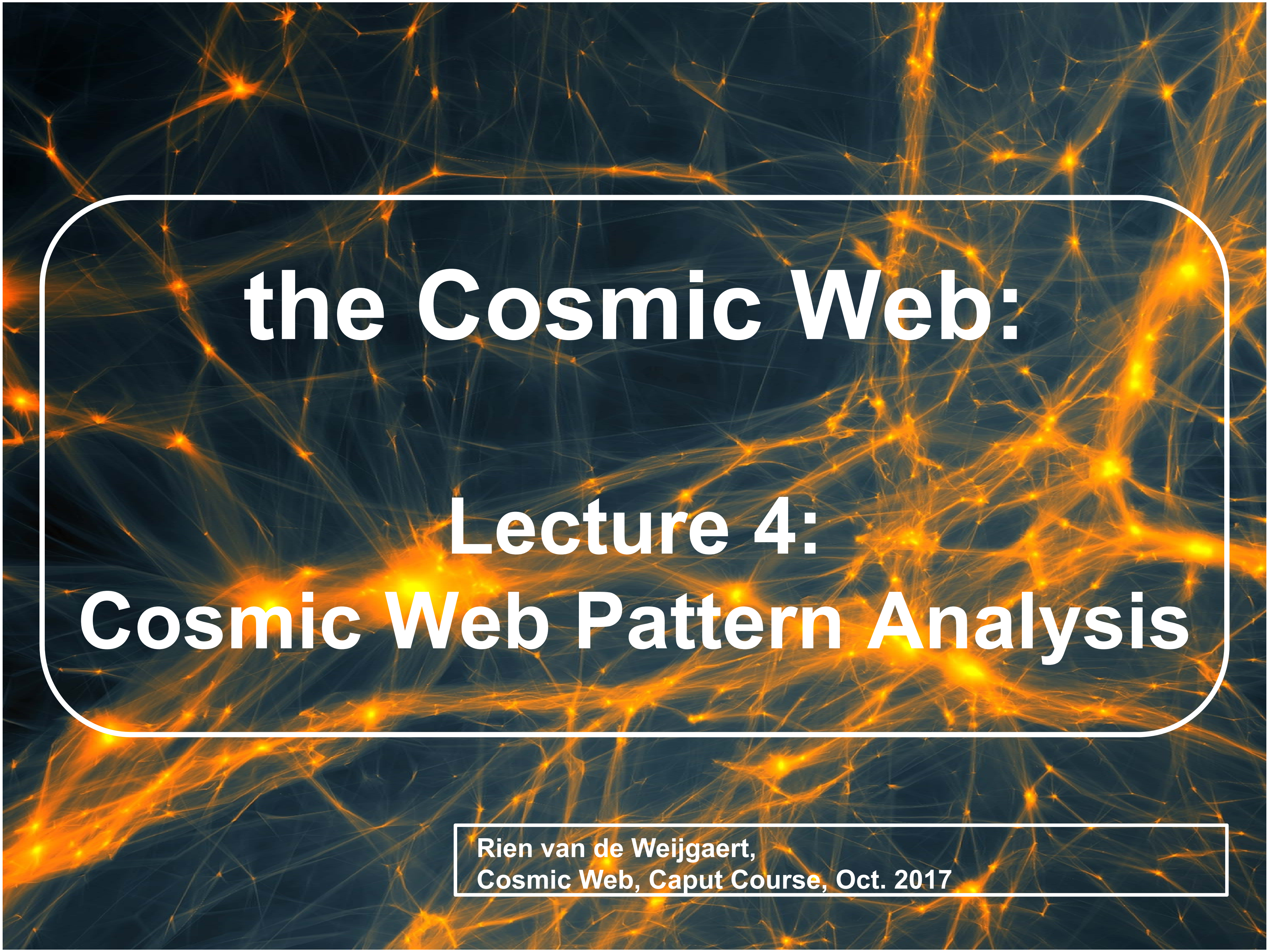


the Cosmic Web:

Lecture 4:
Cosmic Web Pattern Analysis

Rien van de Weijgaert,
Cosmic Web, Caput Course, Oct. 2017

Cosmic Structure Analysis

The
- overwhelming complexity of the individual structures,
- as well as their connectivity,
- the lack of structural symmetries,
- the intrinsic multiscale nature and
- the wide range of densities that one finds in the cosmic matter distribution

has prevented the use of simple and straightforward instruments.

To assess the key aspects of the nonlinear cosmic matter and galaxy distribution:

• multiscale character	⟷	hierarchical structure formation
• weblike network		anisotropic collapse
• volume dominance voids		asymmetry overdense vs. underdense

Despite the multitude of elaborate qualitative descriptions it has remained a major challenge to characterize the structure, geometry and topology of the Cosmic Web.

Quantities as basic and general as the mass and volume content of clusters, filaments, walls and voids are still not well established or defined. Since there is not yet a common framework to objectively define filaments and walls, the comparison of results of different studies concerning properties of the filamentary network -- such as their internal structure and dynamics, evolution in time, and connectivity properties -- is usually rendered cumbersome and/or difficult.

The overwhelming complexity of the individual structures as well as their connectivity, the lack of structural symmetries, its intrinsic multiscale nature and the wide range of densities that one finds in the cosmic matter distribution has prevented the use of simple and straightforward toolbox.

Over the years, a variety of heuristic measures were forwarded to analyze specific aspects of the spatial patterns in the large scale Universe. Only in recent years these have lead to a more solid and well-defined machinery for the description and quantitative analysis of the intricate and complex spatial patterns of the Cosmic Web.

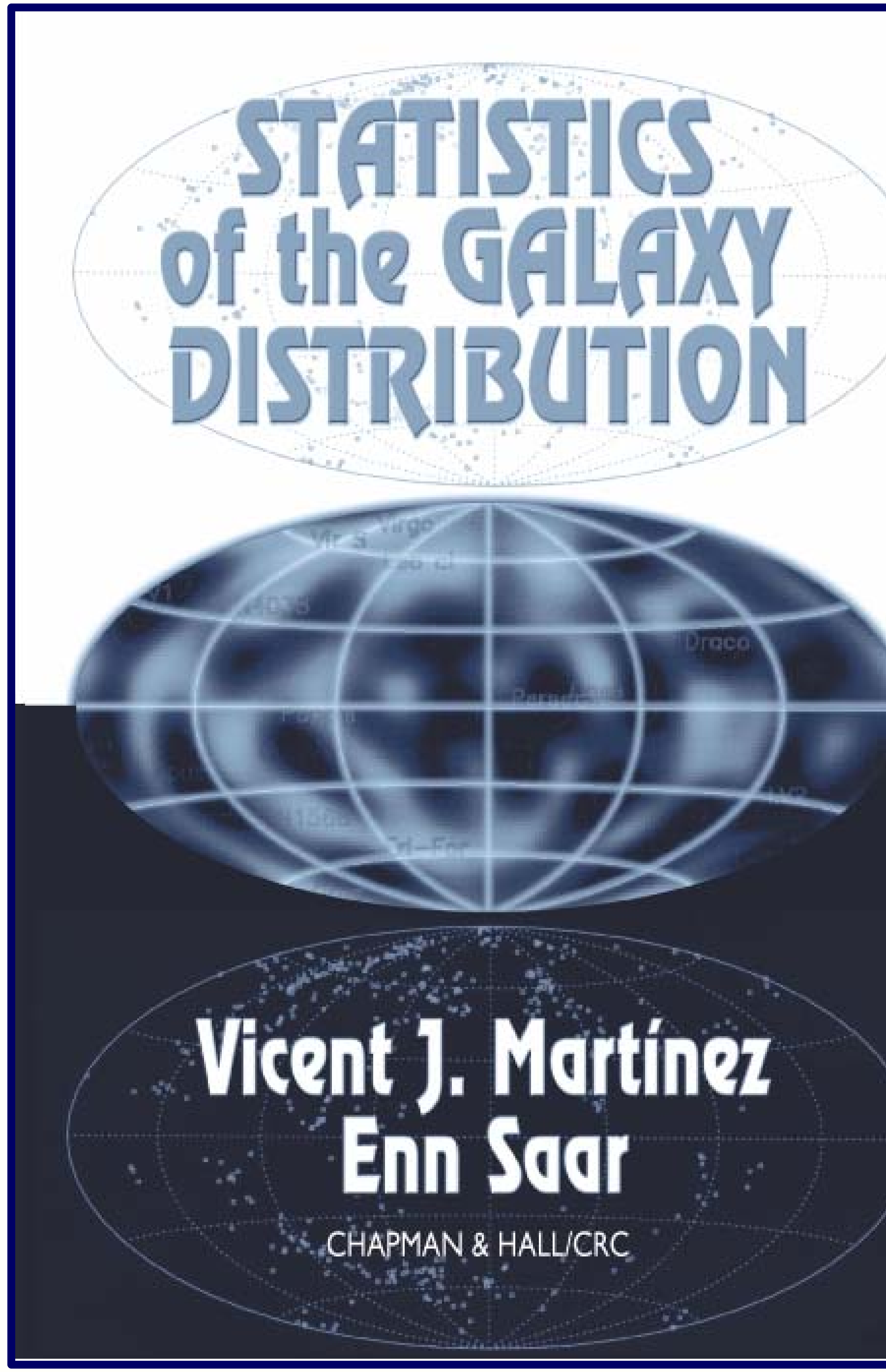
Nearly without exception, these methods borrow extensively from other branches of science such as image processing, mathematical morphology, computational geometry and medical imaging.

Structure Statistics:

Correlation Functions
Power Spectrum, et al.

Standard Reference:

Martinez & Saar



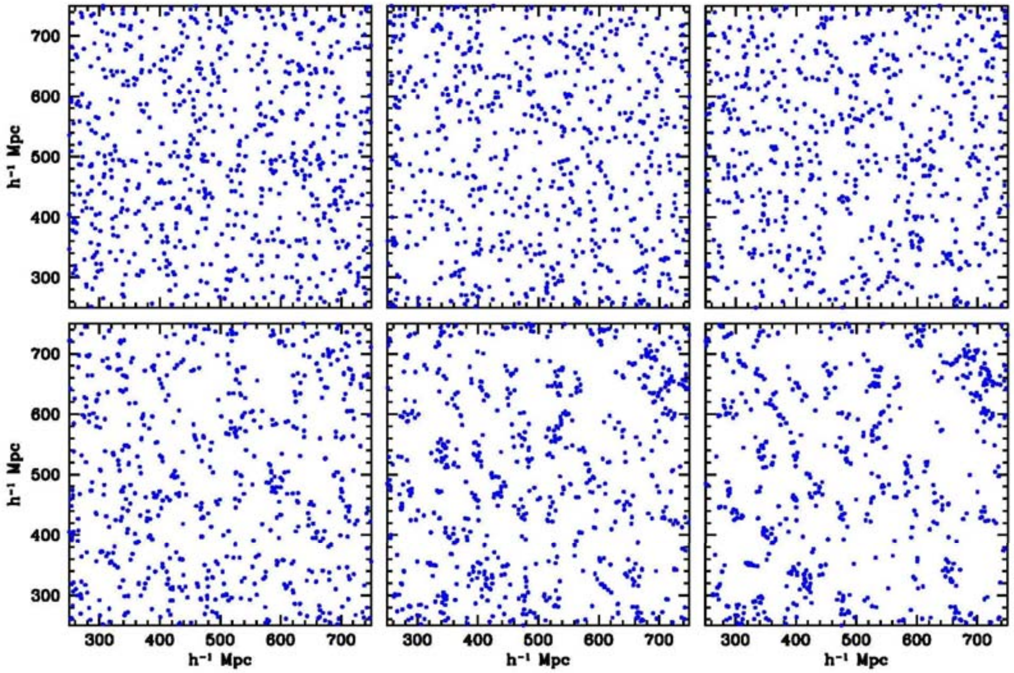
Ergodic Theorem

Ensemble Averages	$\Longleftrightarrow$	**Spatial Averages over one realization of random field**

- Basis for statistical analysis cosmological large scale structure

- In statistical mechanics Ergodic Hypothesis usually refers to time evolution of system, in cosmological applications to <u>spatial distribution</u> at one fixed time

Correlation Functions



Joint probability that
in each one of

the two infinitesimal volumes
dV_1 & dV_2,

at distance r,

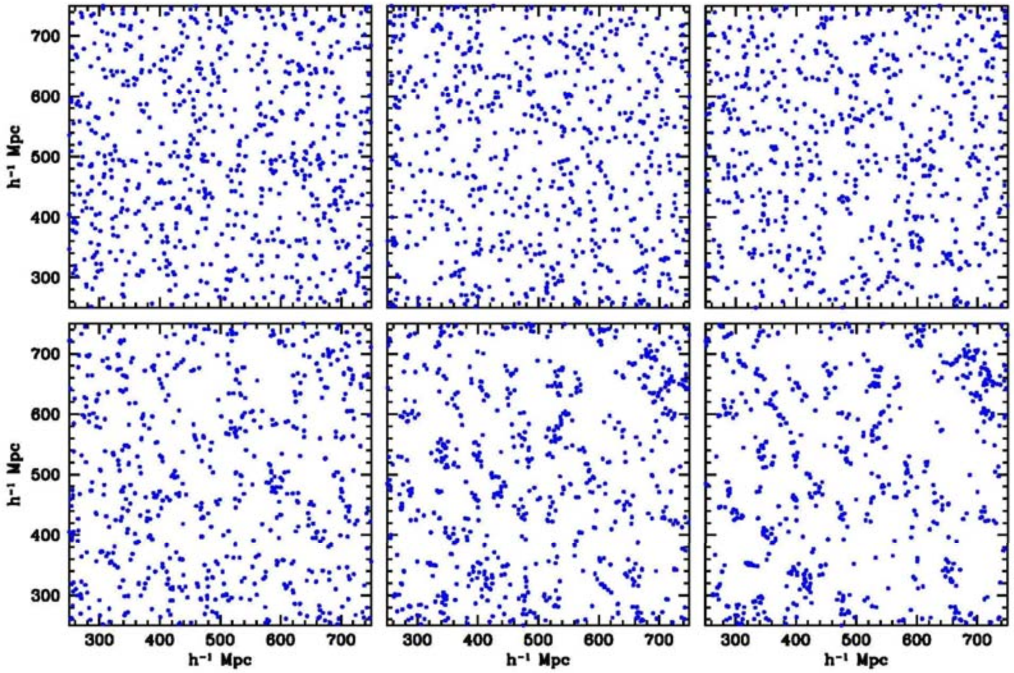
lies a galaxy

Infinitesimal Definition Two-Point Correlation Function:

$$dP(r) = \bar{n}^2 \, (1 + \xi(r)) \, dV_1 dV_2$$

mean density

Correlation Functions



In case of Homogeneous & Isotropic point process

then $\xi(\vec{r})$

only dependent on

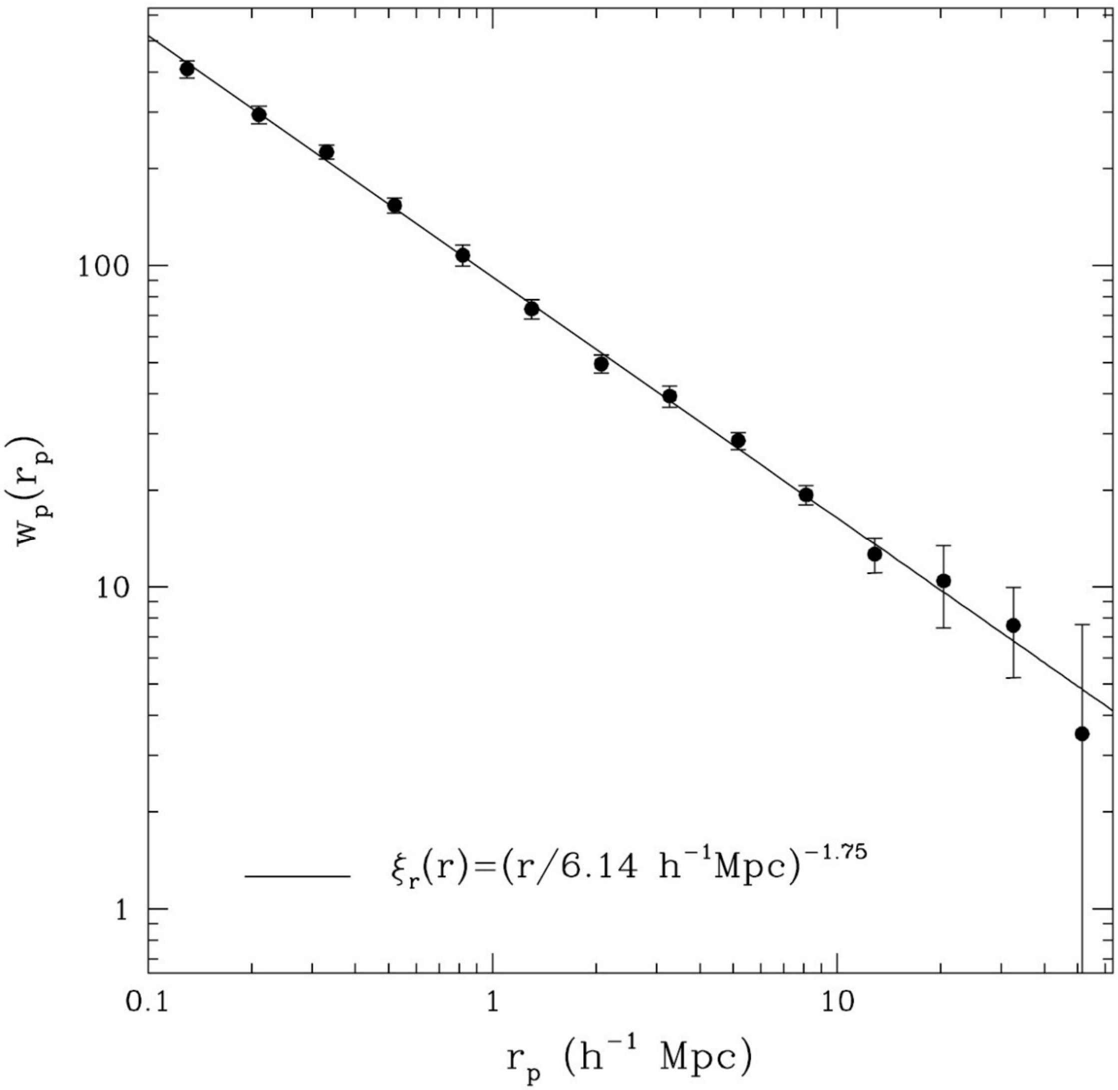
$|\vec{r}| = r$

Infinitesimal Definition Two-Point Correlation Function:

$$dP(r) = \bar{n}^2 \left(1 + \xi(r)\right) dV_1 dV_2$$

mean density

Power-law Correlations



$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma}$$

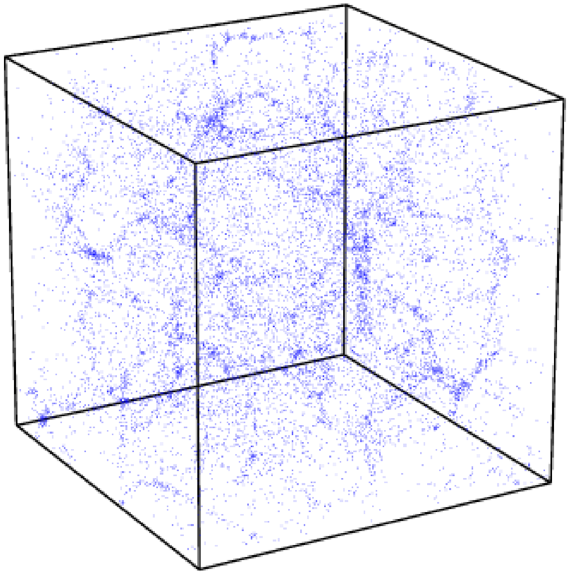
$$\gamma \approx 1.8$$

$$r_0 \approx 5h^{-1}Mpc$$

Totsuji & Kihara 1969

Peebles 1975, 1980, …

Correlation Functions



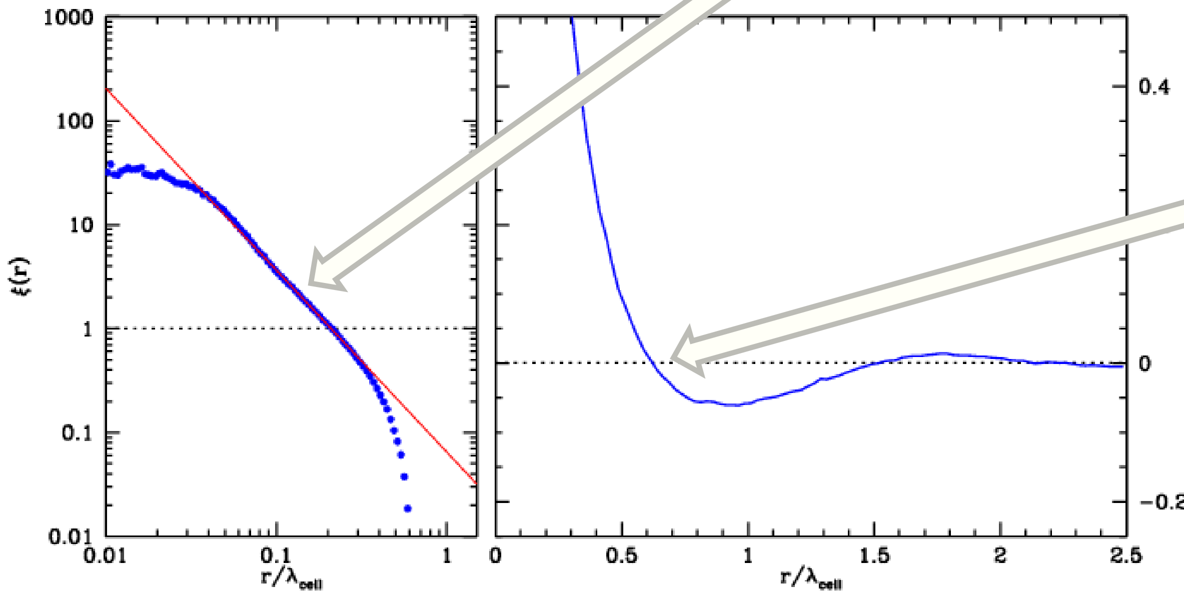
$$\xi_{cc}(r) = \left(\frac{r_o}{r}\right)^{\gamma}$$

$$\xi(r_o) = 1 .$$

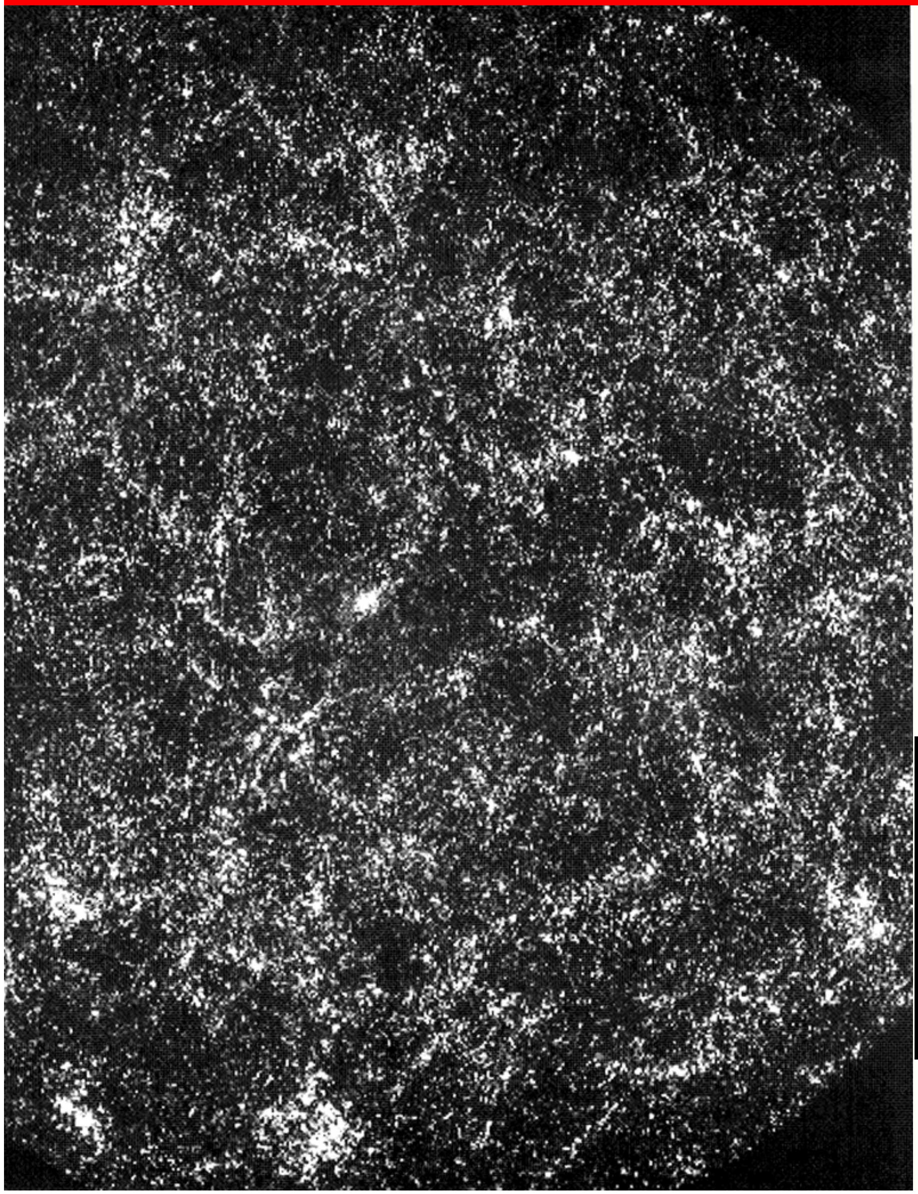
Clustering length/
"Correlation" length

Coherence length

$$\xi(r_a) = 0$$



Angular & Spatial Clustering



$$dP(\theta) = \overline{n}^2 \{1 + w(\theta)\} d\Omega_1 d\Omega_2$$

⇩

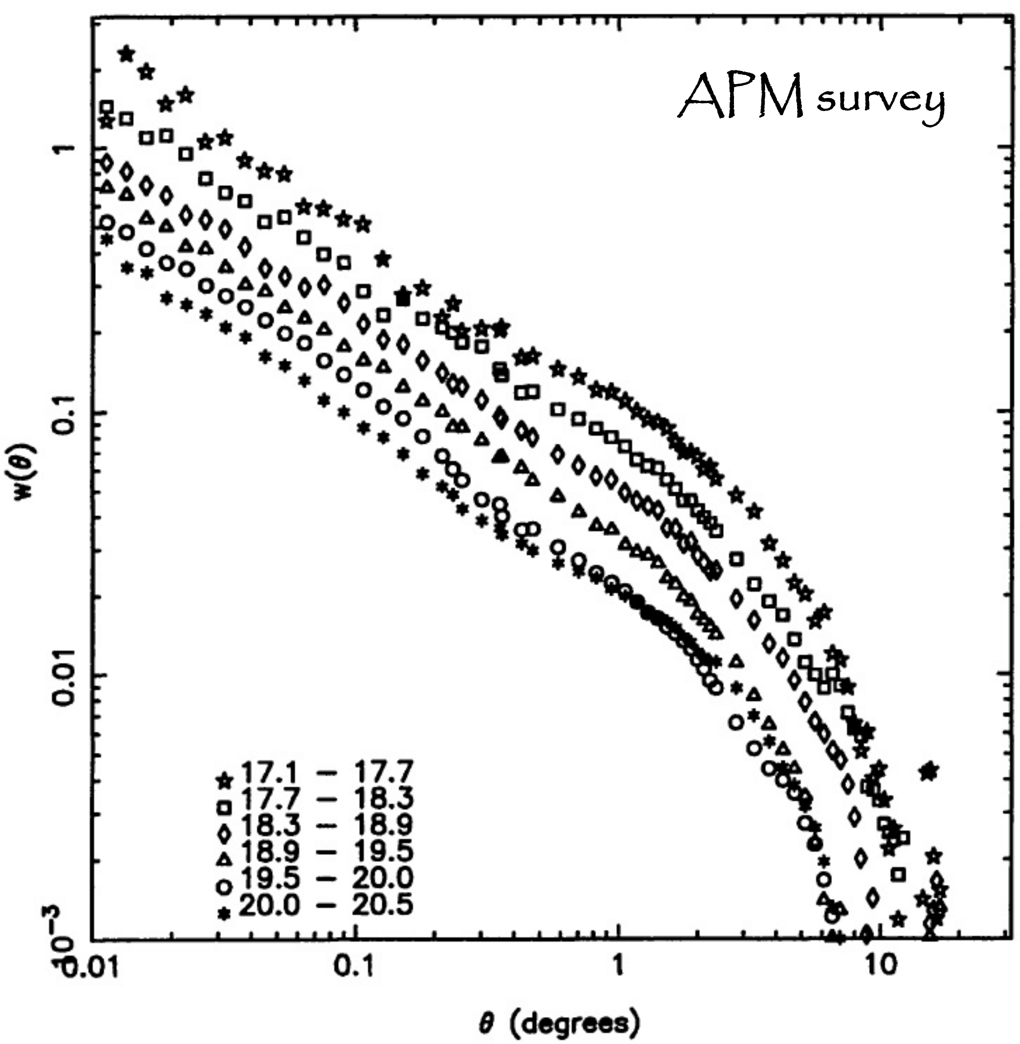
Two-point angular correlation function is the "projection" of $\xi(r)$

Limber's Equation:

$$w(\theta) = \frac{\iint p(\vec{x_1})\, p(\vec{x_2})\, x_1^2 x_2^2\, dx_1 dx_2\, \xi(|\vec{x_1} - \vec{x_2}|)}{\left[\int_0^\infty x^2 p(x)\, dx\right]^2}$$

p(x): survey selection function

Angular Clustering Scaling



Two-point correlation function:

- small angles: power-law

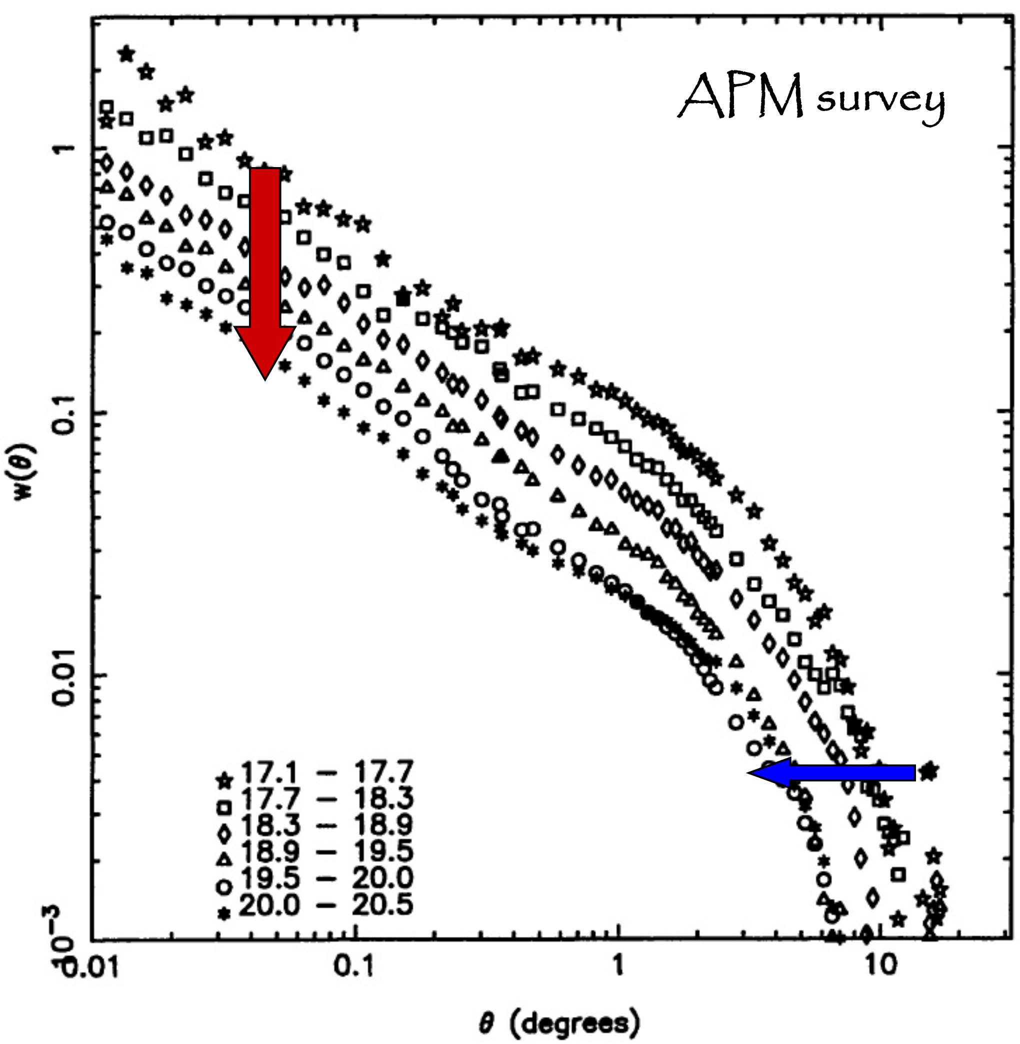
$$w(\theta) = \left(\frac{\theta_0}{\theta}\right)^{\gamma}$$

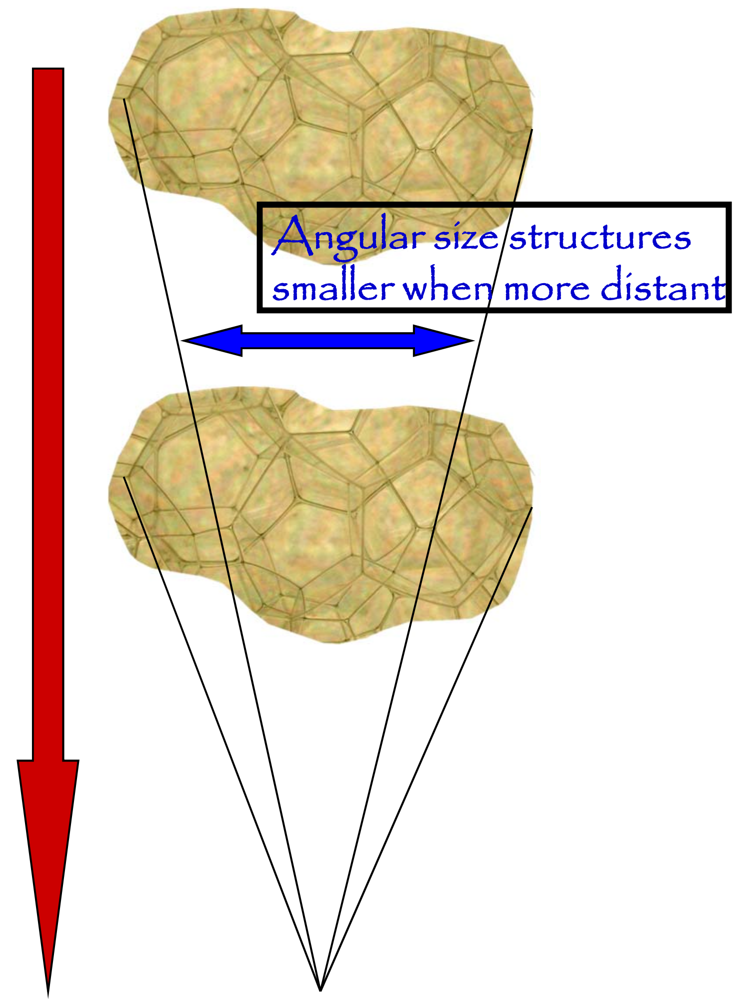
$$\gamma \approx 0.8$$

- large angles → 0

ie. to homogeneity

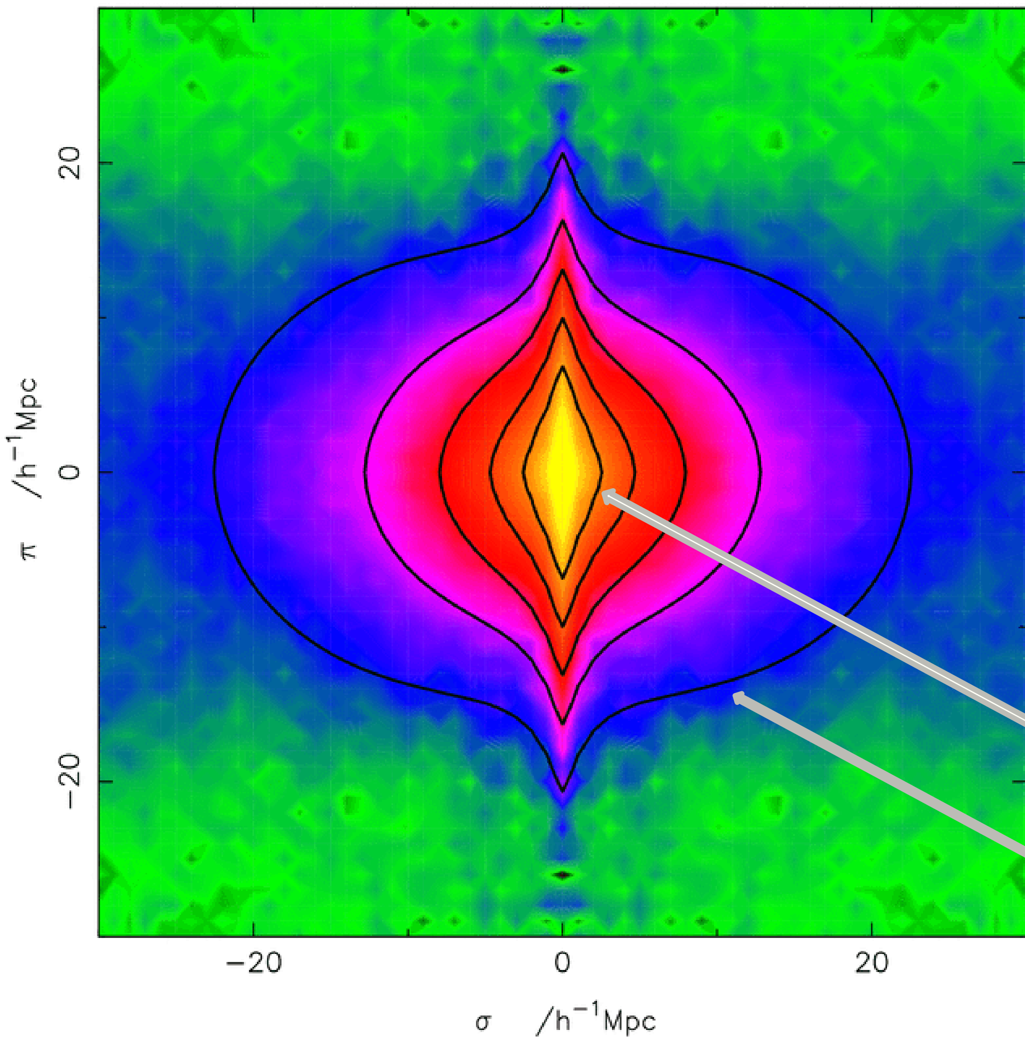
Angular Clustering Scaling





Projection of more layers leads to decreasing amplitude $w(\theta)$

sky-redshift space
2-pt correlation function ?(?,?)



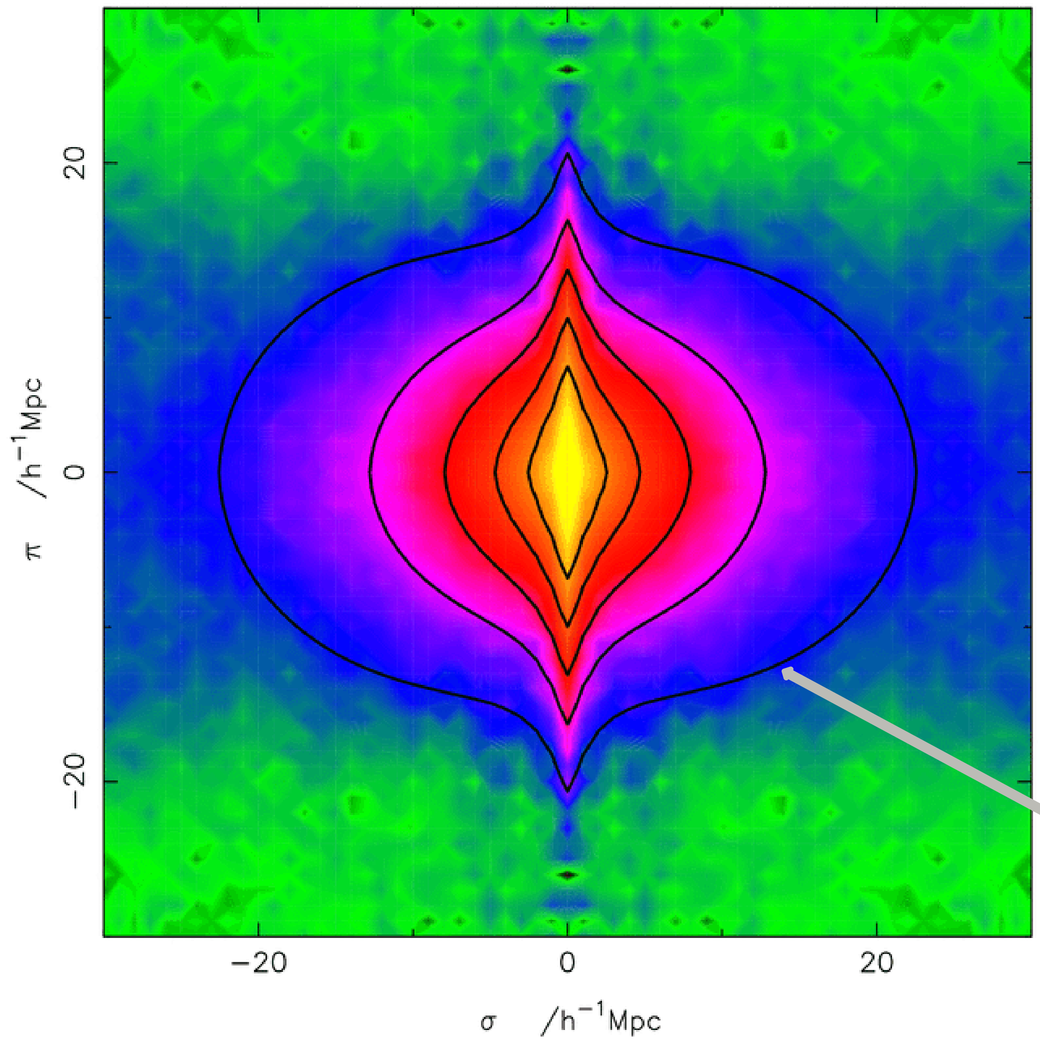
Correlation function determined in sky-redshift space:

$$\xi(\sigma,\pi)$$

sky position: $\sigma = (\alpha, \delta)$
redshift coordinate: $\pi = cz$

Close distances:
distortion due to non-linear Finger of God
Large distances:
distortions due to large-scale flows

Redshift Space Distortions Correlation Function



On average, $\xi_s(s)$ gets amplified wrt. $\xi_r(r)$

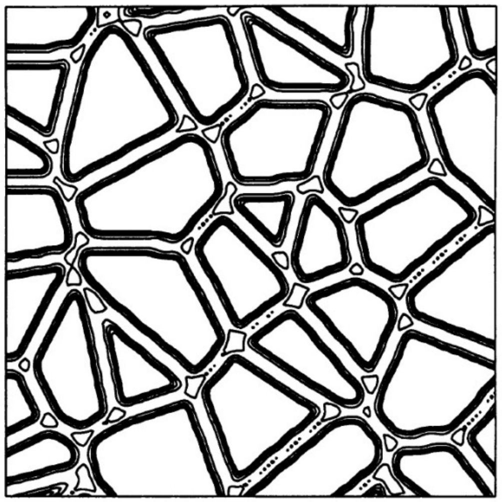
Linear perturbation theory (Kaiser 1987):

$$\xi_s(s) = (1 + \frac{2}{3}\Omega^{0.6} + \frac{1}{5}\Omega^{1.2})\xi_r(s)$$

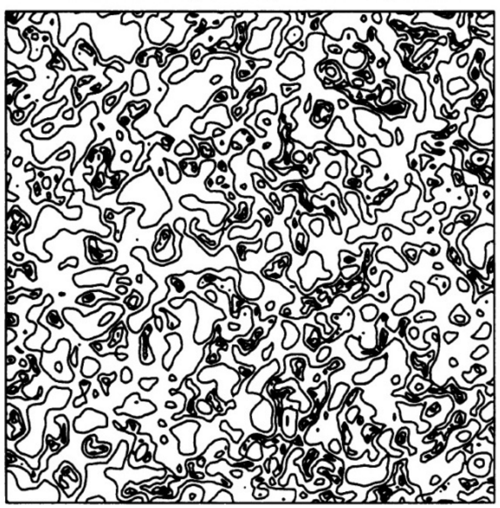
Large distances:
distortions due to large-scale flows

Structural Insensitivity

Voronoi foam, R=1.6, smoothed original



Voronoi foam, R=1.6, random phases



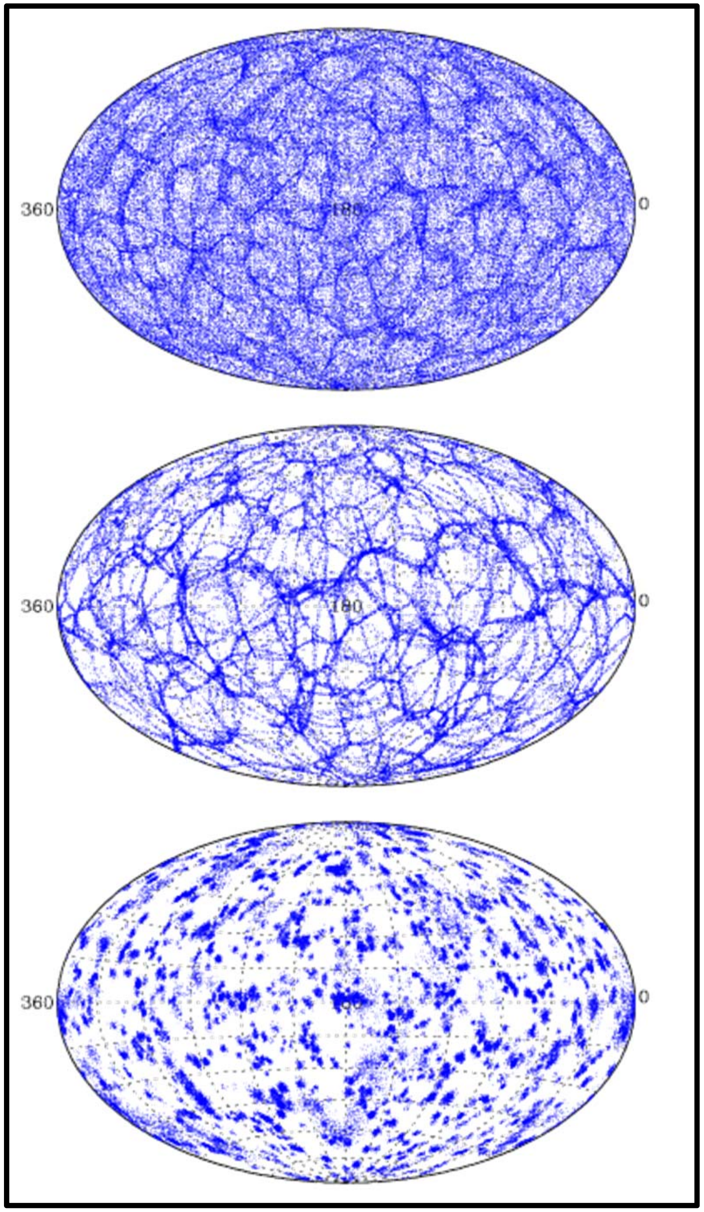
2-pt correlation function is
highly insensitive to the geometry & morphology
of weblike patterns:

compare 2 distributions with same
$\xi(r)$, cq. $P(k)$,
but totally different phase distribution

In practice, some sensitivity in terms of distinction
Field, Filamentary, Wall-like and Cluster-dominated
distributions:

because of different fractal dimensions

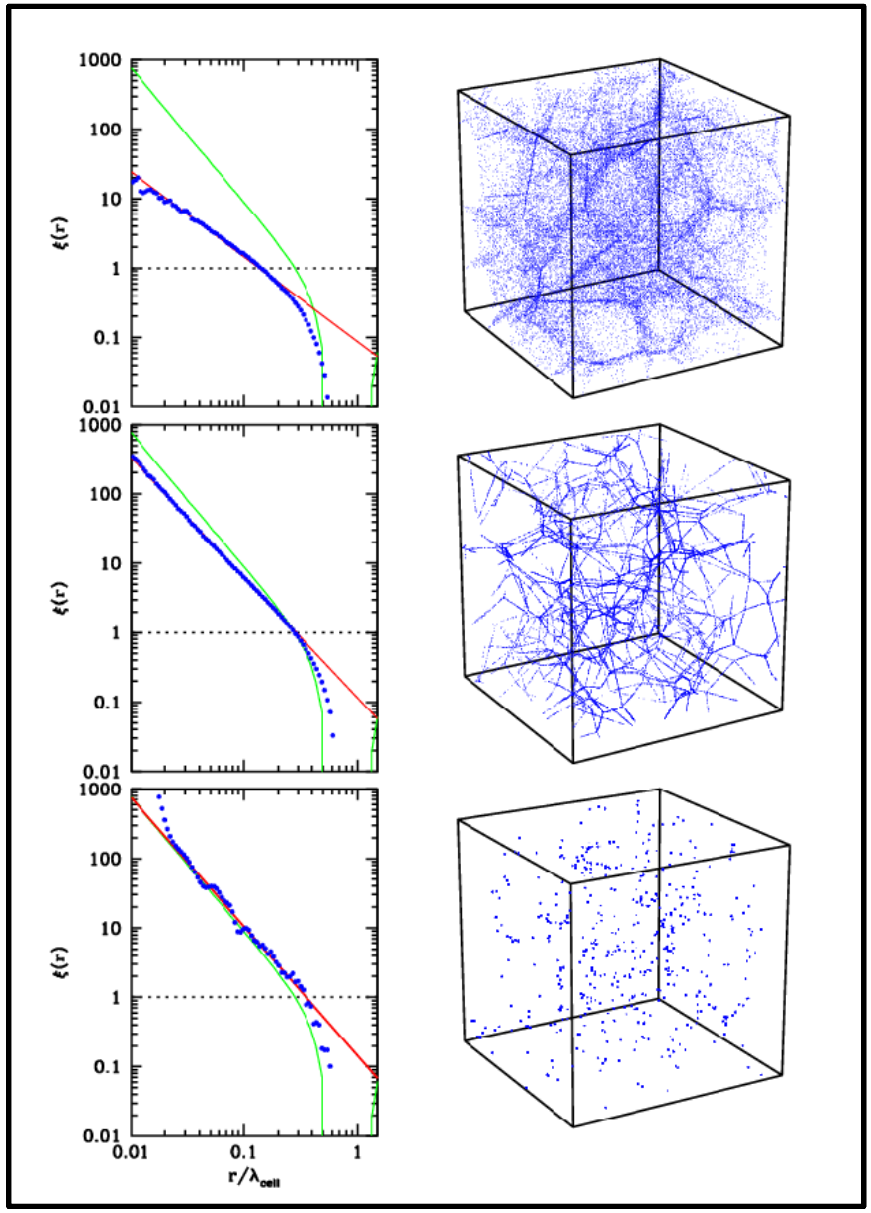
Structural Sensitivity



Wall-dominated

Filamentary

Cluster-like



Power Spectrum

Power Spectrum

P(k) specifies the relative contribution of different scales to the density fluctuation field. It entails a wealth of cosmological information.

