VISUAL HORIZONS IN WORLD-MODELS

W. Rindler

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Summary

This paper seeks to effect a unification and generalization of various particular results on visual horizons scattered in the literature. A horizon is here defined as a frontier between things observable and things unobservable. Two quite different types of horizon exist which are here termed event-horizons and particle-horizons. These are discussed in detail and illustrated by examples and diagrams. The examples include well-known model-universes which exhibit one or the other type of horizon, both types at once, or no horizon. Proper distance and cosmic time are adopted as the main variables, and the analysis is based on the Robertson–Walker form of the line element and therefore applies to all cosmological theories using a homogeneous and isotropic substratum.

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1. *Introduction.*—There has recently occurred a renewed interest in the definition and properties of horizons in world-models, as witnessed by a prolonged correspondence in *The Observatory* (1) and subsequently in *Nature* (2). This discussion was chiefly devoted to the horizon in the de Sitter space-time associated with the model of the steady state theory, but the horizon concept has applications to many other models besides, and cosmologists since de Sitter and Eddington have concerned themselves with it. The above-mentioned correspondence drew attention to the lack of general agreement on even the definition of horizons. Different writers appear to use different horizon concepts, or different descriptions of the same concept, without always making the necessary distinctions. Furthermore, the meanings of many phrases used in discussions of horizons such as, for example, "all particles on one side of the horizon", "crossing the horizon with the speed of light", "in the observer's finite experience", etc. evidently depend critically on the definitions of time and distance whose diversity is notorious. A statement meaningful and valid on one interpretation can be meaningless or even false on another, unintended, interpretation. Consequently there can and did arise certain apparent "paradoxes". These facts seem to indicate the desirability for clear definitions to be laid down and accepted. That there is, moreover, room for a long overdue general study of horizons in cosmology was pointed out by H. Bondi and T. Gold (3) in the course of the *Observatory* correspondence.

In the present paper I seek to supply these needs. The main argument applies to all homogeneous and isotropic model-universes, being based on the Robertson–Walker form of the line element, but certain well-known particular models are singled out for illustrative purposes. As main variables I have chosen proper distance and cosmic time. In terms of these, horizons turn out to be loci of ordinary points, not singularities; in fact, it is now well-known that by means of these variables it is possible to represent the whole of space-time regularly, in spite of what Eddington wrote in 1923*; the description of particles or events beyond the horizon presents no difficulties; and some apparent paradoxes disappear almost automatically. For the sake of clarity a small number of diagrams have also been designed. Their purpose is, however, purely schematic and no attempt was made to draw them to scale.

We shall define a horizon as *a frontier between things observable and things unobservable.* (The vague term *things* is here used deliberately.) There are then two quite different horizon concepts in cosmology which satisfy our definition and to which cosmologists have at various times devoted their attention. The first, which I shall call an *event-horizon*, is exemplified by the de Sitter model-universe. It may be defined as follows: *An event-horizon, for a given fundamental observer A, is a (hyper-) surface in space-time which divides all events into two non-empty classes: those that have been, are, or will be observable by A, and those that are forever outside A's possible powers of observation.* It was this horizon, with particular reference to the steady state theory, which formed the subject of the above-mentioned correspondence. Earlier references to it were made, among others, by Eddington (**5**) in connection with the de Sitter model (it was then called the "mass-horizon") and by E. A. Milne and G. J. Whitrow (**6**) in connection with Page's "uniformly accelerating" equivalence. A very interesting discussion of the event-horizon in the de Sitter model was recently given by Schrödinger (**7**) who greatly developed the geometric technique used by Eddington.

The other type of horizon, which I shall call a *particle-horizon*†, is exemplified by the Einstein–de Sitter model-universe. It may be defined as follows: *A particle-horizon, for any given fundamental observer A and cosmic instant t_0 is a surface in the instantaneous 3-space $t=t_0$, which divides all fundamental particles into two non-empty classes: those that have already been observable by A at time t_0 and those that have not.* W. H. McCrea (**8**) and G. C. McVittie (**9**) have previously discussed the particle-horizon from the point of view of its providing an empirical procedure for the determination of the rate of expansion of the universe. E. A. Milne (**10**) discussed it as an example of the absurdities inherent, in his opinion, in some of the cosmological models of General Relativity.

It must be pointed out that there are models, for example the well-known Lemaître model of General Relativity, which actually possess both types of horizon. Others, like Milne's "uniformly expanding" model possess neither type‡.

The general mechanism of horizons can be simply illustrated by means of the well-known "reduced" model which represents the universe as an expanding balloon. (This, of course, is a representation of a closed universe, but the argument for open universes is similar.) The fundamental particles can be represented by black dots distributed uniformly on the balloon. One particular dot may be marked specially so as to represent a given fundamental particle-observer *A*. Photons can be represented by red dots moving over the balloon along great

* "If de Sitter's form for an empty world is right it is impossible to find any coordinate system which represents the whole of real space-time regularly." (**4**). Our statement above includes de Sitter's form.

† It will be understood that whenever we speak of particles in this context we always mean fundamental particles, i.e. the representations of the nebulae in the world-model.

‡ What is commonly referred to as the horizon in that model is no true horizon in the sense of dividing things observable from things unobservable. The horizon of Milne's model is an accident of a particular choice of coordinates and can be "transformed away". See Section 10.

circle paths and always at constant speed relative to the material of the balloon. An event-horizon will exist for A, and similarly for all other fundamental observers, in models where the rate of expansion is and remains sufficiently great for some of the red dots moving on great circles towards A never to reach A. As Eddington put it, light is then like a runner on an expanding track, with the winning post receding faster than he can run. On the other hand, a particle-horizon will occur if, for example, the balloon is blown up from a volume approximating to a point at an initial rate exceeding the speed of the red dots, so that a finite time will elapse before any given one of them can reach A. *None* will reach A unless the rate of inflation decreases from its initial value. *Some* will never reach A if the rate of inflation, after first decreasing, increases again suitably. This is what happens in a model with both types of horizon.

