2c}{H_0}\, \frac{\Omega_0 z \,+\, (\Omega_0 - 2)\left\{\sqrt{1+\Omega_0 z} \,-\, 1\right\}}{\Omega_0^2(1+z)} \tag{6}$$

or, more convenient for $\Omega_0 \ll 1$,

$$R_0\, S_k(r/R_0) \;=\; \frac{c}{H_0}\, \frac{z}{(1+z)}\, \frac{1 \,+\, \sqrt{1+\Omega_0 z} \,+\, z}{1 \,+\, \sqrt{1+\Omega_0 z} \,+\, \Omega_0 z/2} \tag{7}$$

Exercise 2:
Angular and Luminosity Distance

Using the expressions for Mattig's formulae above,

- Give the general expression for luminosity distance $D_L(z)$ and angular diameter distance $D_A(z)$ in a matter-dominated Universe (use expression eqn. 6).
- Calculate specifically the expression for the angular diameter distance $D_A(z)$ in an Einstein-de Sitter Universe.
- Show that $D_A(z)$ in an EdS Universe has a maximum. Derive the redshift $z_{max}$ at which $D_A(z)$ reaches its maximum. Make a sketch of $D_A(z)$ vs. redshift $z$.
- Repeat the same for an empty $\Omega_0 = 0$ Universe and for a $\Omega_0 = 0.3$ Universe. What difference in behaviour with $z$ do you find ?
- What does this mean for the angular size of an object with a fixed physical size $L$ seen at redshift $z$. Answer this question by plotting the angle $\theta(z)$ as function of $z$ for $\Omega_0 = 1$, $\Omega_0 = 0.3$ and $\Omega_0 = 0.0$.
- For $H_0 = 71$ km/s/Mpc calculate the value of the angular diameter distance for objects at $z = 1089$.

We are now going to look at a very important application, observing the Microwave Background. We want to work out what the angular size is of the horizon of the Universe at recombination/decoupling. In this, we make the simplifying assumption of living in a matter-dominated Universe. The horizon scale at recombination is given by

$$\begin{aligned} R_H &= 3c\,t_{dec} \\ &= \frac{2c}{H_{dec}} \end{aligned} \tag{8}$$

- Show that an approximate expression for $D_a(z)$ at high redshifts $z \gg 1$

$$D_A \approx \frac{2c}{H_0 \Omega_0} \frac{1}{z} \tag{9}$$

- Combing the expression for the horizon distance at decoupling $R_H$ and the approximate expression for $d_A$, what is the angular size $\theta_H$ of a patch on the sky with the size of the horizon at recombination in terms of $H_{dec}$, $z_{dec}$, $\Omega_0$ and $H_0$ ?

- Show that for $z \gg 1$, the Hubble parameter $H(z)$ in a matter-dominated Universe is approximately

$$H^2(z) \approx \Omega_0 H_0^2 \, z^3 \tag{10}$$

- Show that for the recombination/decoupling horizon angle on the sky,

$$\theta_H \approx 1.74^\circ \, \Omega_0^{1/2} \left( \frac{z_{dec}}{1089} \right)^{-1/2} \tag{11}$$

- Given that temperatures on the CMB sky are the same everywhere, what conclusion do you draw from your inference ?

Exercise 3:
Cosmology, the Search for Two Numbers

In a famous 1970 Annual Review of Astronomy and Astrophysics review, the observational cosmologist Allan Sandage described all of cosmology as a "Search for Two Numbers". The two numbers were

$$\begin{cases} H_0 \\ \\ q_0 \end{cases}$$

We are going to explore what the reason was behind this stout statement (which by now is far besides reality ...). And in this we do not need any dynamics ... no assumption about a Friedmann Universe needed (yet),

- On the basis of a Taylor series expansion of the expansion factor $a(t)$, with respect to the current cosmic time $t_0$, show that

$$a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 \tag{12}$$

- By inverting the above expression for $a(t)$ and using the expression for coordinate distance of a source whose radiation was emitted at $t_e$ and has just reached us at $t_0$,

$$d_p(t_0) = r = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{13}$$

  show that

$$d_p(t_0) \approx c(t_0-t_e) + \frac{cH_0}{2} (t_0-t_e)^2 . \tag{14}$$

- Imagine you receive radiation from an object with redshift $z$, which it radiated at cosmic time $t_e$. Show that the approximate relationship between $z$ and $t_e$ is given by

$$z \approx H_0(t_0-t_e) + \left(\frac{1+q_0}{2}\right) H_0^2 (t_0-t_e)^2 \tag{15}$$

- Invert this equation to obtain

$$t_0 - t_e \approx \frac{1}{H_0} \left\{ z - \left(\frac{1+q_0}{2}\right) z^2 \right\} \tag{16}$$

- On the basis of results above show that for $z \ll 1$ we have the following approximate relation between $z$ and coordinate distance $d_p(t_0)$,

$$d_p(t_0) = \frac{c}{H_0} z \left\{ 1 - \frac{1+q_0}{2} z \right\} \tag{17}$$

- Taking along this approximation for $z \ll 1$, we may find an approximation for the luminosity distance of any universe, $d_L \approx (1+z)d_p(t_0)$. Show that an approximate relation for $d_L$ is therefore

$$d + L \approx \frac{c}{H_0} z \left\{ 1 + \frac{1 - q_0}{2} z \right\} \tag{18}$$

- When we observe an object with absolute bolometric magnitude $M_{bol}$ at a redshift $z$, show that in a Universe with acceleration parameter $q_0$,

$$m_{bol} \approx M_{bol} + 5 \log \left[ \frac{c}{H_0} / 10pc \right] + 5 \log z + 1.086(1 - q_0)z + \mathcal{O}(z^2) \tag{19}$$

- Take an empty $\Omega_0 = 0$ matter-dominated Universe (i.e. $q_0 = 0$) as your reference for $(m - M)_)$. Infer what the relation $(m - M)(z) - (m - M)_0(z)$ is for three Universes. One should have $q_0 = 0.5$ (EdS), one $q_0 = 0.15$ ($\Omega_0 = 0.3$ matter-dominated) and one $q_0 = -0.55$ (concordance Universe). Draw a graph of the various predictions.

Note: the latter is exactly what the Supernova Ia experiments have found: an accelerating Universe.