$$\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k) \qquad \Leftrightarrow \qquad P(k) \propto \left\langle \hat{f}(\vec{k})\, \hat{f}^*\left(\vec{k}\right) \right\rangle$$

Formal definition:

$$(2\pi)^3 P(k_1)\, \delta_D\left(\vec{k}_1 - \vec{k}_2\right) = \left\langle \hat{f}\left(\vec{k}_1\right) \hat{f}^*\left(\vec{k}_2\right) \right\rangle$$

$$\Downarrow$$

$$P(k) \propto \left\langle \hat{f}\left(\vec{k}_1\right) \hat{f}^*\left(\vec{k}_2\right) \right\rangle$$

Power Spectrum – Correlation Function

Gaussian random field fully described by 2nd order moment:

- in Fourier space: power spectrum
- in Configuration (spatial) space: 2-pt correlation function

$$(2\pi)^3 P(k_1)\,\delta_D\left(\vec{k}_1-\vec{k}_2\right) = \left\langle \hat{f}\left(\vec{k}_1\right)\hat{f}^*\left(\vec{k}_2\right)\right\rangle$$

$$\xi\left(\vec{r}_1,\vec{r}_2\right)=\xi\left(\left|\vec{r}_1-\vec{r}_2\right|\right)=\left\langle f\left(\vec{r}_1\right) f\left(\vec{r}_2\right)\right\rangle$$

$$P(k)=\int d^3r\ \xi\left(\vec{r}\right)\, e^{i\vec{k}\cdot\vec{r}}$$

$$\xi\left(\vec{r}\right)=\int\frac{d^3k}{\left(2\pi\right)^3}P(k)\,e^{-i\vec{k}\cdot\vec{r}}$$

Random Field Phases

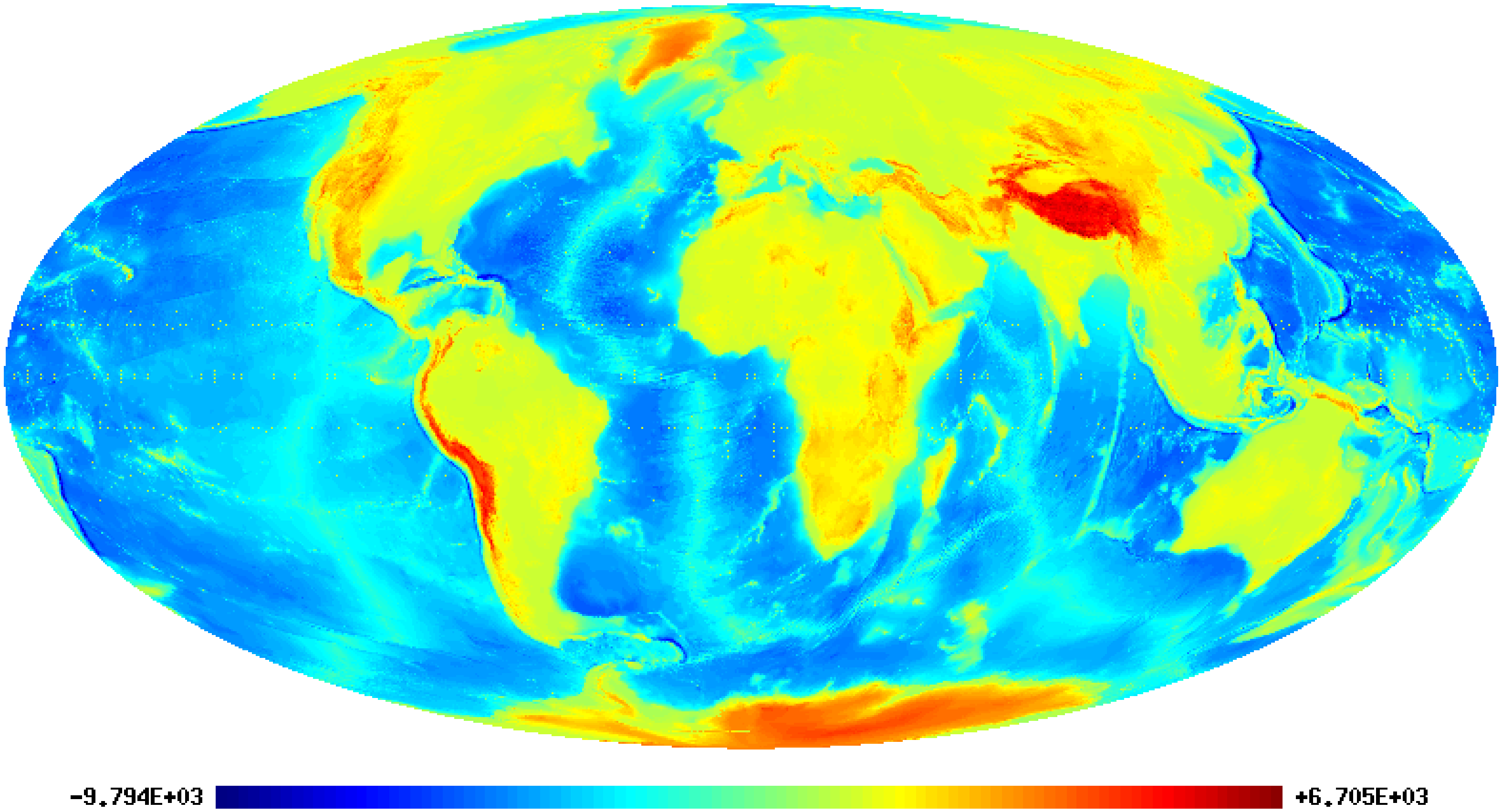
$$f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{f}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

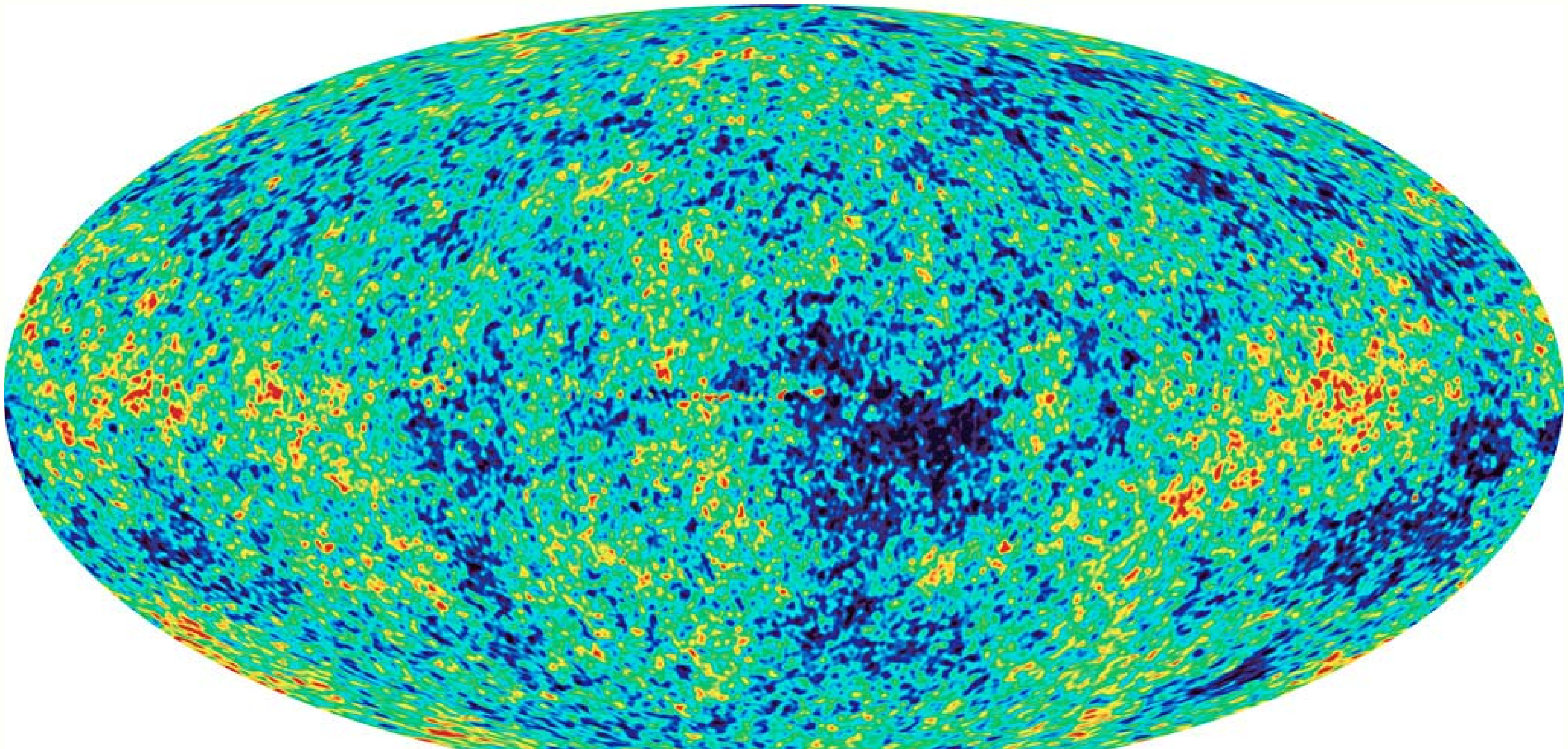
$$\hat{f}(\vec{k}) = \hat{f}_r(\vec{k}) + i\,\hat{f}_i(\vec{k}) = \left|\hat{f}(\vec{k})\right| e^{i\theta(k)}$$

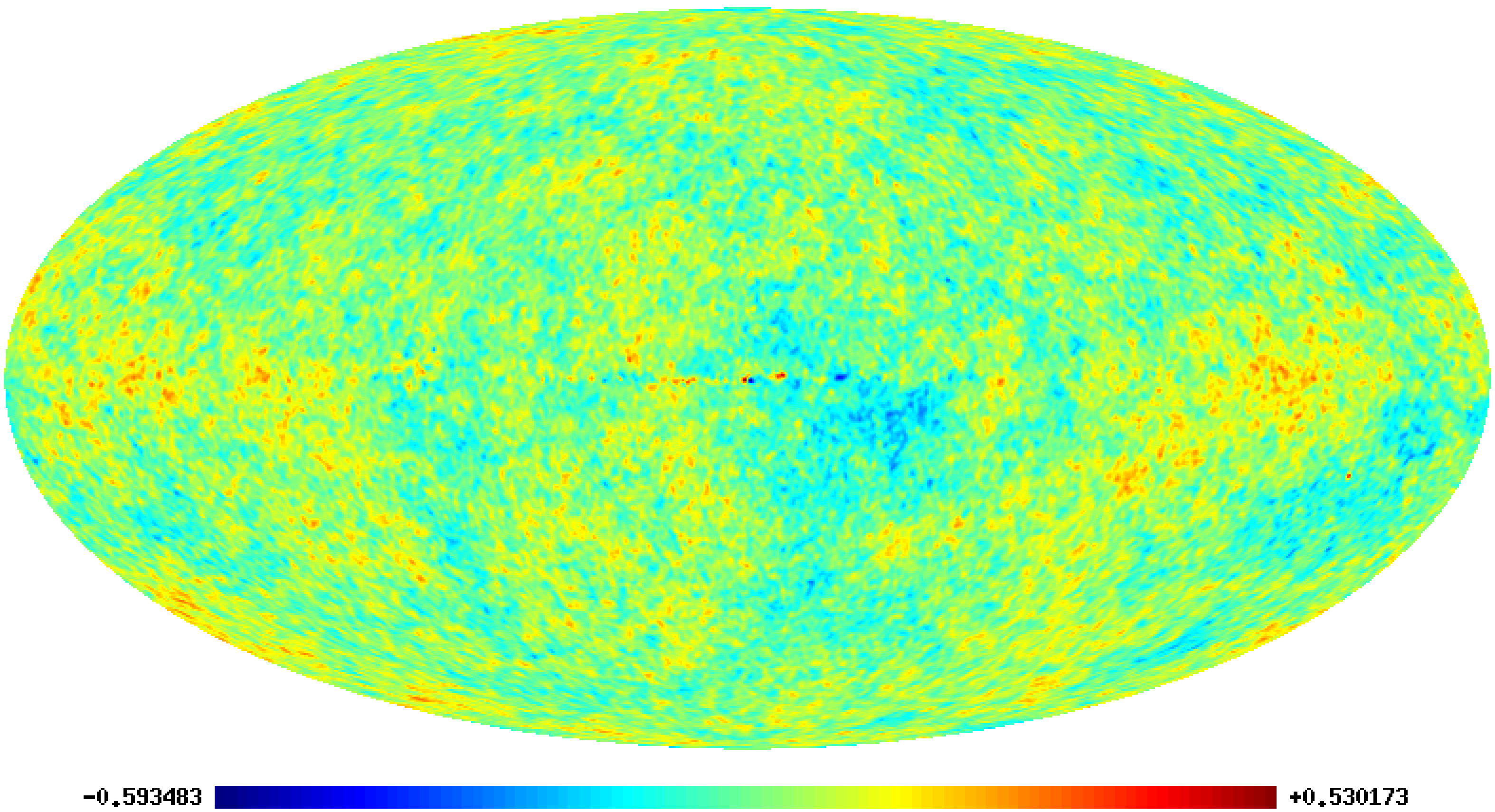
When a field is a Random Gaussian Field, its phases ϕ(k) are uniformly distributed over the interval [0,2π]:

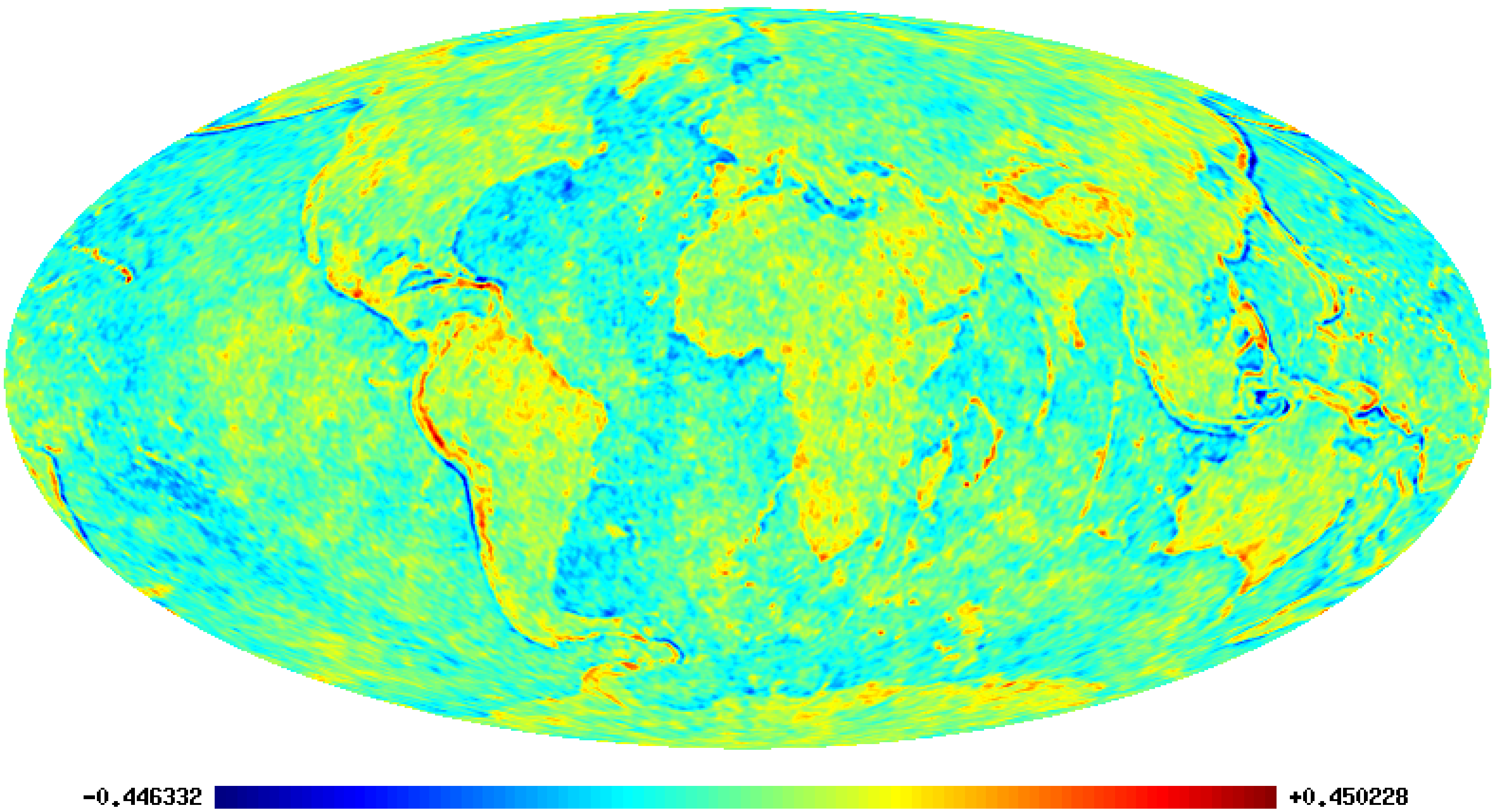
$$\theta(k) \in U[0,2\pi]$$

As a result of nonlinear gravitational evolution, we see the phases acquire a distinct non-uniform distribution.


-9.794E+03
+6.705E+03




-0.593483
+0.530173


-0.446332
+0.450228

DTFE:

Delaunay Tessellation
Field Estimator

Points, Tessellations & Patterns

Schaap & van de Weygaert 2000
Van de Weygaert & Schaap 2007
Cautun & van de Weygaert 2012

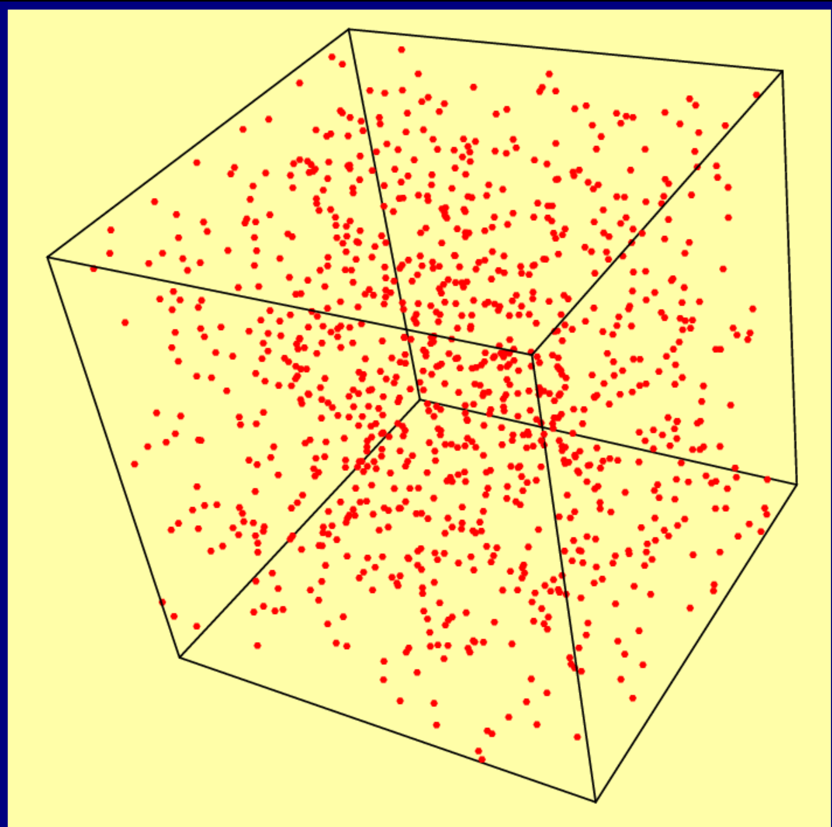
DTFE

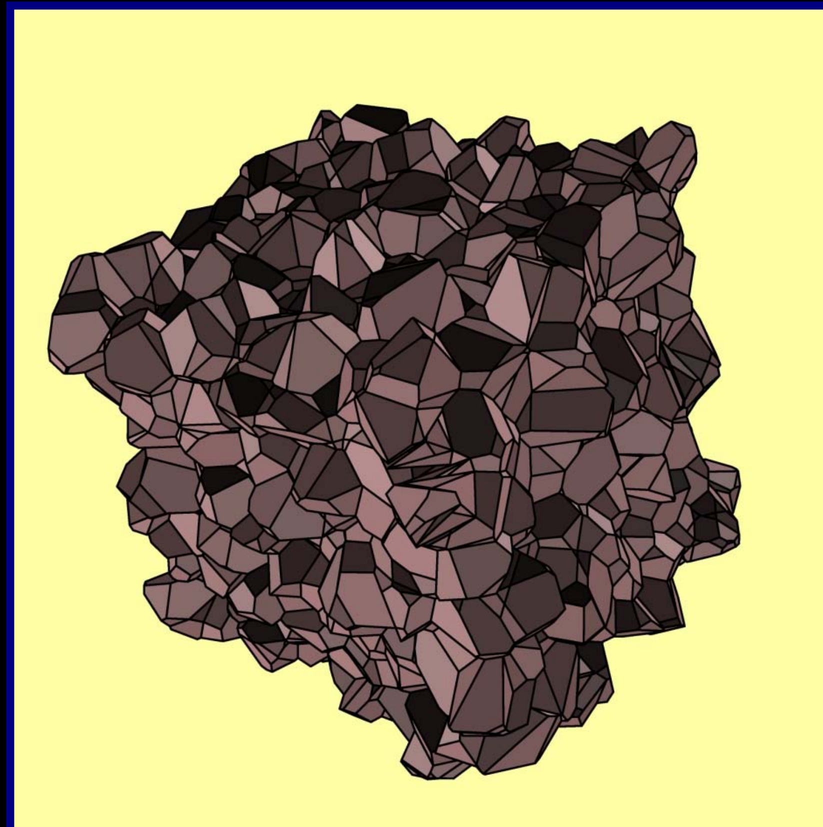
Delaunay Tessellation Field Estimator

- **Density Estimate:**
  Voronoi Tessellation (contiguous)

- **multi-D field interpolation:**
  Delaunay Tessellations

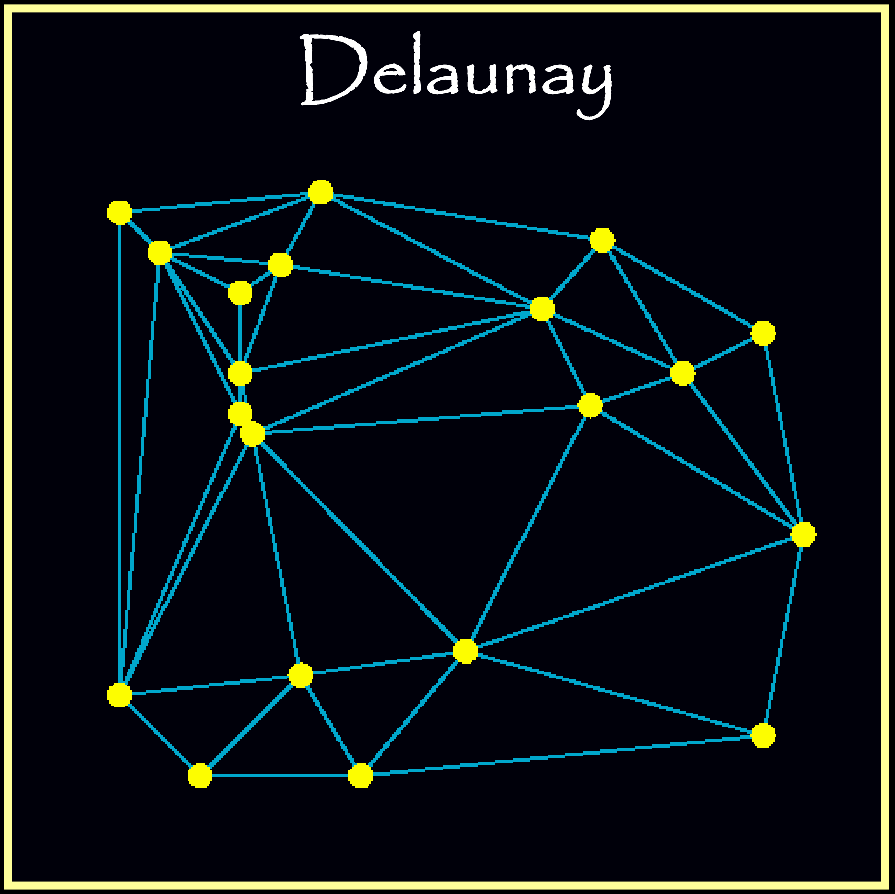
Voronoi Tessellations





$$\Pi_i = \{\vec{x} | d(\vec{x}, \vec{x}_i) < d(\vec{x}, \vec{x}_j) \quad \text{for all } j \neq i\}$$

Dual Tessellations



Voronoi Vertices → Centers Circumscribing Spheres 4 nuclei

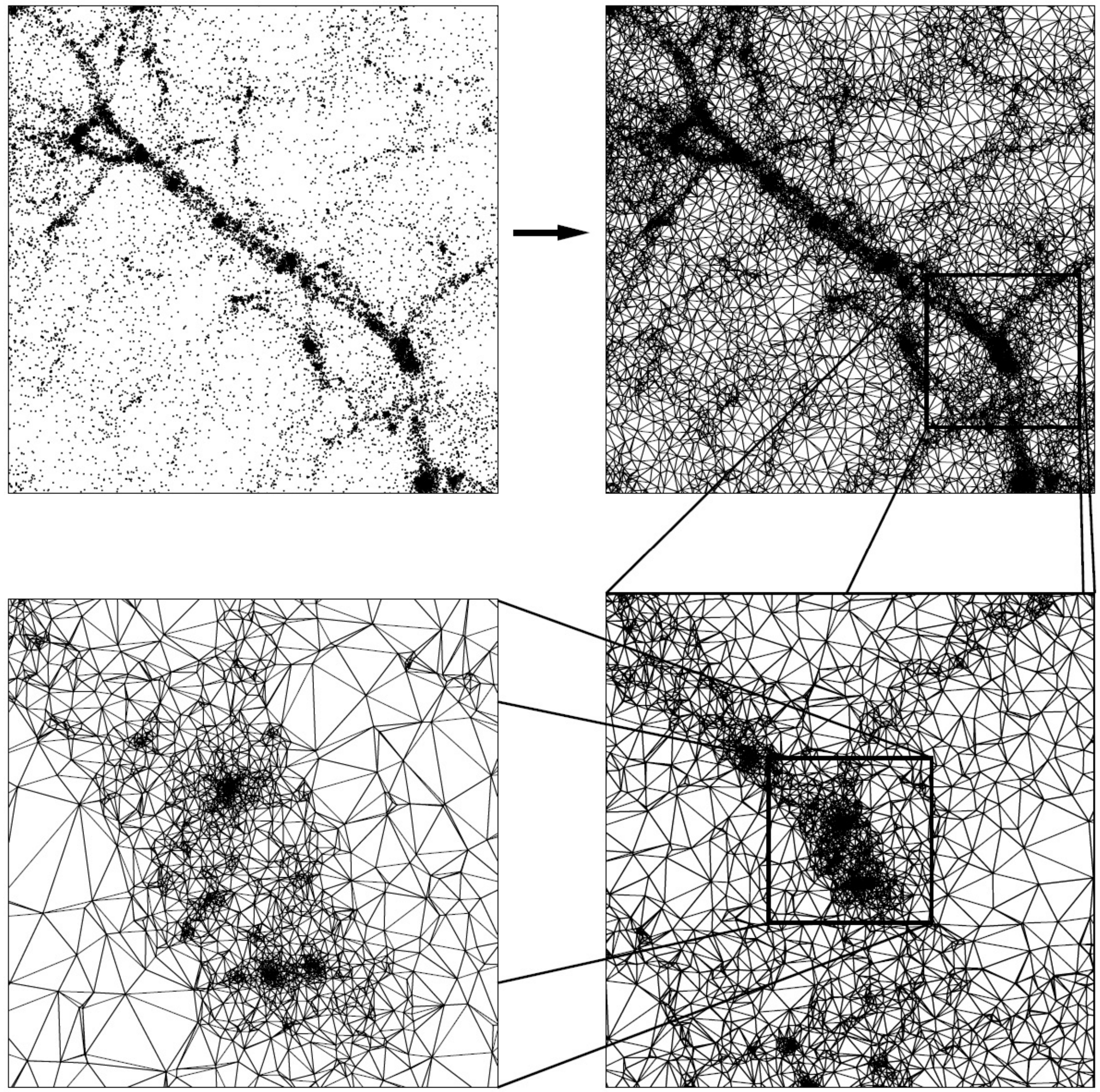
Delaunay Tetrahedron

Sensitivity of Delaunay Tessellations

to weblike geometry of particle distribution:

suggestion for exploiting this to explore the topology of the cosmic mass distribution

DTFE
Alpha Shapes

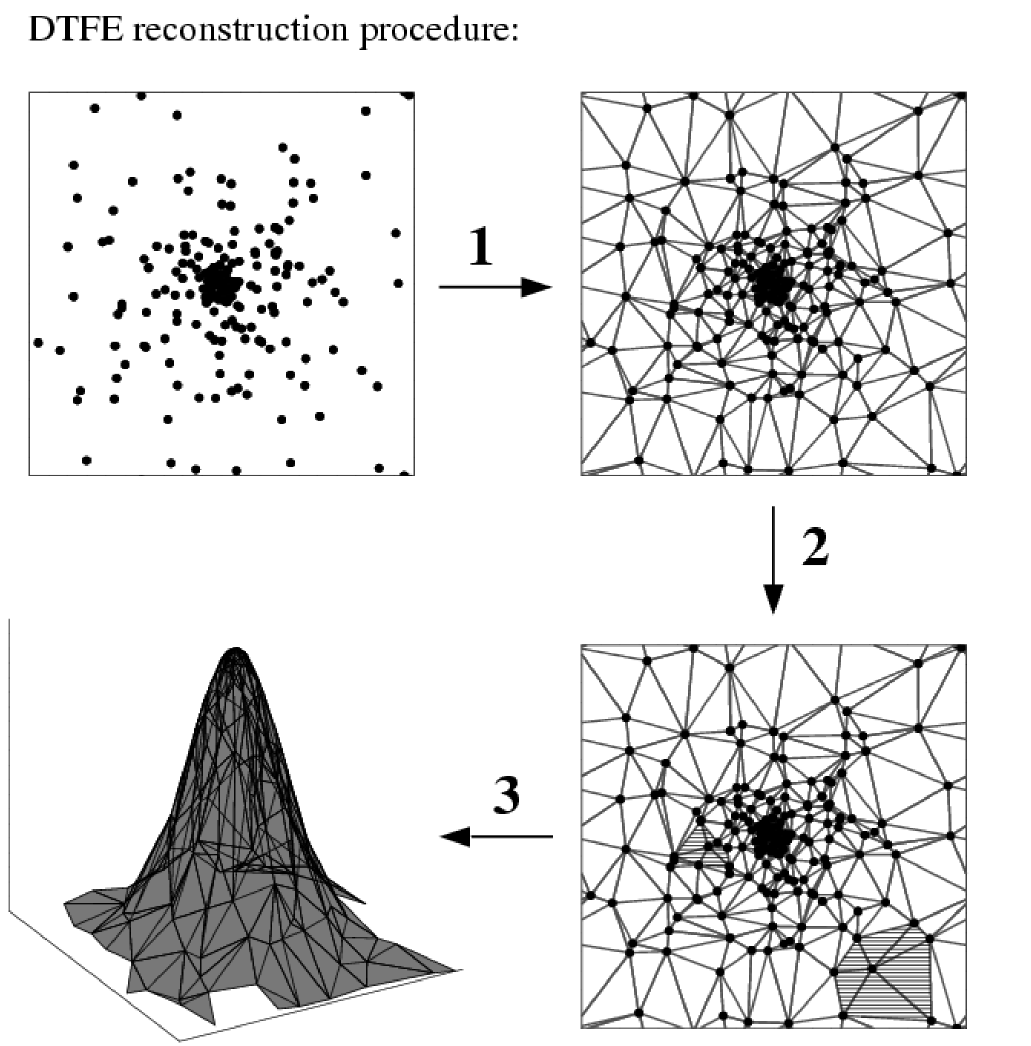


DTFE

- Delaunay Tessellation Field Estimator
- Piecewise Linear representation
  density & other discretely sampled fields
- Exploits sample density & shape sensitivity of
  Voronoi & Delaunay Tessellations
- Density Estimates from contiguous Voronoi cells
- Spatial piecewise linear interpolation by means of
  Delaunay Tessellation

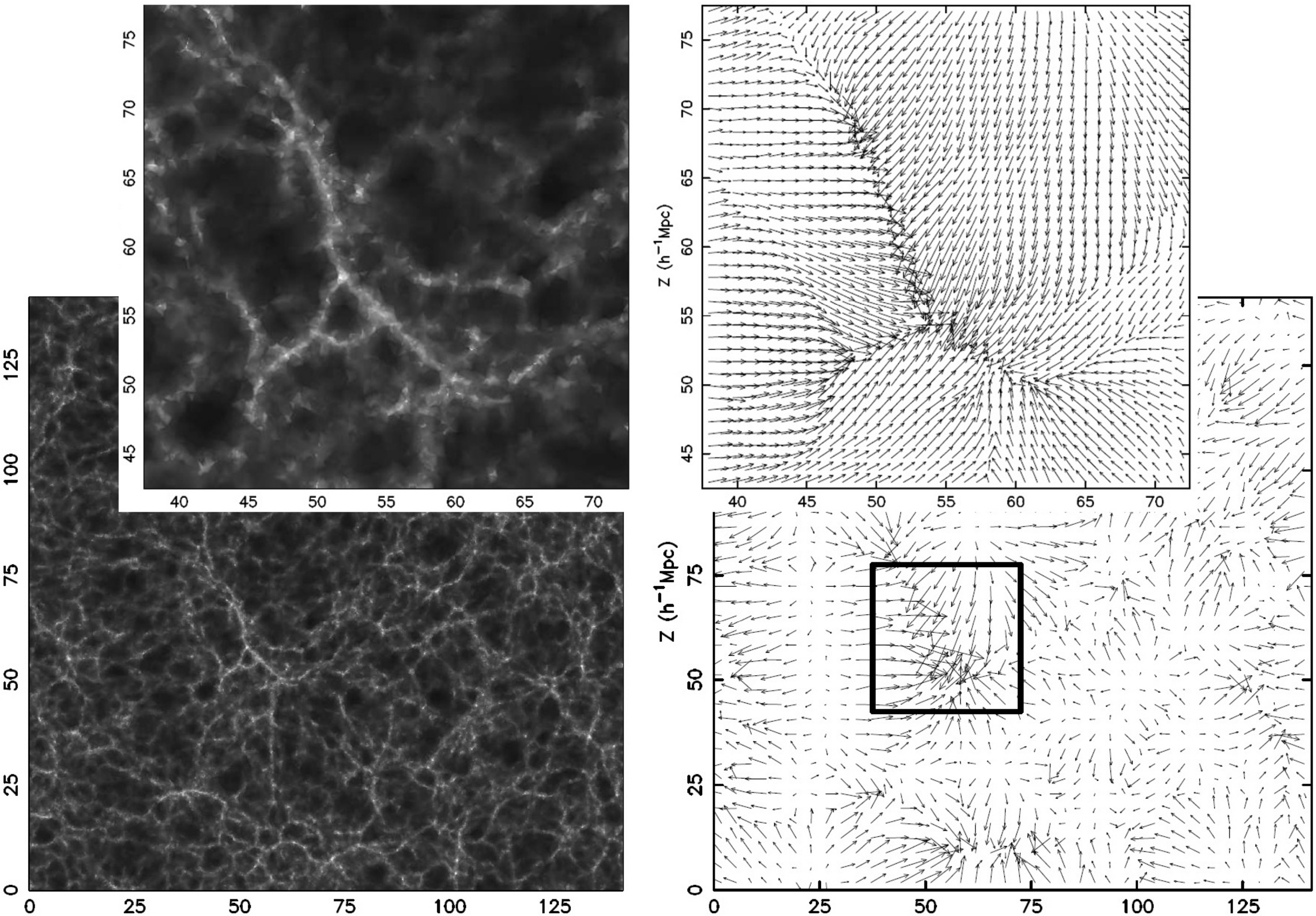
Schaap & vdW 2000
vdW & schaap 2009
Cautun & vdW 2012

DTFE Procedure



Summary

I. Construction Delaunay Tessellation
II. Point Sampling
III. Determination Field Values
IV. Calculation Field Gradient in Delaunay cell
V. - Interpolation to locations x
   - Image construction: interpolation to ordered locations
VI. Processing of field


Z (h^{-1}Mpc)
Z (h^{-1}Mpc)

DTFE website:

http://www.astro.rug.nl/~voronoi/DTFE/dtfe.html

Tracing the Cosmic Web:

Pattern Classification

Tracing the Cosmic Web

Classes   Identification & Classification   procedures

• Graph & Percolation techniques	Minimal Spanning Tree
• Stochastic Methods	Bisous Bayesian sampling geometric configurations
• Geometric, Hessian-based methods	Vweb - velocity shear gradient velocity field Tweb - tidal field Hessian potential field
Scale-space Multiscale Hessian-based methods	MMF/Nexus
• Topological Methods (Morse theory)	Watershed Void Finder / Voboz Disperse Spineweb
• Phase-Space (multistream) structure:	Phase-space sheet & flip-flop Origami Multistream

Nexus/MMF:

Multiscale Morphology of the Cosmic Web

the Formalism

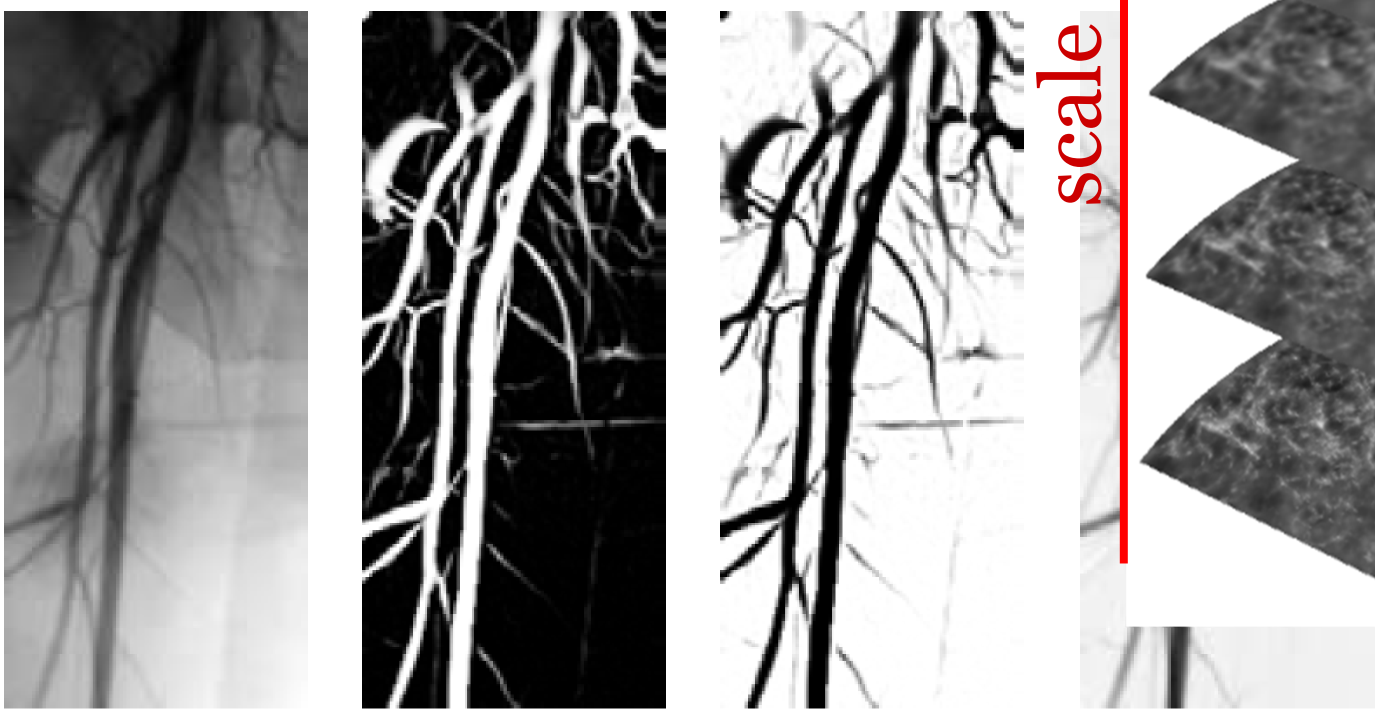
Aragon-Calvo, Jones, vdW, van der Hulst 2007
Cautun, vdW & Jones 2013

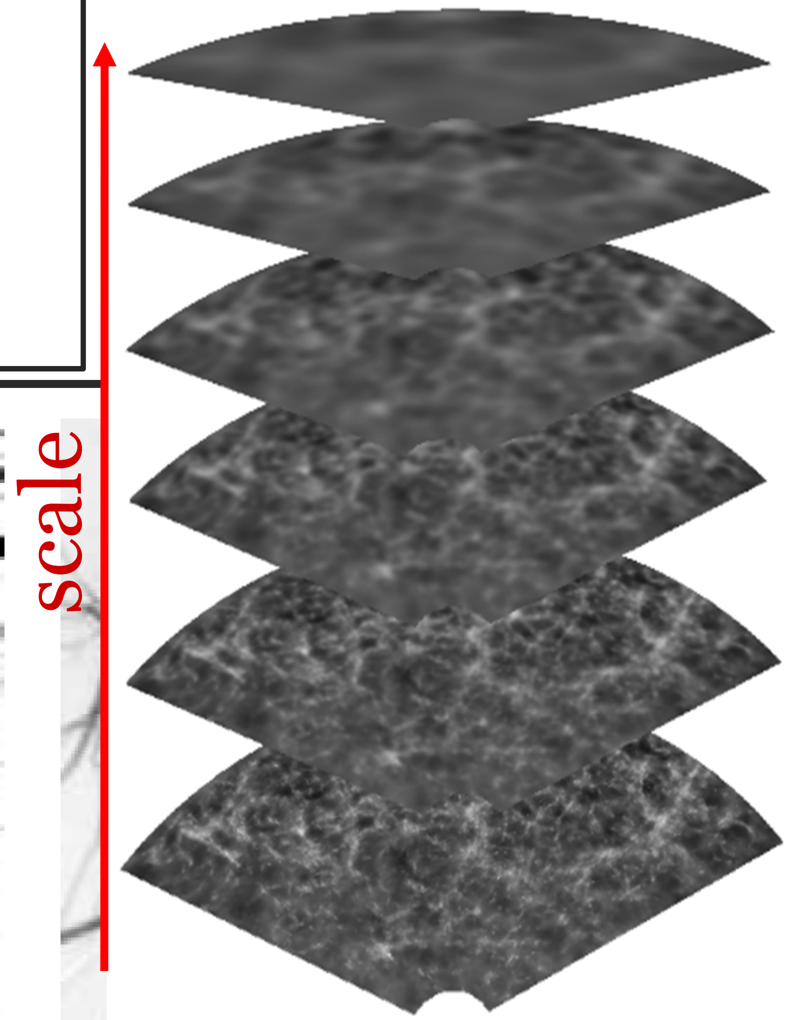
Scale Space Analysis

Inspiration from Medical Imaging:

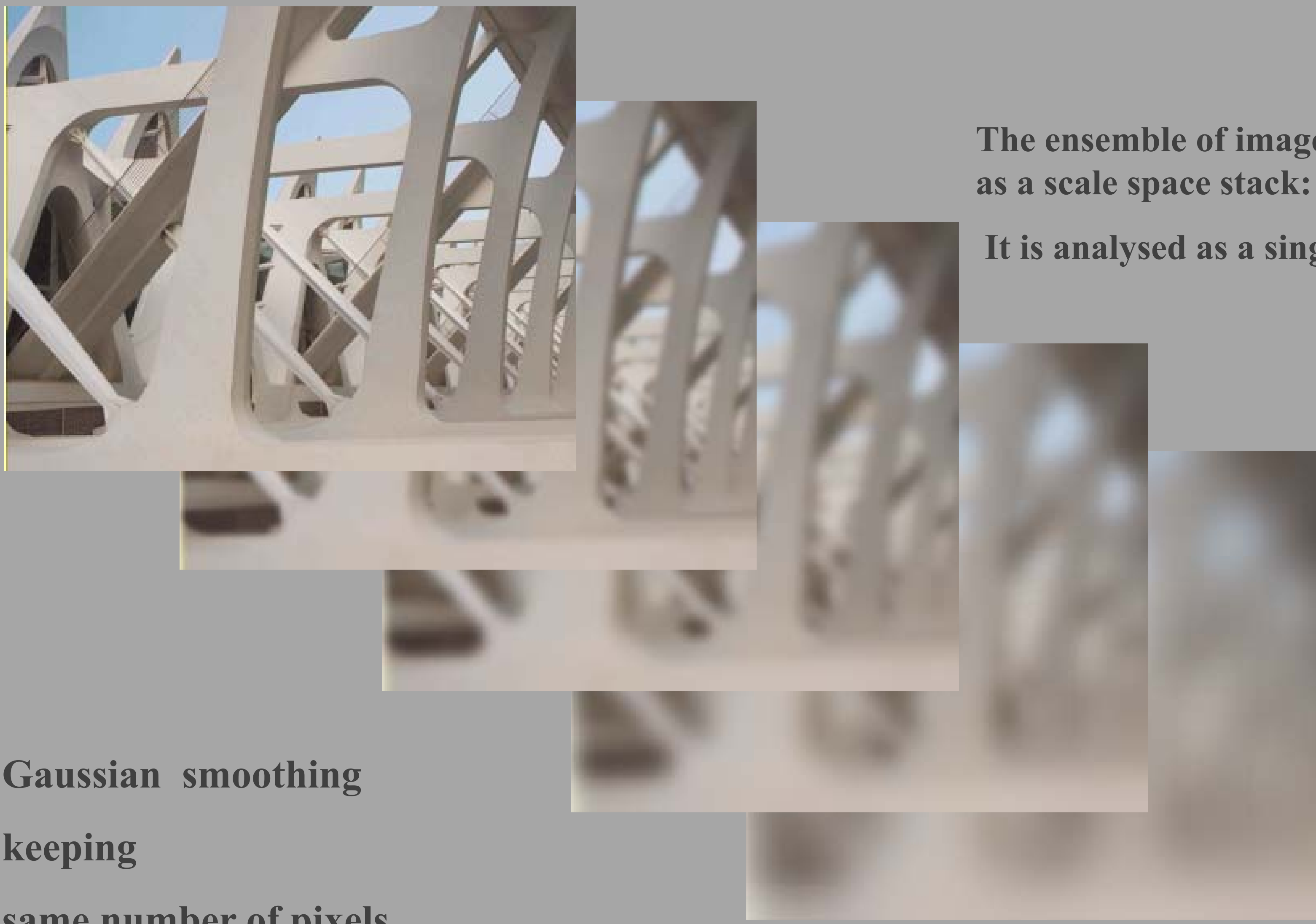
trace blood vessels, tumors, etc.

- Florack, Kuijper et al.; Lindeberg et al.
- Sato et al. 1997; Lorentz et al. 1997
- Frangi et al. 1998 Multiscale vessel enhancement filter





Scale Space Pyramids



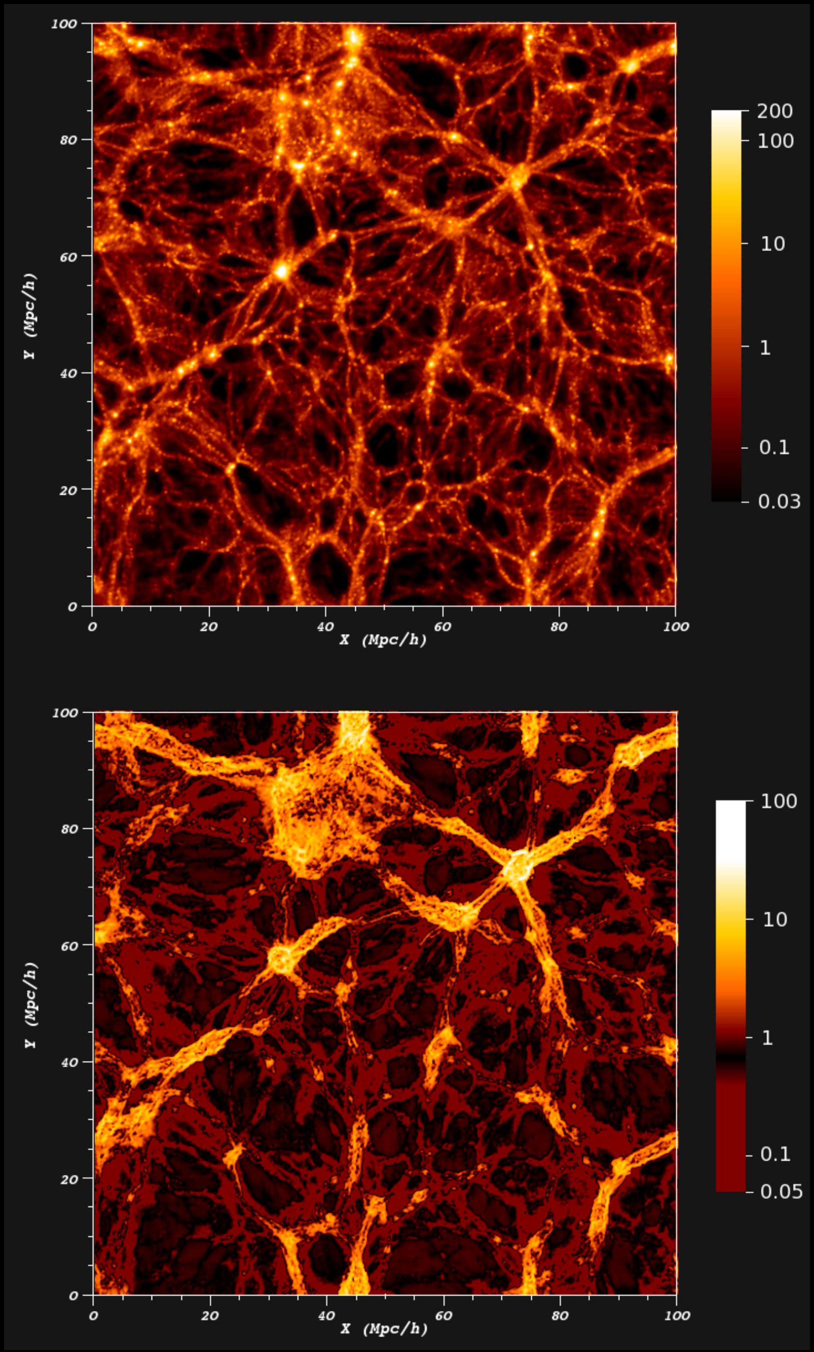
The ensemble of images is referred t
as a scale space stack:

It is analysed as a single object.