A full discussion of these phenomena can, of course, not be given without a mathematical formulation, to which we now proceed.

2. *Mathematical preliminaries.*—As the mathematical basis of our discussion we take the Robertson–Walker line element,

$$ds^2 = c^2\,dt^2 - R^2(t)\left\{\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)}{(1 + kr^2/4)^2}\right\}. \tag{1}$$

In fact, the present paper may in a sense be regarded as a study of some of the properties of this metric. It has been shown to be applicable to all homogeneous and isotropic model-universes (**11**, **12**). Since practically all modern cosmological theories employ such model-universes our discussion will be of sufficient generality. The metric (1) has the following significance: (i) t is a cosmic time coordinate; (ii) θ, ϕ are the usual angular measurements made at the spatial origin $r=0$ (which can be identified with any fundamental particle); (iii) the world-lines of the fundamental particles are the geodesics r, θ, $\phi=$ constant, whence r is a "co-moving" radial coordinate; (iv) light-tracks correspond to the null geodesics of the metric and, in particular, light-tracks through the spatial origin have the equations θ, $\phi=$ constant and

$$\frac{c\,dt}{R(t)} = \pm\frac{dr}{1 + kr^2/4}, \tag{2}$$

the positive sign evidently being required for light travelling away from the origin and the negative sign for light travelling towards it. Additional hypotheses are needed before a particular form can be assigned to the scale function $R(t)$ and a particular value (0, 1 or -1) to the curvature index k, and these are supplied by the various cosmological theories. In any theory which adopts the convention of the constancy of the local speed of light (and which consequently has interdependent time- and distance-scales) that speed is identified with the constant c in (1).

We shall employ proper distance and cosmic time as the main variables throughout this discussion. Due care must be exercised, especially in connection with the state of the universe shortly after the creation event in models where such an event occurs, not to identify these conventional variables too readily with "ordinary" distance and time.

If we introduce a function $\sigma(r)$, which may be regarded as an alternative "co-moving" radial coordinate, by the equation

$$\sigma(r) = \int_0^r \frac{dr}{1 + kr^2/4}, \tag{3}$$

then the proper distance, l, at time t_1 between the spatial origin and a fundamental particle with coordinate r_1 is given by $l=R(t_1)\sigma(r_1)$. (As is well-known, in theories with a conventionally constant speed of light this distance can be interpreted as the sum of the infinitesimal distance measurements made at the world-moment t_1 by a chain of intermediate fundamental observers situated along the space-geodesic joining the particle to the observer. In other theories l must be regarded as purely conventional.) The equation of motion of this fundamental particle is therefore

$$l=R(t)\sigma(r_1). \tag{4}$$

To obtain the equation of motion of a photon emitted at time t_1 at a fundamental particle with coordinate r_1 in the direction towards the origin, we integrate equation (2), choosing the negative sign, between the time and radial coordinate of emission and the current time and radial coordinate, t and r, and find

$$\sigma(r)=\sigma(r_1)-\int_{t_1}^{t}\frac{c\,dt}{R(t)},$$

which, on multiplication by $R(t)$, gives the required equation of motion,

$$l=R(t)\left\{\sigma(r_1)-\int_{t_1}^{t}\frac{c\,dt}{R(t)}\right\}. \tag{5}$$

In the sequel we shall want to assume that $\sigma(r)$ can take all positive values. Reference to (3) shows that this is evidently so when $k=0$ or -1 (in the latter case the radial coordinate r is restricted by $r<2$). When $k=1$, $\sigma(r)$ as originally defined is restricted by $\sigma<\pi$ for all finite values of r, and thus there appears to exist a boundary to the fundamental particles at $l=\pi R(t)$. In fact this boundary is artificial and merely due to the particular definition of the coordinate r which in this case makes $r=\infty$ correspond to the antipole of the origin in the instantaneous spherical 3-spaces $t=$constant. There are, of course, particles beyond this pole on any line of vision. Moreover every particle occupies infinitely many positions on the line of vision, the latter being a closed curve of proper length $2\pi R(t)$. Now it is easily verified, and indeed it follows from the physical significance of proper distance, that σ is an additive coordinate in the sense that the σ of a particle C relative to the origin-particle A equals the σ of an intermediate particle B relative to A plus the σ of C relative to B. We therefore extend the definition of σ to particles beyond the pole by taking the pole as auxiliary origin, evaluating σ from there, and adding π. This process can be continued indefinitely. Analogy with an ordinary circle makes the idea quite clear: σ corresponds to the angular coordinate at the centre. Thus to each σ, however large, there corresponds a particle on the line of vision and each particle corresponds to infinitely many values of σ, all differing by multiples of 2π. A little care must therefore be exercised in the interpretation of horizons in these models, as will be seen in Section 4.

3. *The event-horizon.*—The necessary and sufficient condition for an event-horizon to exist in a given model is that the integral $\int^{\infty}\frac{dt}{R(t)}$ converge to a finite limit. For then, as reference to (5) shows, there exists at any given time t_0 and on any given line of vision a particle determined by

$$\sigma=\sigma_0=\int_{t_0}^{\infty}\frac{c\,dt}{R(t)} \tag{6}$$

such that a photon emitted at that particle towards the origin at time t_0 reaches the origin at time $t = \infty$, i.e. in the infinite future. Photons emitted at the same time, t_0, from all farther particles ($\sigma > \sigma_0$) do not reach the origin at all (l never vanishes) whereas photons emitted at t_0 from all nearer particles ($\sigma < \sigma_0$) reach the origin at some finite time. We shall call the position of the critical particle at time t_0 the *horizon-point* at t_0 on that particular line of vision. If we multiply (6) by $R(t)$ and discard the particularizing suffix we obtain the equation of motion of the horizon-point.

$$l = R(t) \int_t^\infty \frac{c\,dt}{R(t)}. \tag{7}$$

Reference to (5) shows that this is actually the equation of motion of a photon, since (5) reduces to (7) if we put $\sigma(r_1) = \int_{t_1}^\infty \frac{c\,dt}{R(t)}$. Thus the horizon-point on any given line of vision may be identified with one particular photon travelling towards the origin. It follows that the event-horizon, which we now define as the aggregate of all horizon-points, is a closed light-front travelling towards the origin, and at proper distance from it given by (7).

It is perhaps useful to retain a dual picture of the event-horizon: (i) as a surface (geodesic sphere) in the observer's instantaneous 3-spaces $t = \text{constant}$, whose proper radius may or may not change with time, depending on the model, and (ii) as the corresponding hypersurface (a pseudo cylinder) in space-time. Where a distinction is necessary, the former might be called the space event-horizon, and the latter the space-time event-horizon.

Events occurring beyond this horizon are evidently for ever outside the possible powers of observation of the origin-observer A. On the other hand, *particles that have at some time been visible to A remain so for ever*. For, by equation (5), light emitted at time t_1 from a particle with coordinate r_1 reaches the origin at time t given by

$$\sigma(r_1) = \int_{t_1}^t \frac{c\,dt}{R(t)}.$$

If for some particular value $t = t_0$ this equation for t_1 furnishes a solution then it furnishes a solution for any $t > t_0$. Thus there is then a signal from the particle that arrives at the origin at any specified later time t. As $t \to \infty$, t_1 approaches a limiting value which evidently is the time at which the particle crosses A's horizon. Hence, although no particle can "pass out of view", its history as observed by A becomes more and more dilated, the event of its crossing the horizon being visible to A only in the infinite future, and all events at the particle after that event will never be visible to A. Again, *all fundamental particles other than A itself that are at some time within A's event-horizon, if such a horizon exists, must eventually pass beyond this horizon at the speed of light as measured locally.* These statements follow easily from equations (4) and (7), but they can also be seen in the following way. Consider any line of vision originating from a particle A and let B be another particle on this line, *within* A's horizon at some given time t. Then the proper distance of A's horizon-point P from A is the same as that of B's horizon-point Q from B, both distances being given by (7). Thus Q lies beyond P, whence P is a photon within B's horizon moving towards B, and will therefore reach B at a finite time. Moreover, like all other photons, it will reach B with the speed of light. This means that B will pass A's horizon-point, and pass it with the speed of light.

We may note that a contracting model-universe cannot have an event-horizon, since for the convergence of $\int^{\infty} \frac{dt}{R(t)}$ it is necessary that $R(t)$ increase to infinity. Models oscillating between singular states must logically be treated as existing only for a finite time (the time of one oscillation) and so the concept of an event-horizon would appear to be irrelevant to them.

4. *Examples of models possessing an event-horizon.*—As an important first example we consider the de Sitter model-universe. In this model $R(t)=e^{t/T}$, where T is the reciprocal of Hubble's coefficient and has the dimensions of time, and $k=0$. The model evidently satisfies the condition given at the beginning of Section 3 for the existence of an event-horizon. From (4) the equations of motion of fundamental particles in the de Sitter model are

$$l=\sigma(r_1)e^{t/T}, \tag{8}$$

which correspond to similar exponential curves on an l–t diagram (see Fig. 1) referring to a single semi-infinite line of vision. All these curves have the t-axis as asymptote. From (5) we find that the equations of motion of photons moving towards the origin are given by

$$l=cT+\gamma e^{t/T}, \quad \gamma=\alpha(r_1)-cTe^{-t_1/T}. \tag{9}$$

These equations also correspond to similar exponential curves, but they have the line $l=cT$ for asymptote. They lie above or below this asymptote according as γ is positive or negative. (There is no contradication in the fact that a photon travels towards A, yet its proper distance from A increases: cf. Eddington's runner mentioned in the Introduction.) It is evident that the line $l=cT$ represents

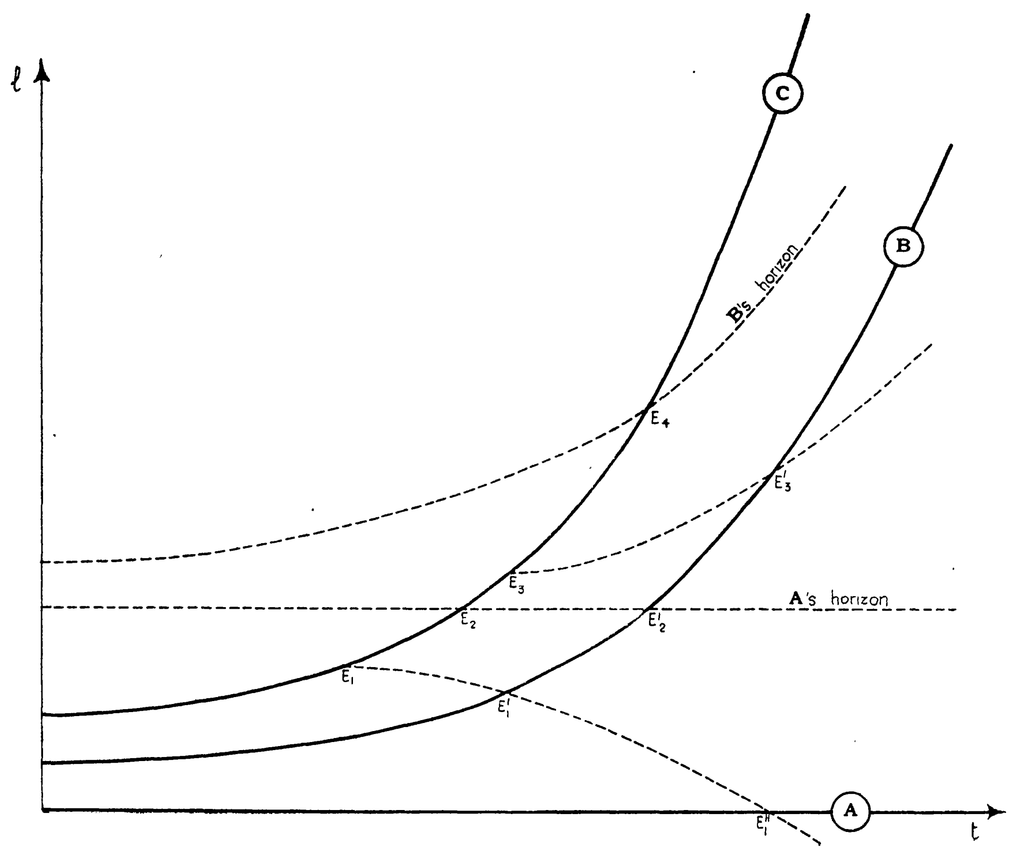

FIG. 1.