Gaussian smoothing

keeping

same number of pixels.



Nexus Fields

- Nexus/Nexus+
  fields relevant for cosmic web dynamics

- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:

- Density
- Log(Density)
- Tidal field
- Velocity Divergence
- Velocity Shear
- Nexus+ - log(density)

- from: Cautun et al. 2013

Scale Space Analysis

- Smooth the field over the range of relevant scales

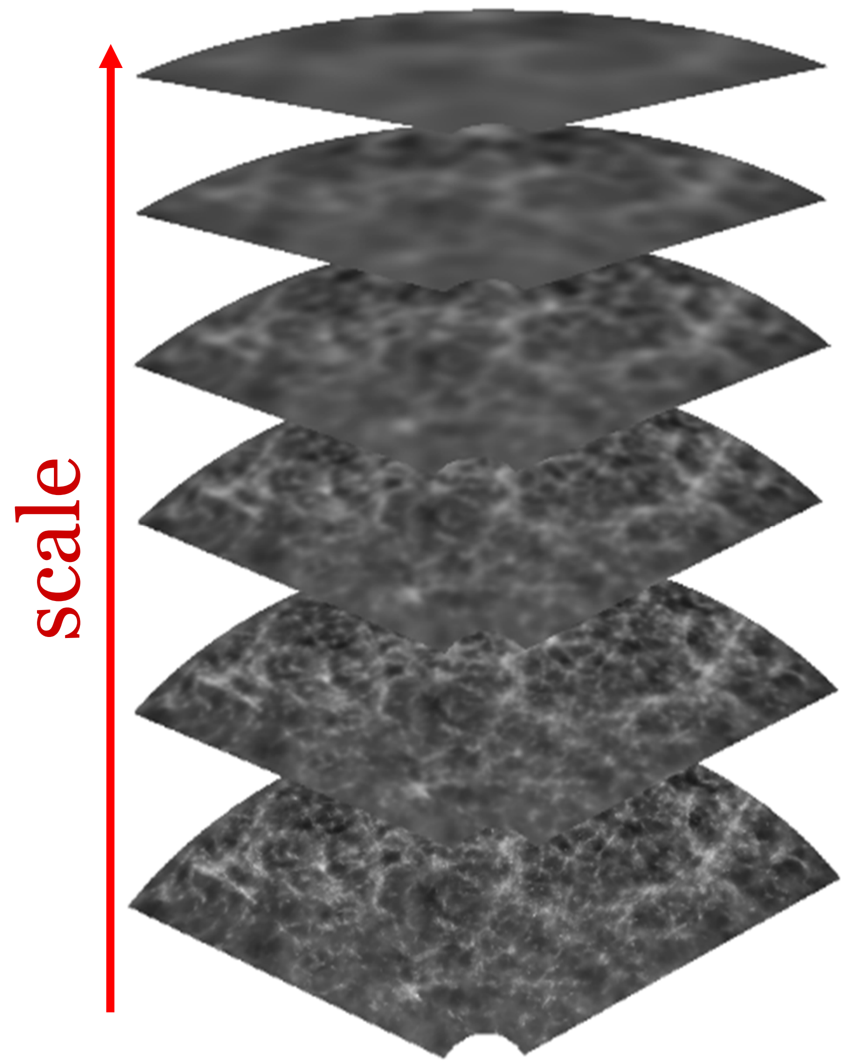
$$f_n(\vec{x}) = \int d\vec{y}\ \ f_{DTFE}(\vec{y})\, W_n(\vec{y},\vec{x})$$

- with Gaussian filter

$$W_n(\vec{y},\vec{x}) = \frac{1}{(2\pi R_n^{\ 2})^{3/2}} \exp\left(-\frac{|\vec{y}-\vec{x}|^2}{2R_n^{\ 2}}\right)$$

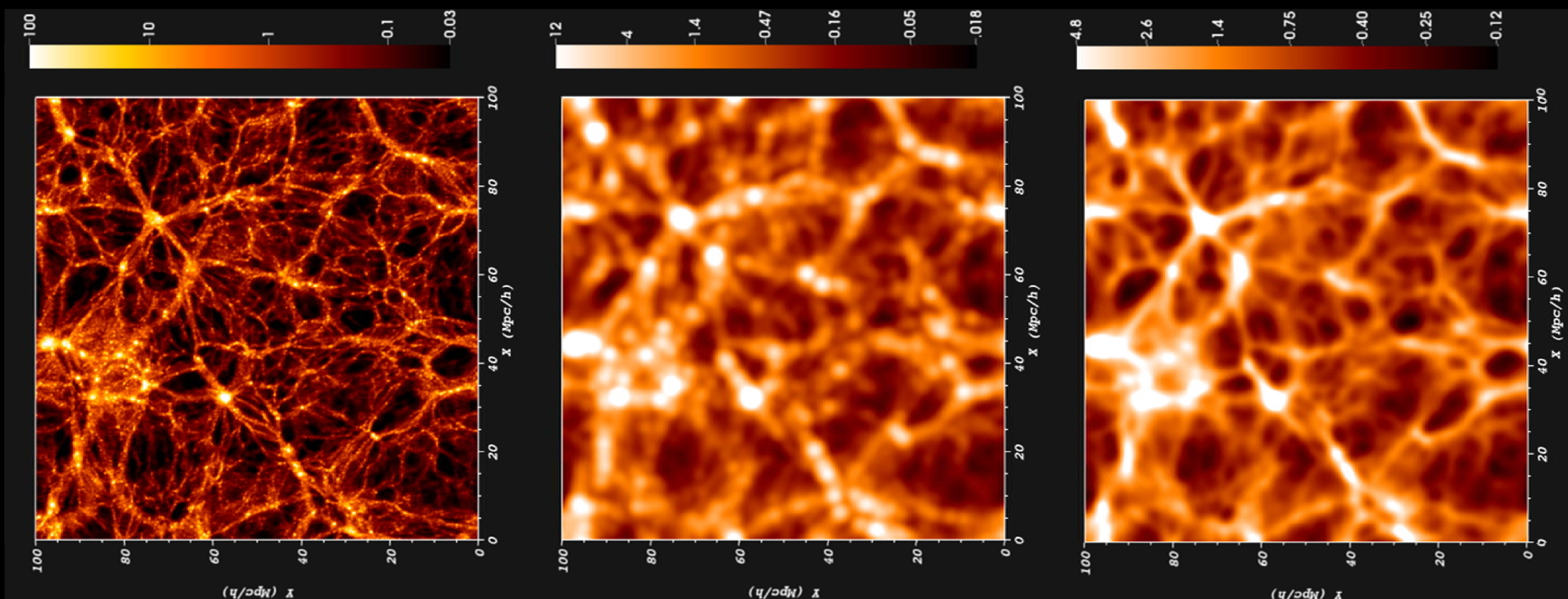
- Scale space:
  stacking density maps f_n

$$\Phi = \bigcup_{levels\ n} f_n$$



Nexus/Nexus+ Scale Space

Input field **Gaussian smoothing** **Log-Gaussian smoothing**



Scale Space Analysis

- **Smooth the field over the range of relevant scales**

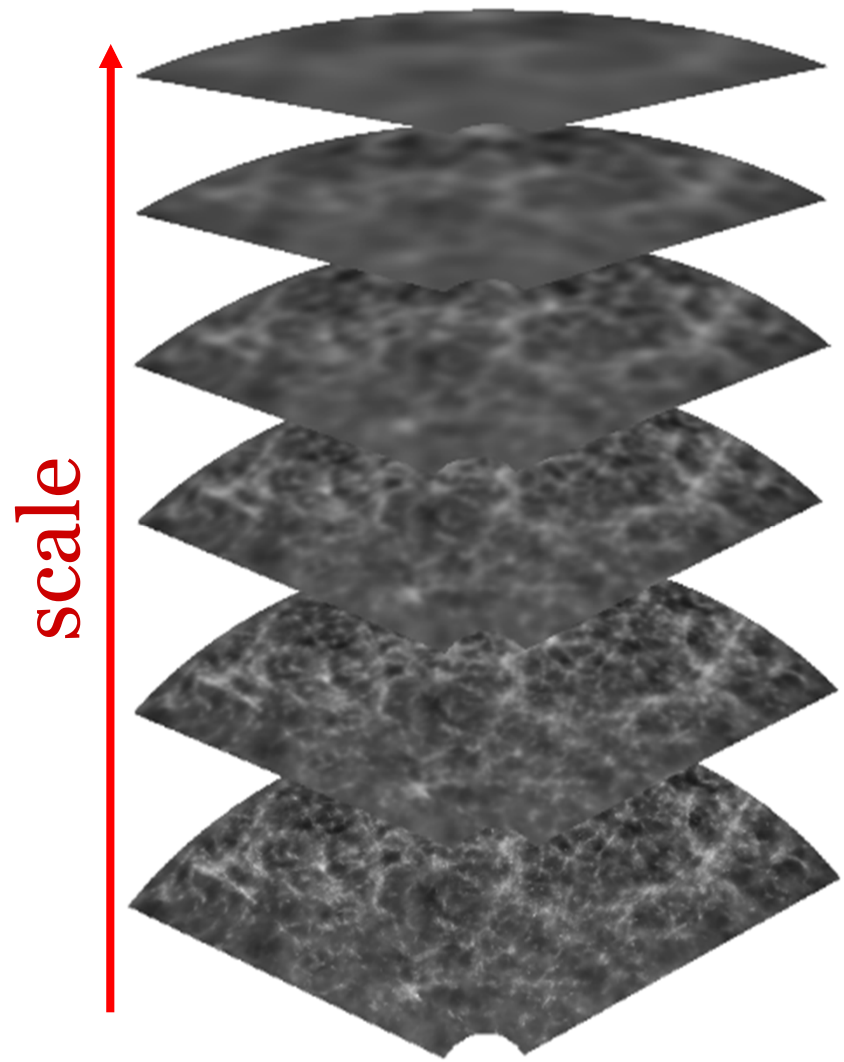
- **Density field around location to 2nd order determined by Hessian:**

$$f(\boldsymbol{x}_0 + \boldsymbol{s}) = f(\boldsymbol{x}_0) + \boldsymbol{s}^T \nabla f(\boldsymbol{x}_0) + \frac{1}{2}\boldsymbol{s}^T \mathcal{H}(\boldsymbol{x}_0)\boldsymbol{s} + \ldots$$

- **Hessian filtered Density field:**

$$\frac{\partial^2}{\partial x_i \partial x_j} f_S(\boldsymbol{x}) = f_{\mathrm{DTFE}} \otimes \frac{\partial^2}{\partial x_i \partial x_j} W_G(R_S)$$

$$= \int \mathrm{d}\boldsymbol{y}\, f(\boldsymbol{y}) \frac{(x_i - y_i)(x_j - y_j) - \delta_{ij} R_S^2}{R_S^4} W_G(\boldsymbol{y}, \boldsymbol{x})$$

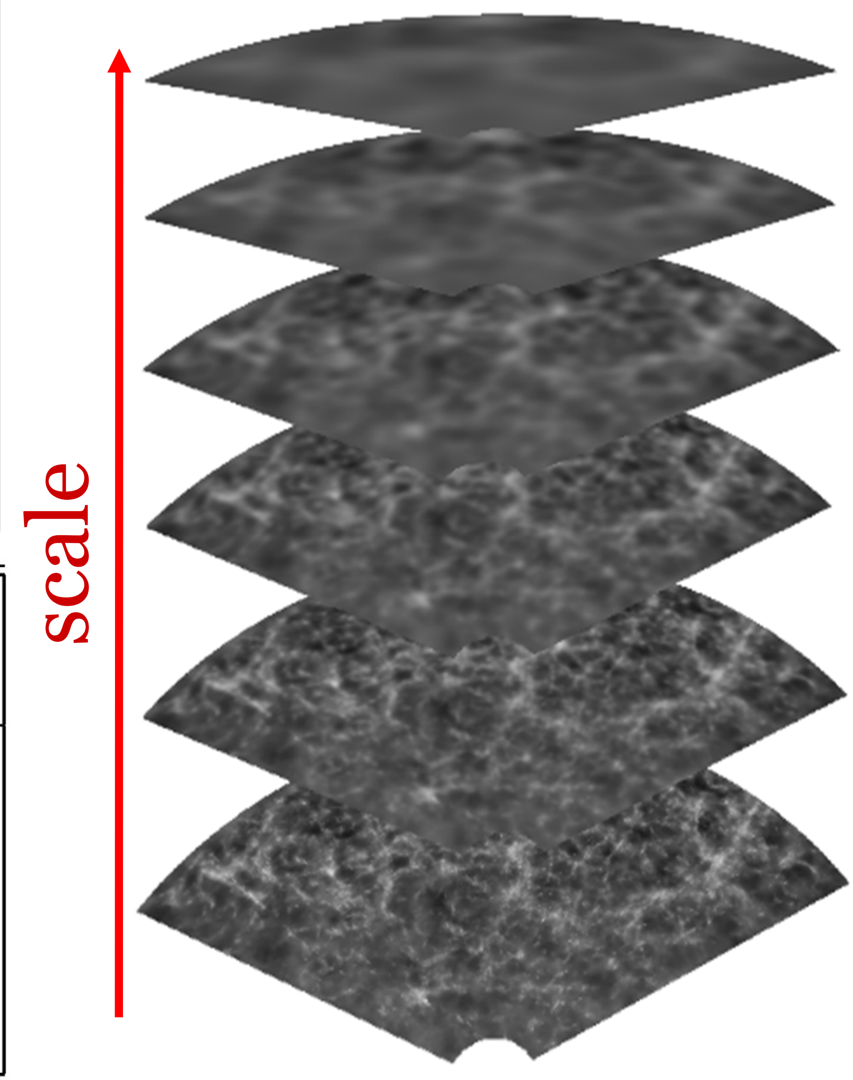


Scale Space Analysis

- **Morphology determined by eigenvalues of Hessian:**
- $\left| \dfrac{\partial^2 f_n(\vec{x})}{\partial x_i \partial x_j} - \lambda_a(\vec{x})\delta_{ij} \right| = 0, \quad a = 1,2,3$

$\lambda_1 > \lambda_2 > \lambda_3$

Structure	λ ratios	λ constraints
Blob	$\lambda_1 \simeq \lambda_2 \simeq \lambda_3$	$\lambda_3 < 0$; $\lambda_2 < 0$; $\lambda_1 < 0$
Line	$\lambda_1 \simeq \lambda_2 \gg \lambda_3$	$\lambda_3 < 0$; $\lambda_2 < 0$
Sheet	$\lambda_1 \gg \lambda_2 \simeq \lambda_3$	$\lambda_3 < 0$



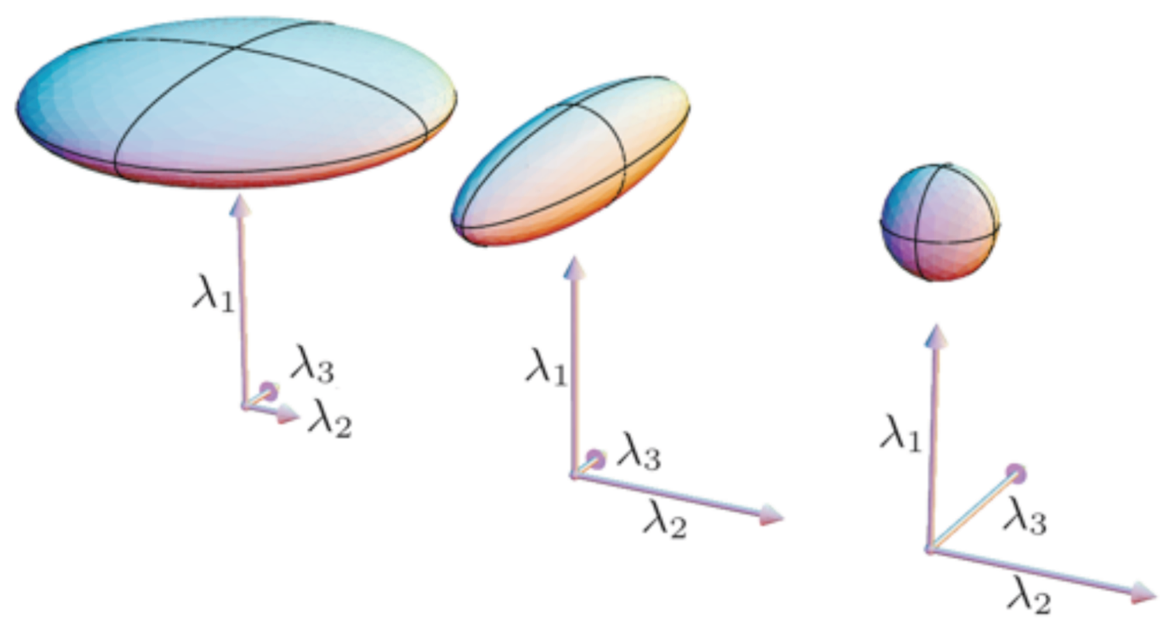
Scale Space Analysis

- Smooth the field over the range of relevant scales
- Select the characteristic scale of a particular (local) morphological element

Structure	λ ratios	λ constraints
Blob	$\lambda_1 \simeq \lambda_2 \simeq \lambda_3$	$\lambda_3 < 0\ ;\ \lambda_2 < 0\ ;\ \lambda_1 < 0$
Line	$\lambda_1 \simeq \lambda_2 \gg \lambda_3$	$\lambda_3 < 0\ ;\ \lambda_2 < 0$
Sheet	$\lambda_1 \gg \lambda_2 \simeq \lambda_3$	$\lambda_3 < 0$

Nexus/MMF formalism:

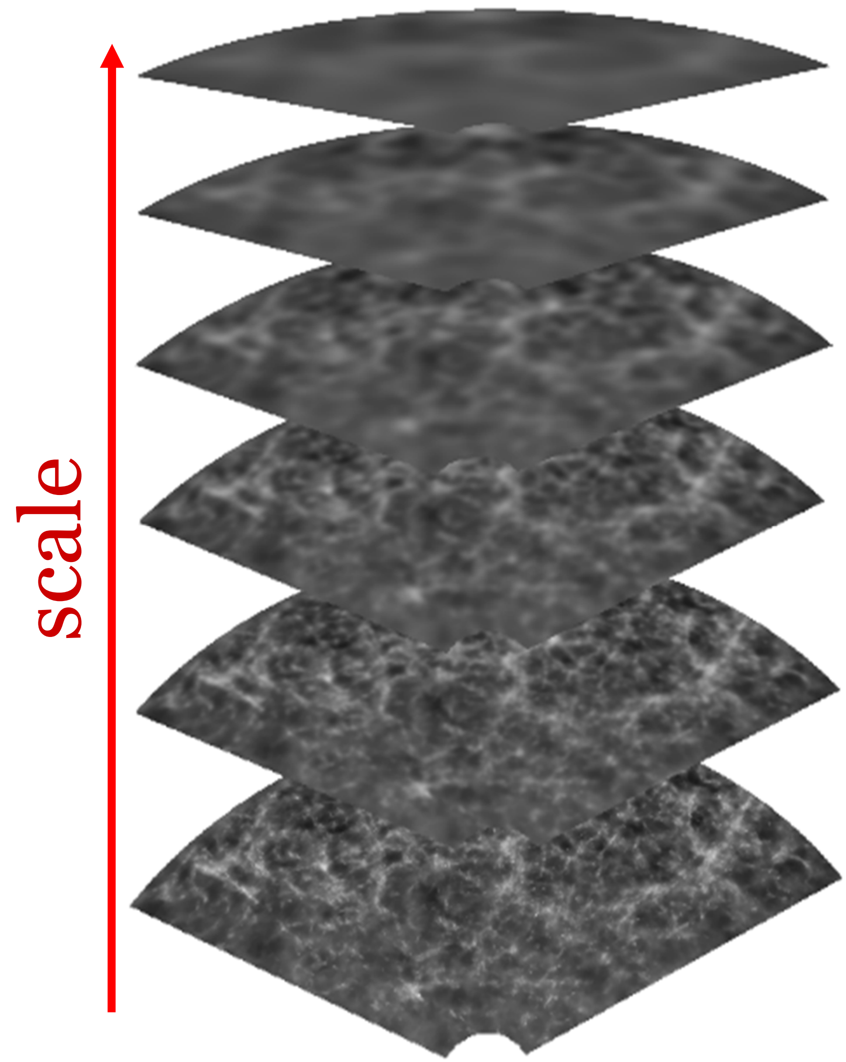
Aragon-Calvo et al. 2007
Aragon-Calvo et al. 2010
Cautun et al. 2013
Cautun et al. 2014



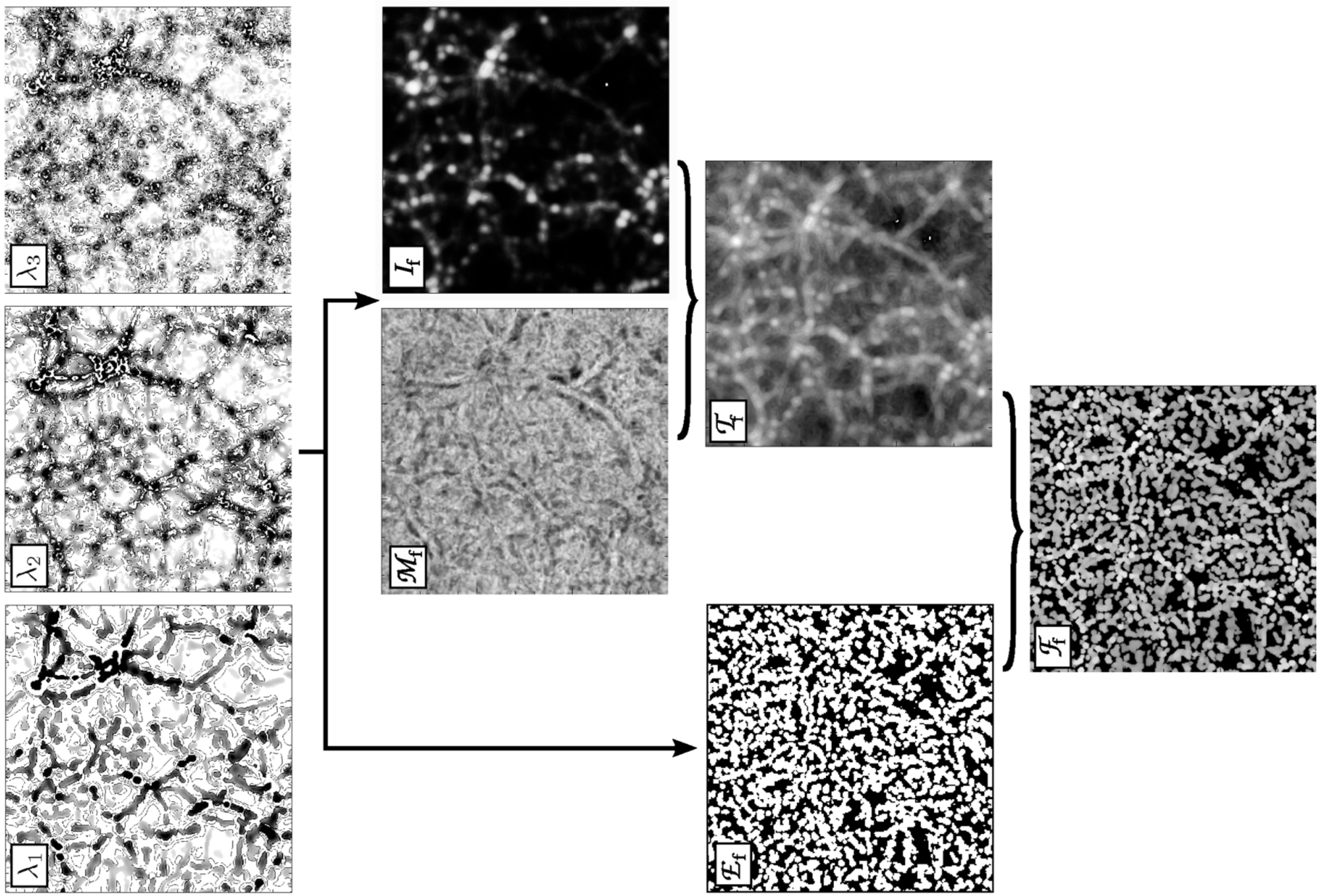
Scale Space Analysis

Nexus/MMF Procedure

- ***Smooth*** **the field over the range of relevant scales**
- ***Hessian*** **filtered density field**
- **Morphological characterization in terms of *eigenvalues Hessian***
- **Select the *characteristic scale* of a particular (local) morphological element**
- **Nexus/MMF morphology Filter Bank**



Nexus: MMF Filter Bank



Scale Space Analysis

- ***Scale Space Map Stack***

$$\Psi(\vec{x})$$

maximum morphology response across full range of scales

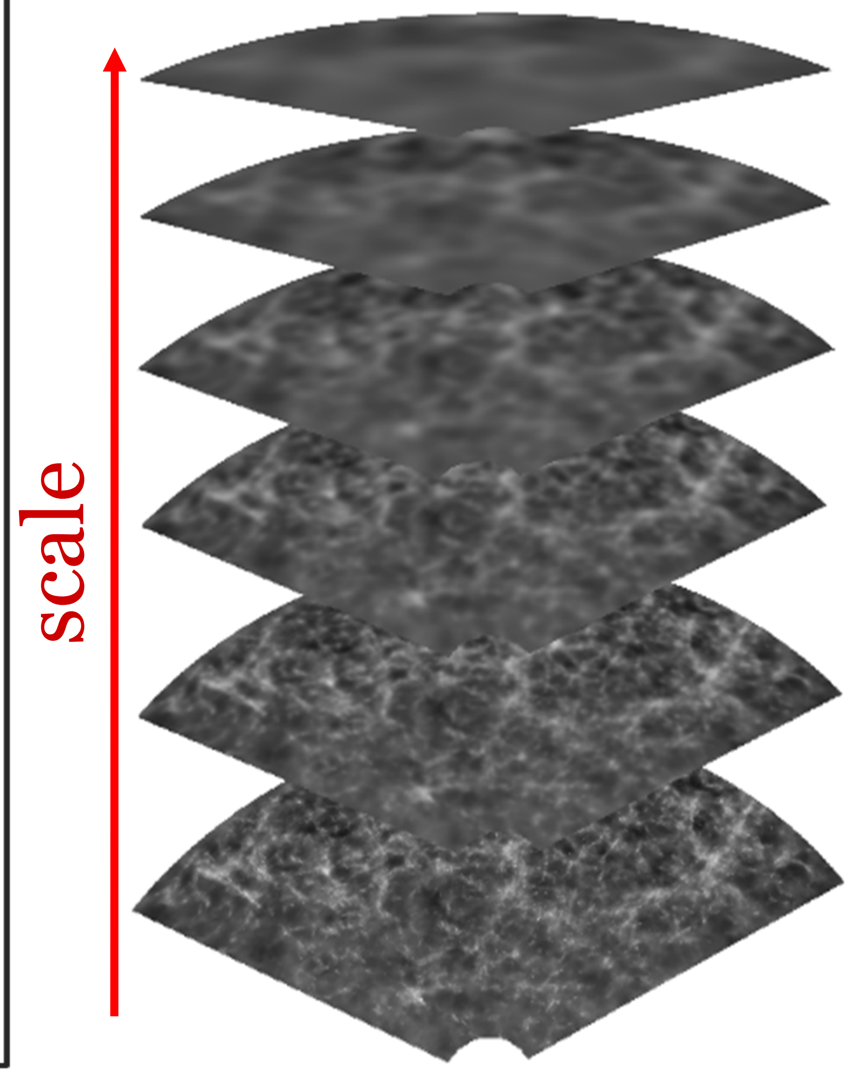
- **To filter out morphology noise: *morphology dependent thresholds*,**

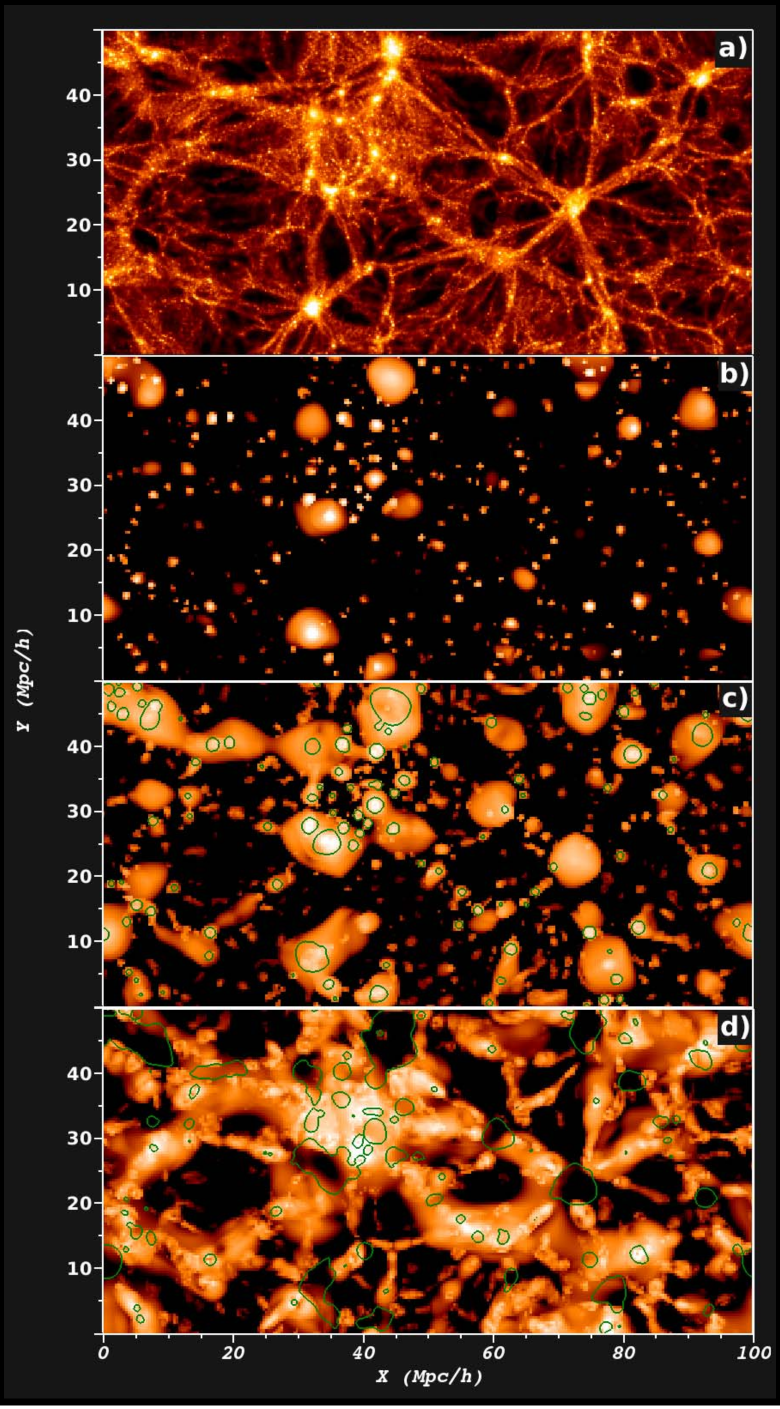
$$\tau_c, \ \tau_f, \ \tau_w$$

value dependent on dynamical and/or structural (percolation) considerations

- ***Object Map***

$$O(\vec{x})$$



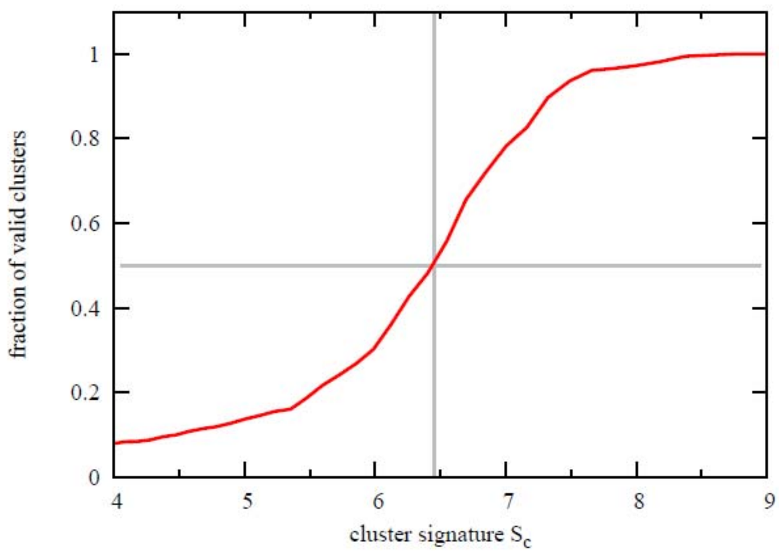


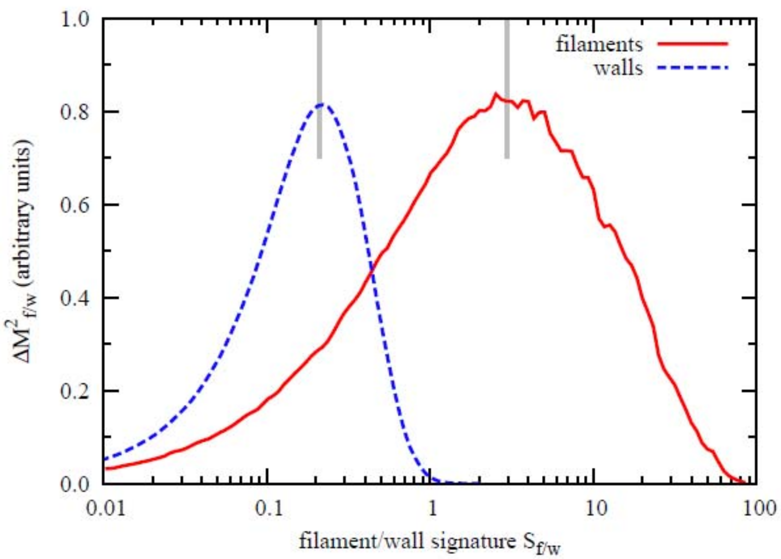
Nexus
Morphological Signatures

- a) density field
- b) blob/cluster node signature
- c) filament signature
- d) wall signature

- from: Cautun et al. 2013

Nexus Signature & Thresholds

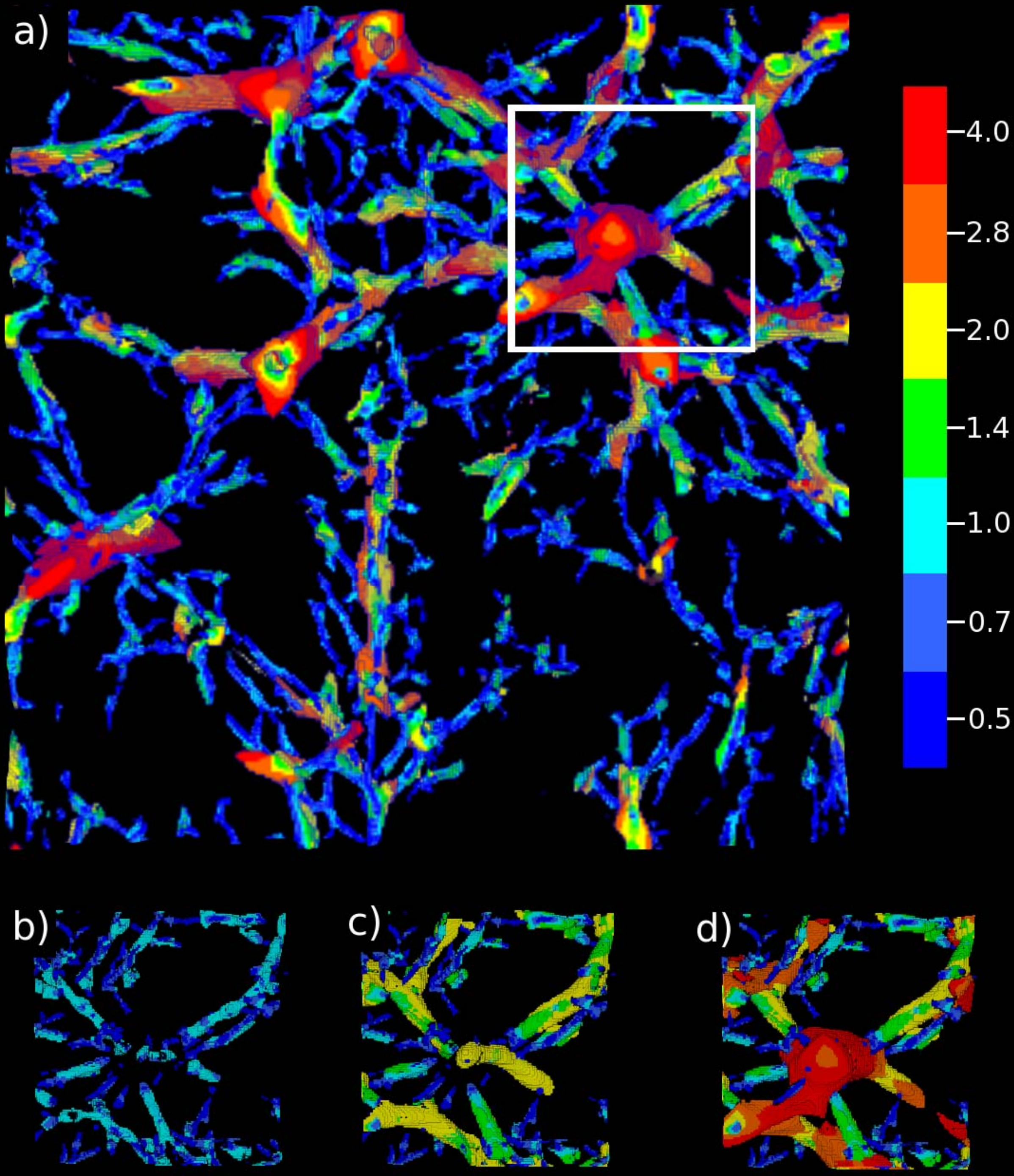




Nexus:

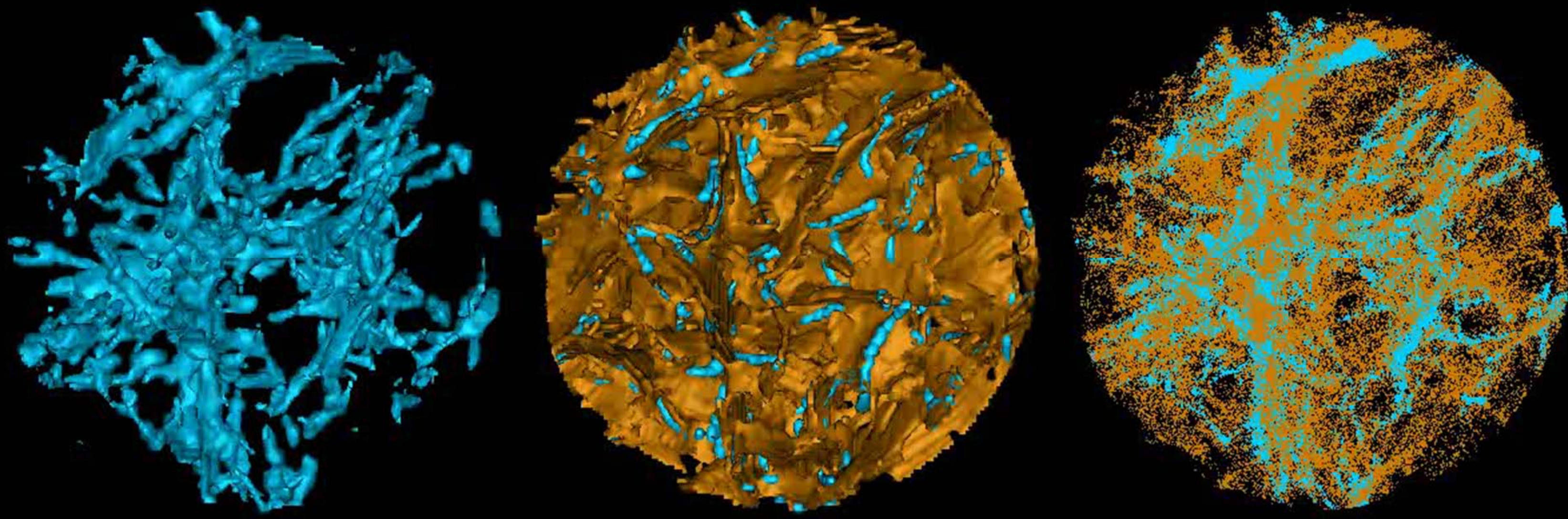
Multiscale Morphology Identification

Filaments



Colouring :
Local scale filament

Nexus Cosmic Web



MMF/Nexus
Cautun et al. 2013, 2014

Stochastic Spatial Pattern

- Clusters,
- Filaments &
- Walls

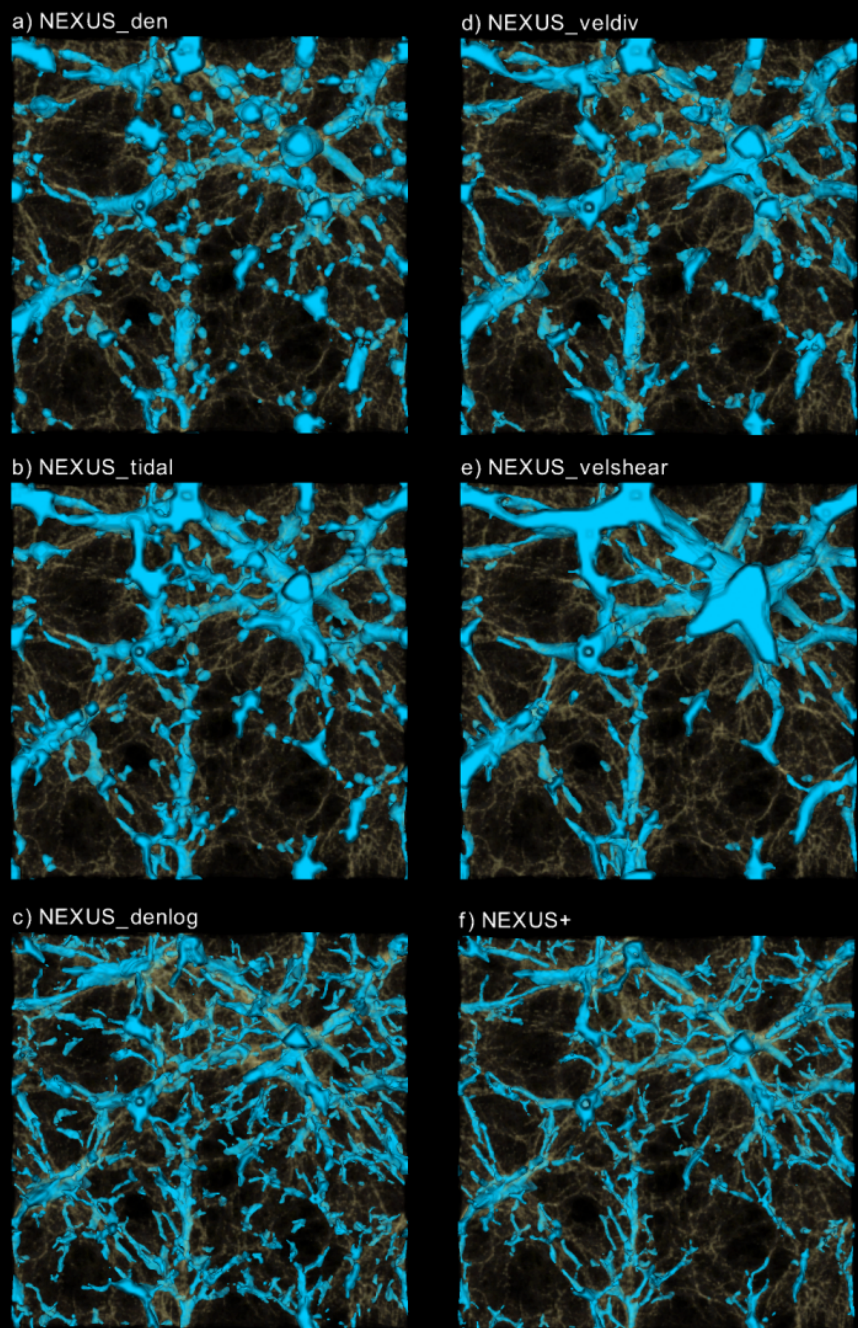
around

- Voids

in which matter & galaxies

have agglomerated

through gravity

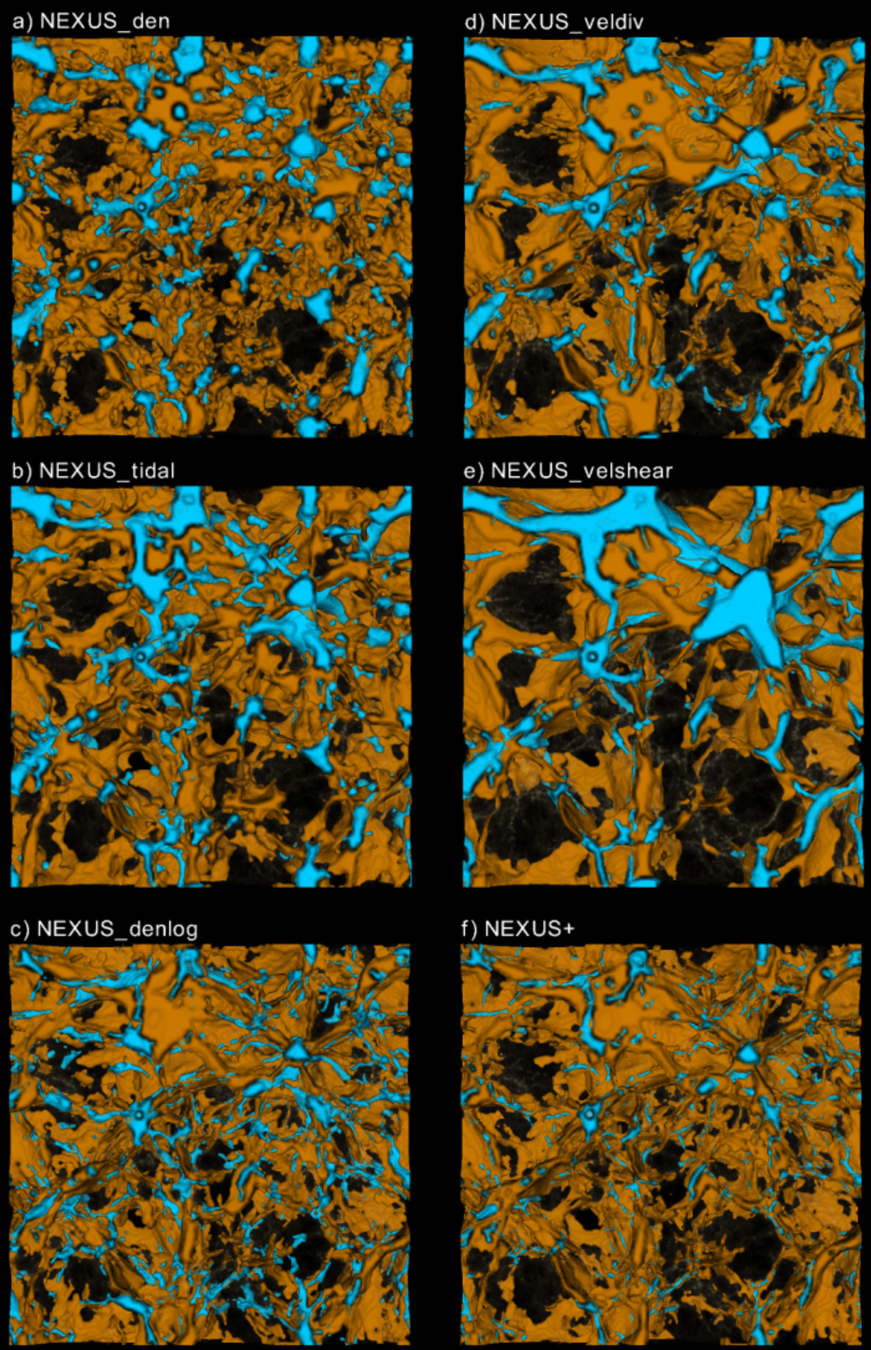


Nexus Filaments

- Nexus identification of filaments (blue)

- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:

- Density	- Velocity Divergence
- Tidal field	- Velocity Shear
- Log(density)	- Nexus+



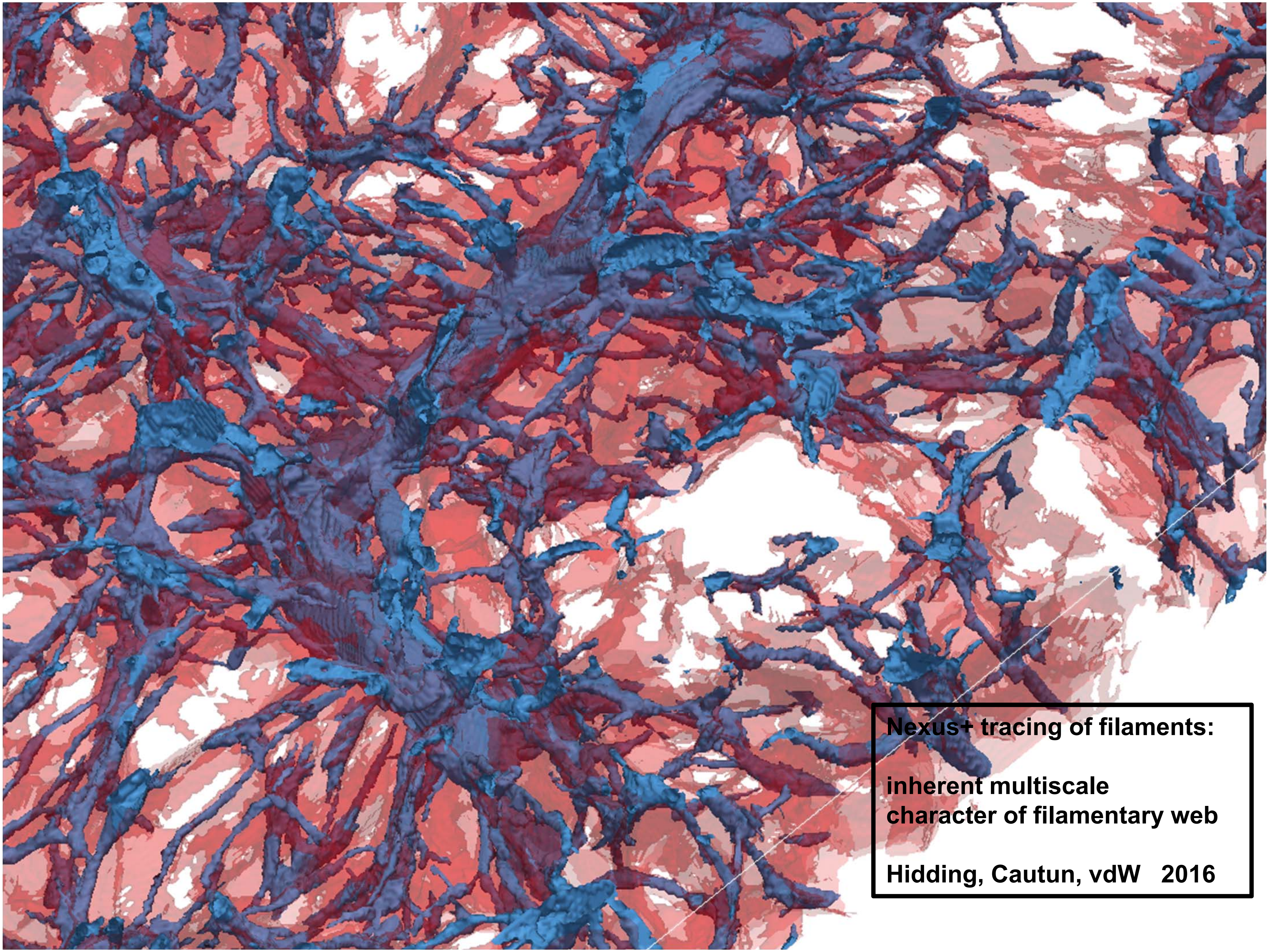
Nexus Walls

- Nexus identification of walls (orange)