the event-horizon, and this naturally agrees with (7). In Fig. 1 only three particle paths have been drawn (full lines)—those of the origin-particle A, and of two other particles, B and C. The stippled curves represent light-paths corresponding to photons travelling towards the origin and two of these, in particular, are the paths of the horizon-points of A and B. (It can easily be verified, and indeed it is obvious from symmetry considerations, that light-paths directed *away* from the origin-particle correspond to similar exponential curves with the line $l=-cT$ for asymptote. However, none of these has been marked in the diagram.) E_1, E_2, E_3, E_4 are events, in that order, at C. E_1 is observed at B at E_1' and at A at E_1''. E_2 is the event of C's crossing A's horizon, and this is observed at B at E_2', but the corresponding event E_2'' at A occurs in the infinite future. E_3, which occurs beyond A's horizon but within B's, will be observed at B at E_3' but never at A. Events after E_4 (the crossing of B's horizon by C) are unobservable also at B.

The same diagram can be used to illustrate the significance of a horizon in models with positive space-curvature ($k=1$) which present some complications as was pointed out at the end of Section 2. Let us consider a model which has $R(t)=e^{t/T}$ and $k=1$. The above calculations are equally valid for this model, only the interpretation differs somewhat. We shall now suppose that A, B and C are representations of the same particle (at $\sigma=0$, 2π, 4π respectively). Thus, for example, the segment E_1, E_1', E_1'' represents a light-track going twice round the universe. If E_1' and E_3 are the same event, then so are E_1'' and E_3'. E_2' and E_4 are evidently identical. All events occurring before t_2' (the time of E_2') are effectively observable at A, that is, at A or B or C etc. At time t_2' each particle crosses its own horizon for the last time. Before t_2' the expansion of the model is such that each photon is able to make at least one complete circuit of the universe, but for photons emitted after t_2' no more complete circuits are possible. The domain of unobservable events for A is thus not given by $l>cT$ alone but by $l>cT$ and $t>t_2'$.*

As a second example we consider the class of models characterised by $R(t)=at^n$, where a is a constant. This class contains several well-known models. For example, the Einstein–de Sitter model with $n=2/3$, Dirac's model with $n=1/3$, Milne's "uniformly expanding" model with $n=1$ and Page's "uniformly accelerating" model with $n=2$ (**13**). Evidently the necessary and sufficient condition for this type of model to have an event-horizon is $n>1$. Consequently the Einstein–de Sitter model, Dirac's model and Milne's model do not have an event-horizon. From (7) the equation of the horizon, when it exists, is

$$l=\frac{ct}{n-1}, \qquad (10)$$

which shows that the horizon expands uniformly in all cases. For illustration we may choose the typical case of Page's model which has $R(t)=at^2$. From (4) the particle paths are $l=a\sigma(r_1)t^2$ and from (5) the light-paths are $l=\gamma t^2+ct$, where $\gamma=a(\sigma(r_1)-c/t_1)$. On an l–t diagram the former are parabolas touching the t-axis at the origin, whereas the latter are parabolas passing through the origin with gradient c and curving away from or towards the t-axis according as γ is

* Events unobservable from this one direction may, however, be observable from the opposite direction. Only after A's antipode $\bar{A}(\sigma=\pi)$ crosses A's horizon for the last time, does there develop around $\bar{A}$ an ever-growing region whose events are truly unobservable by A.

positive or negative. The line $l=ct$ corresponding to $\gamma=0$ divides these two classes and thus evidently represents the event-horizon of the origin-observer, which is in agreement with (10).

5. *The particle-horizon.*—The necessary and sufficient condition for a particle-horizon to exist in a given model is the convergence of the integral $\int_0 \frac{dt}{R(t)}$, or of $\int_{-\infty} \frac{dt}{R(t)}$ in cases where the definition of $R(t)$ extends to negatively unbounded values of t. The analysis is quite similar in the two cases and we therefore assume the former condition to obtain, since the latter implies the less probable situation of a universe contracting from a state of infinite expansion. It then follows from (5) that at any given time t_0 all those particles for which $\sigma > \int_0^{t_0} \frac{c\,dt}{R(t)}$ have not yet been observed by the origin-observer A, whereas all others have. Hence the surface

$$\sigma = \int_0^{t_0} \frac{c\,dt}{R(t)} = \phi(t_0), \tag{11}$$

where ϕ is defined by this equation, divides all particles into two non-empty classes: those that have been observed by A at or before t_0, and those that have not. This surface we call A's particle-horizon at time t_0. It is evidently a geodesic sphere in the observer's instantaneous 3-space $t=t_0$ and its radius may or may not change with time, depending on the model. Occasionally we shall wish to regard the particle-horizon as the section $t=t_0$ of the hypersurface

$$\sigma = \phi(t) \tag{12}$$

in space-time, which might be called A's space-time particle-horizon.

As $R(t)$ is by its nature positive and finite, $\phi(t)$ is an increasing function of t when it exists, and thus, by (11), *more and more particles become visible to A as time goes on*, since σ is a "co-moving" coordinate. Whether *every* particle eventually becomes visible to A or not depends on whether $\phi(t)$ tends to infinity with t or not. If $\phi(t)$ approaches a finite limit then all those particles for which $\sigma > \int_0^{\infty} \frac{c\,dt}{R(t)}$ are entirely outside A's possible powers of observation and the model evidently possesses an event-horizon in addition to its particle-horizon. In any case, as before, a particle once seen remains for ever visible. Each particle is first seen at its "birth" or, in other words, the first signal received from the particle in question was emitted there at $t=0$ (or $t=-\infty$, as the case may be). However, in models which postulate a unique creation event in the finite past ($R(0)=0$) each particle comes into view with an infinite Doppler ratio. For the latter is given by the well-known formula (14)

$$\frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)},$$

where λ_1, t_1 and λ_0, t_0 are the wave-lengths and times of emission and reception, respectively, of the light-signal whereby the particle is observed, and in our case $R(t_1)=R(0)=0$. This would, in practice, prevent the observation of the actual creation process, which was one of the objections Milne (*loc. cit.*) made to models possessing a particle-horizon. On the other hand, in models which contract from infinite extension in the infinite past and possess a particle-horizon (e.g. a model with $R(t)=e^{-t/T}$), particles first come into view with zero wave-length!

The unique creation event (point-creation) is a singularity in all models in which it occurs, in the following sense: there is one particle world-line, and only one, through each event except the creation event, through which there passes an infinity of such world-lines. But in models with point-creation and a particle-horizon, the creation event is a singularity in yet another sense: each event determines a unique light-cone with vertex at that event except the creation event at which there originates an infinity of different light-cones, one corresponding to each particle. For consider the equation

$$l = R(t)\left\{\sigma(r_1) \pm \int_{t_1}^{t} \frac{c\,dt}{R(t)}\right\}, \tag{13}$$

which is obtained from (2) in the same way as (5) but with the retention of both signs. This is an equation of the light-cone whose vertex is at a particle characterized by r_1 at time t_1. Now if $\int_0 \frac{dt}{R(t)}$ is divergent, the light-cone of the creation event ($t_1 = 0$) has equation $t = 0$, the same for all particles. But if this integral is convergent the light-cone of the creation event is evidently not unique but depends on which particle emits the light.

Let us consider now the $l-t$ equation of the space-time particle-horizon. We obtain this by multiplying (12) by $R(t)$ and using (11),

$$l = R(t)\int_0^t \frac{c\,dt}{R(t)}. \tag{14}$$

Comparison of (14) with (13) shows that *the space-time particle-horizon of any observer is the boundary of his creation-light-cone.* Thus a particle B becomes visible to the particle-observer A the moment it enters A's creation-light-cone. This is not surprising, for, by the symmetry that obtains between the two particles, A at that same instant enters B's creation-light-cone. Fig. 2 illustrates this situation.

In the same way that we identified the event-horizon with a geodeso-spherical light-front converging on the observer and reaching him at time $t = \infty$, so from our present discussion of the particle-horizon we see that this latter can be identified with a geodeso-spherical light-front diverging from and emitted by the observer at time $t = 0$ (or at $t = -\infty$, as the case may be), being the section t = constant of the space-time particle-horizon. It is therefore evident that fundamental particles entering this horizon do so at the speed of light, as measured locally. This is why the particle-horizon has sometimes been described, perhaps misleadingly, as expanding with the speed of light. Its proper distance from the observer, given by (14) or the corresponding formula with an integral from $-\infty$ to t, may remain constant or even decrease.

6. *Examples of models possessing a particle-horizon.*—Of the class of models characterized by $R(t) = at^n$, where a is a constant, evidently only those models which have $n < 1$ possess a particle-horizon which, by (14), has equation

$$l = \frac{ct}{1-n}. \tag{15}$$

Thus, in particular, Milne's uniformly expanding model ($n = 1$) and Page's uniformly accelerated model ($n = 2$) do not possess a particle-horizon, whereas Dirac's model ($n = 1/3$) and the Einstein–de Sitter model ($n = 2/3$) do possess one. I shall choose this latter model for illustration since it is typical. From (4) we obtain the equations of the particle paths, $l = a\sigma(r_1)t^{2/3}$, and from (13) we find the equations of the paths of photons emitted at an arbitrary particle with coordinate r_1 at the creation event ($t_1 = 0$) and directed towards and away from the origin,

$$l = t^{2/3}\{a\sigma(r_1) \pm 3ct^{1/3}\}. \tag{16}$$

In Fig. 2 the world-lines of only two particles (one of which is taken to be the origin-particle) are shown, together with their corresponding creation-light-cones. We note that these cones have no overlap near their common vertex, and this is typical for models with a particle-horizon. At time t_0 each of the two particles enters the creation-light-cone of the other, and thus they become visible to each other for the first time.

It may be of interest to quote Milne's description (*loc. cit.*) of the horizon in the Einstein–de Sitter model: "The frontier...of observability moves onward with the speed of light; it contains always the particles just being created and it leaves in its wake a spray of decelerating newly created particles." Although Milne's deliberately absurd deductions can to some extent be blamed on the subjective coordinates he used, certain physical difficulties seem to be inherent in models possessing a particle-horizon: if the model postulates point-creation we have material particles initially separating at speeds exceeding those of photons; if the model postulates collapse from infinite rarification there is the extraordinary phenomenon, noted above, of particles coming into view with zero wave-length; lastly, models postulating a beginning at a finite epoch with finite separation between particles, whilst possessing neither of these crass difficulties, seem hardly less artificial.