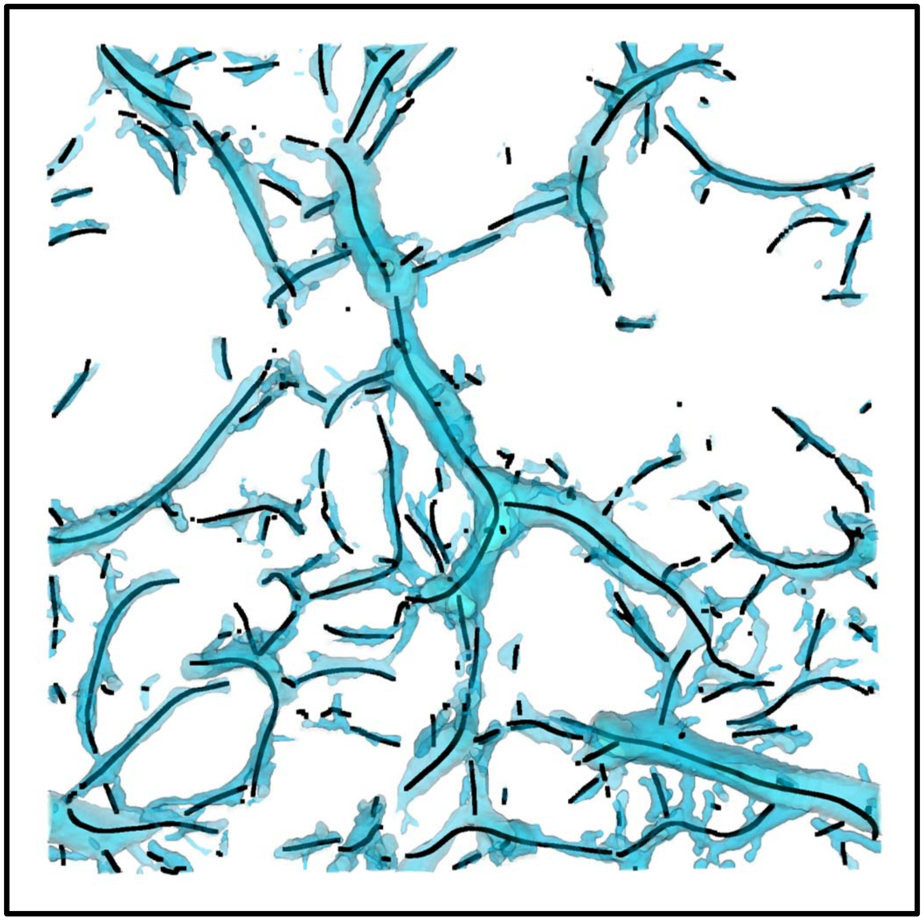
- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:

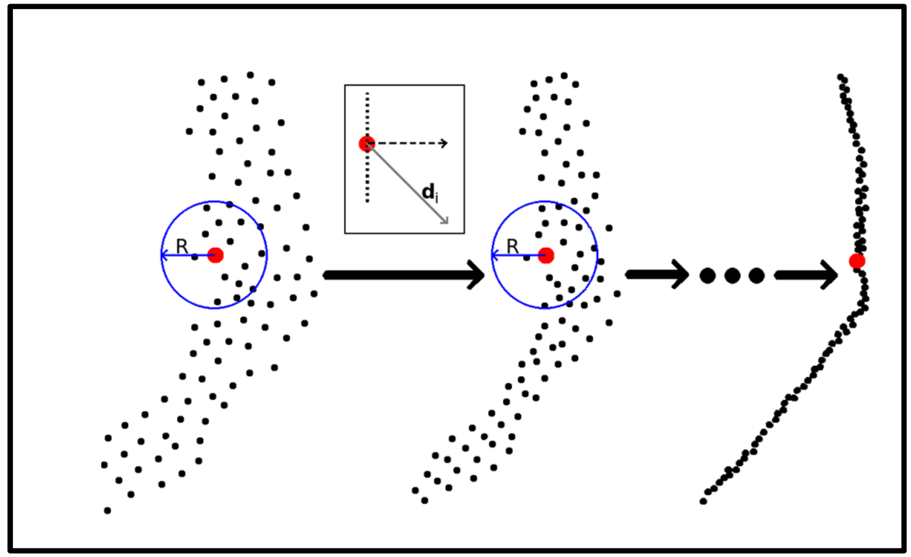
- Density - Velocity Divergence
- Tidal field - Velocity Shear
- Log(density) - Nexus+

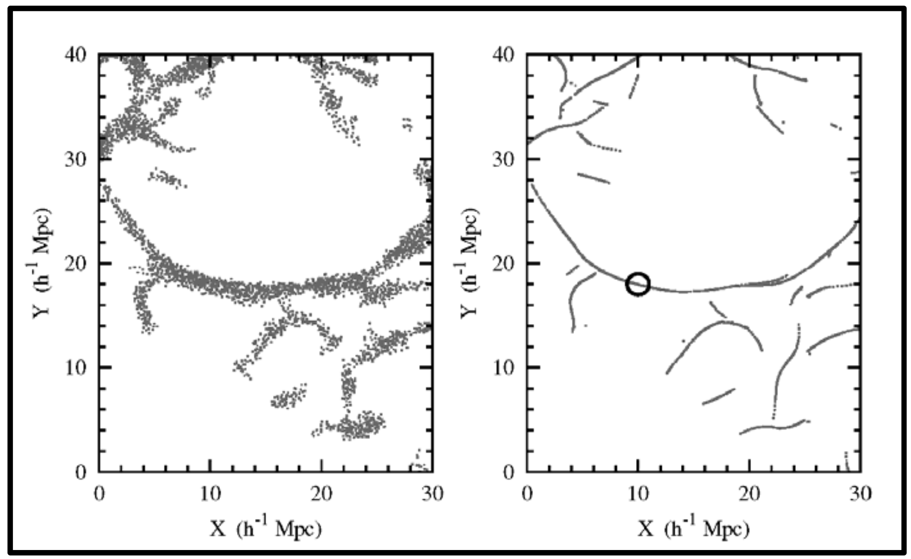
- from: Cautun et al. 2013

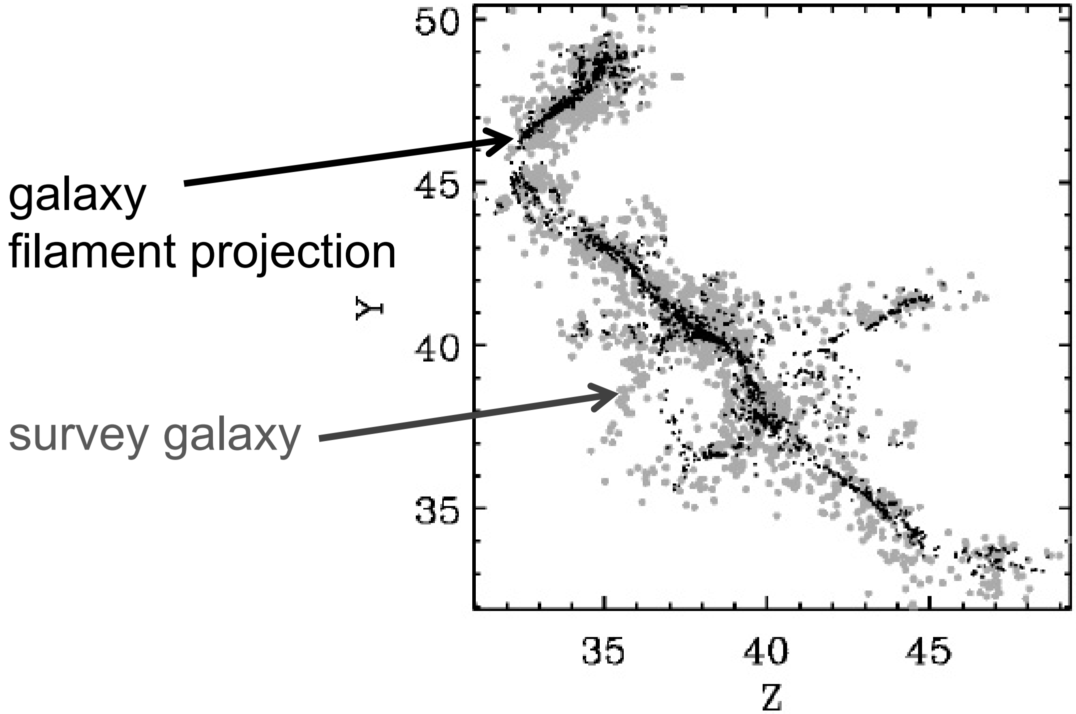

Nexus+ tracing of filaments:
inherent multiscale
character of filamentary web
Hidding, Cautun, vdW 2016

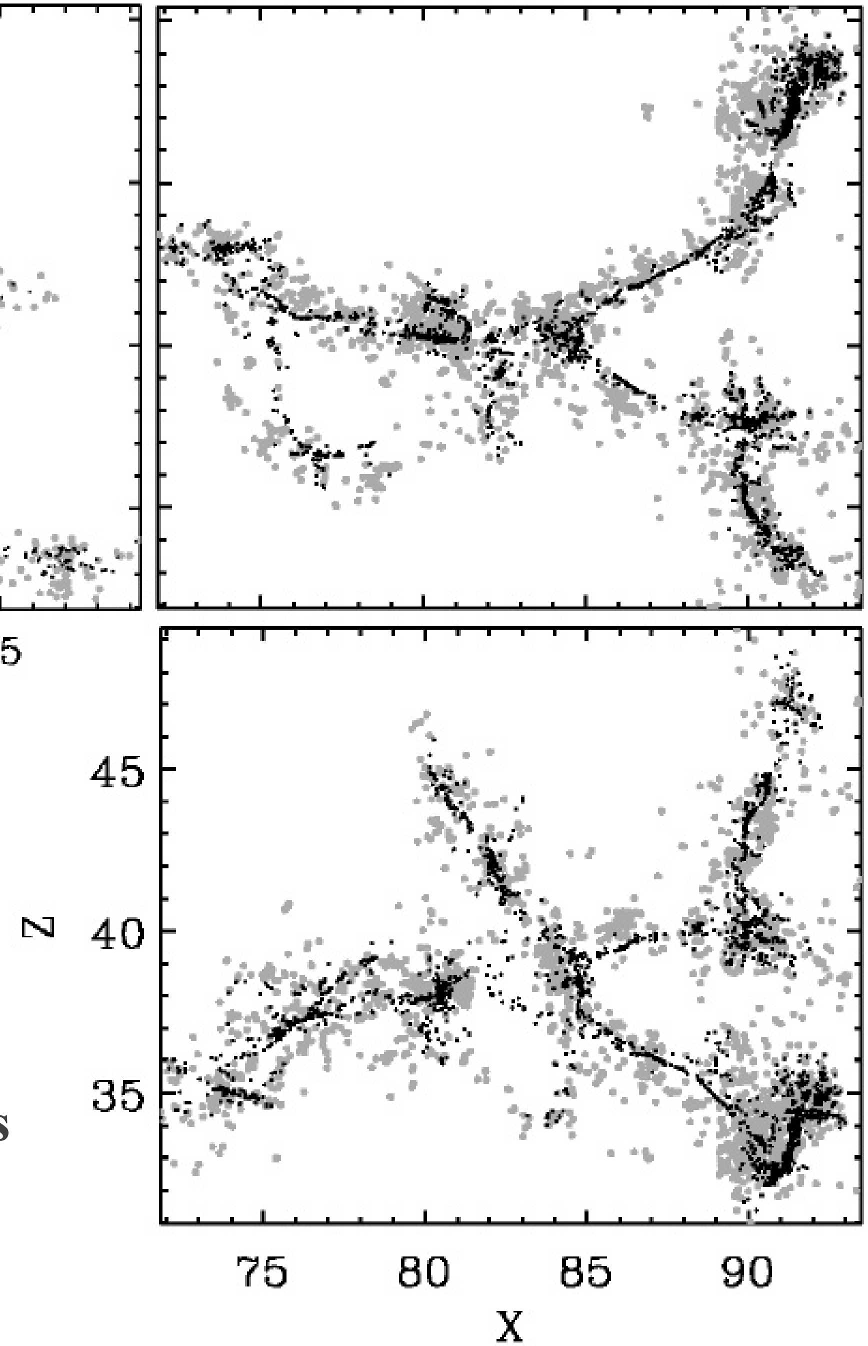
Spine of the Cosmic Web











- **Additional Analysis Step:**

  projection of filament galaxies
  to Medial Axis of Nexus/MMF filaments

Nexus/MMF:

Cosmic Web Characteristics

Aragon-Calvo, vdW & Jones 2010
Cautun, vdW, Jones & Frenk 2014

Cosmic Web:

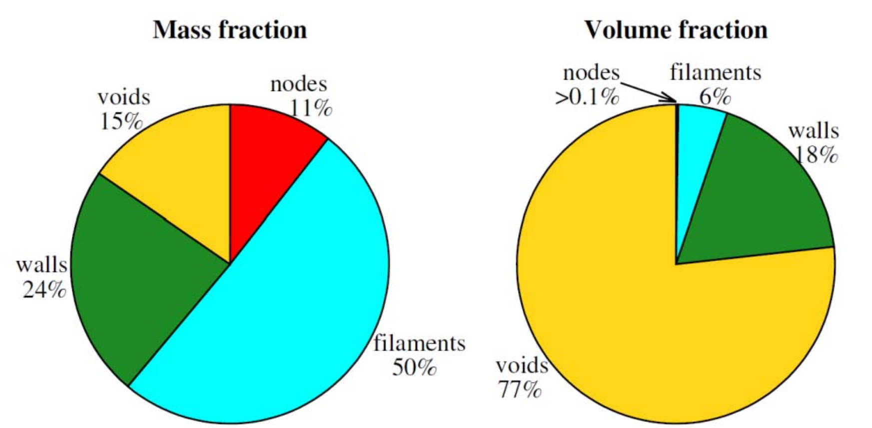
Density-Morphology Connection

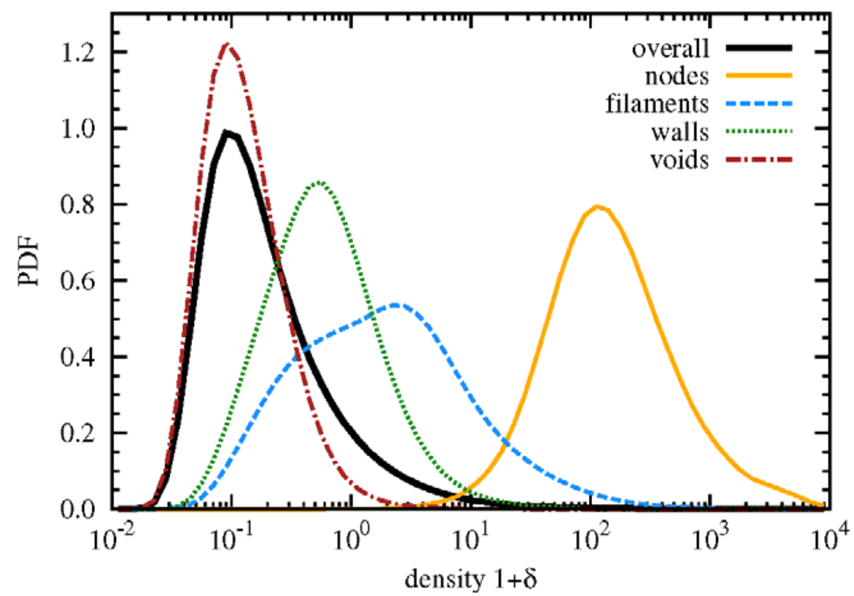
Mass & Volume content

Web morphologies

Density distribution

Individual morphologies

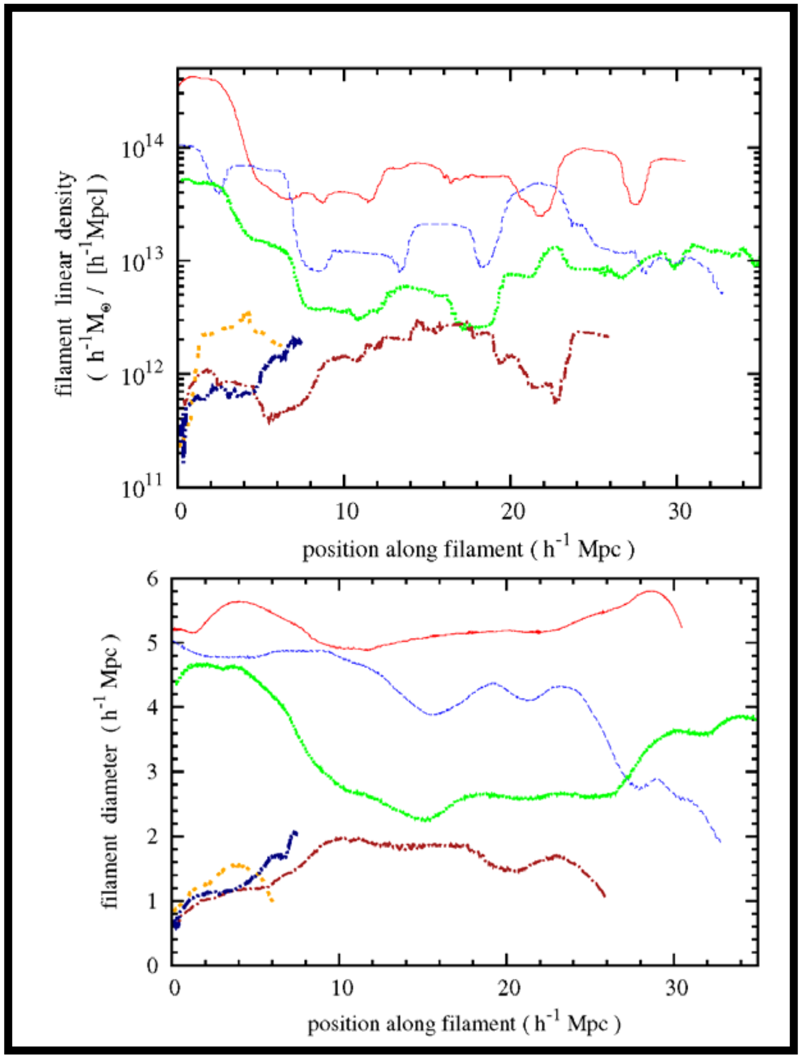


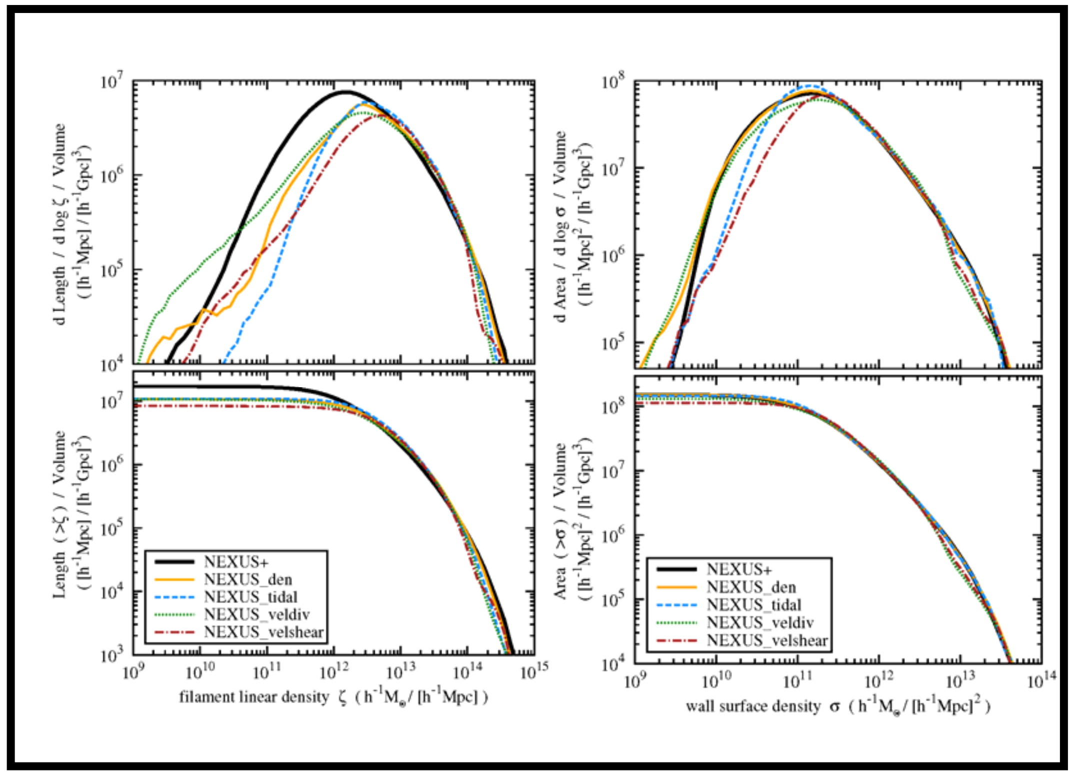


Cautun et al. 2014

Walls & Filaments

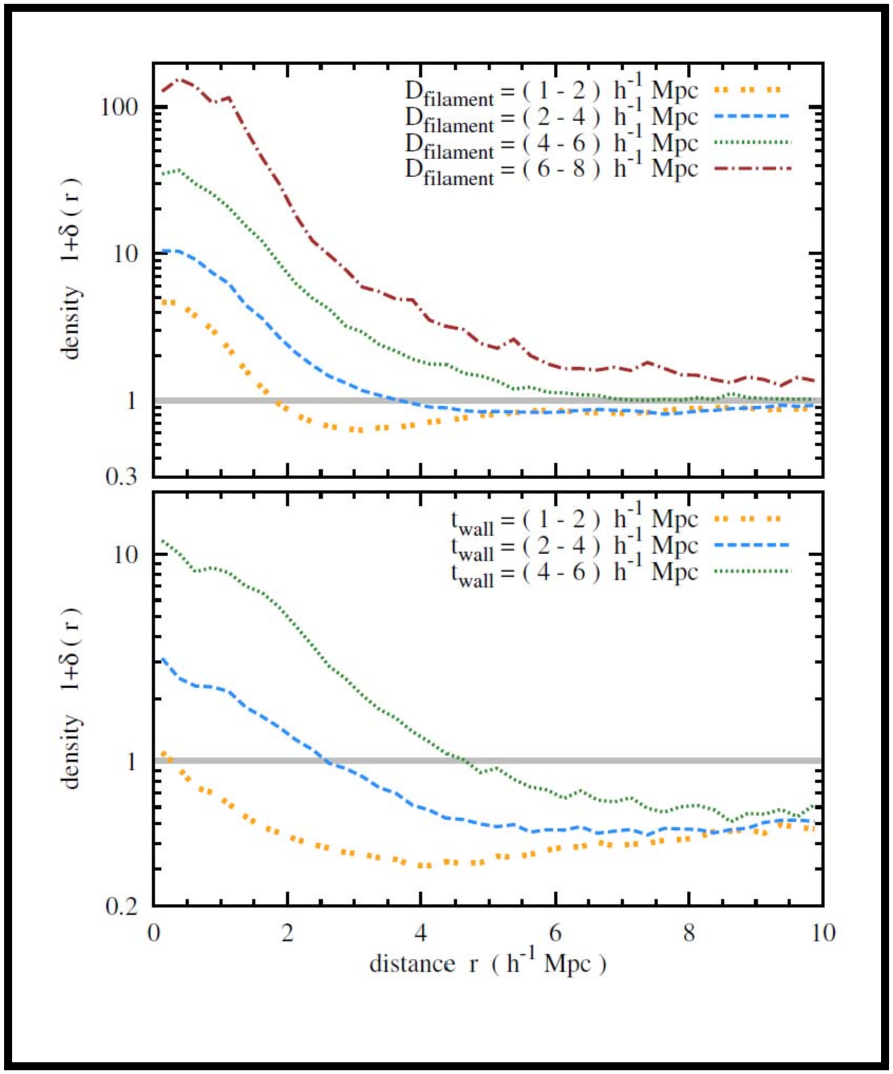
Internal Diameter & Density Distribution



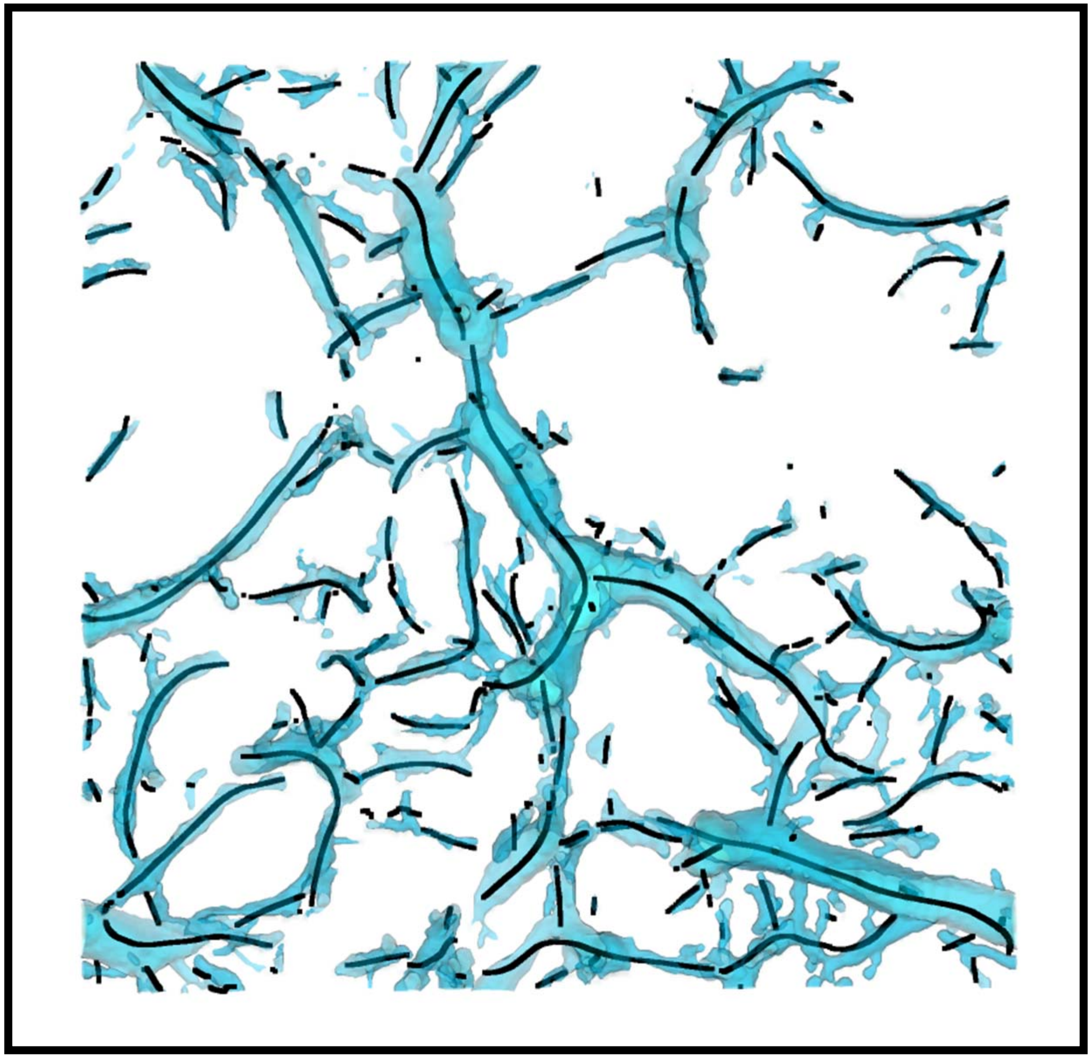


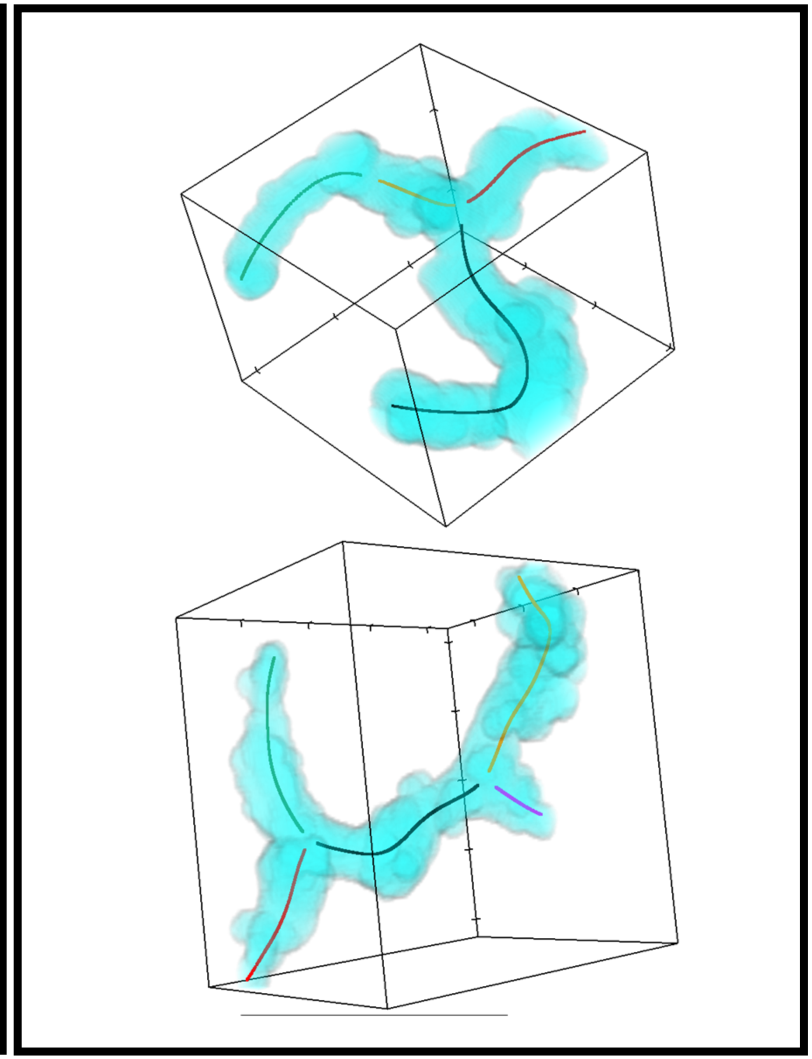
Walls & Filaments

Density Profiles



Filament Segmentation





V-web:

velocity (shear) flow field of the Cosmic Web

Hoffmann et al. 2012
Libeskind et al. 2013, 2014

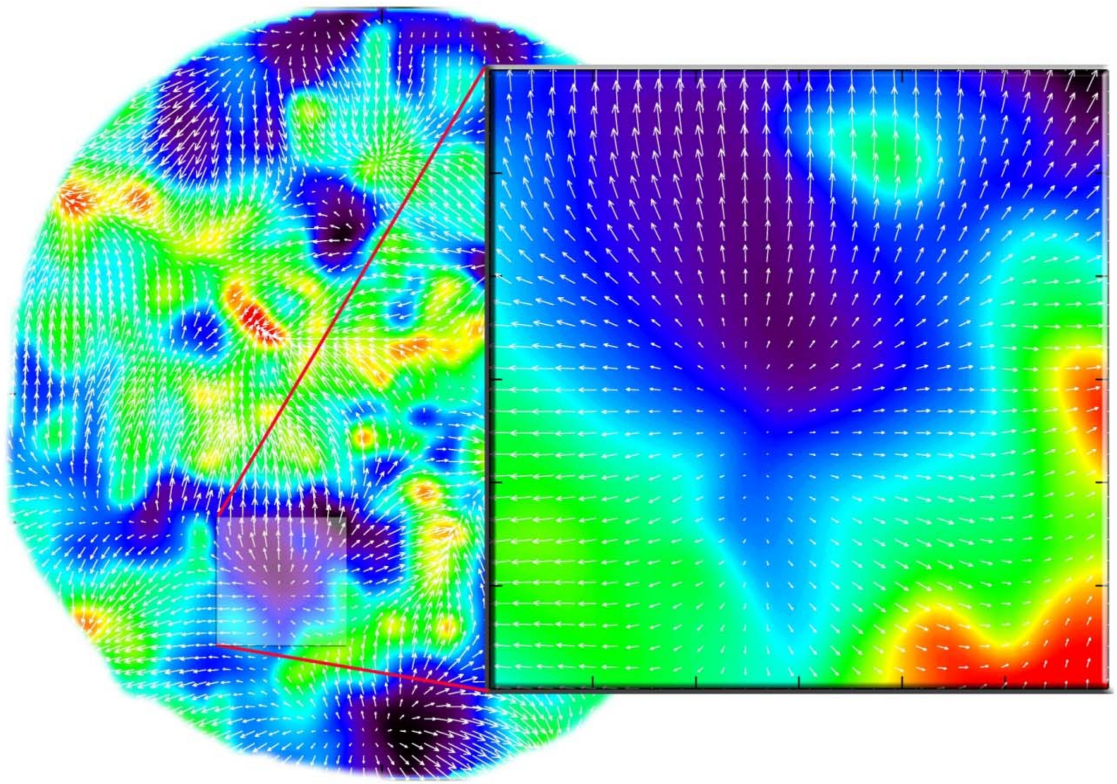
Large Scale Flows

Large-Scale Flows:

- Structure buildup accompanied by displacement of matter:
  - Cosmic flows
- On large (Mpc) scales, structure formation still in linear regime
- Directly related to cosmic matter distribution
- Note: redshift space distortion

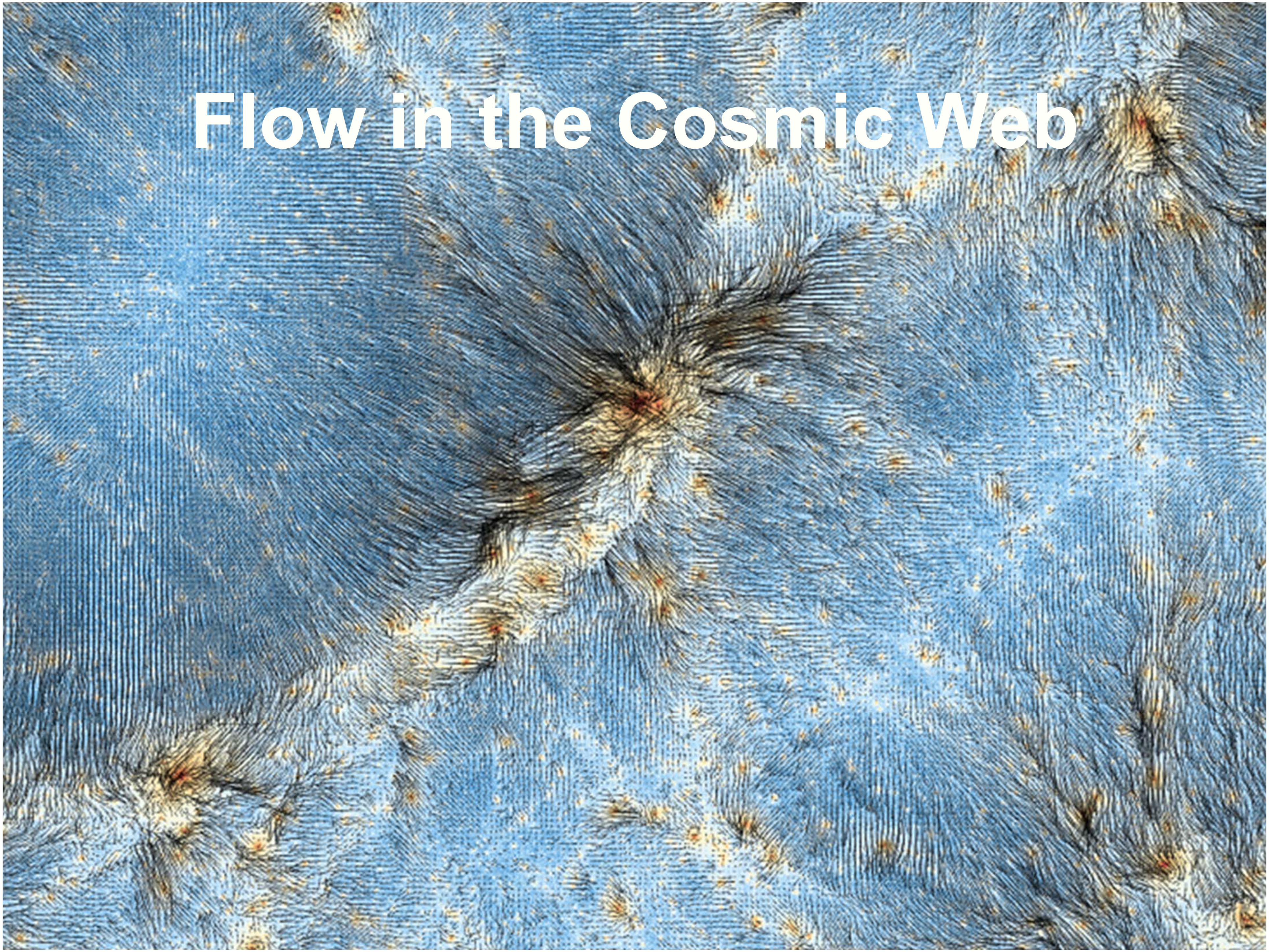
  $cz = Hr + v_{pec}$

  In principle possible to correct for this distortion, ie. to invert the mapping from real to redshift space
- Condition: entire mass distribution within volume should be mapped



$$\mathbf{v}(\mathbf{x},t) = \frac{H}{4\pi}\frac{f(\Omega_m)}{b}\, a \int d\mathbf{x}'\, \delta_{gal}(\mathbf{x}',t)\, \frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}$$

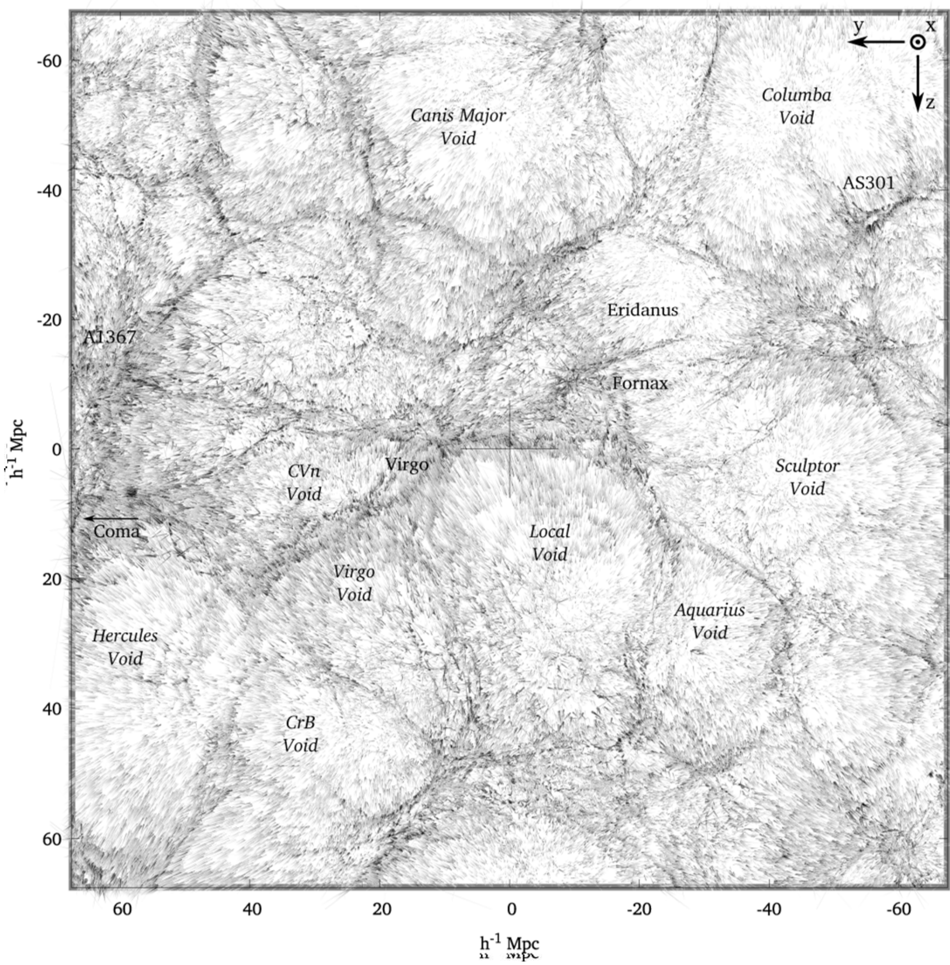
Flow in the Cosmic Web

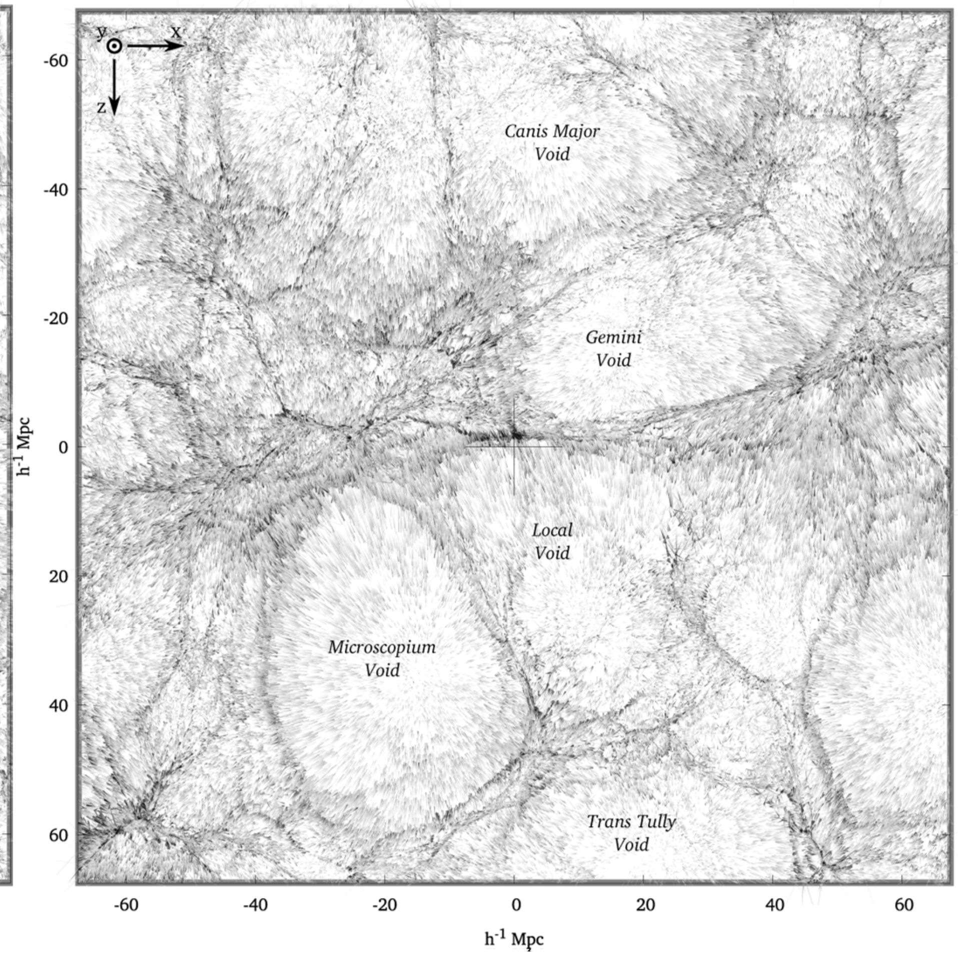


Supergalactic Plane

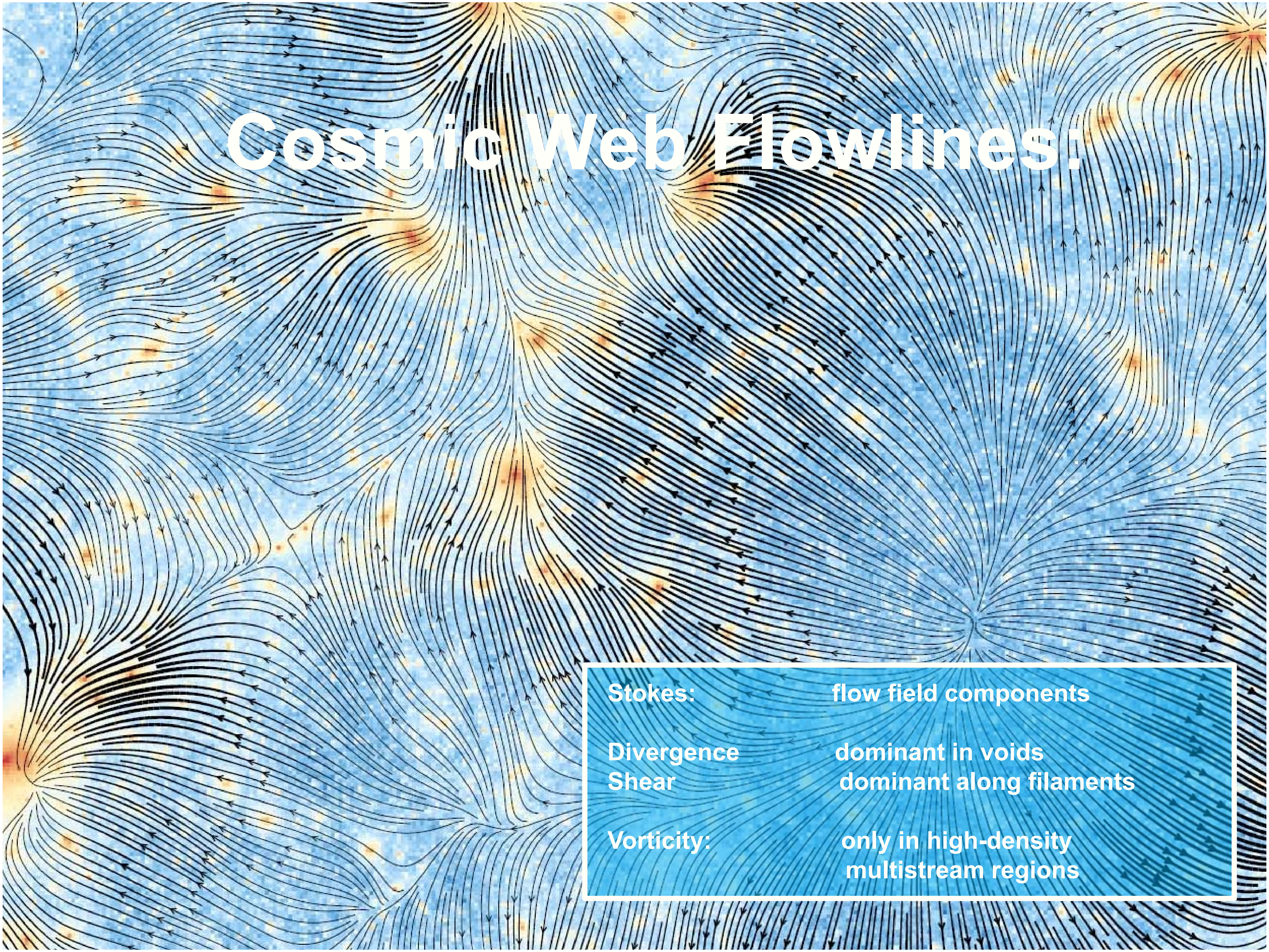
mean KIGEN - adhesion reconstruction

Hidding, Kitaura, vdW & Hess 2016/2017

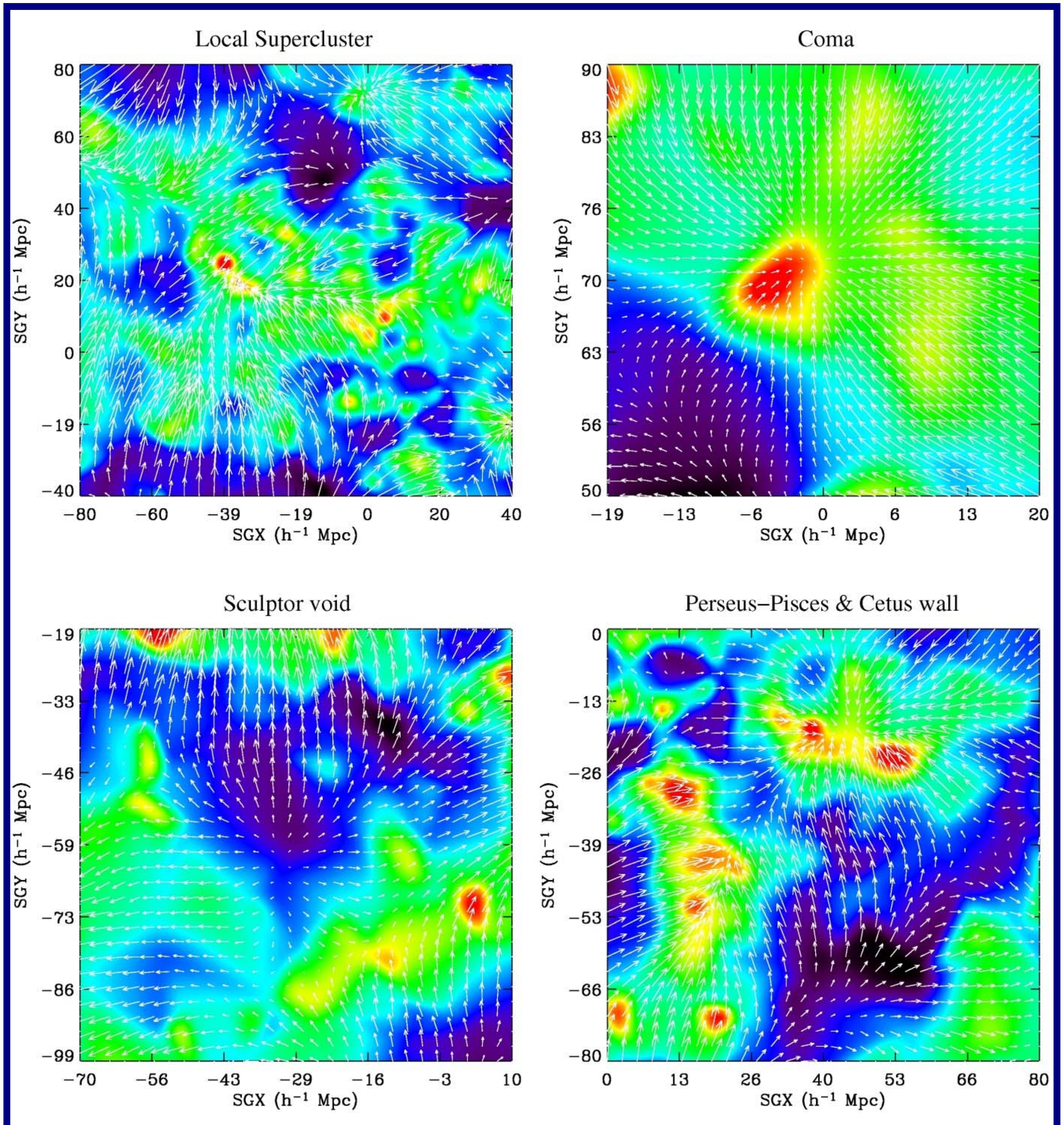




Cosmic Web Flowlines:

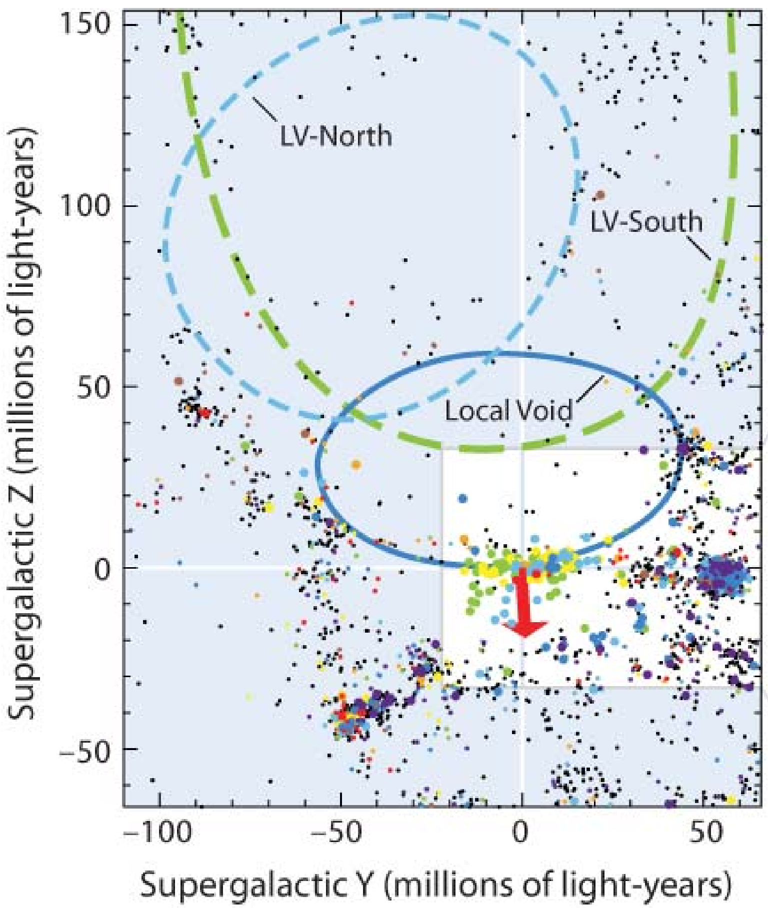


Stokes:	flow field components
Divergence	dominant in voids
Shear	dominant along filaments
Vorticity:	only in high-density multistream regions

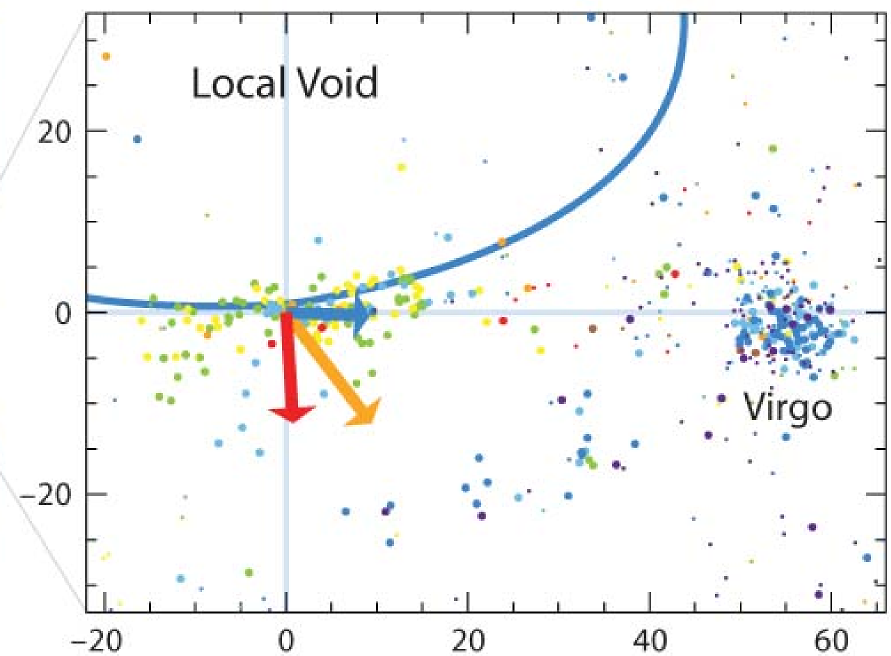


Romano-Diaz & vdW 2007

Push of the Local Void



Our motion with the respect to galaxies in the Local Supercluster *Tully et al. 2008, ApJ, 676, 184*



Tully et al. 2008:
Local Void pushes with ~260 km/s against our local neighbourhood

Stokes' Flow Theorem

A velocity field can be decomposed into 3 basic components, according to the gradient of the flow field:

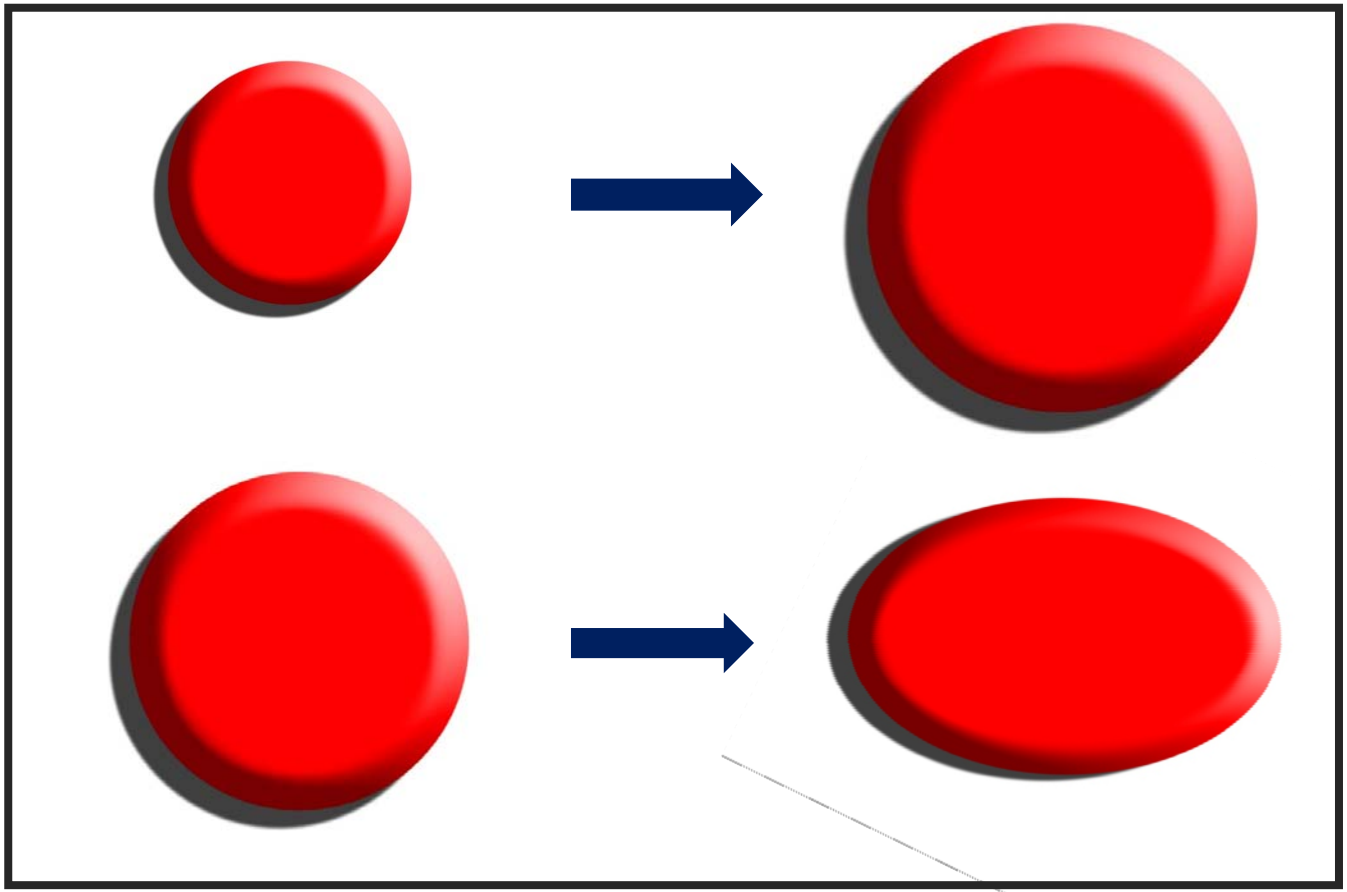
the velocity divergence, shear and vorticity in each tetrahedron.

$$\theta = \frac{1}{H}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$	**Divergence**
$$\sigma_{ij} = \frac{1}{2}\left\{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right\} - \frac{1}{3}(\nabla \cdot \mathbf{v})\,\delta_{ij}$$	**Shear**
$$\omega_{ij} = \frac{1}{2}\left\{\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right\}$$	**Vorticity**

Stokes' Flow Theorem

A velocity field can be decomposed into 3 basic components, according to the gradient of the flow field:

the velocity divergence, shear and vorticity in each tetrahedron.



Divergence:

Expansion/ Contraction

Shear:

Deformation

Shear Tensor:
Eigenvalues & Deformation directions

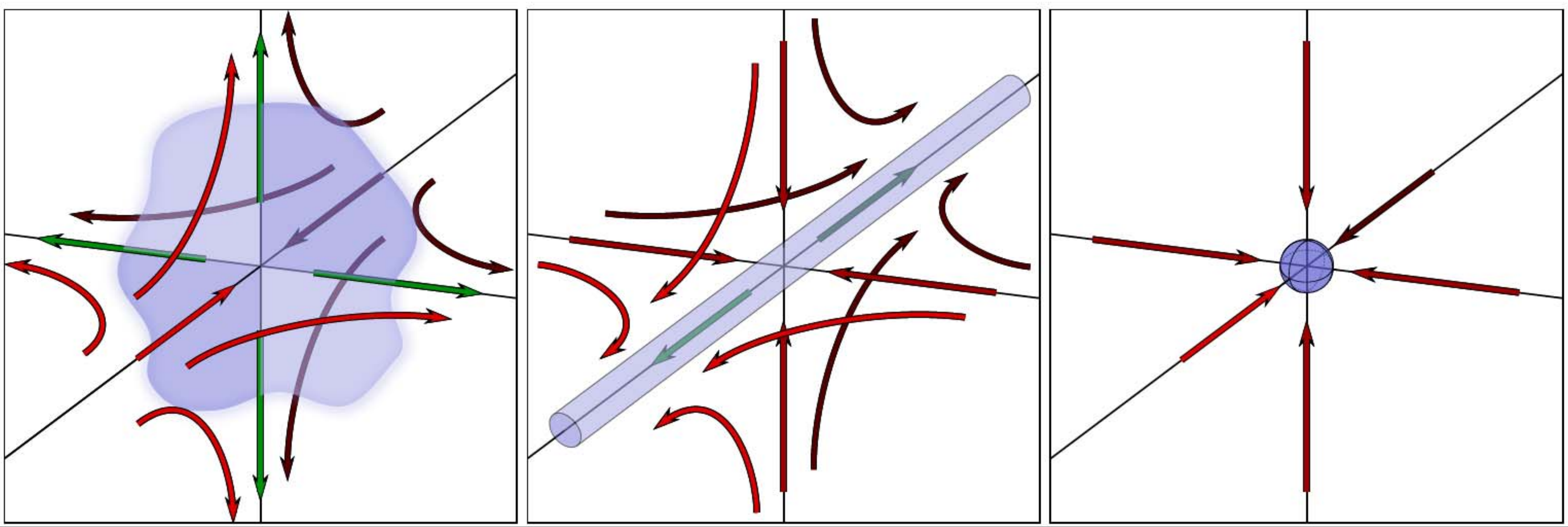
$$\sigma_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\left(\vec{\nabla} \cdot \vec{u}\right)\delta_{ik}$$

$$\Rightarrow \quad \sigma_1\,,\ \sigma_2\,,\sigma_3$$

Wall
Inflow: 1 direction
Outflow: 2 directions

Filament
Inflow: 2 directions
Outflow: 1 direction

Cluster node
Inflow: 3 direction

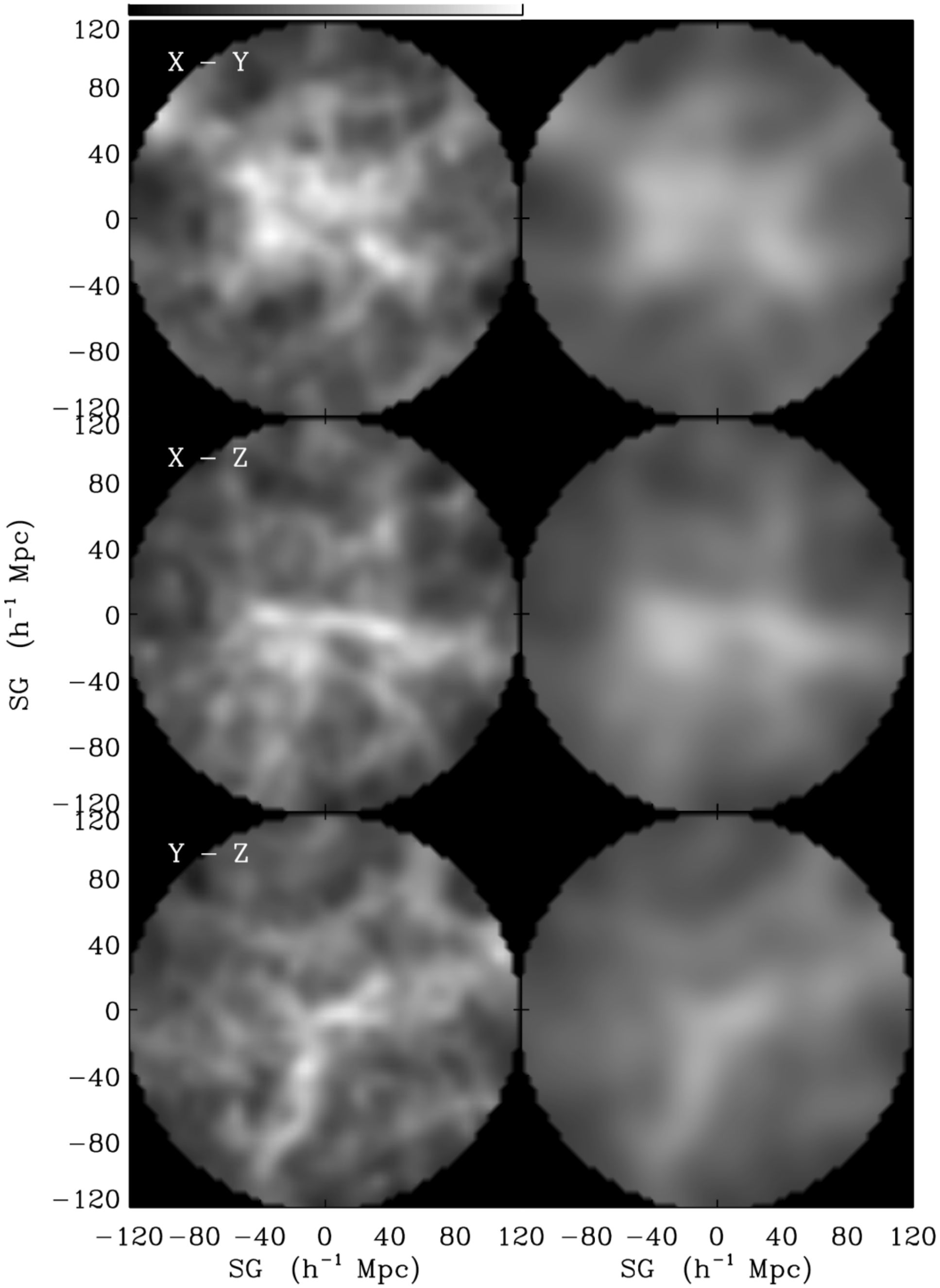


PSCz
Divergence & Shear

Romano-Diaz & vdW 2007

Velocity Shear

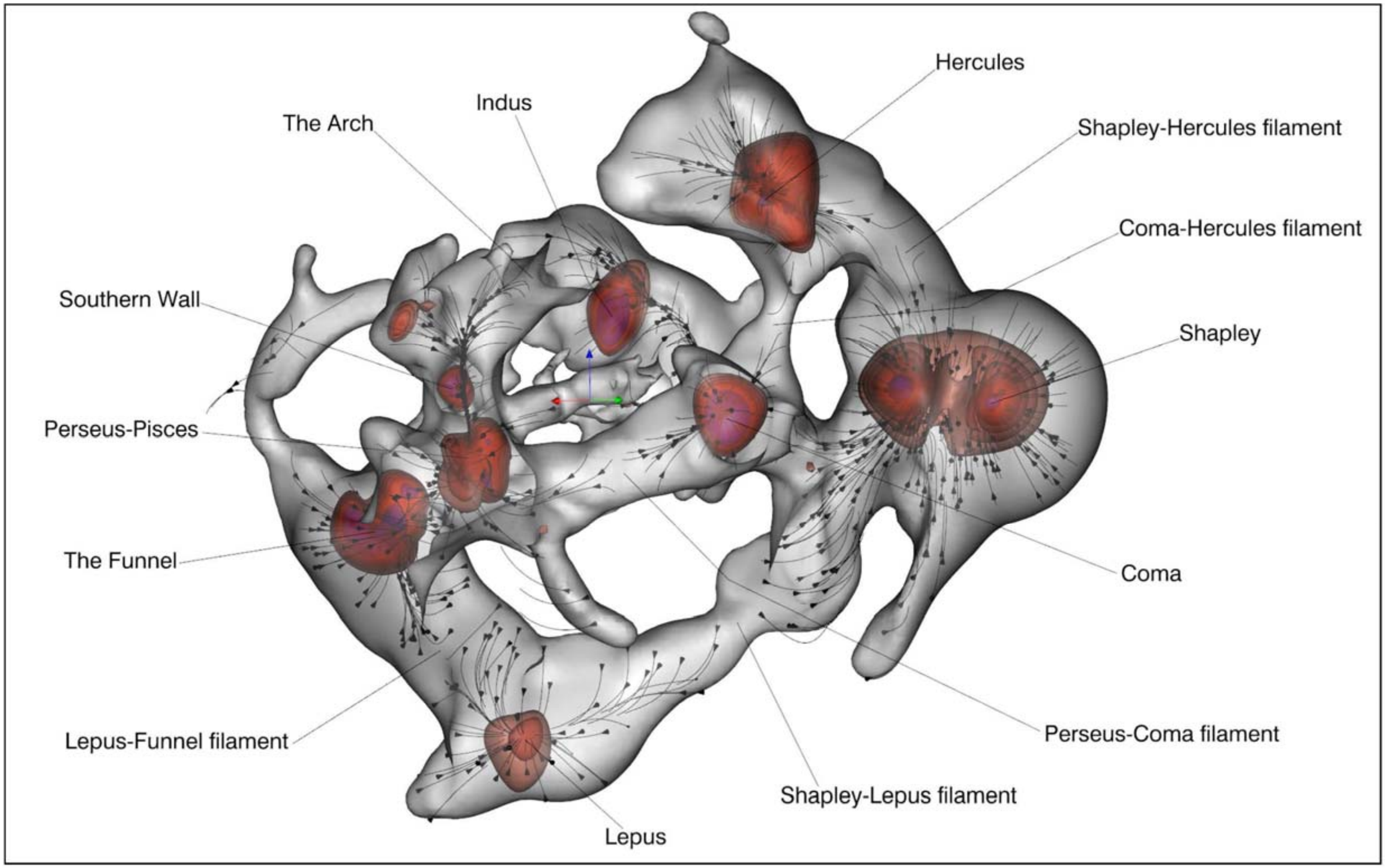
Field



Resolution:
R_G= 3.0h^{-1} Mpc (left)
R_G=10.0h^{-1} Mpc (right)

Romano-Diaz & vdW 2007

CosmicFlows-3



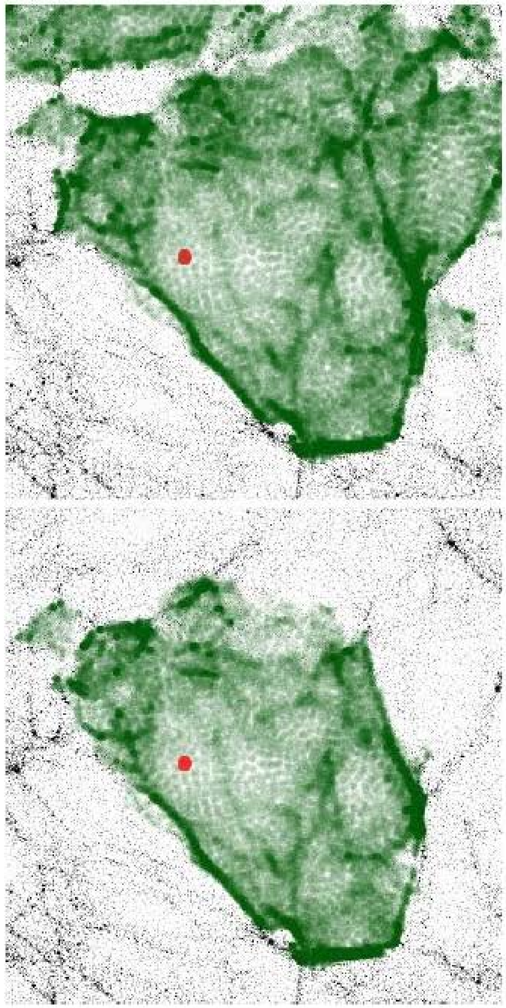
Cosmic Web morphology:
velocity shear based V-web identification flow pattern in cosmic web
(Pomarede et al. 2017)

Watershed

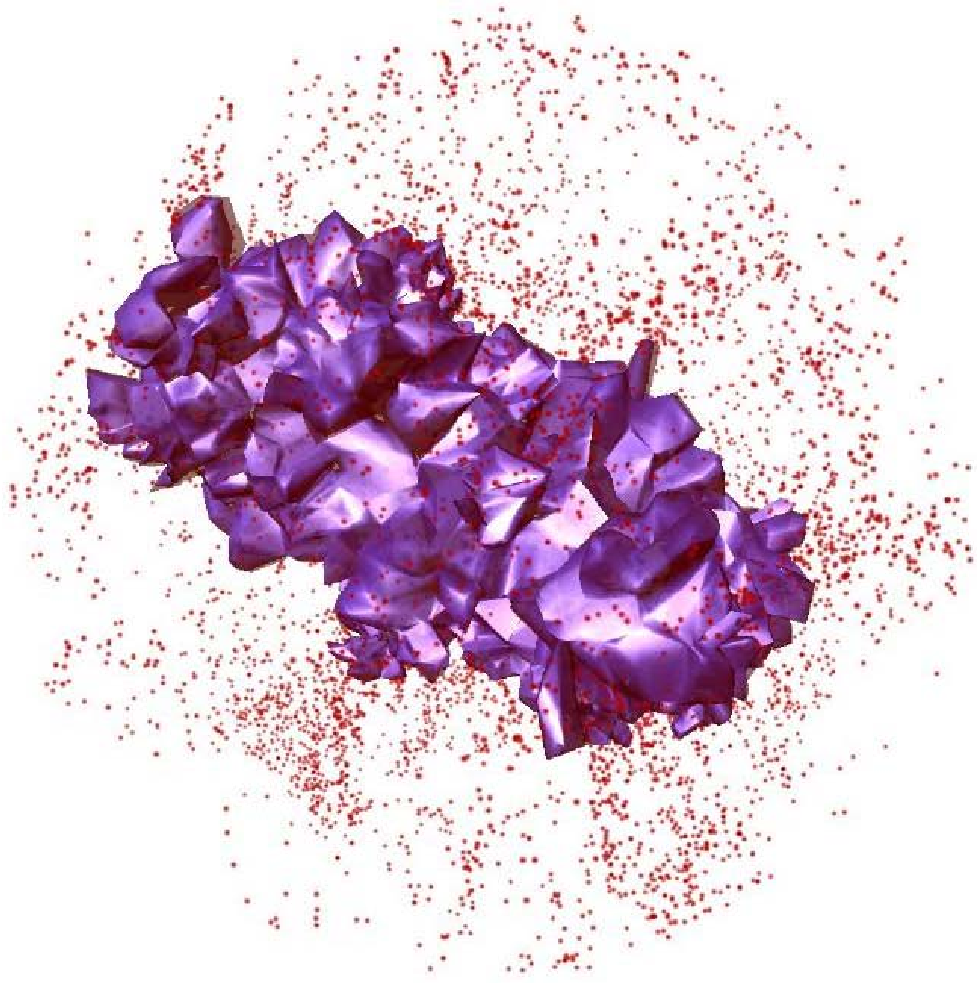
Void Identification

Platen, vdW & Jones 2007

Definition of voids (Voronoi density & watershed)



WVF: Platen et al. 2007
ZOBOV: Neyrinck 2008



Sutter, Lavaux, Wandelt, Weinberg 2012

The Multiscale Watershed Void Finder

No exact definition of a void!
→ broad range and variety of void detection techniques

Our void finder:

- closely follows real geometry cosmic web
- no assumptions geometry void
- no user defined parameters

→ Watershed Void Finder by *Platen et al.*, 2007.

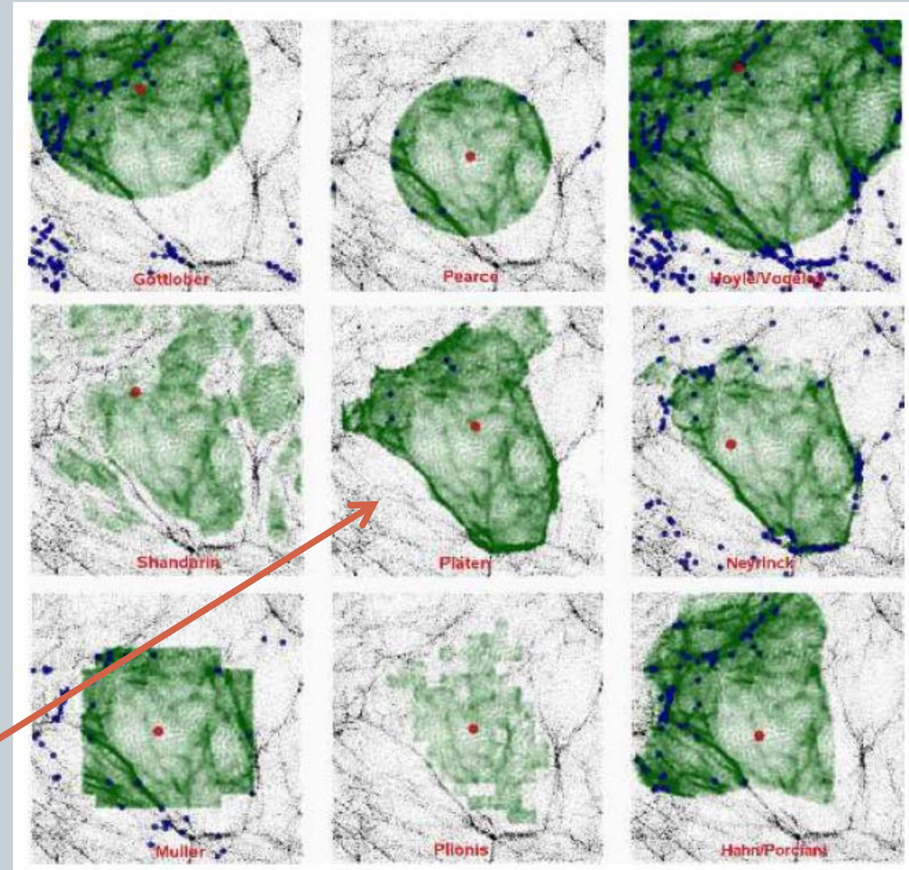
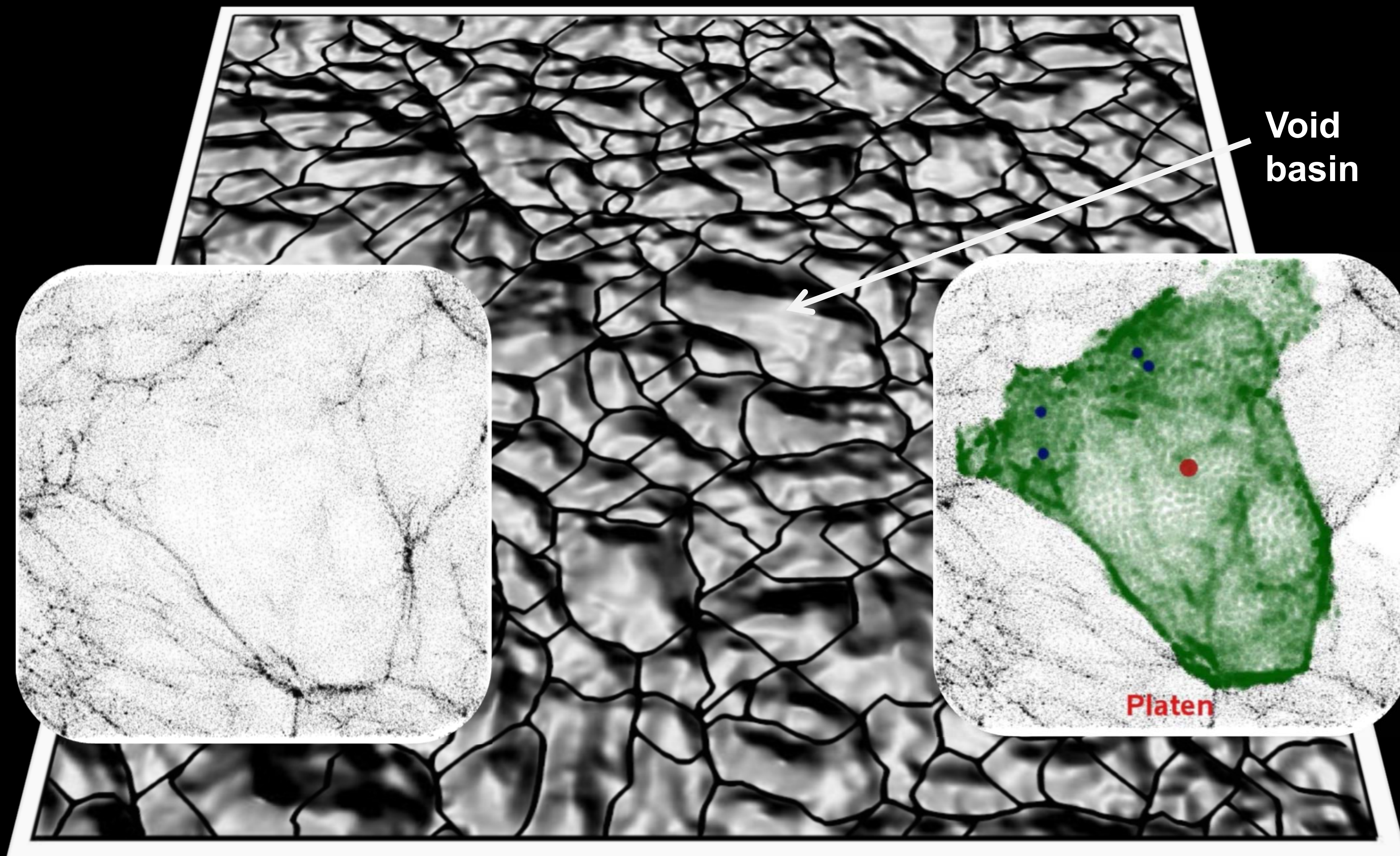


Figure from *Colberg et al.*, 2008

Watershed Void Identification



Watershed Void Transform

Segmentation:

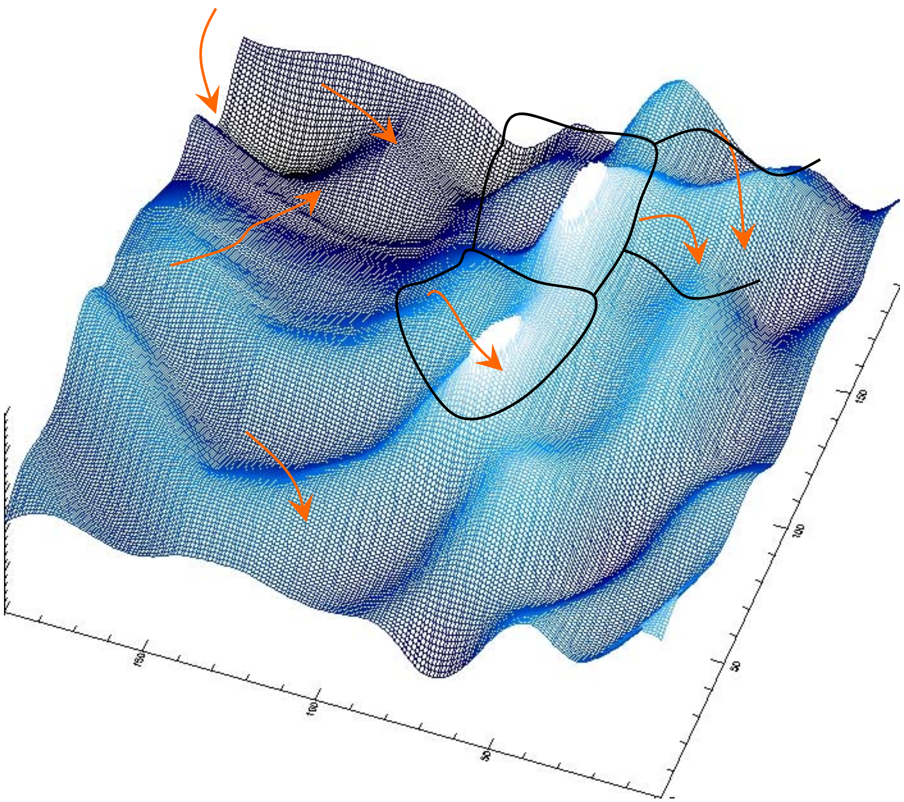
A division of space in individual cells

WATERSHEDS:

A cell is the union of points that are topological closer to a certain minimum

Topological Distance:

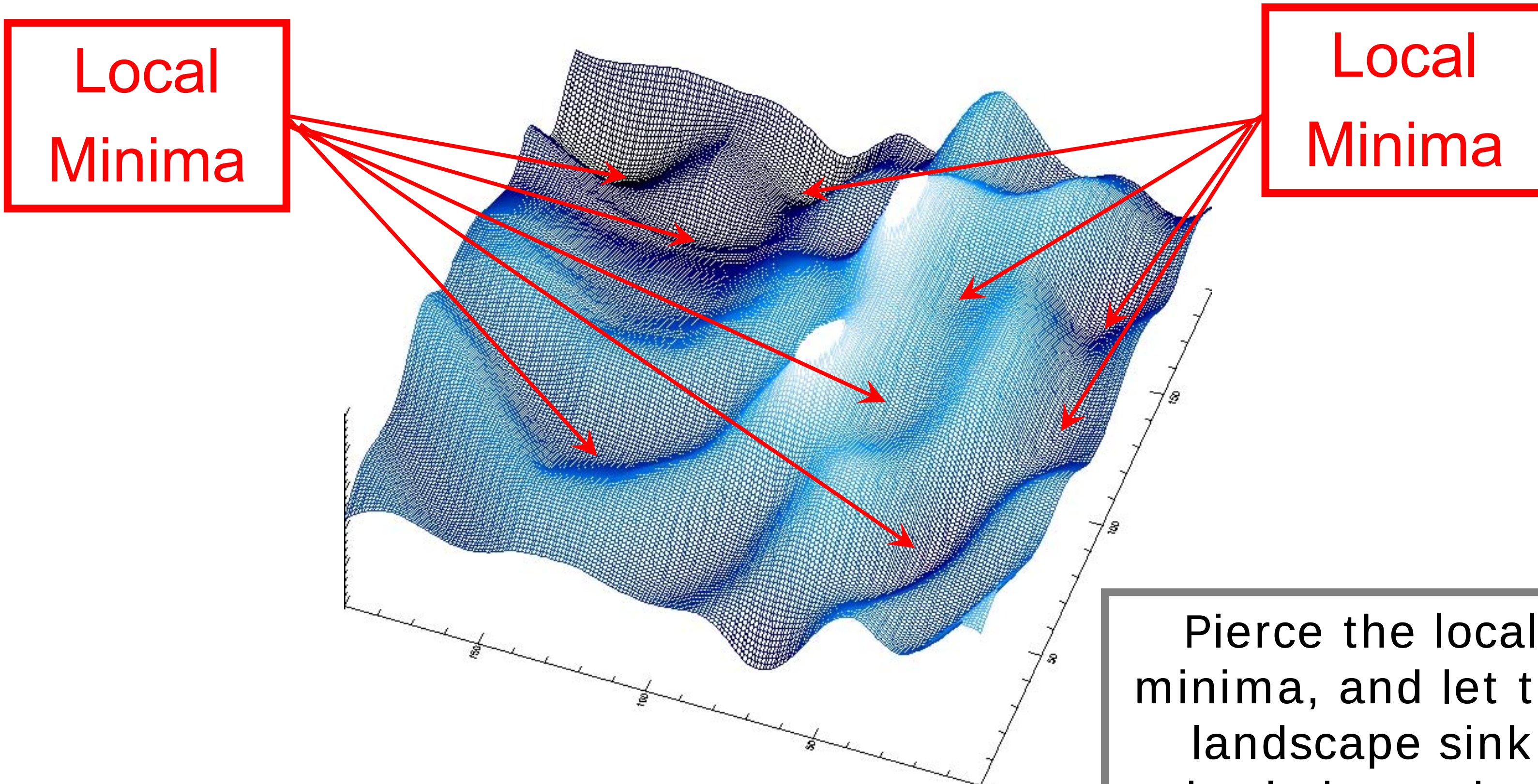
The path that connects two points via the steepest slope: the path a water-droplet would take, when running down a landscape



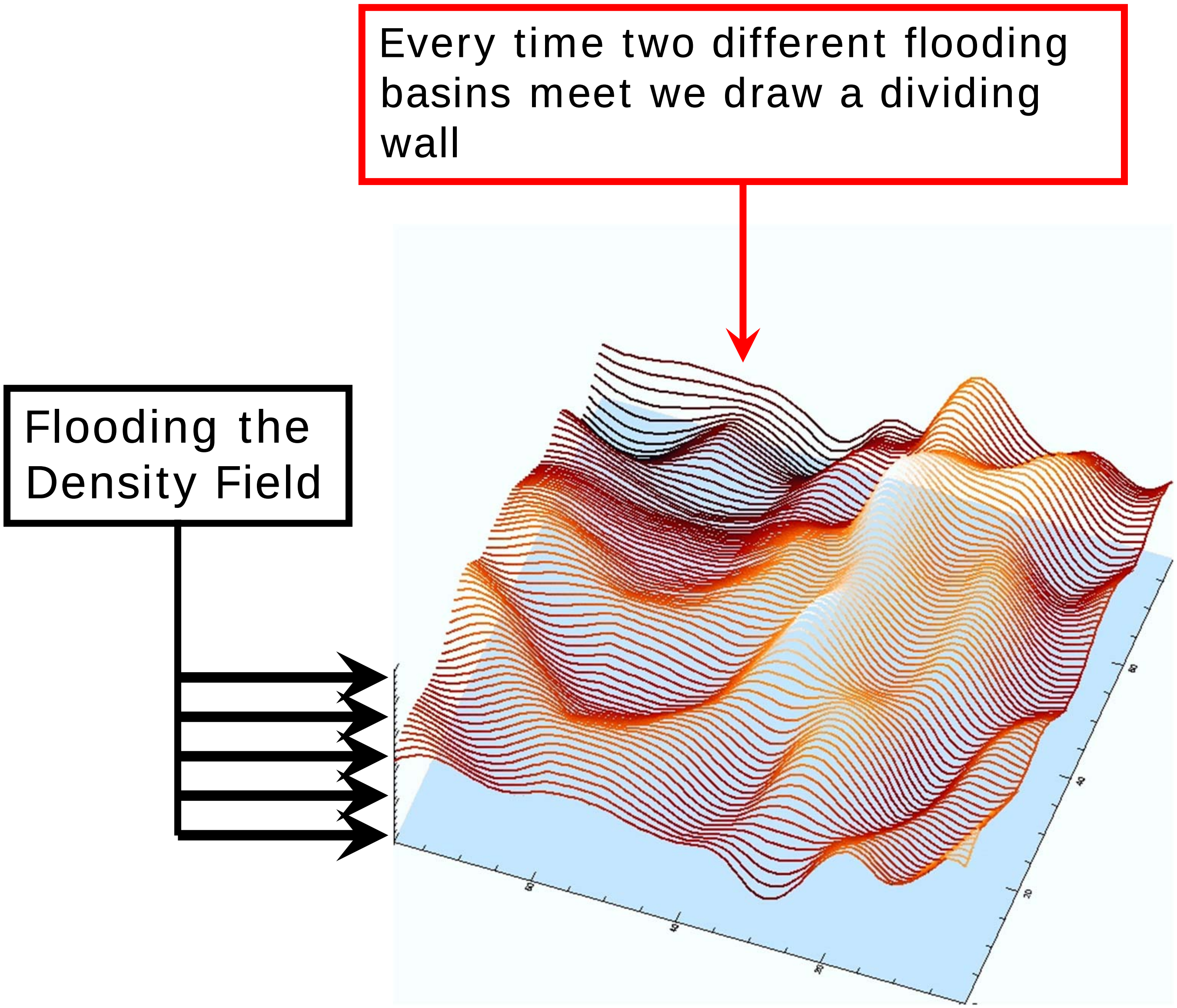
Following the water-flow into the distinct catchment basins.

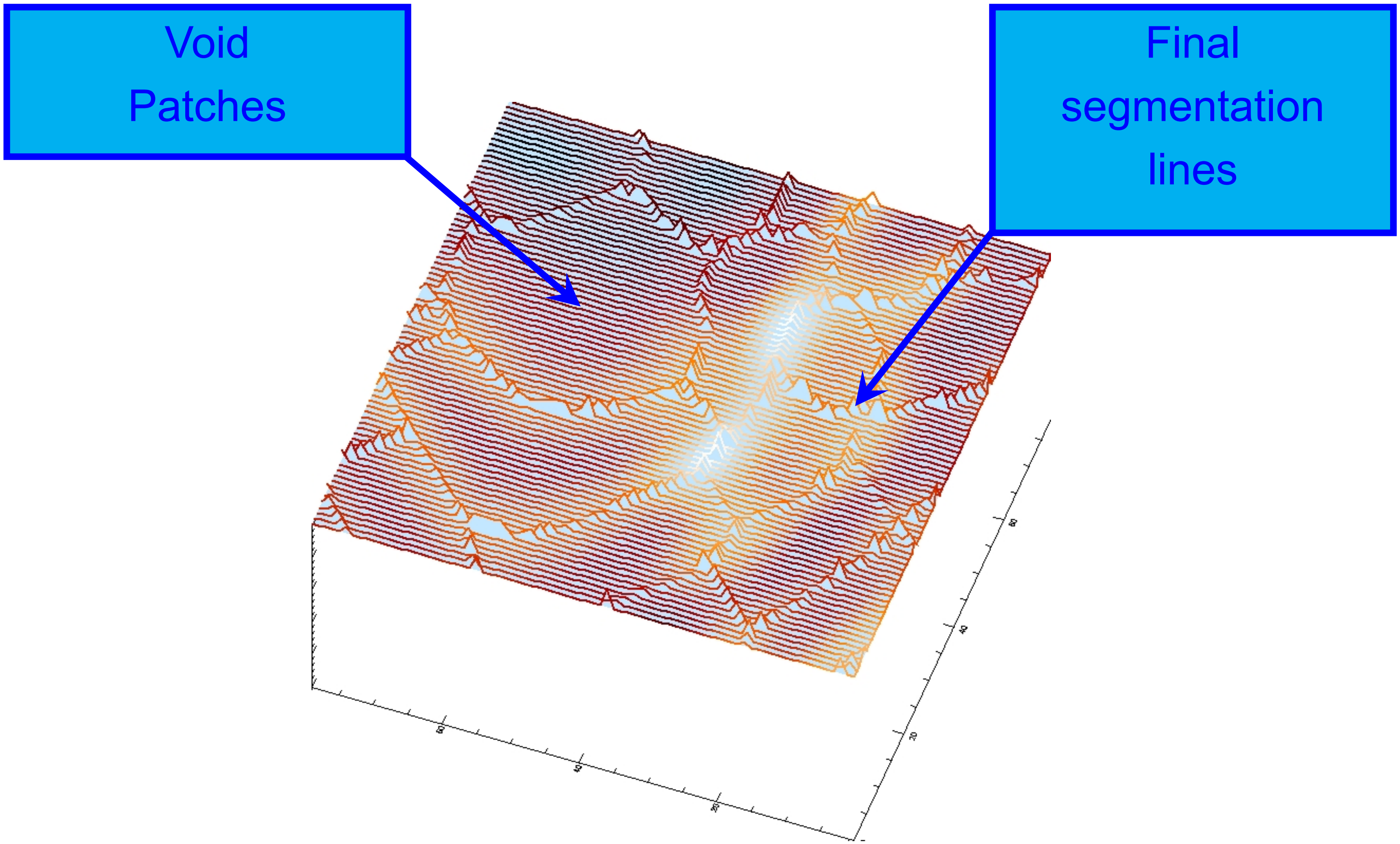
Each basin belonging to one individual minima defines one region

Surface of Density Field

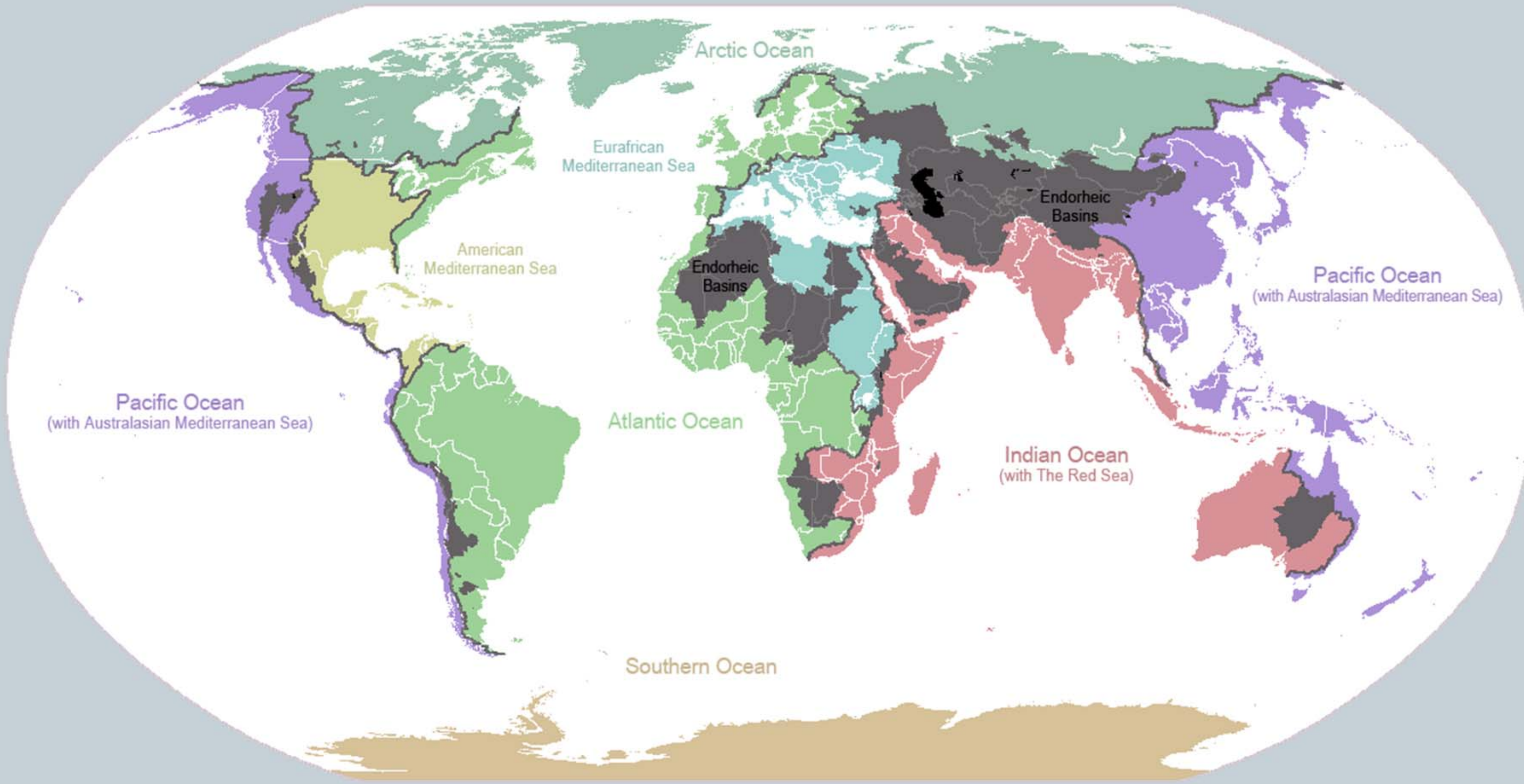


Pierce the local minima, and let the landscape sink slowly in a tub of water


Every time two different flooding basins meet we draw a dividing wall
Flooding the Density Field

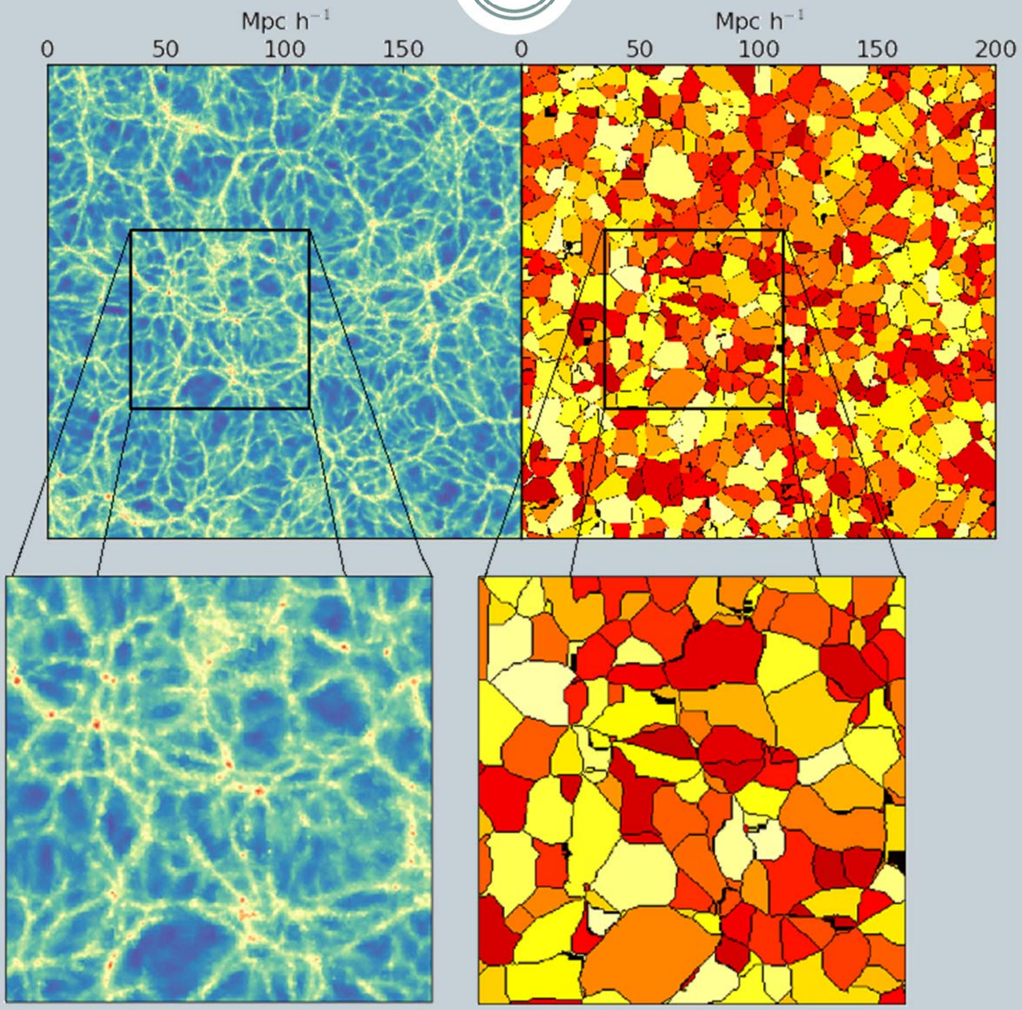

Void
Patches
Final
segmentation
lines

The Multiscale Watershed Void Finder



- local height → local density
- mountain ridges → walls and filaments
- watershed basins → voids

WVF: Watershed Void Finder




COSI Columbus Science Center:
Hands-On Voids by Watershed

Void persistence and merger trees

Adhesion model

Void evolution in idealized adhesion model:

- self gravity of walls and filaments modelled by artificial viscosity ν
- discards nonlinear evolution on smaller scales
- models hierarchical evolution very good

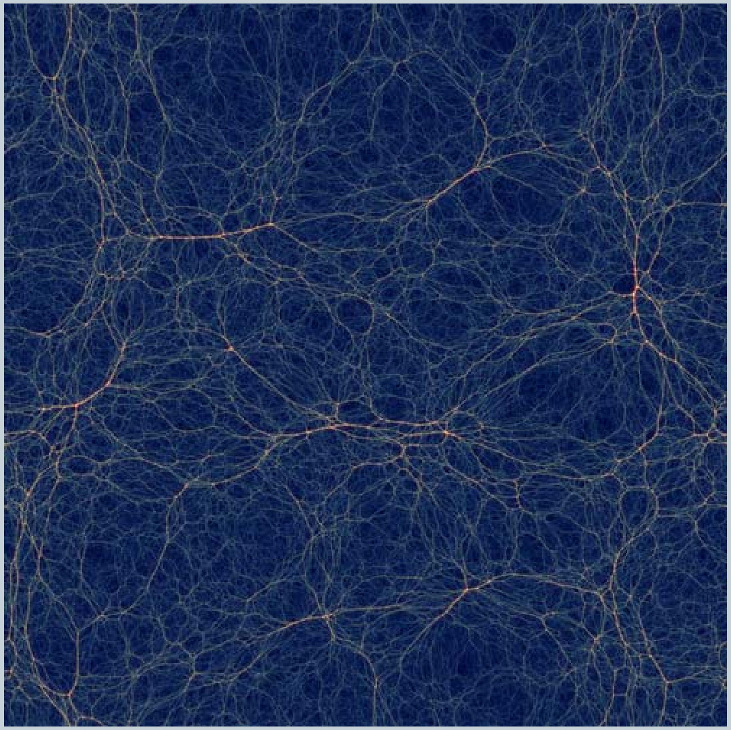
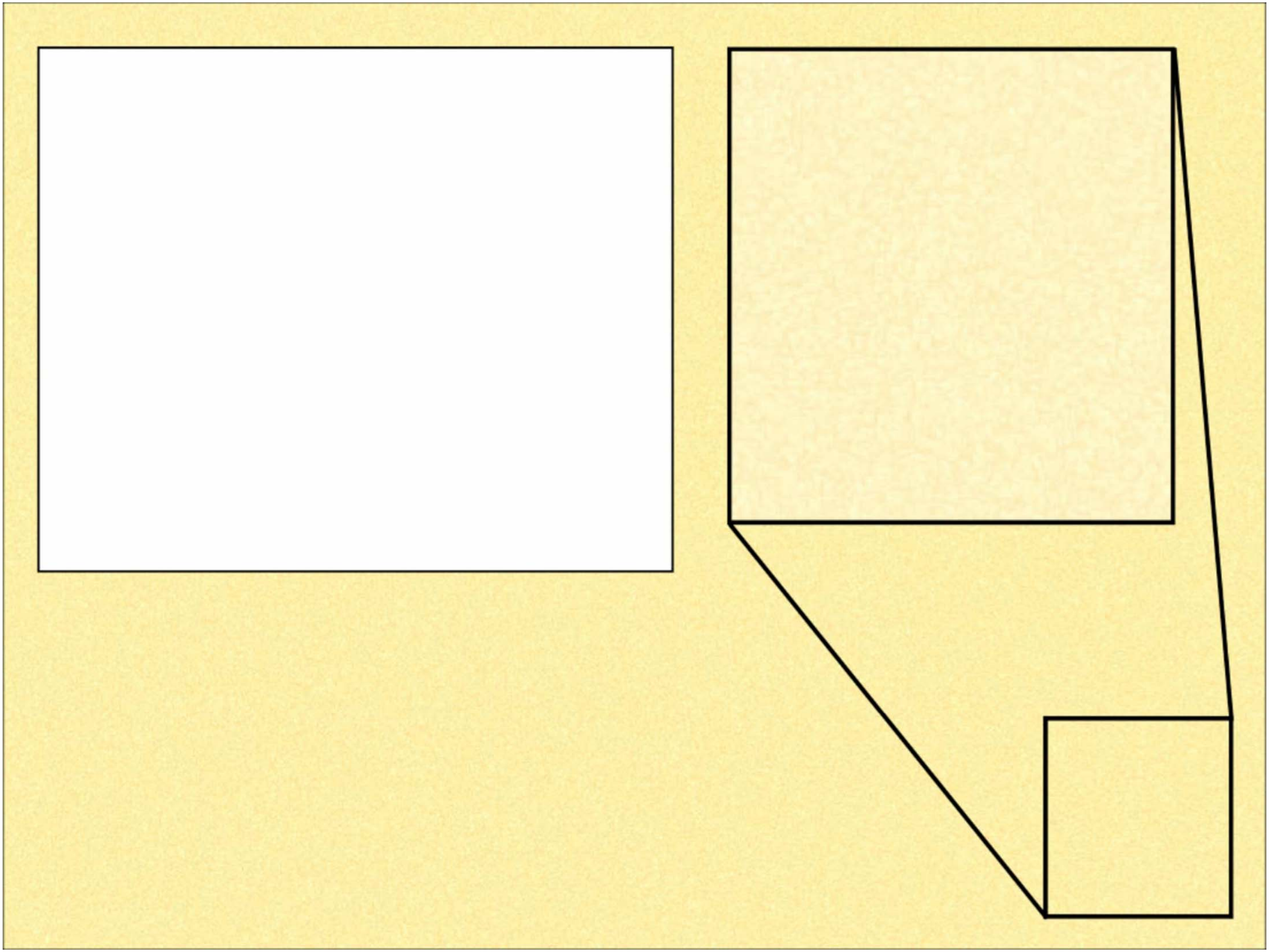


Image courtesy: Johan Hidding

2 adhesion models

- $P(k) \propto k^1$
- $P(k) \propto k^{-1}$

Zel'dovich, 1970
Gurbatov, Saichev and Shandarin, 1989
Hidding et al., 2012



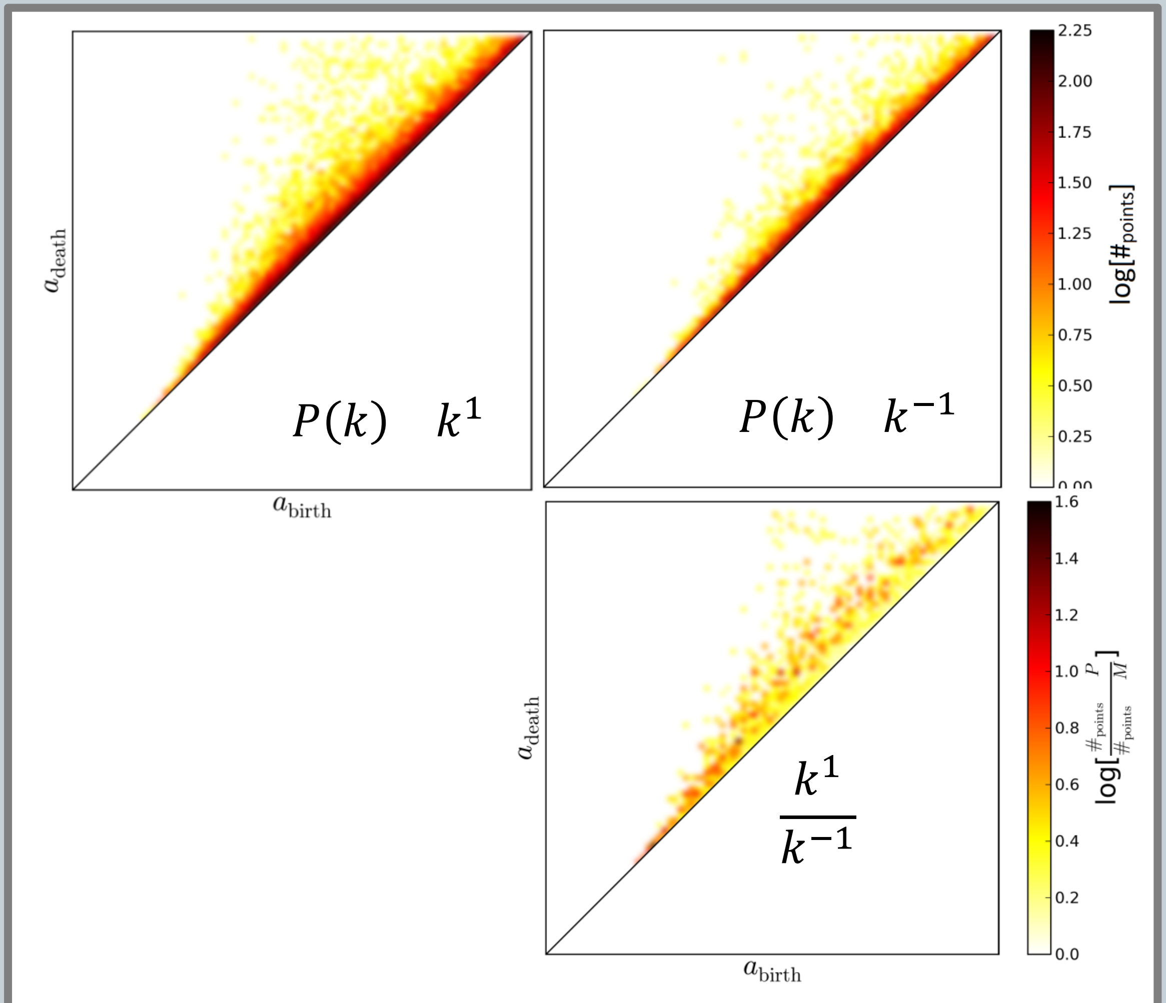
Void persistence and merger trees

- Merger tree is only based on one parent void!
- Combine information of all merger trees into

  Persistence Diagram

  (Edelsbrunner et al. 2000)

- Information w.r.t. formation and disappearance of voids due to hierarchical evolution
- Not only mathematical principle.

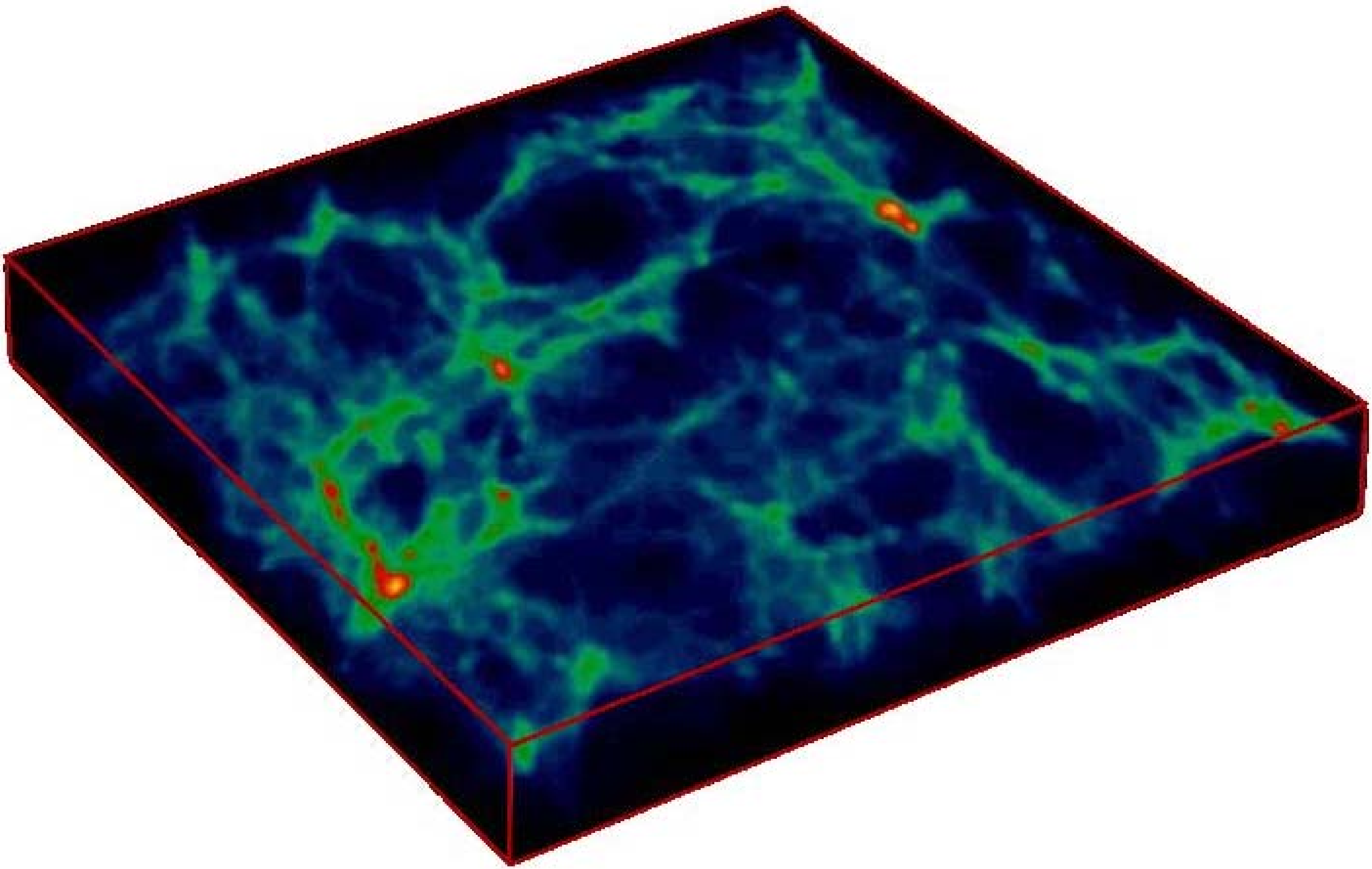


SpineWeb

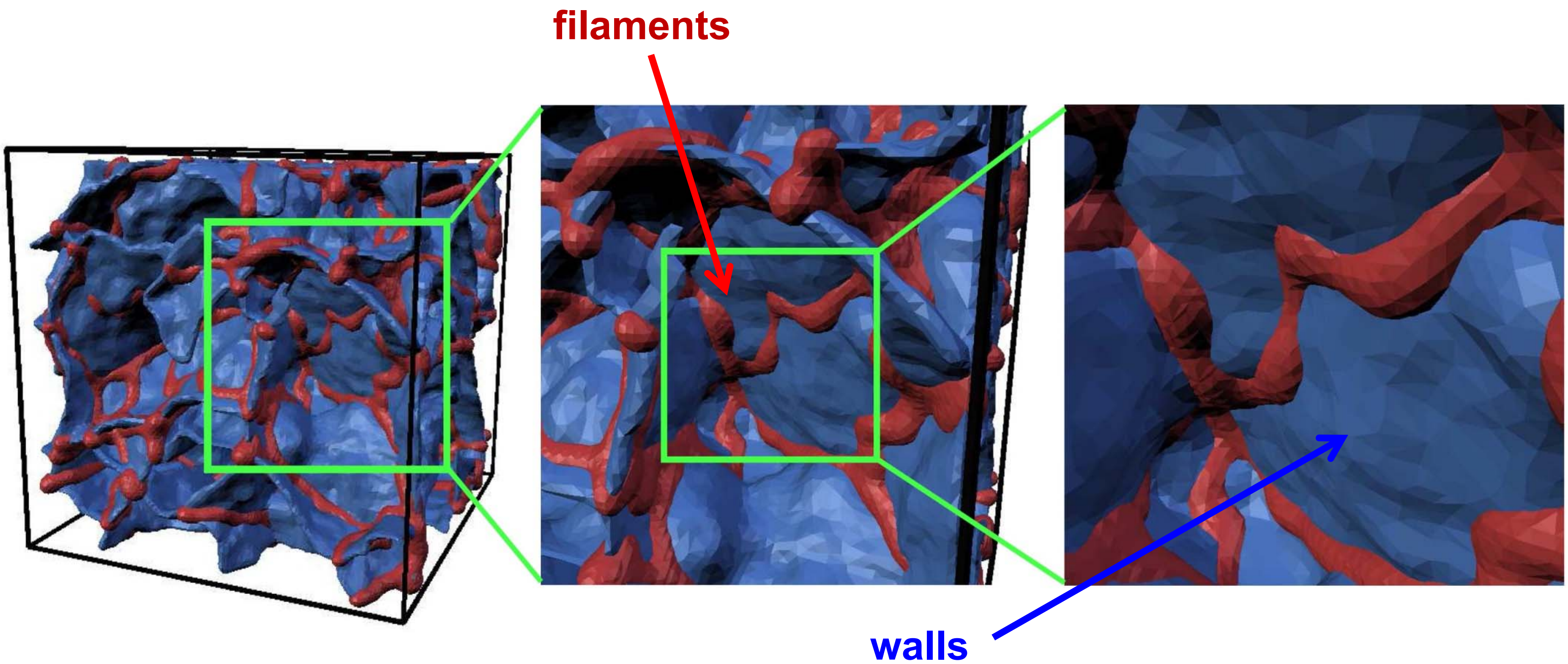
Morse Smale & Watershed

Aragon-Calvo, Platen, vdW et al. 2010

Spine of the Cosmic Web



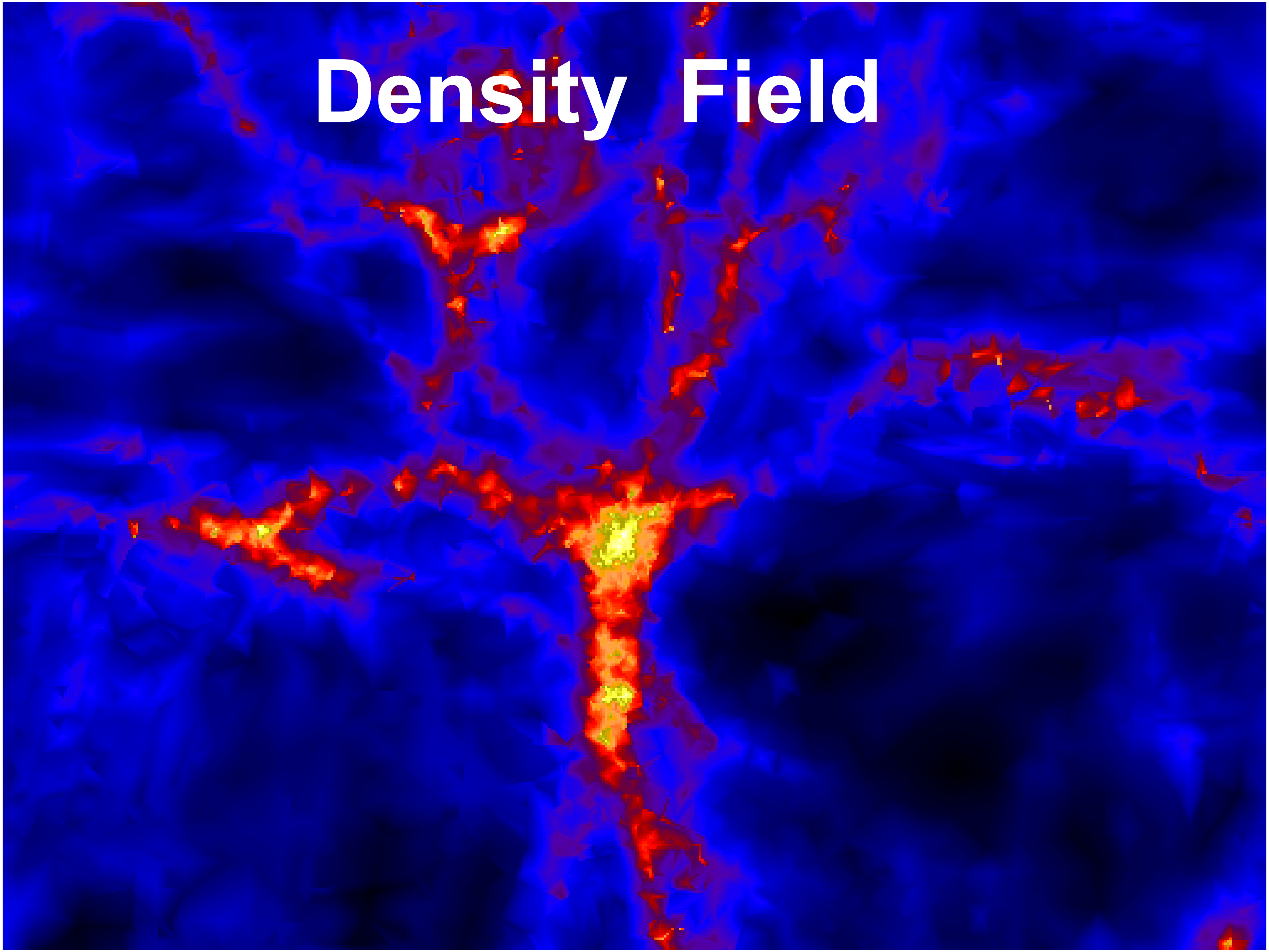
SpineWeb

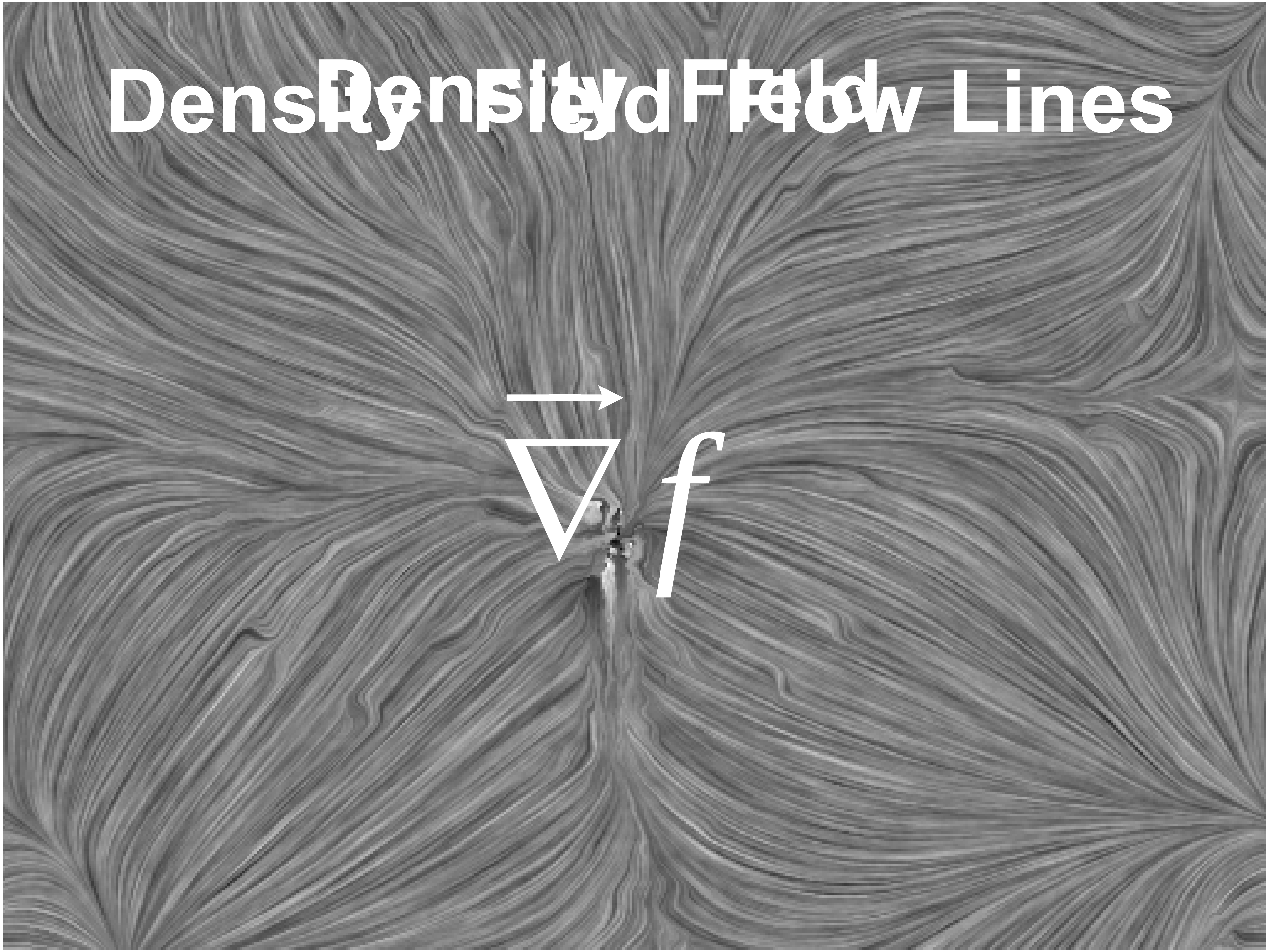


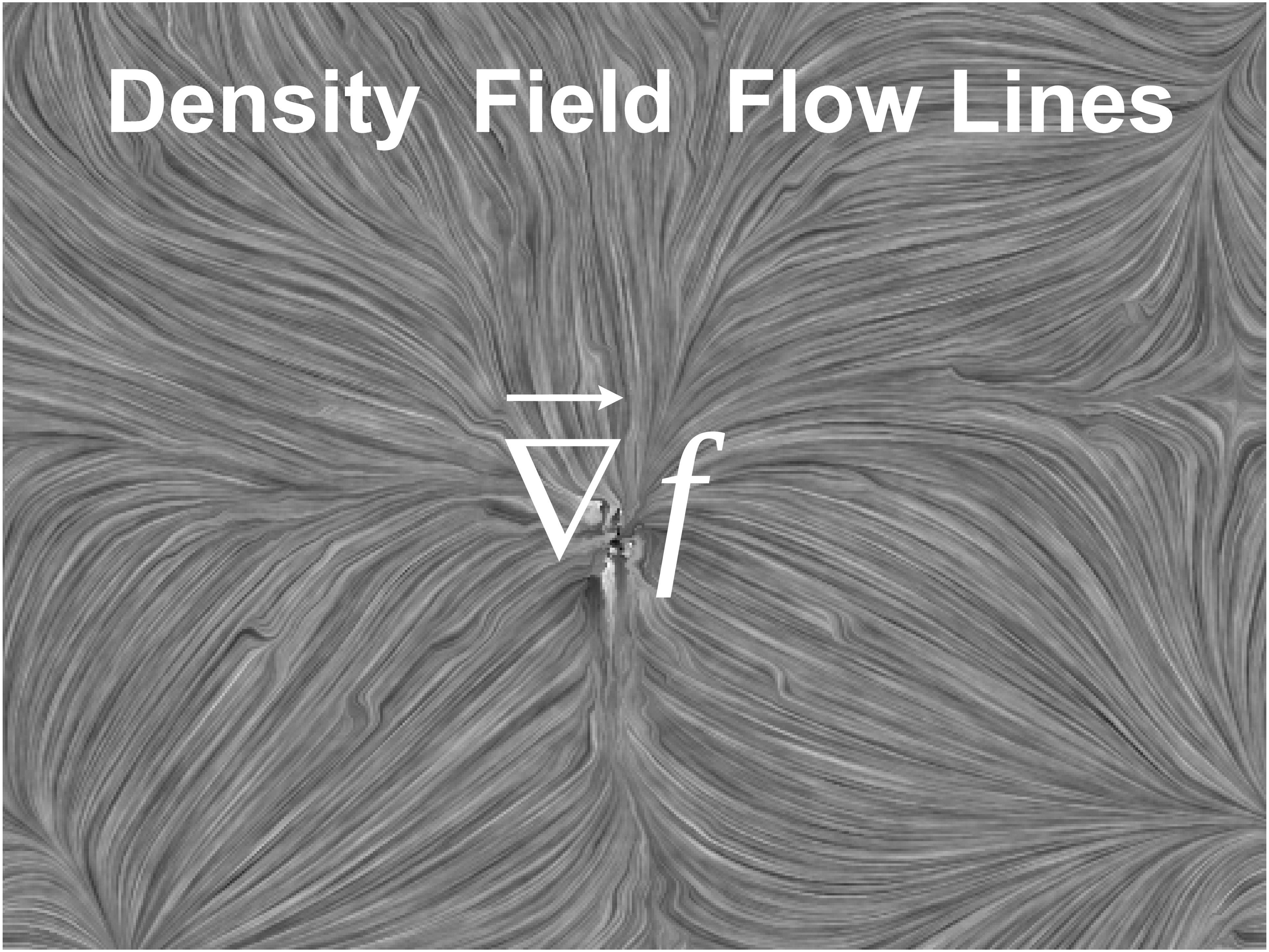
Cosmic Spine

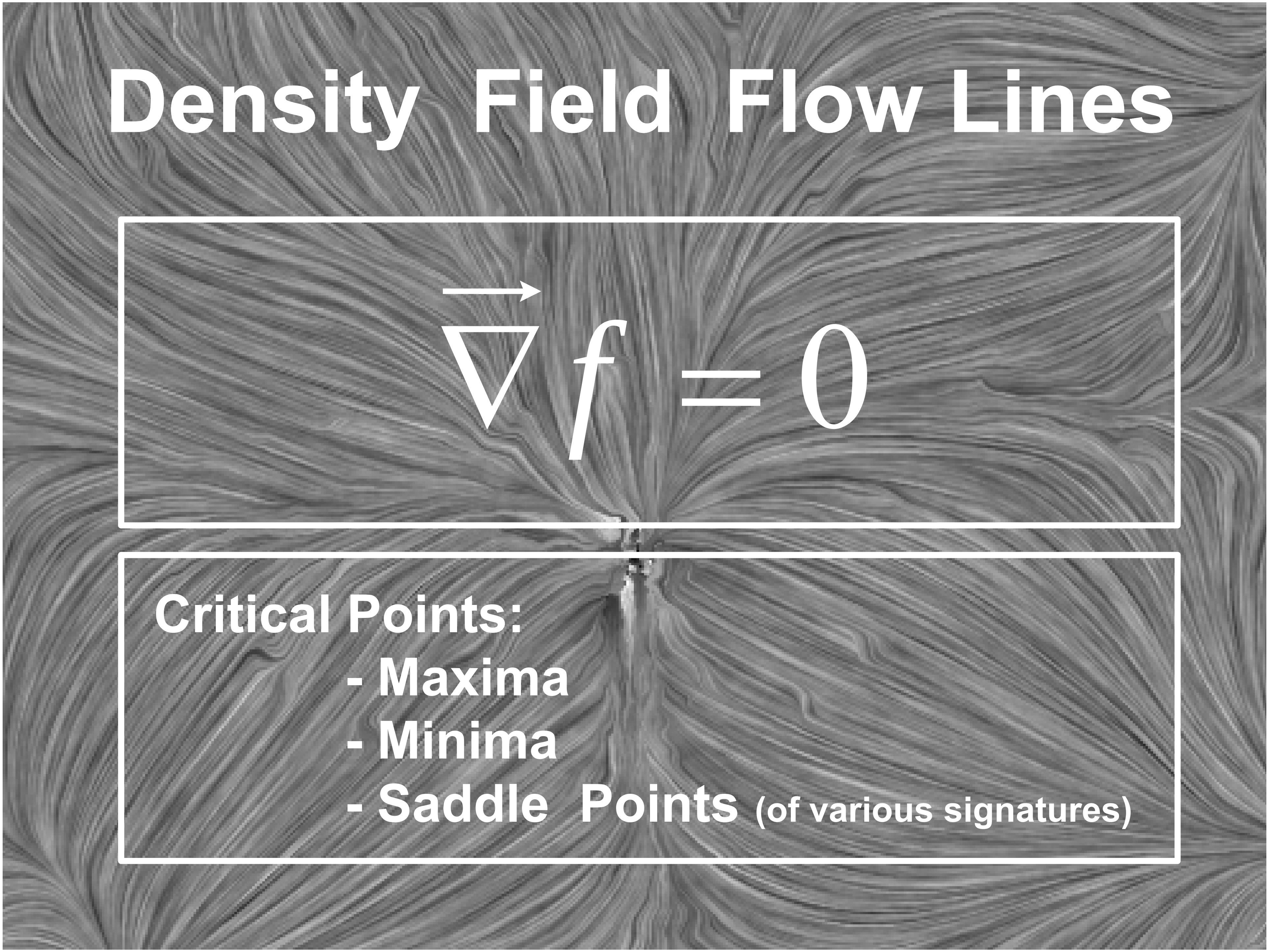
Cosmic Spine:

- Network of filamentary edges & sheetlike walls
- Connection of Cluster Nodes via filamentary bridges


Density Field


Density Field
Density Flow Lines
$\vec{\nabla} f$


Density Field Flow Lines
$\vec{\nabla} f$

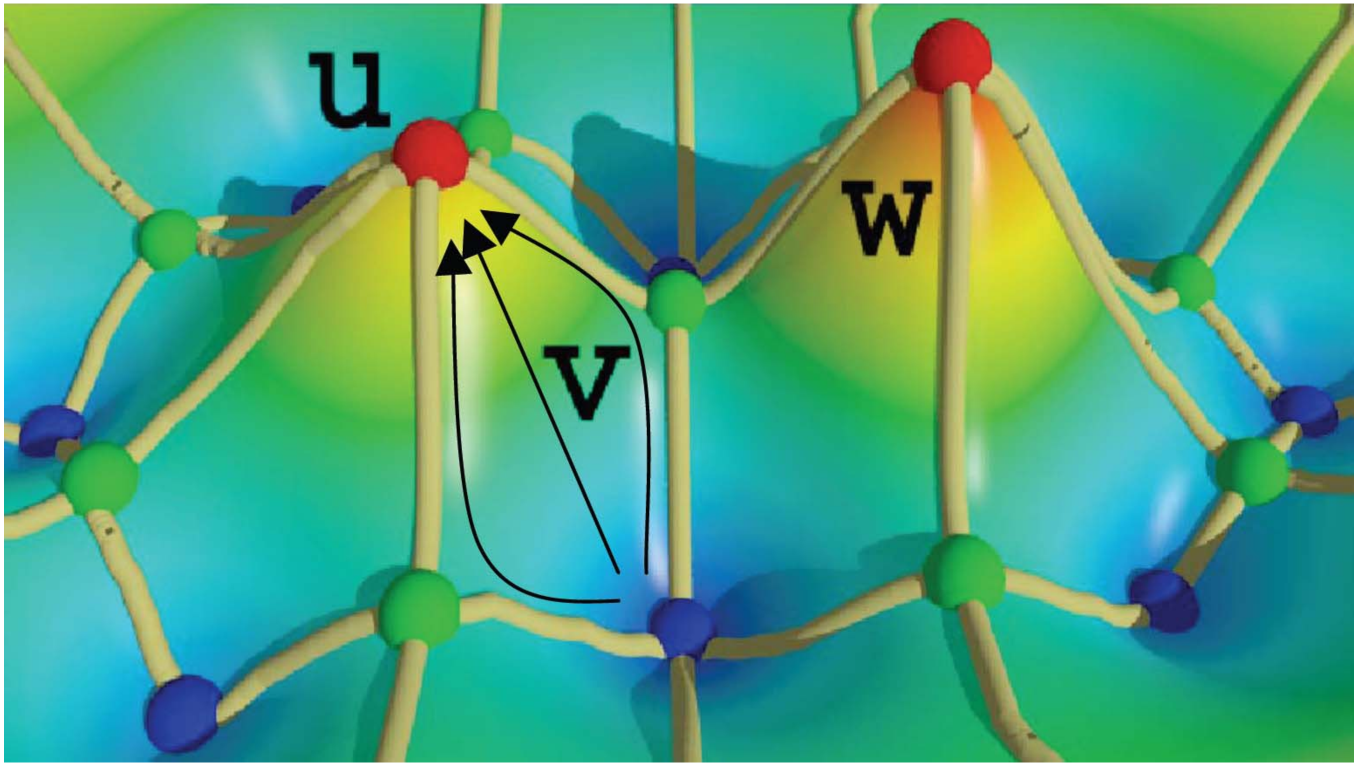


Density Field Flow Lines

$$\vec{\nabla} f = 0$$

Critical Points:

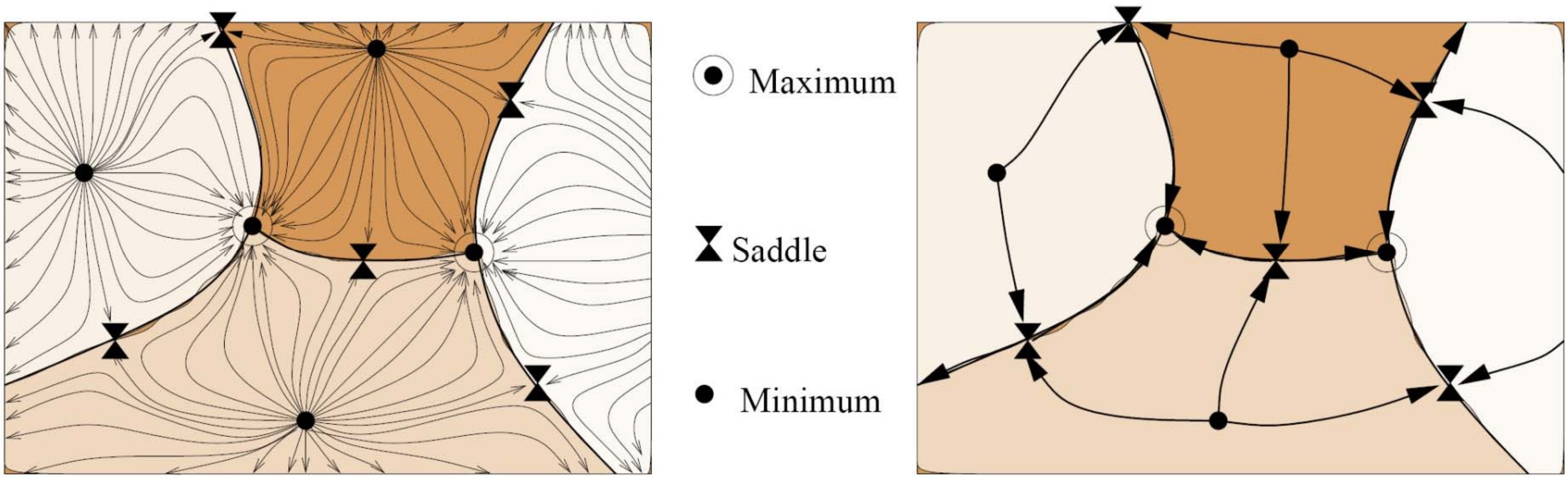
- **Maxima**
- **Minima**
- **Saddle Points** (of various signatures)



Density Field Critical Points:

Ridges:

Connections Saddles-Maxima



Morse Complex & Field Singularities

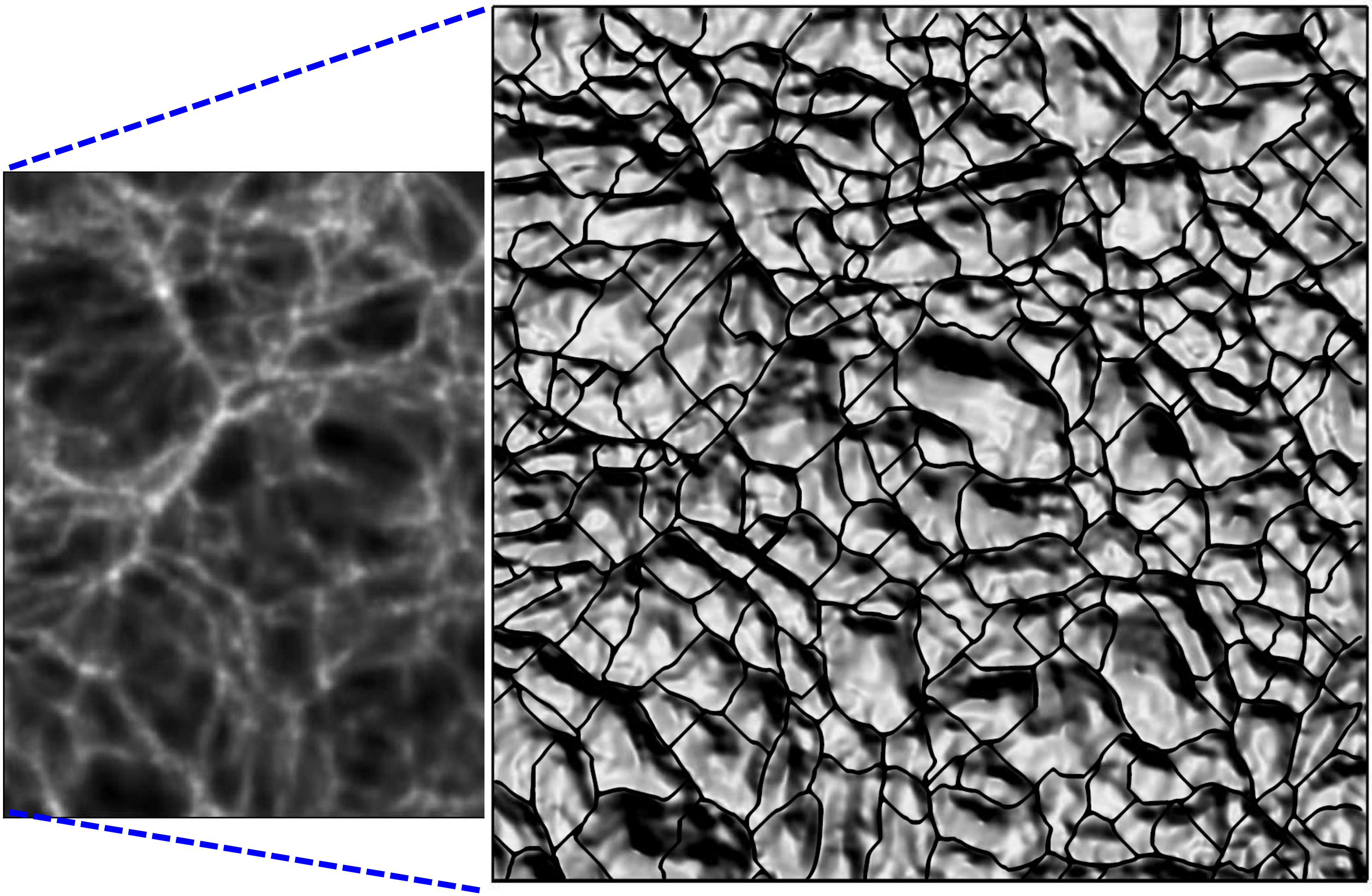
Topological structure well-behaved C^2 field:

- "flow" field
- singularities - minima, maxima, saddles
- critical integral lines: connection singularities
- saddles-maxima: spine of field - filaments, sheets
- basin minima: voids

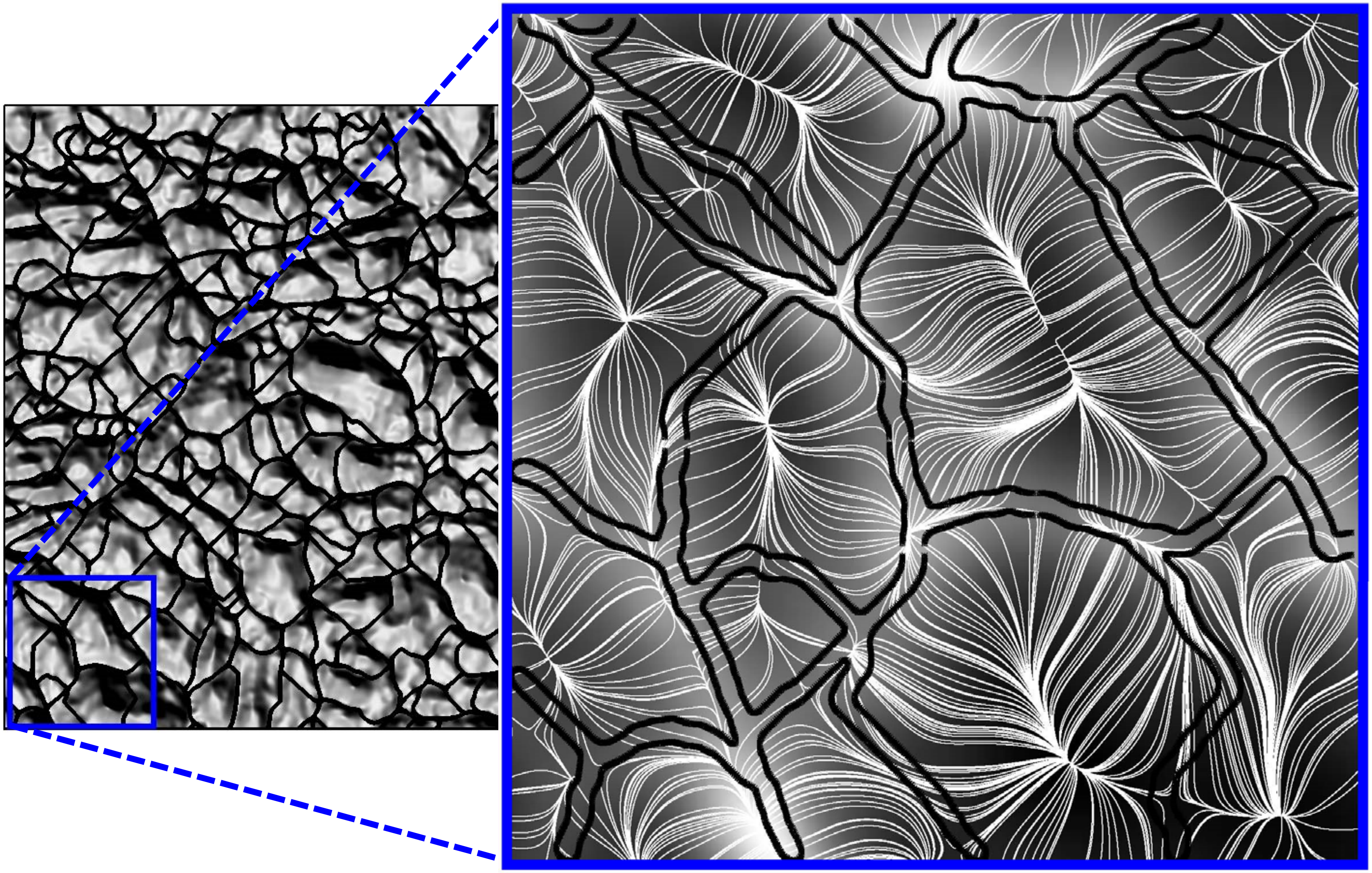
Practical Computation:

- **Watershed Transform**
- Pseudo Morse complex !!!!