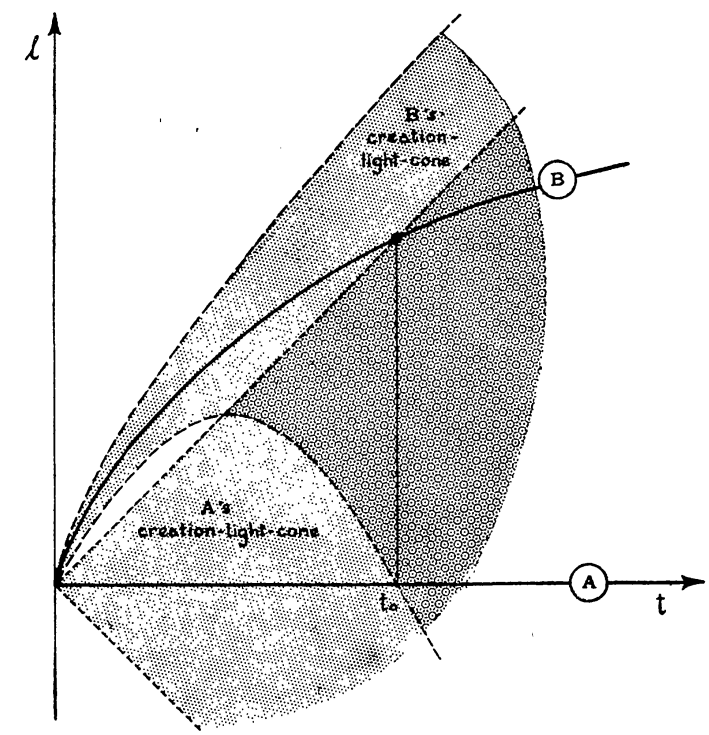

FIG. 2.

As an example of a model-universe possessing both a particle-horizon *and* an event-horizon I shall choose that General Relativity model which is characterized by $k = 0$ and a positive cosmological constant, and has

$$R(t) = a(\cosh bt - 1)^{1/3},$$

where a and b are constants (15). This model is somewhat similar in general behaviour to the better known Lemaître model, but has the advantage, for illustrative purposes, of possessing a simple functional form of $R(t)$. We need not here enter into the exact calculations for the light-paths, but we note that $R \sim t^{2/3}$ for small values of t, and $R \sim \exp(bt/3)$ for large values of t, whence the model is seen to approximate asymptotically to the Einstein–de Sitter model for small values of t and to the de Sitter model for large values of t. The properties of this model are illustrated schematically in Fig. 3. The origin-observer is denoted by A. B is an observer on a typical particle which becomes visible to A at creation-time t_1 (when A and B enter each other's creation-light-cones) and which passes beyond A's event-horizon at time t_2, so that events at B after t_2 are outside A's possible powers of observation. C is the critical particle which becomes visible to A only at $t = \infty$. C's creation-light-track towards A is that of the unique photon which reaches A at $t = \infty$, and which we have already identified with A's event-horizon. And in the same way that A approaches asymptotically the boundary of C's creation-light-cone, so C approaches that of A's creation-light-cone. Evidently all particles beyond C are entirely outside A's cognizance. In the diagram only the portions near the vertices of the creation-light-cones have been shaded in.

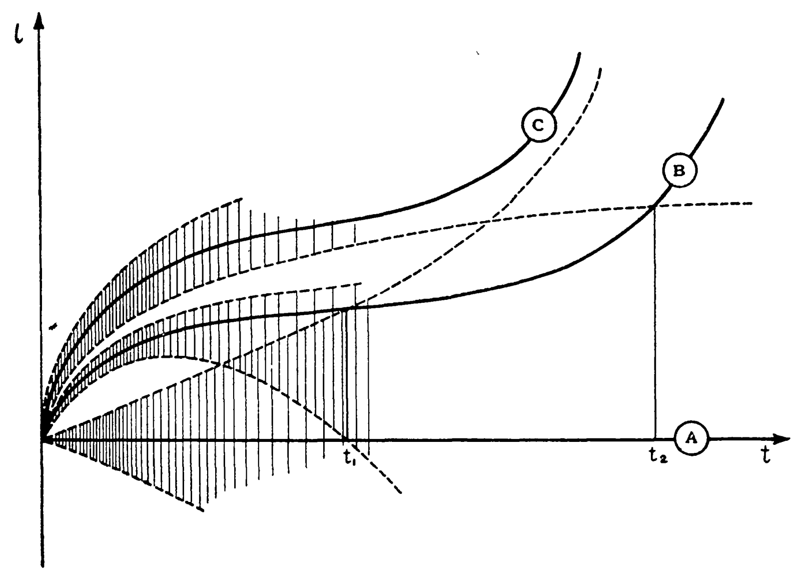

FIG. 3.

We may note that the existence of a critical particle with properties analogous to those of C above, one on each line of vision of each fundamental observer, is a general feature of all models possessing both types of horizon. From equations (4) and (5) we easily find that the σ-coordinate of this particle relative to the observer is given by

$$\sigma_c = \int_0^\infty \frac{c\,dt}{R(t)} \quad \left(\text{or} \quad \sigma_c = \int_{-\infty}^\infty \frac{c\,dt}{R(t)}, \text{ as the case may be}\right).$$

7. *Horizons for non-fundamental observers.*—In connection with the de Sitter model Eddington (16) made the assertion that "events before $t = -\infty$ (i.e. events outside the event-horizon) may produce consequences in the neighbourhood of the observer and he might even see them happening..." Schrödinger (17) in his recent book concurred with this assertion and amplified it to some

extent. Evidently the observer contemplated by these authors cannot be a fundamental observer. In defining the event-horizon of a particular fundamental observer in an arbitrary model we assumed that the observer in question remained attached to his original fundamental particle. If we discard this restriction the class of events observable by him is naturally increased, and we shall now briefly examine this new situation.

By a generally accepted relativistic principle an observer can theoretically reach and be present at any preassigned event within his forecone. This is merely equivalent to saying that matter can attain any velocity short of the velocity of light, both velocities being measured locally and with respect to the substratum. Thus if at some instant t_0 an observer can detach himself from his fundamental particle and journey with sufficient energy into space, thereby ceasing to be a fundamental observer at least for a certain period, all events within his forecone at t_0 become attainable to him and all events whose forecones intersect his own become observable by him. We shall now prove the following statements, in so far as they have not already been proved:

(i) If a model possesses no event-horizon then all events are observable sooner or later by any fundamental observer.

(ii) If a model possesses an event-horizon but no particle-horizon then any observer can be present at any one preassigned event provided he is willing if necessary to detach himself from his original fundamental particle and provided also he does so soon enough. Nevertheless, if two events are preassigned it will, in general, be impossible for any one observer either to be present at, or ever to observe, both events.

(iii) If a model possesses an event-horizon *and* a particle-horizon then for any observer originally attached to a given fundamental particle there exists a class of events absolutely beyond his cognizance, no matter how he journeys through space. We may here speak of an *absolute horizon.*

Statement (i) has been justified in Section 3. To prove the first part of (ii) we must show that any event E lies in at least one forecone of any given fundamental observer A. Let E occur at a fundamental particle B at cosmic time t_0. Then if E lies in none of A's forecones, A has not been observable at B at or before time t_0, and this contradicts our assumption that there is no particle-horizon. Thus E lies in one of A's forecones and the statement is proved. In support of the second part of (ii) we shall show that corresponding to any event E there exists a class S of events such that no event of S can be observed by any observer who ever observes E. Consider the closed light-front π converging on the fundamental particle B associated with E and at the time t_0 of E at proper distance $2\rho_0$ from B, where ρ_0 is the radius of the event-horizon at t_0. Any two photons emitted towards each other at t_0 a proper distance $2\rho_0$ apart will meet at $t=\infty$. Hence photons emitted at E meet π at $t=\infty$. This proves that the forecones of events occurring beyond π will not intersect the forecone of E. These events evidently constitute the class S. For an observer seeing an event must be situated in the forecone of that event and cannot ever get out of it again. We may note that, conversely, the forecones of all events occurring within π do intersect the forecone of E and hence any one of these events can be observed jointly with E.

Lastly we prove (iii). We remember that in any model possessing both types of horizon there exists on any line of vision of any given fundamental observer A

a critical particle C with coordinate σ_c which enters A's creation-light-cone at $t=\infty$ (cf. Section 6). Consider now the particle P with coordinate $2\sigma_c$ on the same line of vision. Clearly C enters P's creation-light-cone also at $t=\infty$ and therefore the creation-light-cones of A and P intersect at $t=\infty$. Thus it is impossible for A ever to observe the particle P at a finite time, even if he detaches himself from his original fundamental particle. He can therefore have absolutely no cognizance of events occurring at P and, *a fortiori*, of events occurring beyond P. All events occurring nearer than P can, on the other hand, be observed by A provided he can if necessary detach himself from his fundamental particle and journey within his creation-light-cone. Thus the path of P, which has equation

$$l=2\sigma_c R(t), \tag{17}$$

constitutes an absolute horizon.

8. *Effects of reversing the direction of time.*—We have noted the phenomenon that in certain models (those possessing a particle-horizon) fundamental particles suddenly come into the view of fundamental observers at a finite time. We have also noted that the reverse phenomenon, viz. particles disappearing from view at a finite time, cannot occur under any circumstances. A very loose argument might seem to suggest that if we reverse the direction of time in any model with a particle-horizon we ought to obtain a model in which particles disappear from view. Although this is by no means the case it is of some interest briefly to consider such time reversals, especially in view of the fact that in some models (e.g. all the cosmological models of General Relativity) the direction of time can be reversed without violating the hypotheses on which the model is constructed. In any case there is nothing to prevent us from contemplating the dual of any given model formed in this way. The one result that is of interest in this connection is that an event-horizon transforms into a particle-horizon and vice versa.

Let us first consider a model with a scale-function $R(t)$ defined over the whole range $-\infty<t<\infty$ and possessing an event-horizon whose equation will be, as we saw in (7),

$$l=R(t)\int_t^\infty \frac{c\,dt}{R(t)}. \tag{18}$$

The dual of this model has the scale-function $R(-t)$ and thus possesses a particle-horizon whose equation, by analogy with (14), is given by

$$l=R(-t)\int_{-\infty}^t \frac{c\,dt}{R(-t)}. \tag{19}$$

This equation also results if we change the free variable t in (18) into $-t$ and thus our assertion is proved. Let us also consider a model with point-creation in the finite past and a particle-horizon. On time reversal the point-creation event transforms into a point-annihilation event in the finite future. The particle-horizon transforms into an event-horizon in the sense that events occurring beyond it will not be observed in the finite stretch of time left to the observer before annihilation. With similar modifications our result applies in all cases.

We may also note that in the dual model all Doppler ratios are inverted. If a light-signal from an event E_1 to an event E_2 in the original model exhibits a Doppler ratio $D=\frac{\lambda_2}{\lambda_1}$, where λ_1 and λ_2 are the wave-lengths as measured at E_1

and E_2 respectively, then the corresponding signal from E_2 to E_1 in the dual model exhibits a Doppler ratio $1/D$. This is easily seen by considering two successive light-signals whose time difference is dt_1 at E_1 and dt_2 at E_2. Then the Doppler ratios in the original model and the dual are $\frac{dt_2}{dt_1}$ and $\frac{dt_1}{dt_2}$ respectively.

9. *General time transformations.*—It is one of the achievements of the theory of Kinematic Relativity that it brought into prominence the relativity of clock rates. No one clock rate is any longer regarded as absolute. Cosmological theories other than steady state theories must recognize the possibility that even different physical clocks, e.g. "atomic" clocks and "dynamic" clocks, though synchronous momentarily will not necessarily go on keeping the same time indefinitely. Again, Kinematic Relativity showed how it was possible to adopt a conventionally constant speed of light by linking the unit of distance to the unit of time. Thus the only standard unit required is the unit of time. If we change from one standard clock to another whose rate, let us assume, accelerates relative to the first, then the corresponding new unit of length decreases relative to the old one. Distances which in our first reckoning remained constant will now be regarded as increasing with time. Thus by means of a suitable time transformation it is possible to transform a static model-universe into an expanding one, or indeed to transform a model with any given rate of expansion into one with any other given rate of expansion. The detailed formulation and proof of this result for models described as "kinematic equivalences" is sometimes referred to as the "main theorem on equivalences" (**18**). I have shown elsewhere (**13**) that this theorem applies equally to models described in terms of the metric (1). The question arises whether, if no one clock rate can be regarded as absolute, all models are indeed equivalent to each other. This is an impression one sometimes gains from Milne's writings.