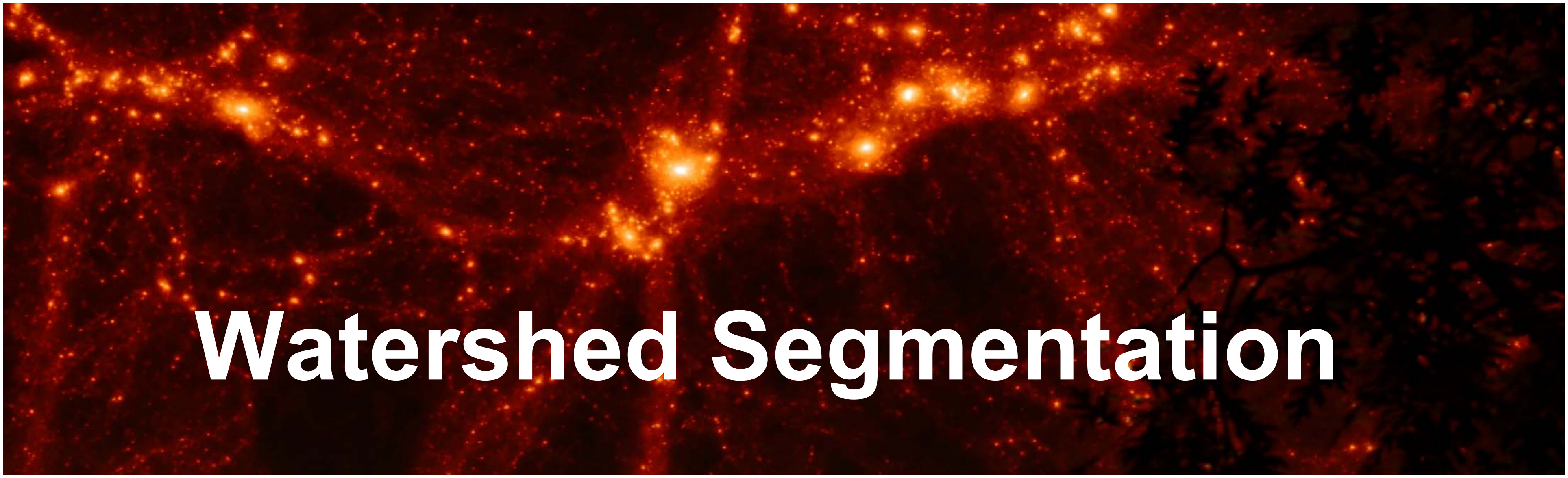
Density Field & Landscape

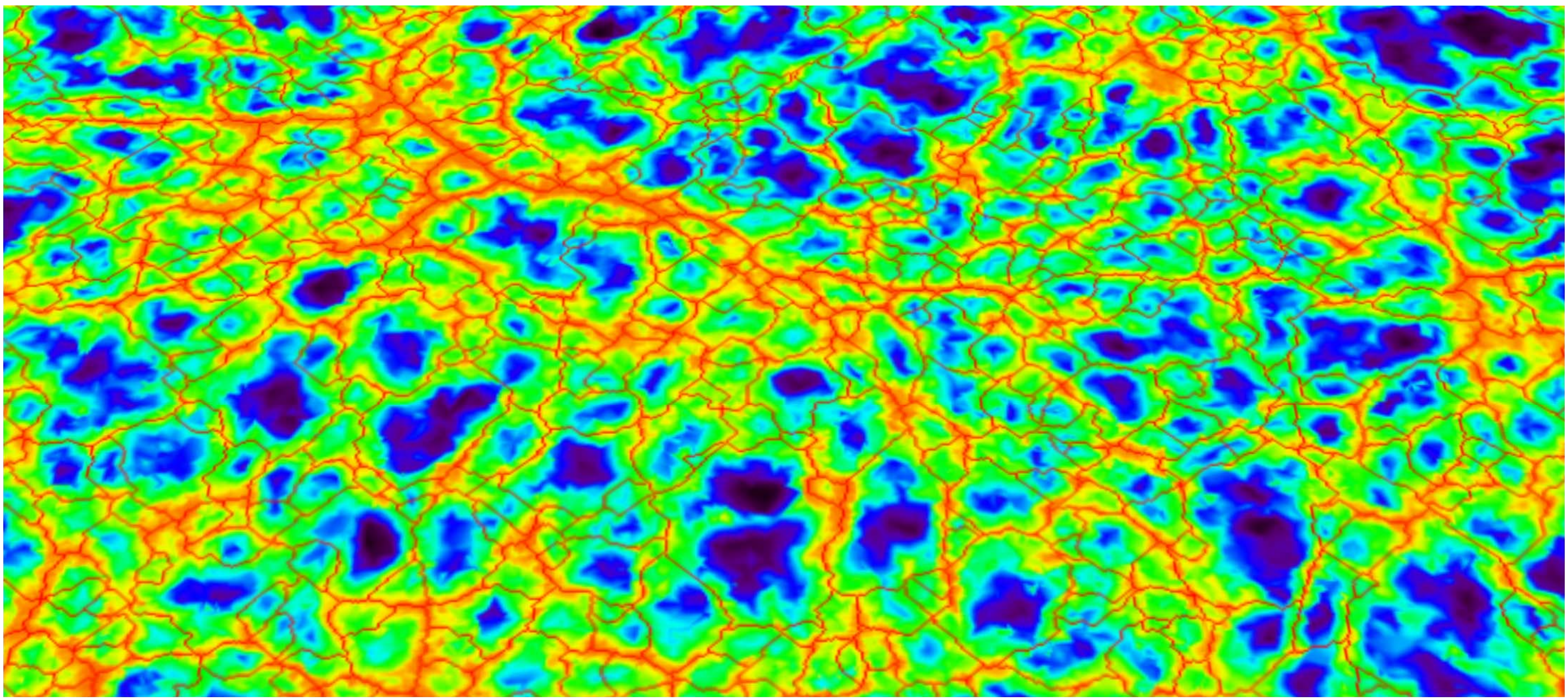


Segmentation & Flowlines

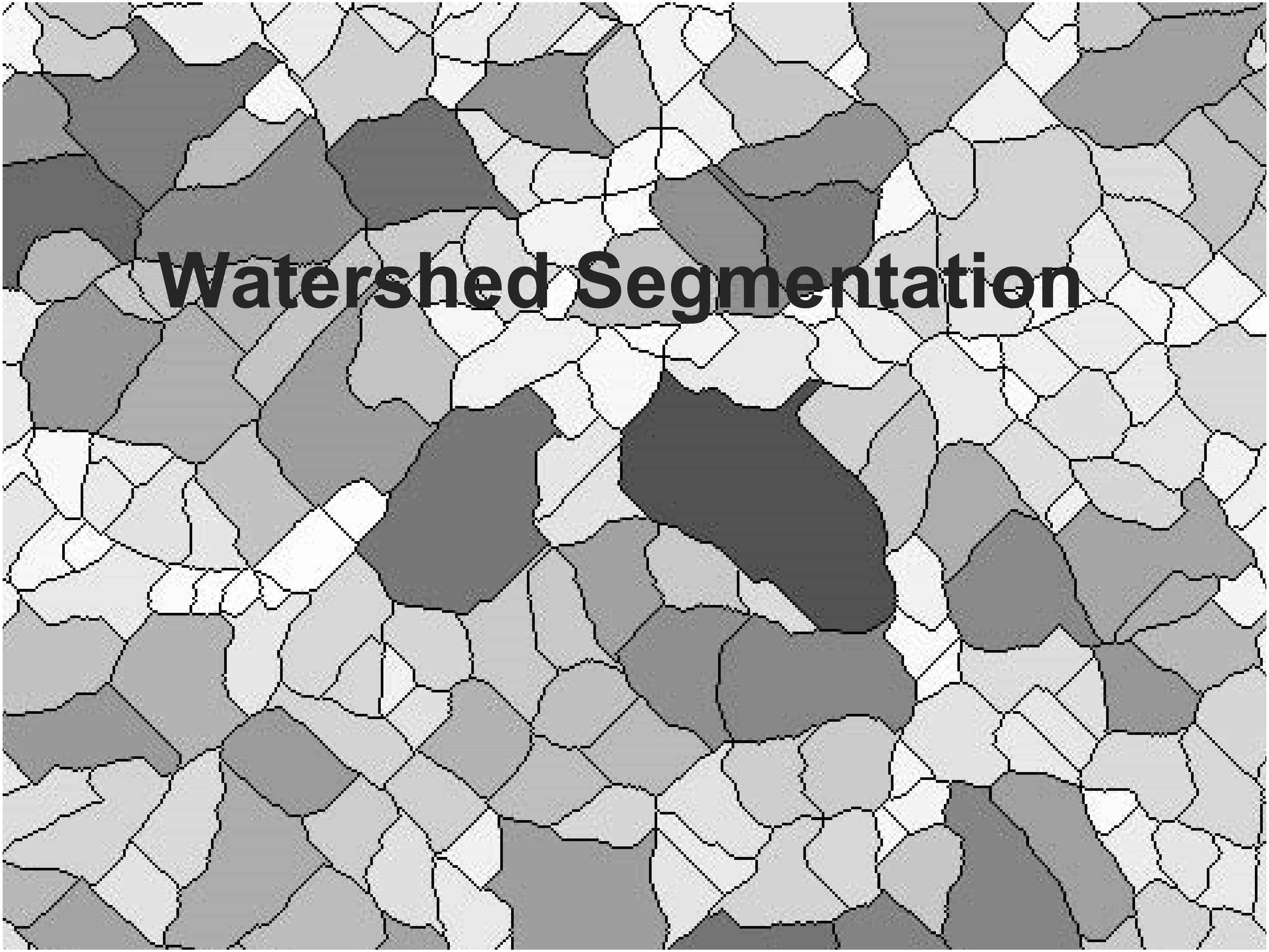


Watershed Segmentation





Watershed Segmentation

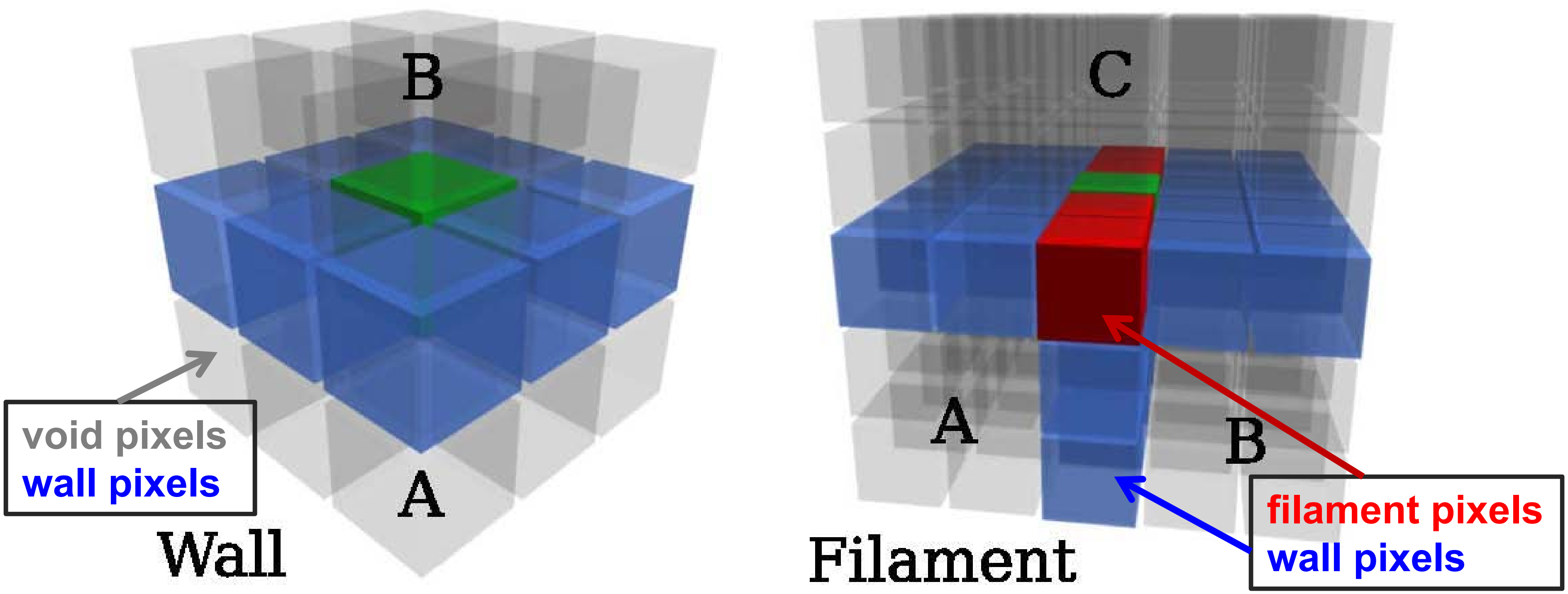


SpineWeb Formalism

Extension of Watershed Transform:

- determination boundary regions between the watershed basins (the “voids”).
- Identification of boundary pixels
- Topological Identity determined
  on the basis of # neighbouring voids/basins,

SpineWeb Procedure



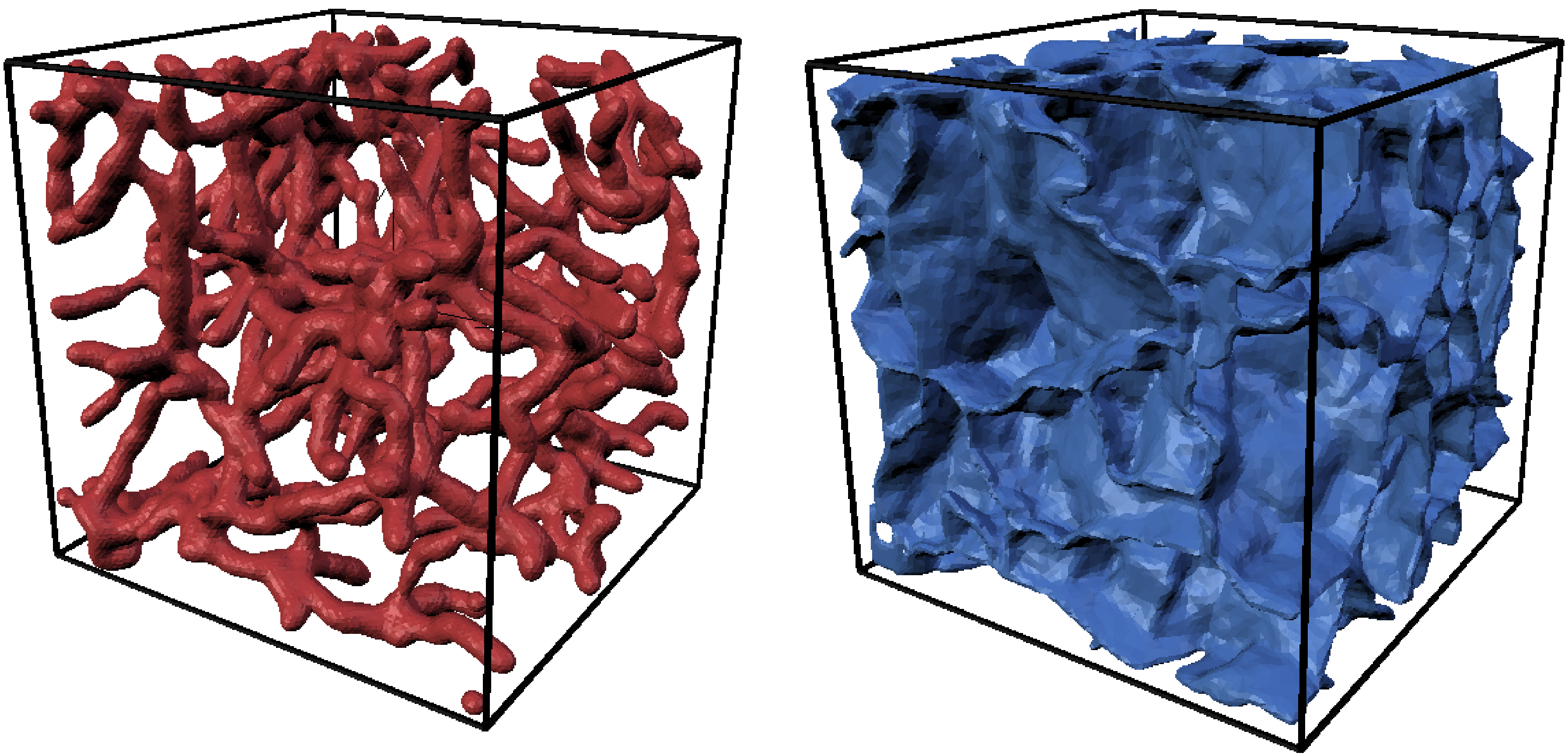
Local Neighbourhood:

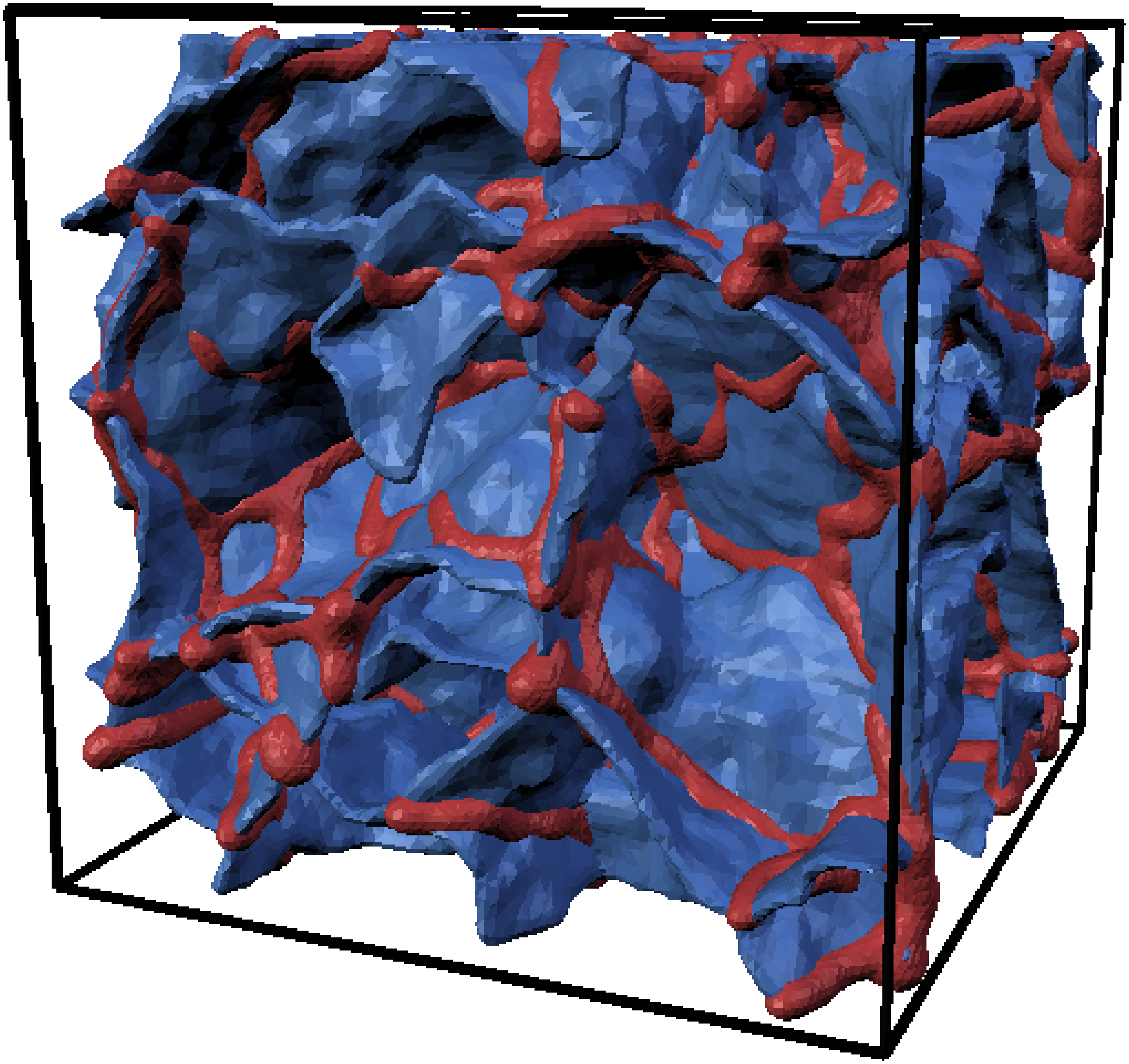
Counting Number Adjacent Voids:

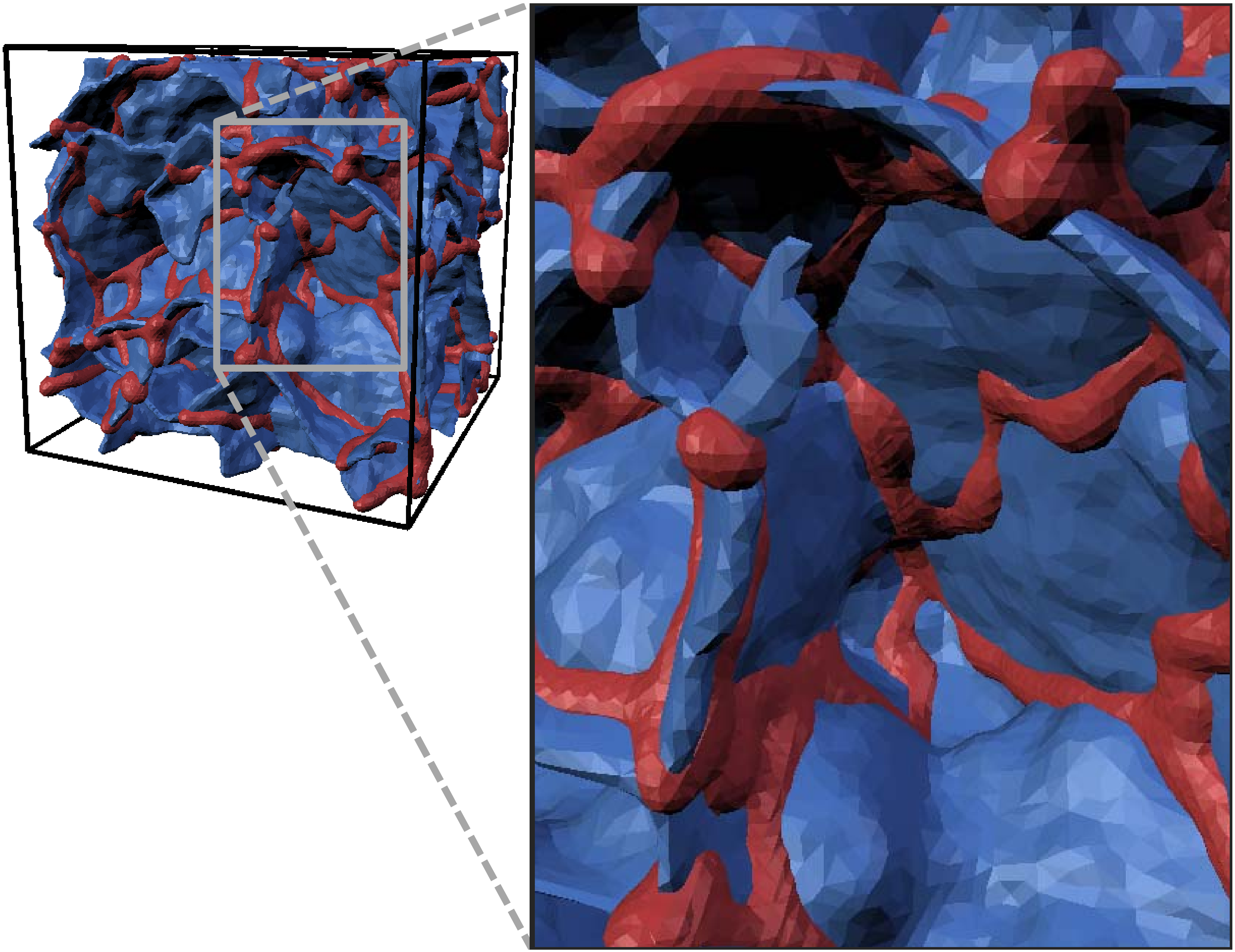
$N_v = 2$ wall

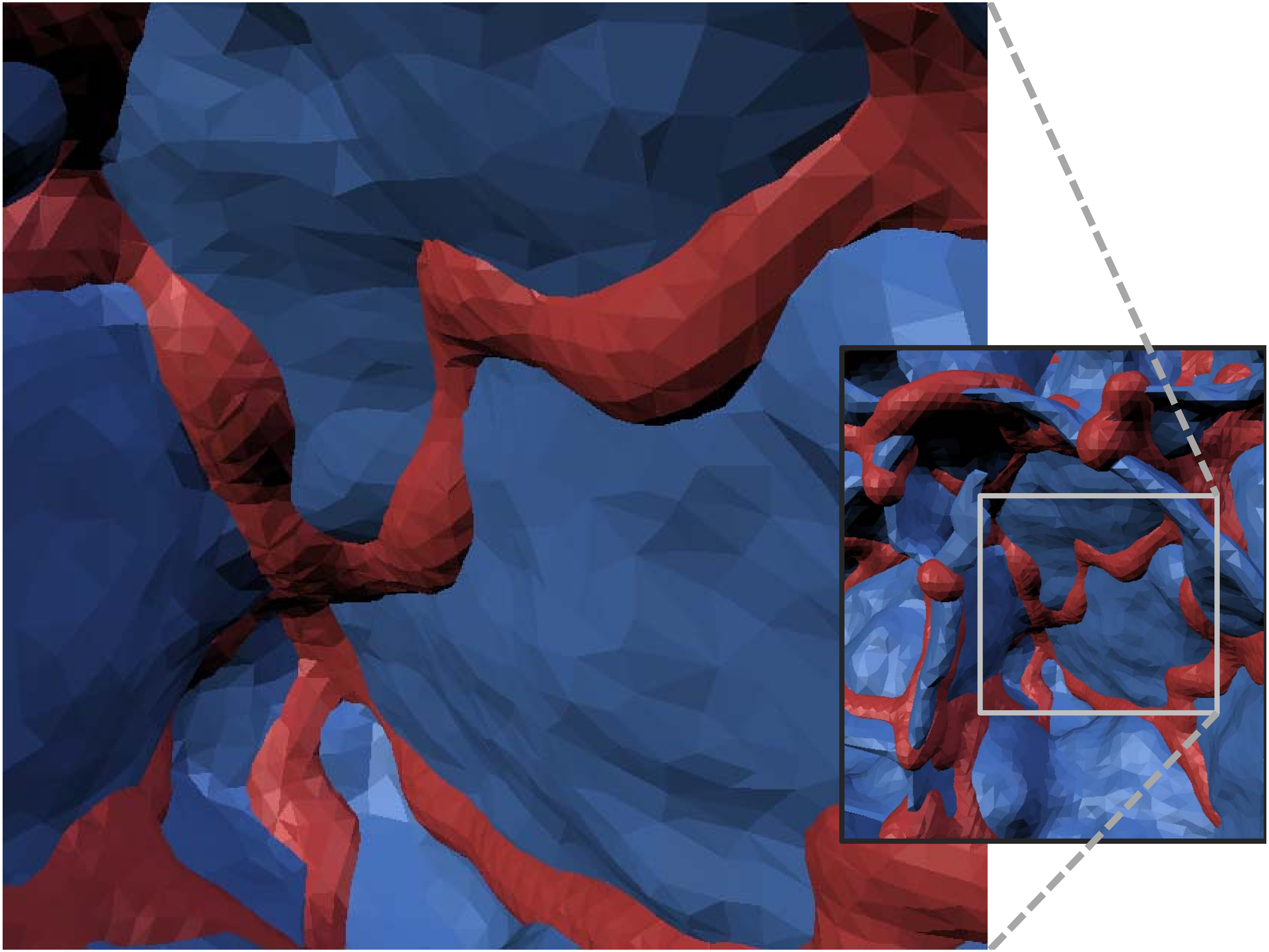
$N_v = 3$ filament

SpineWeb Morphology Dissection

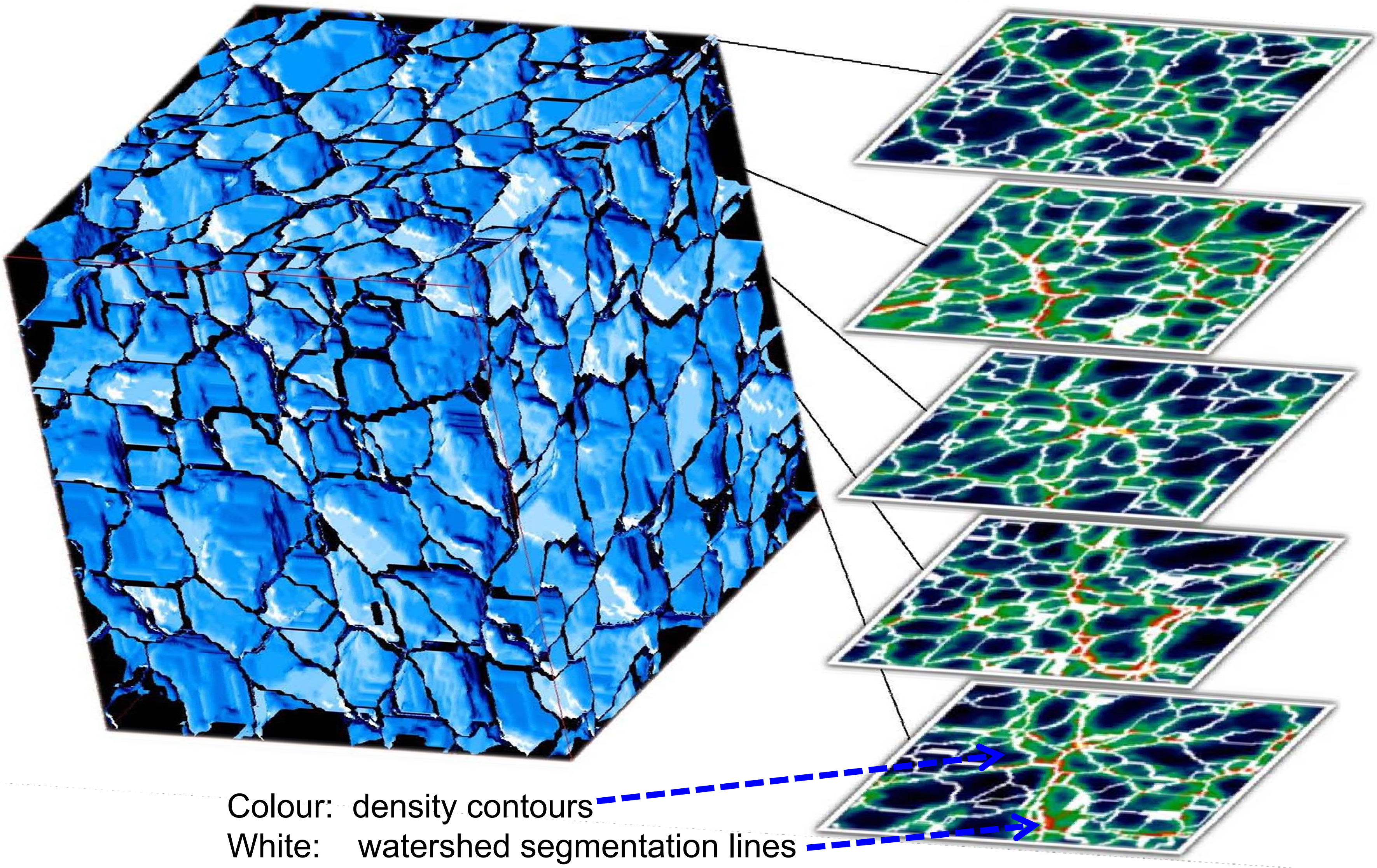






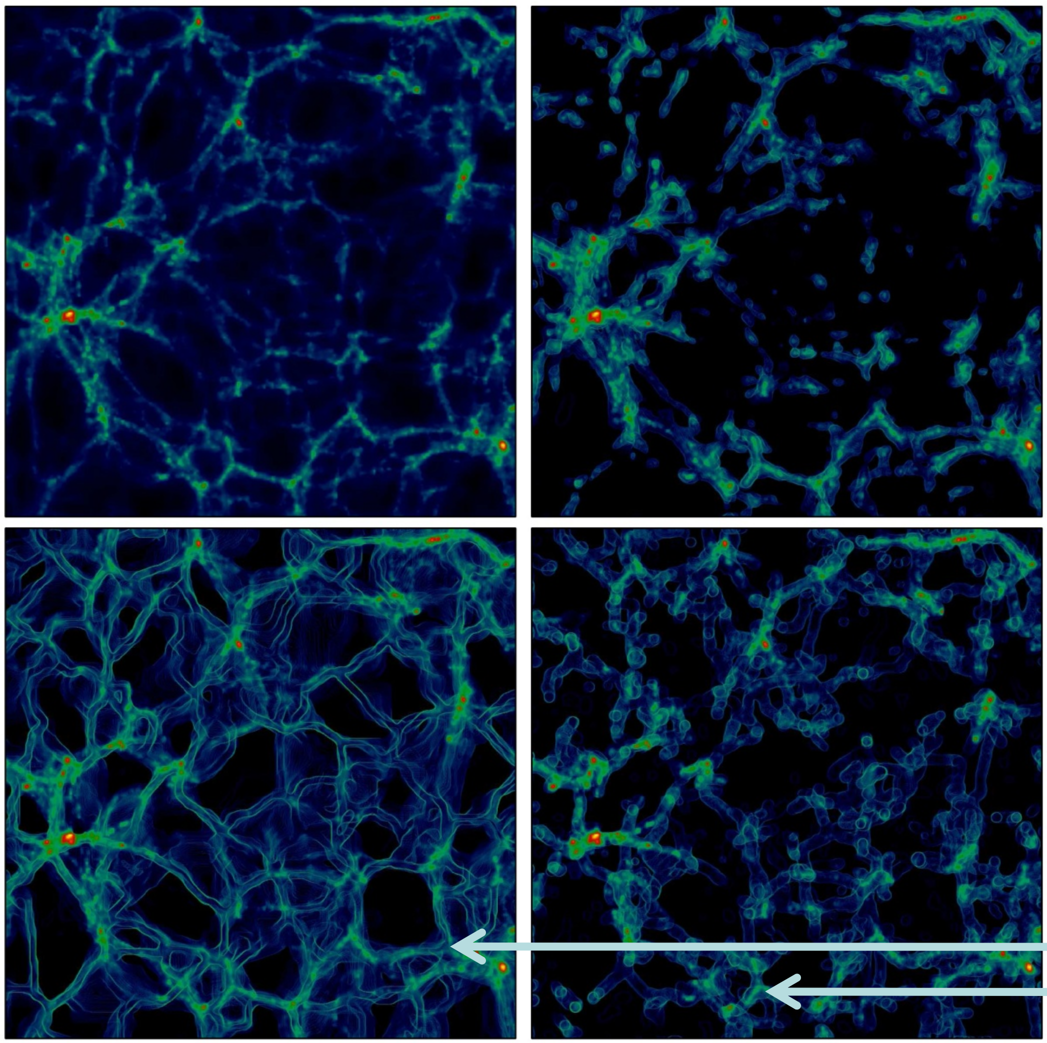


3-D SpineWeb Segmentation



Colour: density contours
White: watershed segmentation lines (cosmic spine)

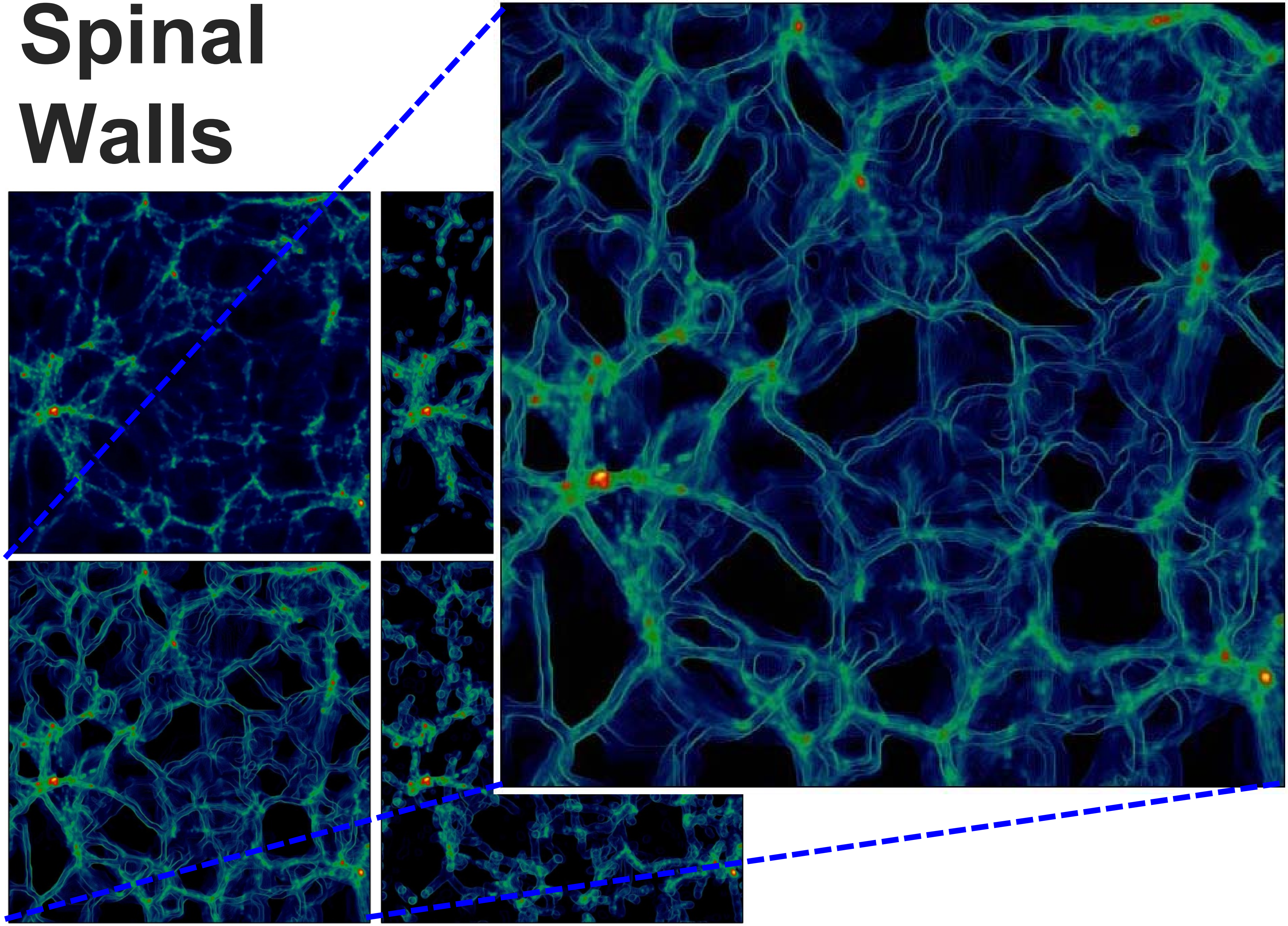
Density Levels vs. Spine



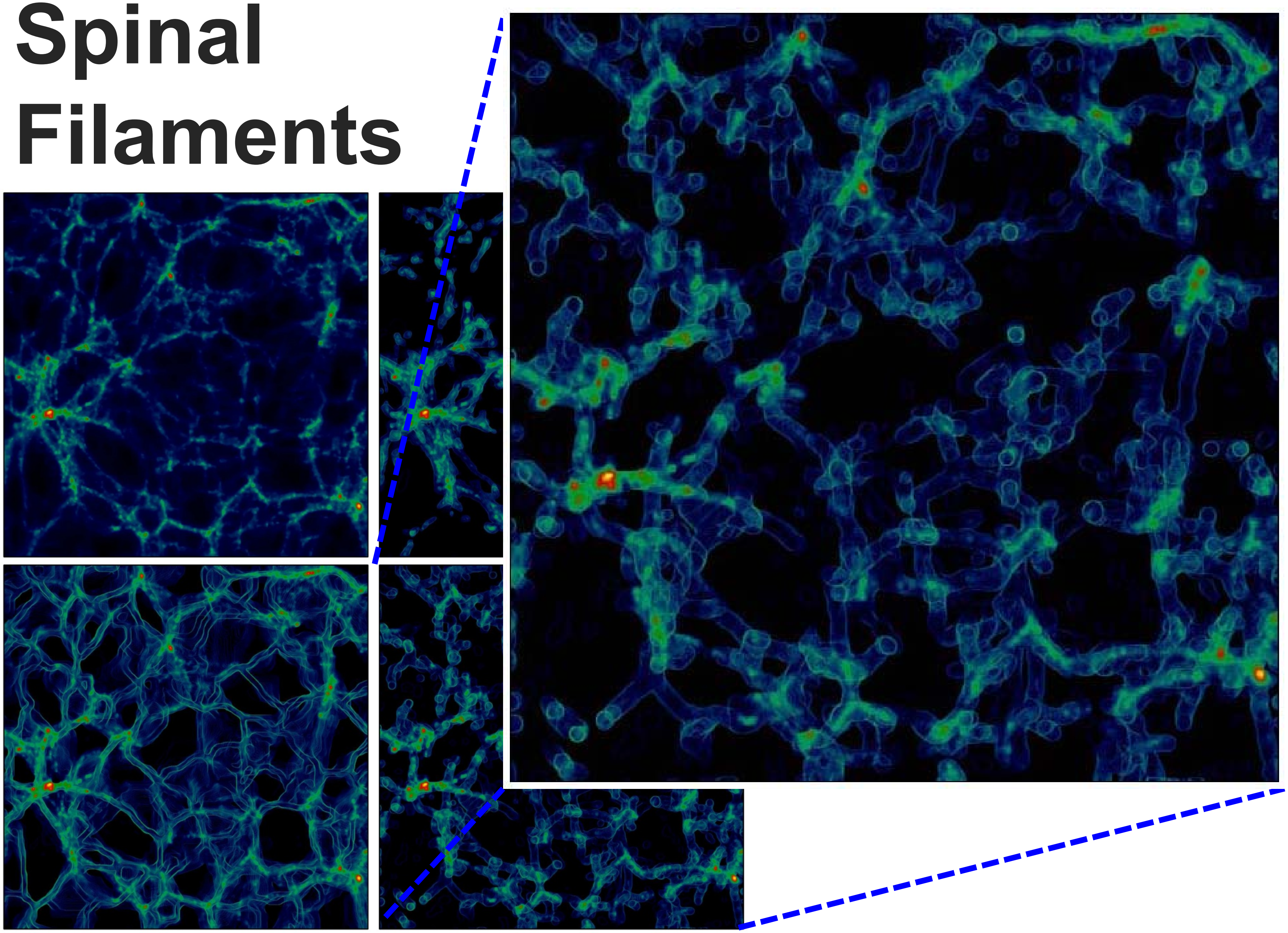
density field level sets

Spinal Components:
- walls
- filaments

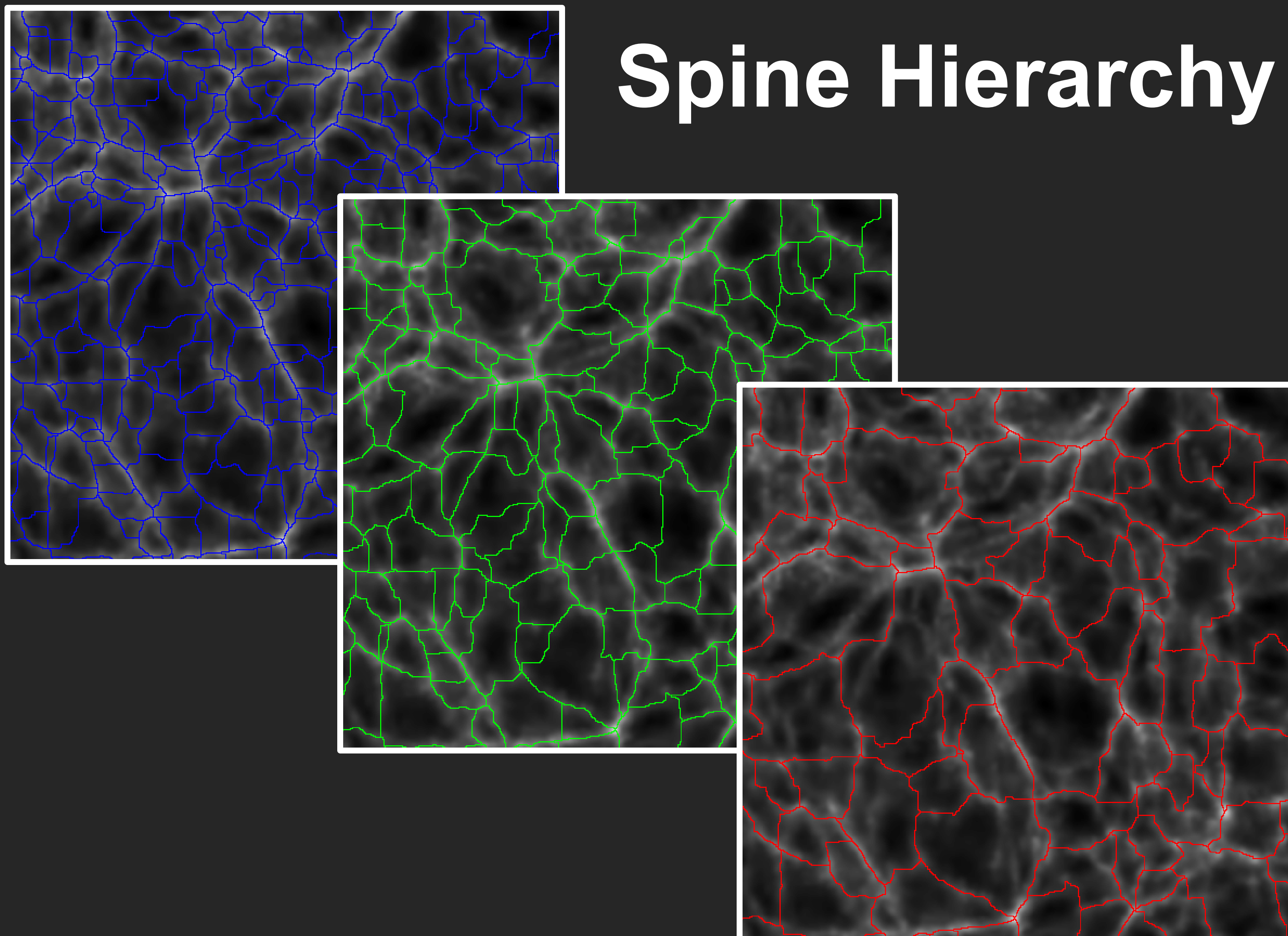
Spinal Walls



Spinal Filaments

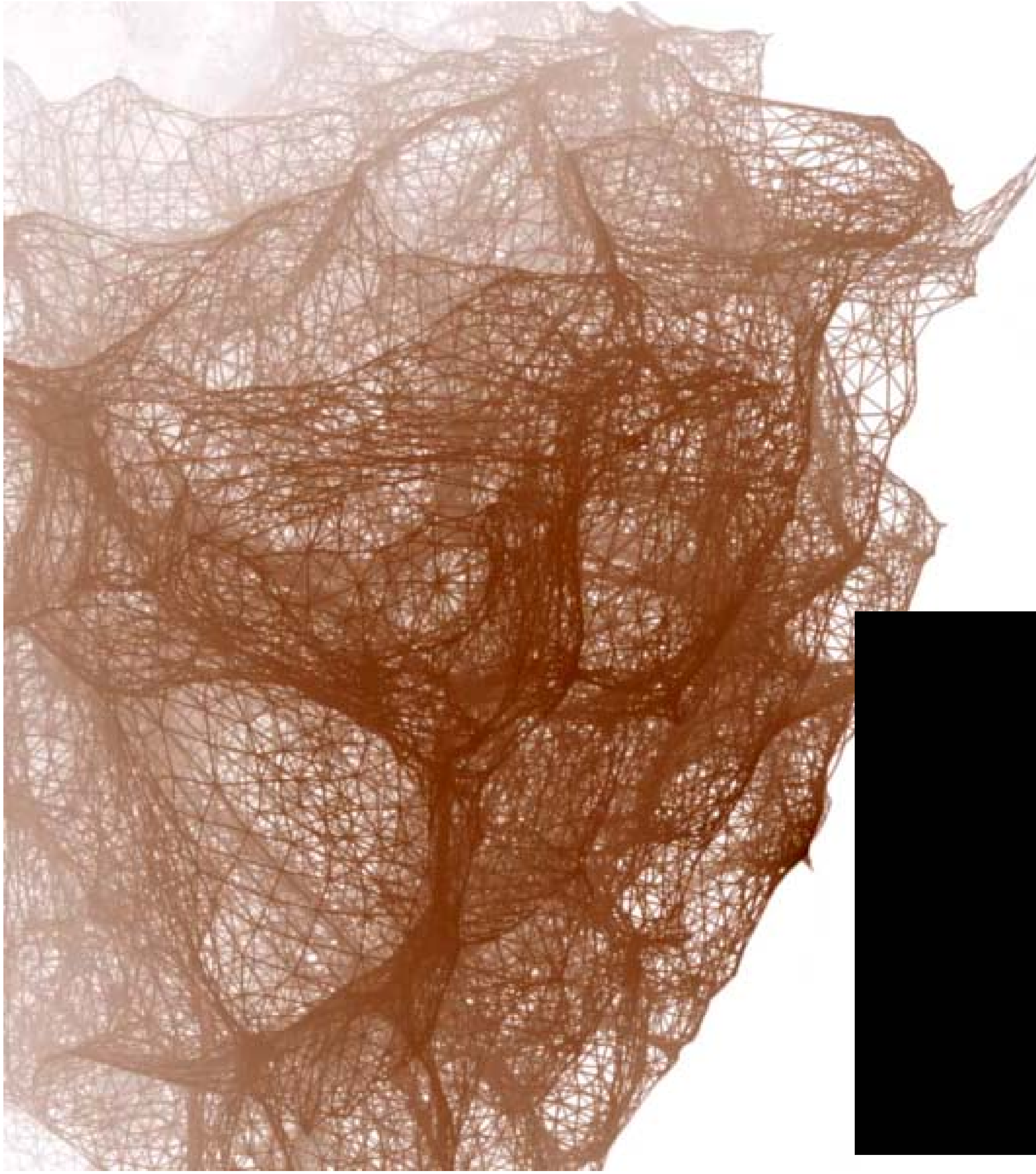


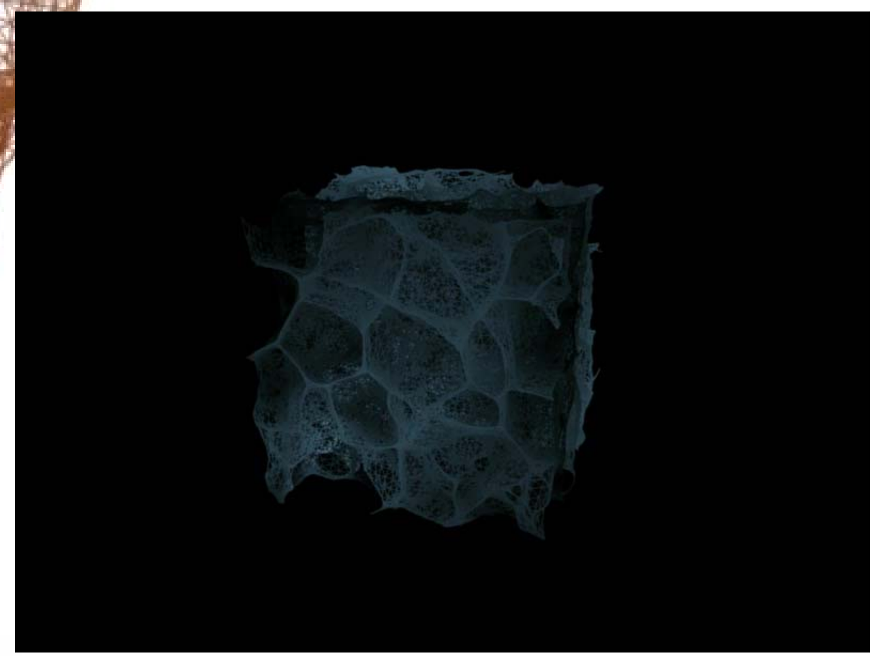
Spine Hierarchy



Spine of the Cosmic Web:

directly on Delaunay grid





Cosmo Topology

Topology

Study of the

(multiscale)
shapes, complexity and connectivity

of the Cosmic Web

Geometry & Topology

- Conventional Cosmological Topology Measure:

(Reduced) Genus

- # holes - # connected regions
- (Gott et al. 1986; Hamilton et al. 1986; Choi et al. 2010)

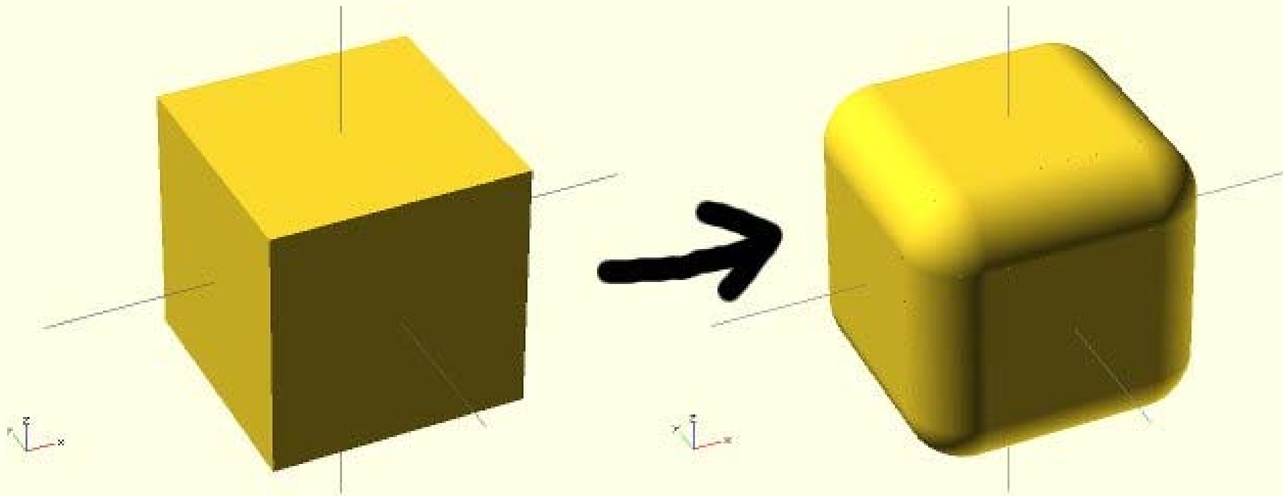
- Complete quantitative characterization of local geometry in terms of

Minkowski Functionals

- Minkowski Functionals:
  - Volume
  - Surface area
  - Integrated mean curvature
  - Genus/Euler Characteristic

- (Mecke, Buchert & Wagner 1994)

Minkowski functionals



- **Weyl's Tube formula:**

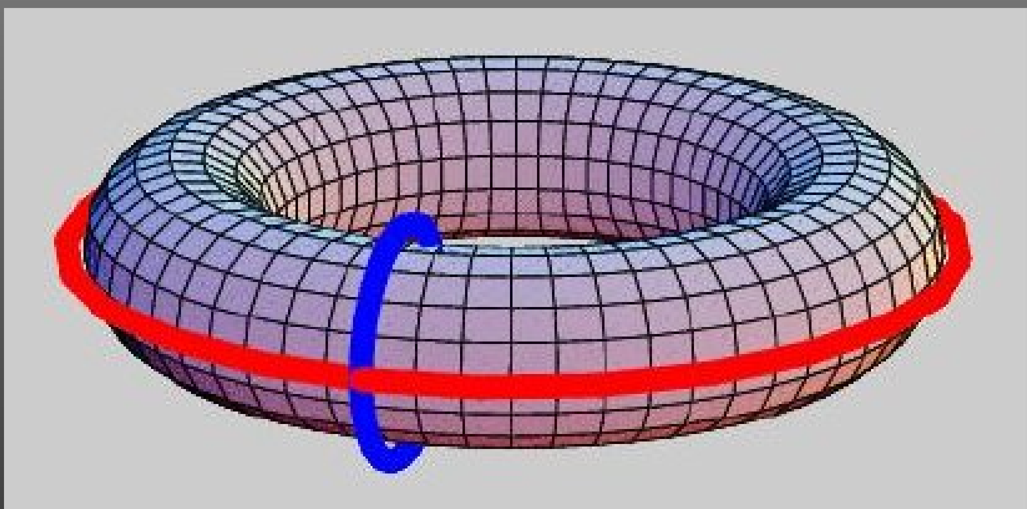
  Minkowski functionals Q_k are the parameters specifying the contribution of volumes r^k to the volume of a cube M^r with rounded edges of radius r:

$$\mathrm{Vol}\left(M^r\right) = Q_0 + Q_1 r + Q_2 r^2 + Q_3 r^3$$

Topology, Homology & Cycles

Topology:
Study of connectivity and spatial relations that remain invariant under homeomorphisms (= continuous mapping between two topological objects)

Homology:
Description of topology of a space in terms of the relationship between cycles and boundaries.