There are physical objections to this view: the clocks implicit in our empirical work on the expansion of the universe are atomic clocks and not arbitrary clocks (**13**) and the model we seek is unique. However, even theoretically there is a limitation to the formal equivalence of different models, and horizons provide the clue to this limitation. Eddington (**19**) considered an essentially similar problem and may even have anticipated Milne in his interrelation of the units of time and distance. Eddington concluded that a reduction of the de Sitter model to rest by means of a time transformation leads to an absurdity, namely the sudden arrest of all proper motions at a finite future time. We can now say that this result is characteristic of all models possessing an event-horizon. For the clocks that give rise to the static description of the model can be identified with rays of light oscillating between neighbouring fundamental particles, successive reflections defining equal units of time. We know that if the model possesses an event-horizon there will come a time when the signal can no longer pass from one of these particles to the other. Evidently each photon in the static description of the model moves with uniform speed towards a limiting particle and then stops dead. All other proper motions must cease at that instant also. Hence models with event-horizon when transformed to the static form exhibit properties startlingly different from those of genuinely static models which exist indefinitely. Evidently no model with an event-horizon can be equivalent to one without, and a similar remark holds for models with particle-horizons. Transformations of one of these types of model into another are possible only partially, a fact already noted by Milne and Whitrow (6) in connection with Page's and Milne's models.*

10. *The boundary in Milne's model.*—Milne's model is often described in coordinates different from the cosmic time and proper distance employed here, namely as an expanding spherical aggregate of particles satisfying the postulates of homogeneity and isotropy and embedded in the space-time of the special theory of relativity. Each fundamental particle-observer may consider himself to be at the centre of the sphere, whose boundary expands with the speed of light and which therefore has radius ct at time t, in the coordinates of the special theory of relativity. It is well known that this description of the model is equivalent to its description by means of the metric (1) with $R(t)=ct$ and $k=-1$, and we saw that that model possesses neither an event-horizon nor a particle-horizon. Nevertheless, using the first description, it has seemed quite natural to Milne to call the boundary of the sphere a "horizon". What is the intrinsic significance of this "horizon" when we leave out of account the particular coordinates employed? It is simply that each observer considers the distances of fundamental particles from himself to possess an unattained upper bound. It appears to him at time t that this bound is at a distance $\frac{1}{2}ct$ from him (world-picture) but taking into account the finite velocity of light he calculates that the bound is actually at distance ct (world-map). In order to associate an intrinsic significance with the distance concept we note that in the special theory of relativity distance from parallax is identical with coordinate distance. Thus we can finally characterize the situation in Milne's model as follows: All fundamental particles are visible at all times and there exists at any time t a finite upper bound $\bar{P}(t)$ to the apparent distances by parallax of these particles. When we inquire whether this state of affairs is in any way peculiar to Milne's model we find that this is by no means the case.

The distance by parallax, P, is defined by the equation $P\,d\theta=dl$, where $d\theta$ and dl are measured at the observer and represent angular difference and distance respectively between two neighbouring rays diverging from the event in question. It can be shown (20) that in a model referred to the metric (1) the distance by parallax of an event occurring at coordinate r_1 and observed at time t_0 is given by

$$P=\frac{R(t_0)r_1}{1-\frac{1}{4}kr_1^2+R'(t_0)r_1/c}. \tag{20}$$

* The mathematical formulation is straightforward: to transform a model with metric (1) into another with the same curvature index k but a different scale-function, say $G(t)$ instead of $R(t)$, let us apply the clock transformation $t=g(T)$. The metric becomes

$$ds^2=(g'(T))^2\left\{c^2\,dT^2-\left(\frac{R(g(T))}{g'(T)}\right)^2 d\rho^2\right\},$$

where $d\rho^2$ is the { } of (1). The conformality factor is discarded and the Robertson–Walker metric of the new model is $ds^2=c^2\,dT^2-G^2(T)\,d\rho^2$, provided $\dfrac{R(g(T))}{g'(T)}=G(T)$, which gives the required $g(T)$. This equation also yields the relation

$$\int_{t_0}^{t_1}\frac{dt}{R(t)}=\int_{T_0}^{T_1}\frac{dT}{G(T)}$$

for corresponding time intervals t_1-t_0, T_1-T_0. Suppose the R model possesses an event-horizon so that the left integral converges for $t_1\to\infty$. If the G model does not possess such a horizon evidently $t_1=\infty$ transforms into a *finite* time T_1 and greater values of T do not correspond to any values of t. The situation is similar when only one of the models possesses a particle-horizon.

In Milne's model we therefore have

$$P = \frac{ct_0 r_1}{(1+\frac{1}{2}r_1)^2}, \tag{21}$$

whence P approaches the value $\bar{P}(t_0) = \frac{1}{2}ct_0$ as r_1 approaches its limiting value of 2 (for the most distant particles), which agrees with our previous assertion.

Let us next consider the general class of models characterized by

$$R(t) = ct^n \quad (n>1), \quad k = -1.$$

From (20) we find that, for this class,

$$\bar{P}(t_0) = \frac{ct_0{}^n}{1+nt_0{}^{n-1}}, \tag{22}$$

and thus all these models possess a Milne type of boundary which, moreover, to a first approximation expands uniformly. When we consider a model of the above general class but with $n<1$ we find the situation somewhat changed: although there still exists a finite upper bound to the distances by parallax of all visible particles, not all particles are visible at any given time, as we saw in Section 6.

We may note, finally, that the de Sitter model ($R(t) = e^{t/T}$, $k=0$) also possesses a Milne type of boundary with constant radius $\bar{P} = cT$, as is easily verified.

11. *Acknowledgments.*—Certain portions of this work were first presented as a chapter of a Ph.D. thesis submitted by me to the University of London in April 1956. In this connection I wish to thank my supervisor, Dr G. J. Whitrow, and my external examiner, Professor H. Bondi, for their interest and discussion. I also wish to express my thanks to Mrs Margaret Henderson for the patience and skill with which she executed the diagrams.

Department of Mathematics,
Cornell University, Ithaca, New York, U.S.A.:
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