Torus:	one	0-cycle:	rank group H_0:	1
	two	1-cycles:	rank group H_1:	2
	one	2-cycle	rank group H_2:	1

p-chain: sum of p-simplices
p-cycle: boundary of (p+1) chain

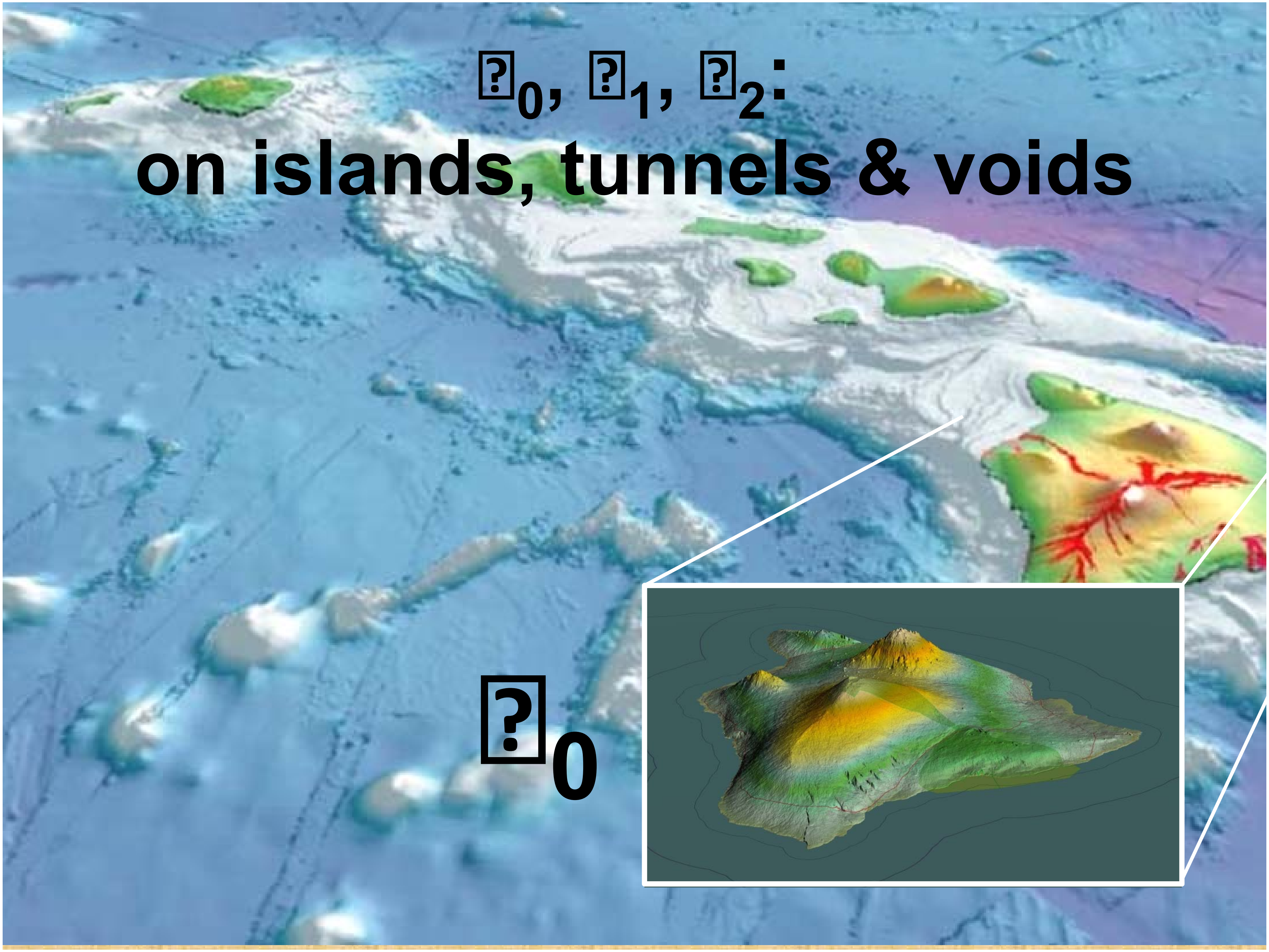
0-cycle: closed component
1-cycle: closed loop of edges, or finite union
2-cycle: closed surface, or finite union

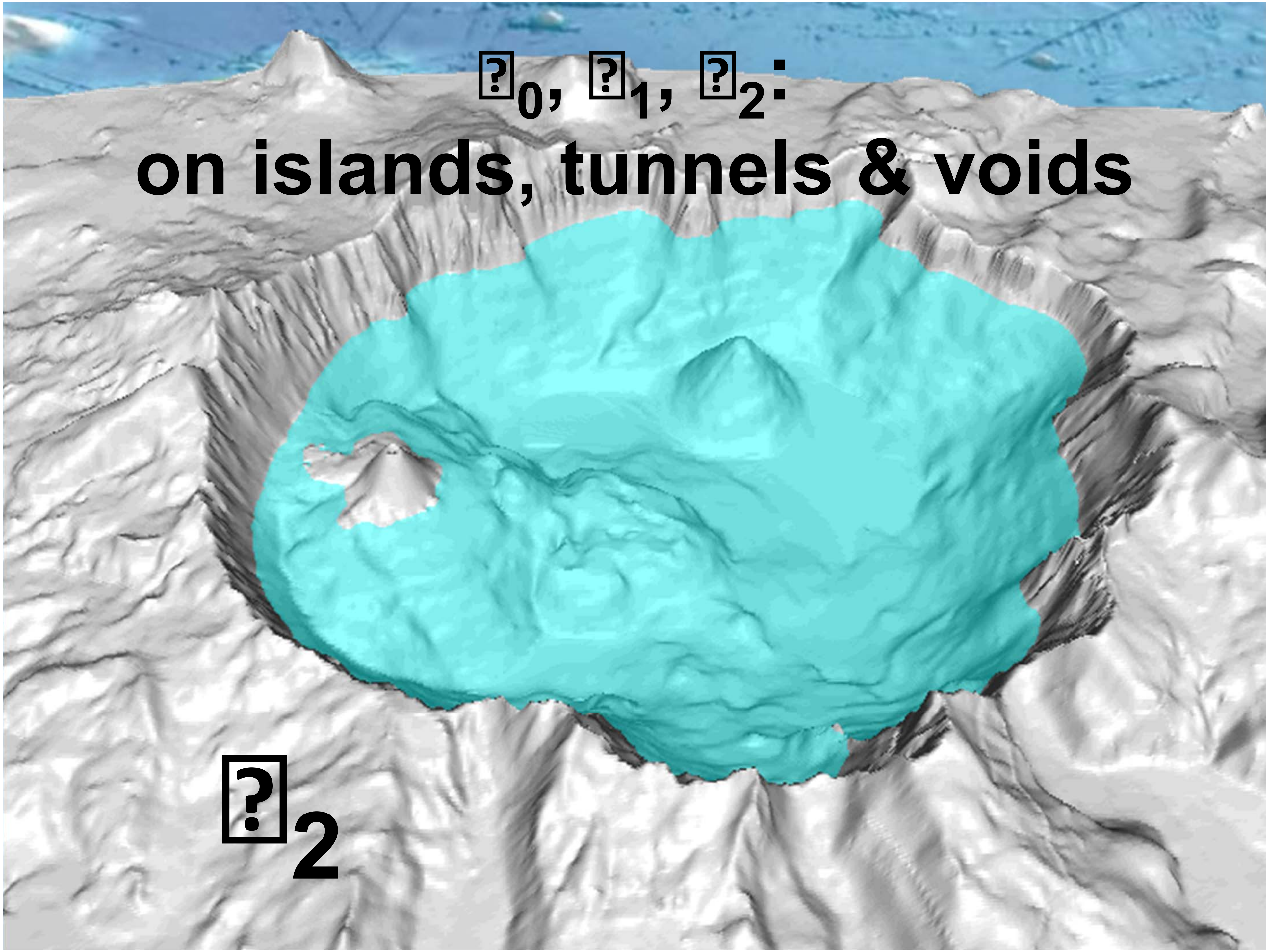
adding two p-cycles → p-cycle

⇩

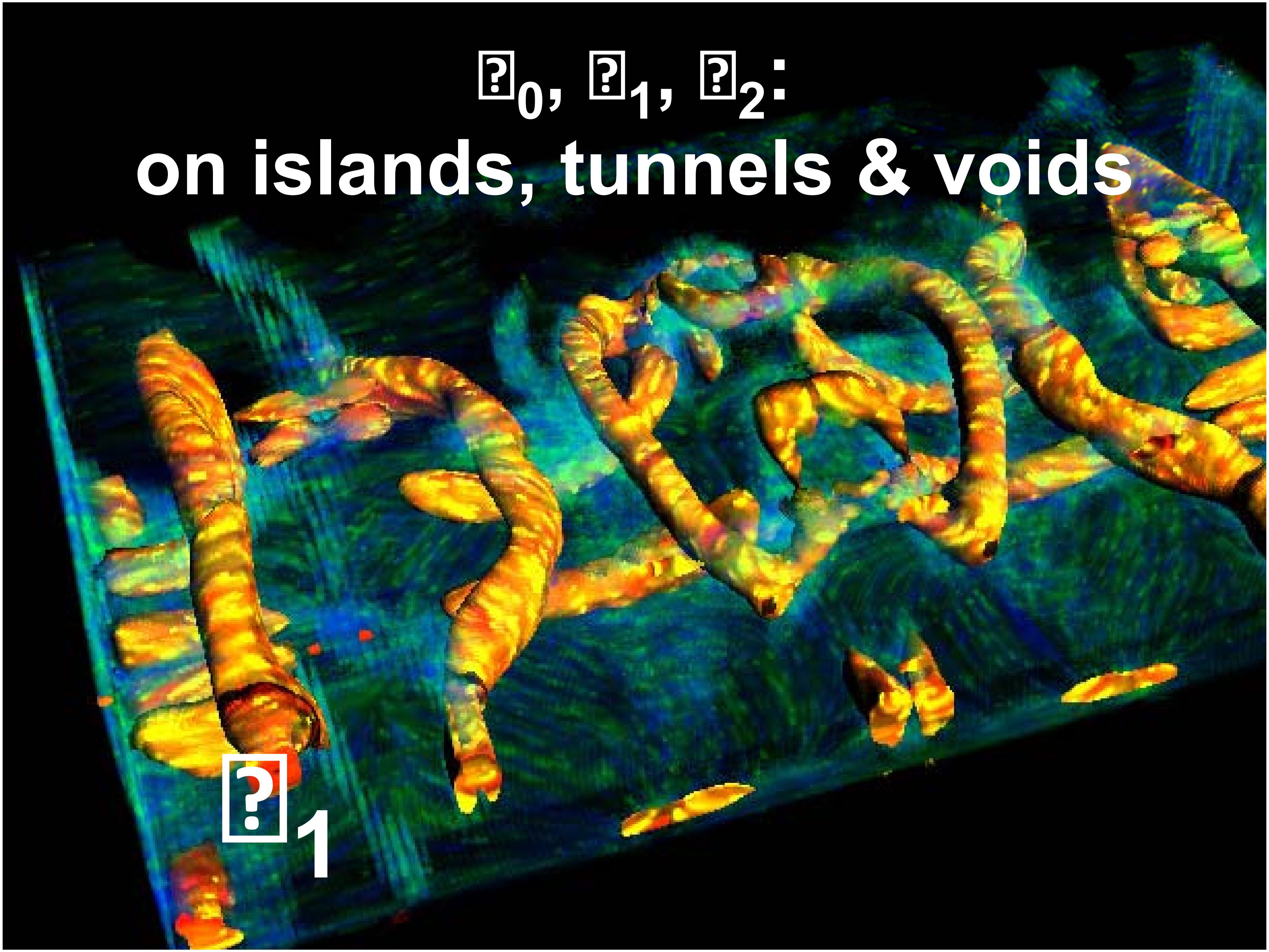
Group of p-cycles:

β_0, β_1, β_2: on islands, tunnels & voids





$?_0$, $?_1$, $?_2$:
on islands, tunnels & voids
$?_2$


$?_0$, $?_1$, $?_2$:
on islands, tunnels & voids
$?_1$

Euler-Poincare

Euler Characteristic ? is alternating sum of Betti Numbers

3-manifold ?:

$$\chi\left(M\right) = \beta_0 - \beta_1 + \beta_2 + \beta_3$$

$$\approx \beta_0 - \beta_1 + \beta_2$$

boundary 2-manifold ??:

$$\chi\left(\partial M\right) = \beta_{0b} - \beta_{1b} + \beta_{2b}$$

the Rule of Euler

from: Robert Adler

SIMPLICIAL TOPOLOGY

Simplices, complexes,
cycles, numbers of simplices,
Betti numbers

$$\sum_k (-1)^k \#\{k\text{-dimensional simplices}\}$$

INTEGRAL GEOMETRY

Convexity, convex ring
kinematic formulae
Minkowski functionals

$$\mathcal{M}_k(M) = c_{dk} \int_{\mathrm{Graff}(d,d-k)} \chi(M \cap V)\, d\mu^d_{d-k}(V)$$

$$\chi$$

$$\sum_k (-1)^k \beta_k$$

$$\sum_k (-1)^k \#\{\text{critical points of index } k\}$$

$$\int_M \mathrm{Tr}(R^{m/2}) \mathrm{Vol}_g$$

ALGEBRAIC TOPOLOGY

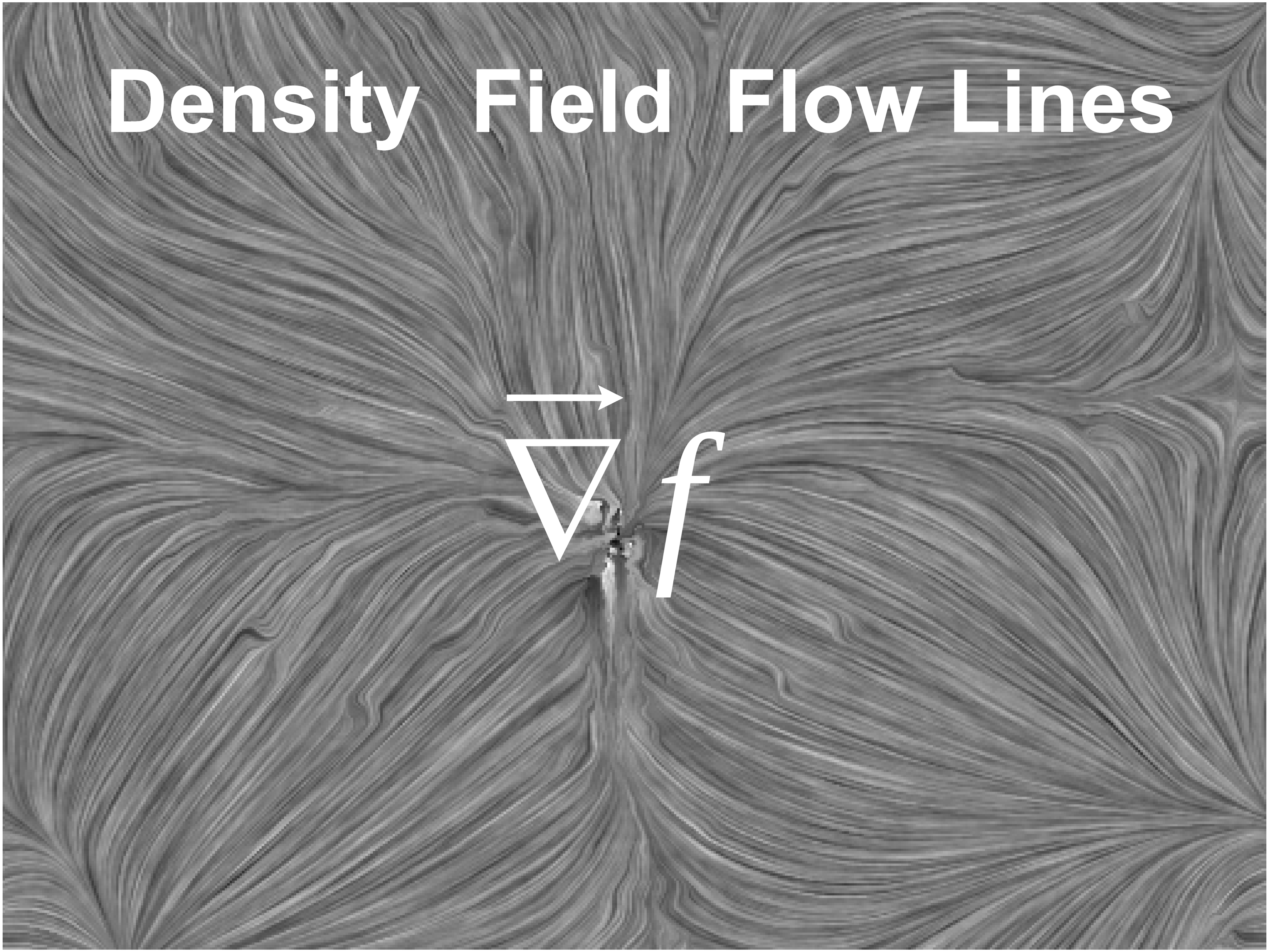
Homology, homotopy,
dimensions of groups,
Betti numbers, persistence

DIFFERENTIAL TOPOLOGY

Curvature, forms, Betti numbers,
Morse theory, integration,
Lipschitz-Killing curvatures

Random Field Topology:

Morse Complex


Density Field Flow Lines
$\vec{\nabla} f$

Density Field Flow Lines

$$\vec{\nabla} f = 0$$

Critical Points:

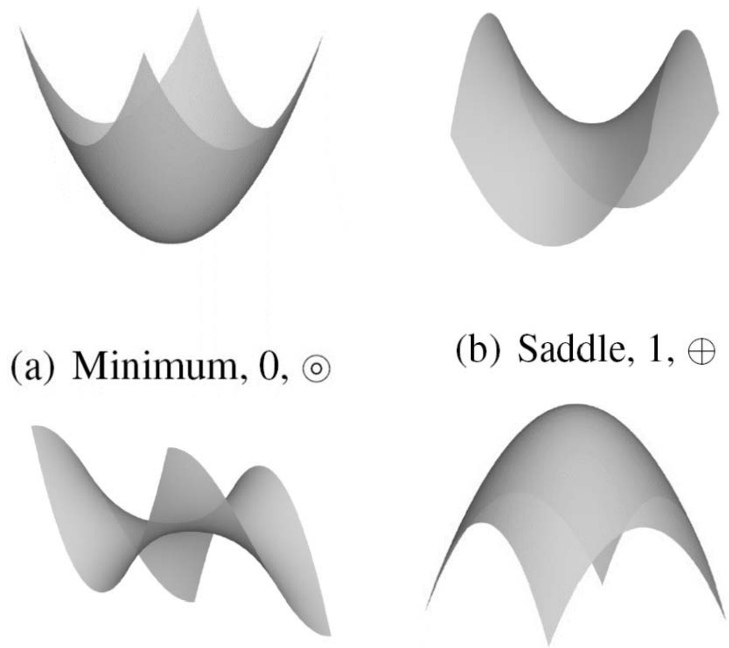
- Maxima
- Minima
- Saddle Points (of various signatures)

Betti & Morse

Relation to Morse Theory:

Topological Structure Continuous Field determined by **singularities:**

- **maxima**
- **minima**
- **saddle points**

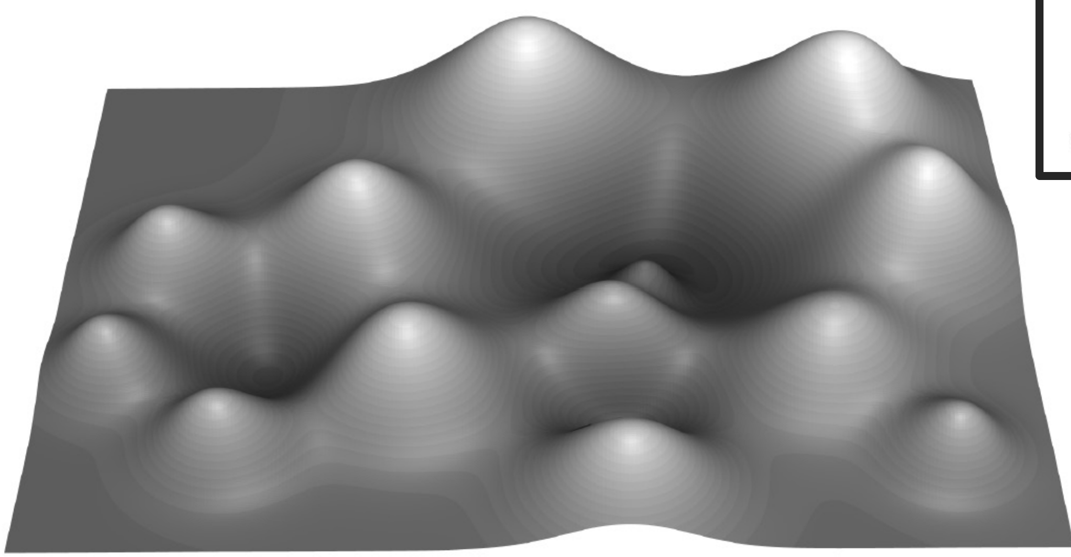


(a) Minimum, 0, ⊚

(b) Saddle, 1, ⊕

(d) Monkey Saddle, ⊛

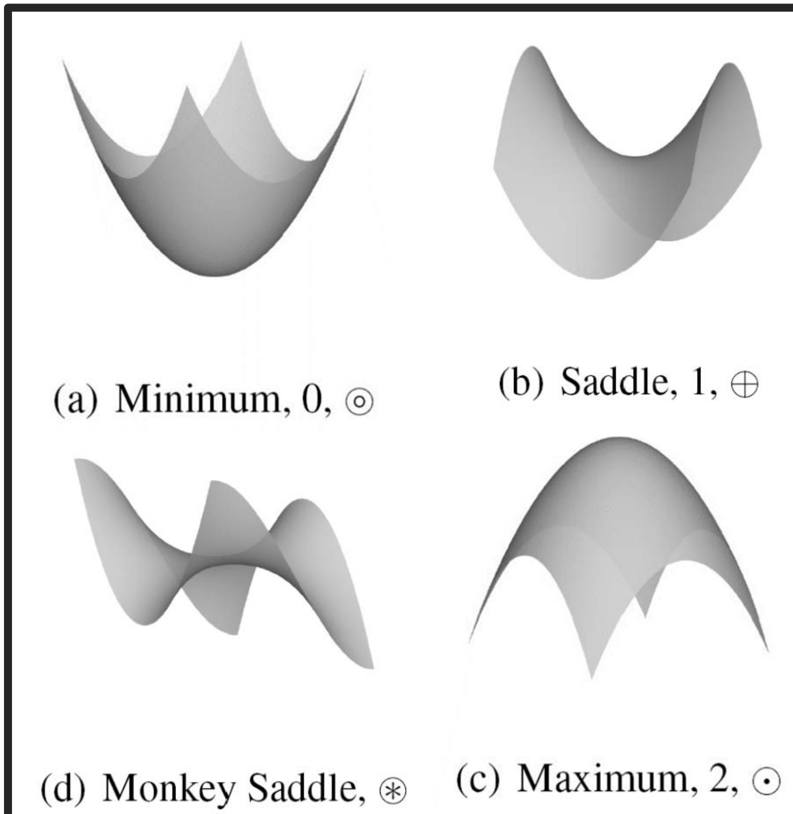
(c) Maximum, 2, ⊙

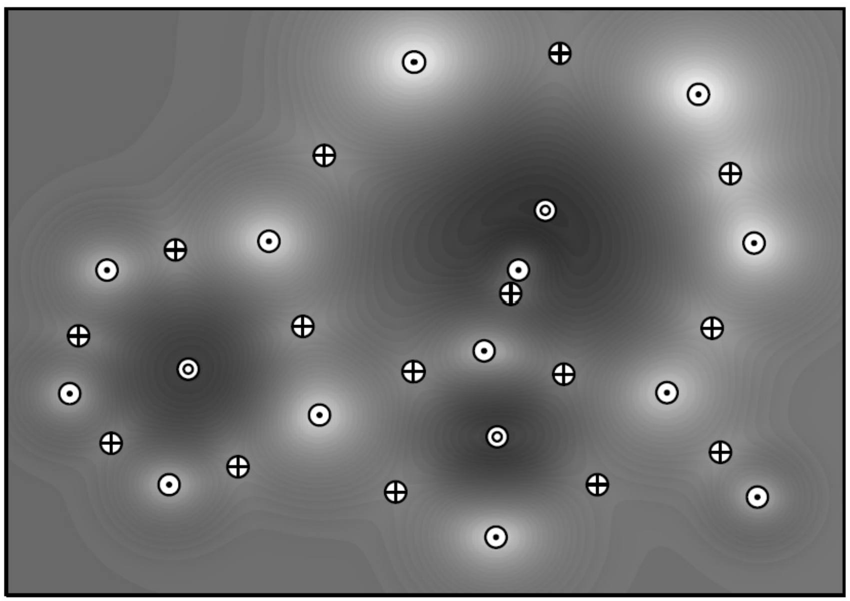


Betti & Morse

Number of singularities in field determines Euler characteristic:

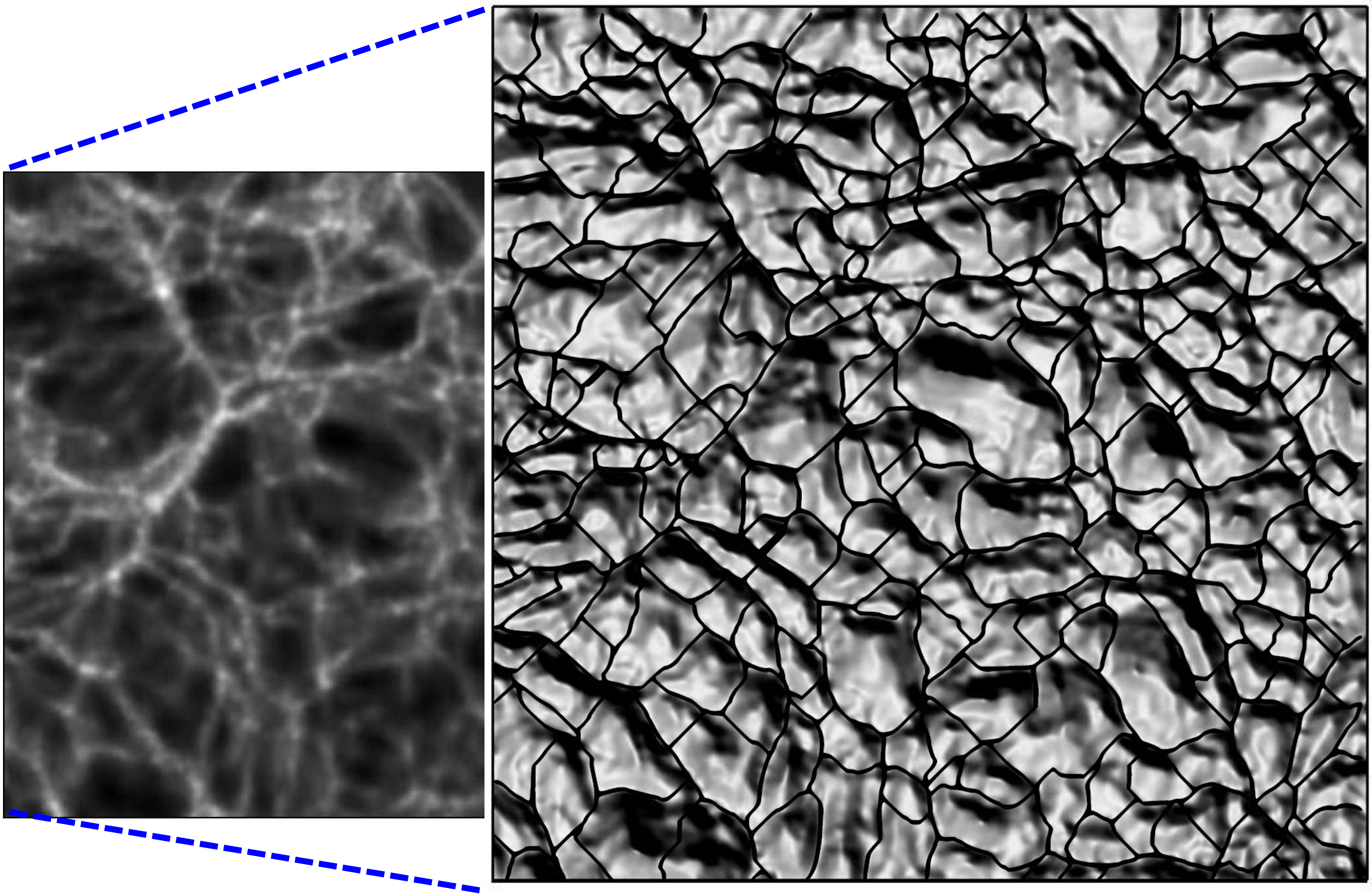
ζ_0 : **minima**
ζ_1 : **saddle 1**
ζ_2 : **saddle 2**
ζ_3 : **maxima**



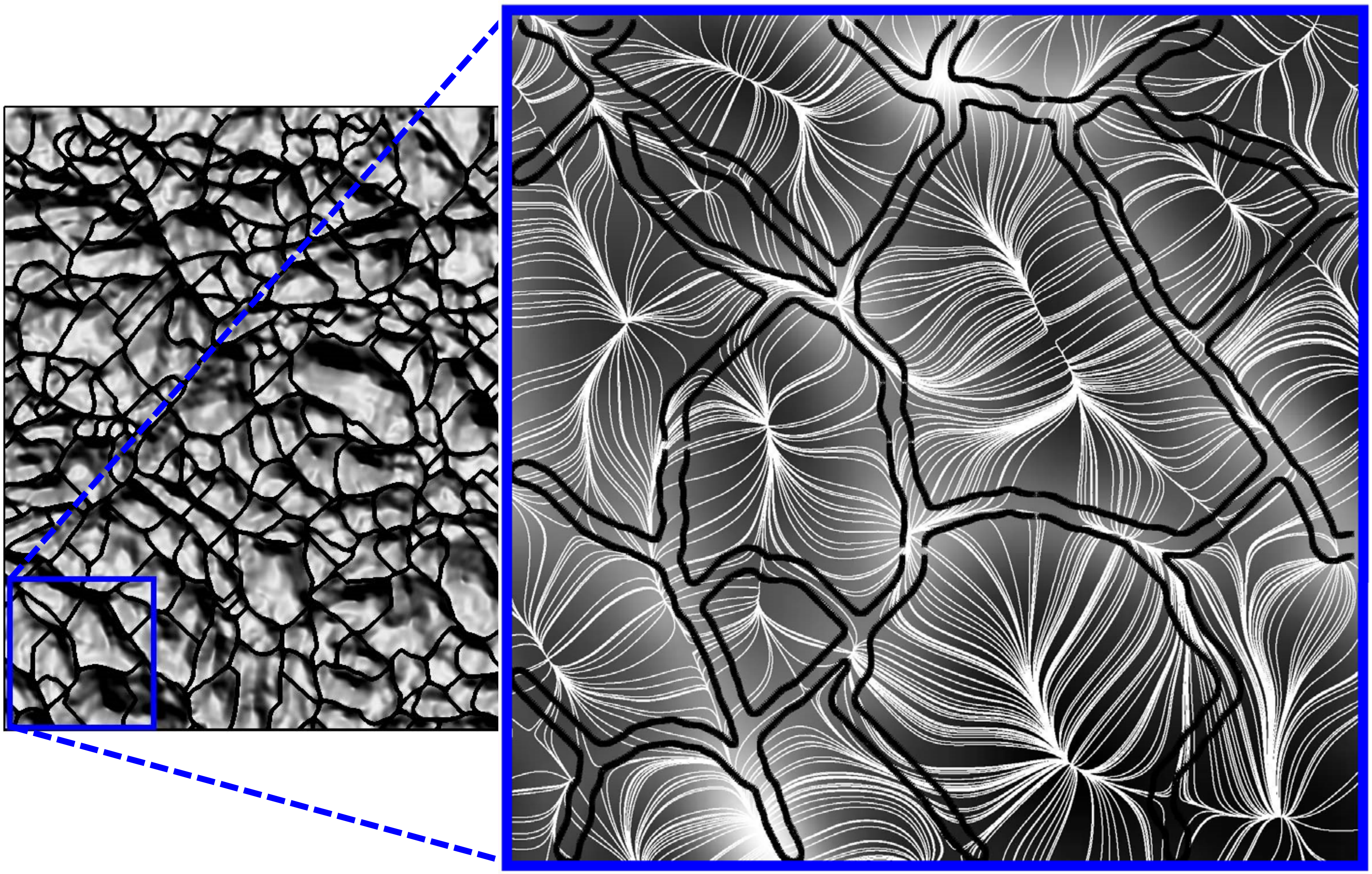


$$\chi = \sum_{k=0}^{d} (-1)^k \zeta_k$$

Density Field & Landscape



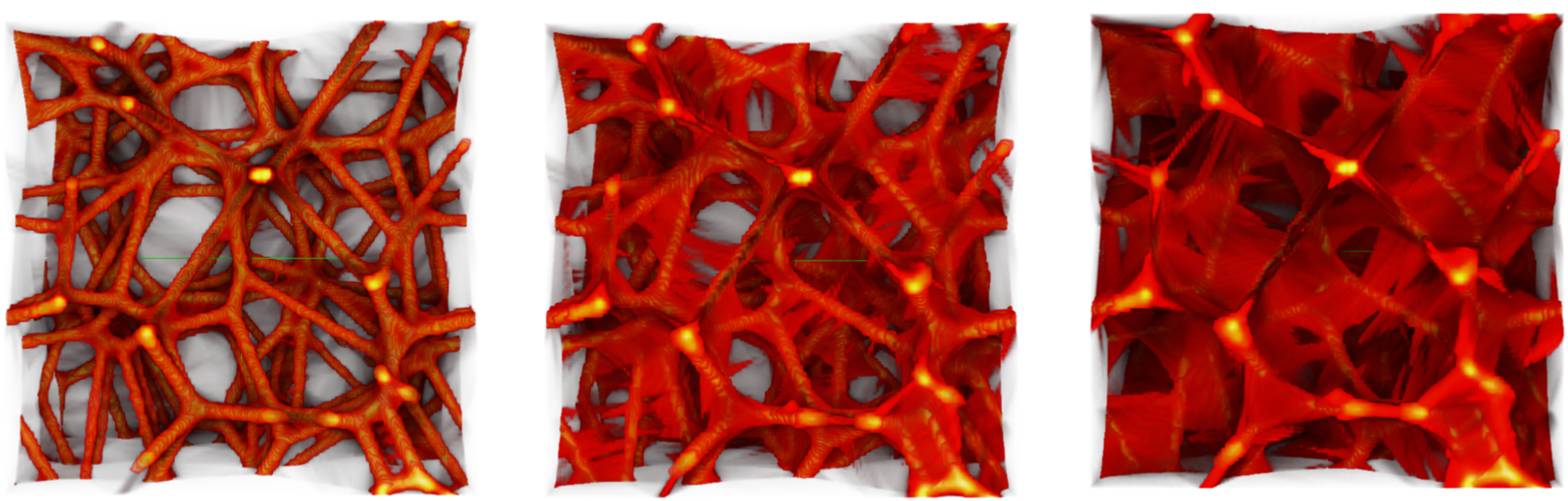
Segmentation & Flowlines



Topological Hierarchy: Excursion Sets & Filtrations

Superlevel Sets

$$\mathfrak{M}_\nu = \left\{ \vec{x} \in \mathfrak{M} \mid f_s\left(\vec{x}\right) \in [f_\nu, \infty) \right\}$$
$$= f_s^{-1}[f_\nu, \infty)$$



Pranav et al. 2013a

Filtrations

Important source of information about topology of a field/point distribution:

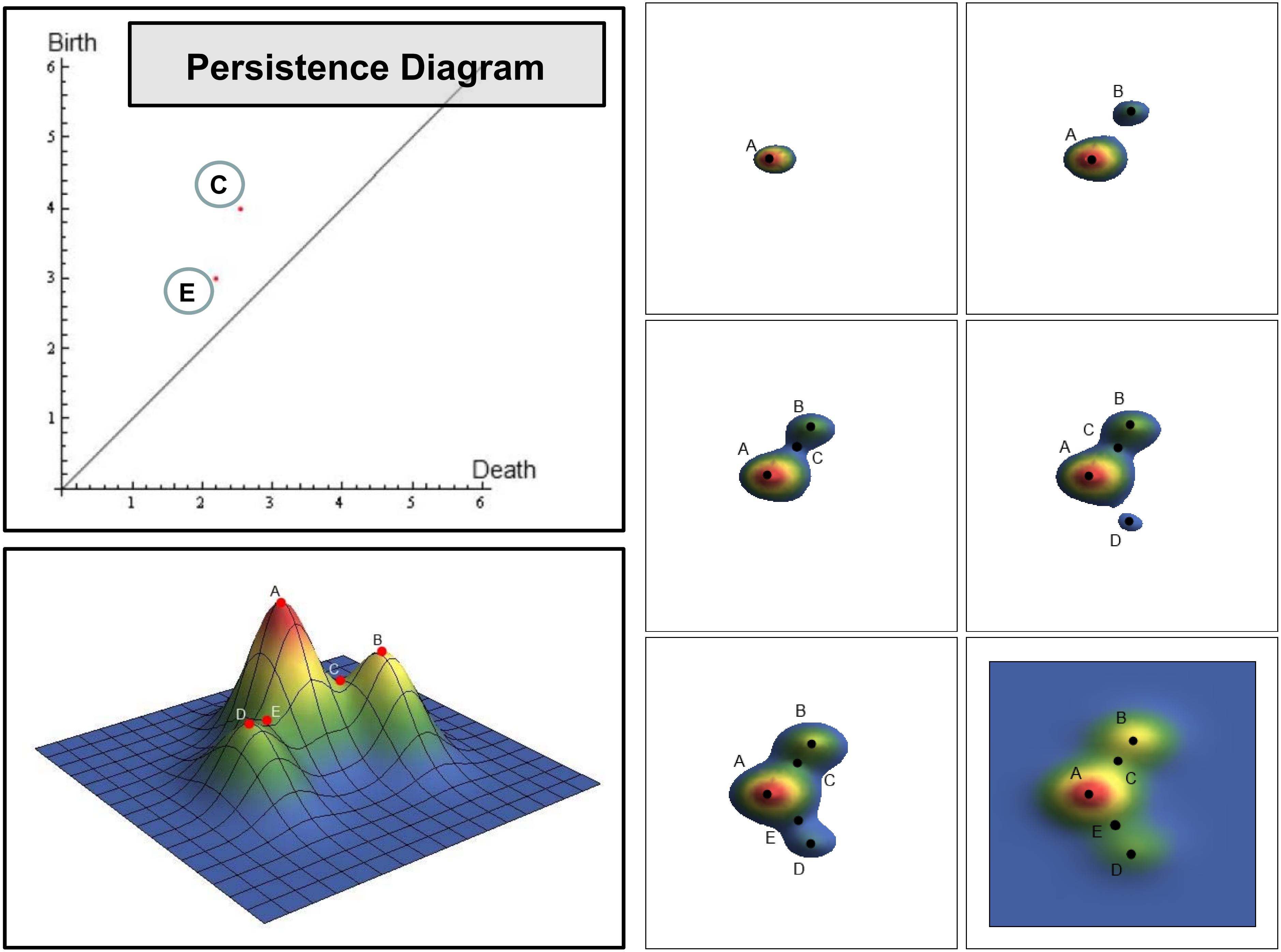
Filtrations

Filtration provides view of topology as a function of scale.

Formally, given a space 𝔐, a filtration is a nested sequence of subspaces

$$\varnothing = \mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}_2 \subseteq \cdots \mathfrak{M}_m = \mathfrak{M}$$

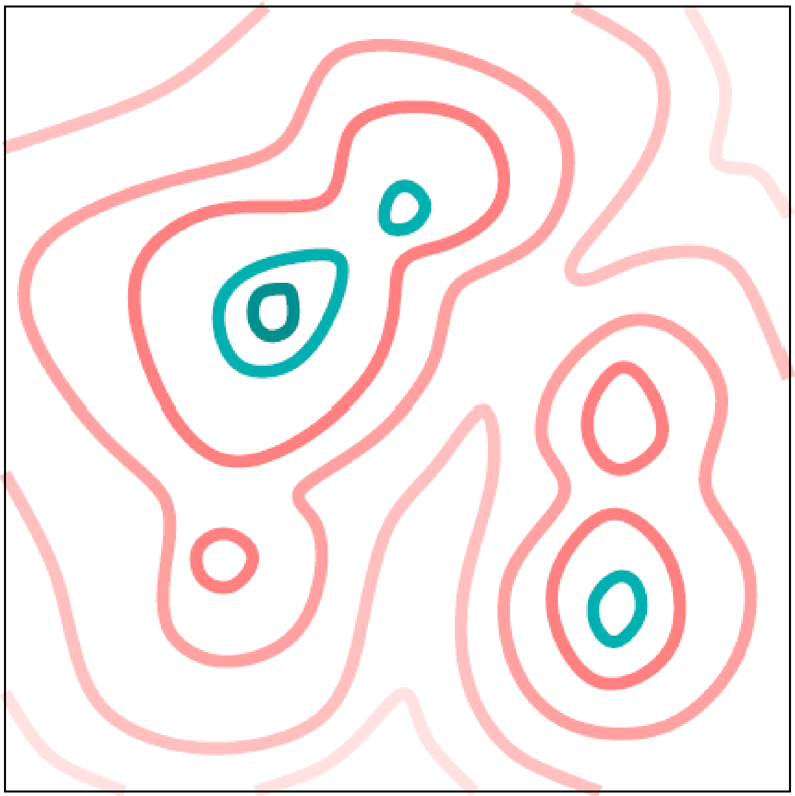
Nature of filtrations depends (amongst others) on representation of the mass distribution.



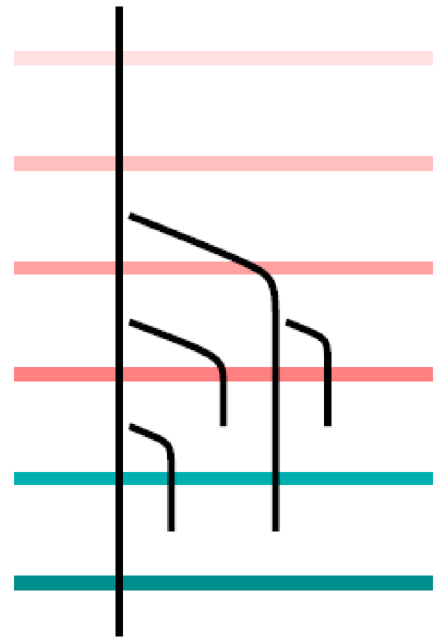
Persistence Diagram
Birth
Death
1
2
3
4
5
6
C
E
A
B
C
D
E

Topological Hierarchy

Persistent Homology:
"Cycling" over density filtration



Topology Tree



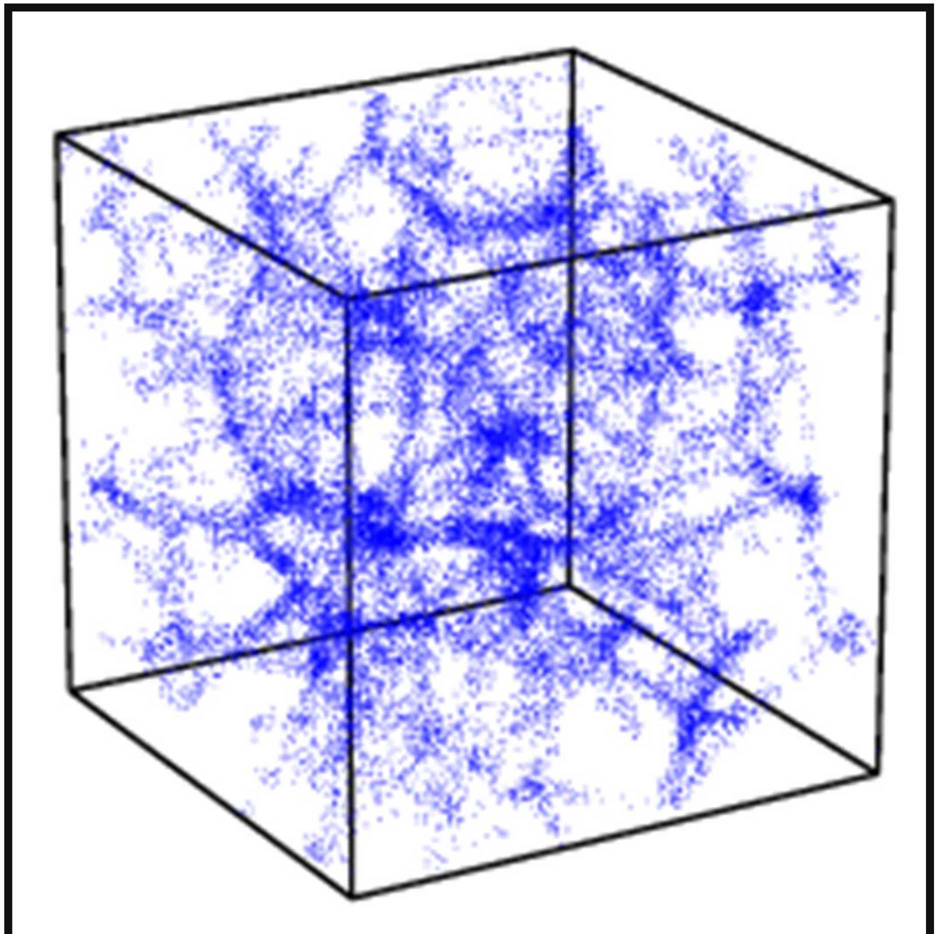
Edelsbrunner & Harer 2010

field value filtration
tree hierarchy

Cosmic Web

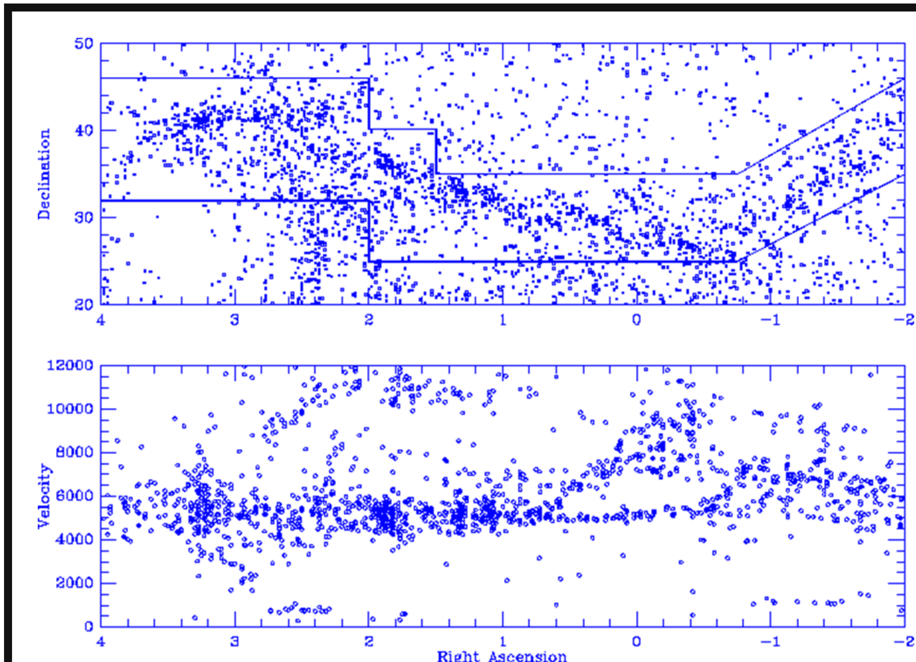
Homology & Persistence

Voronoi Elements: Filaments

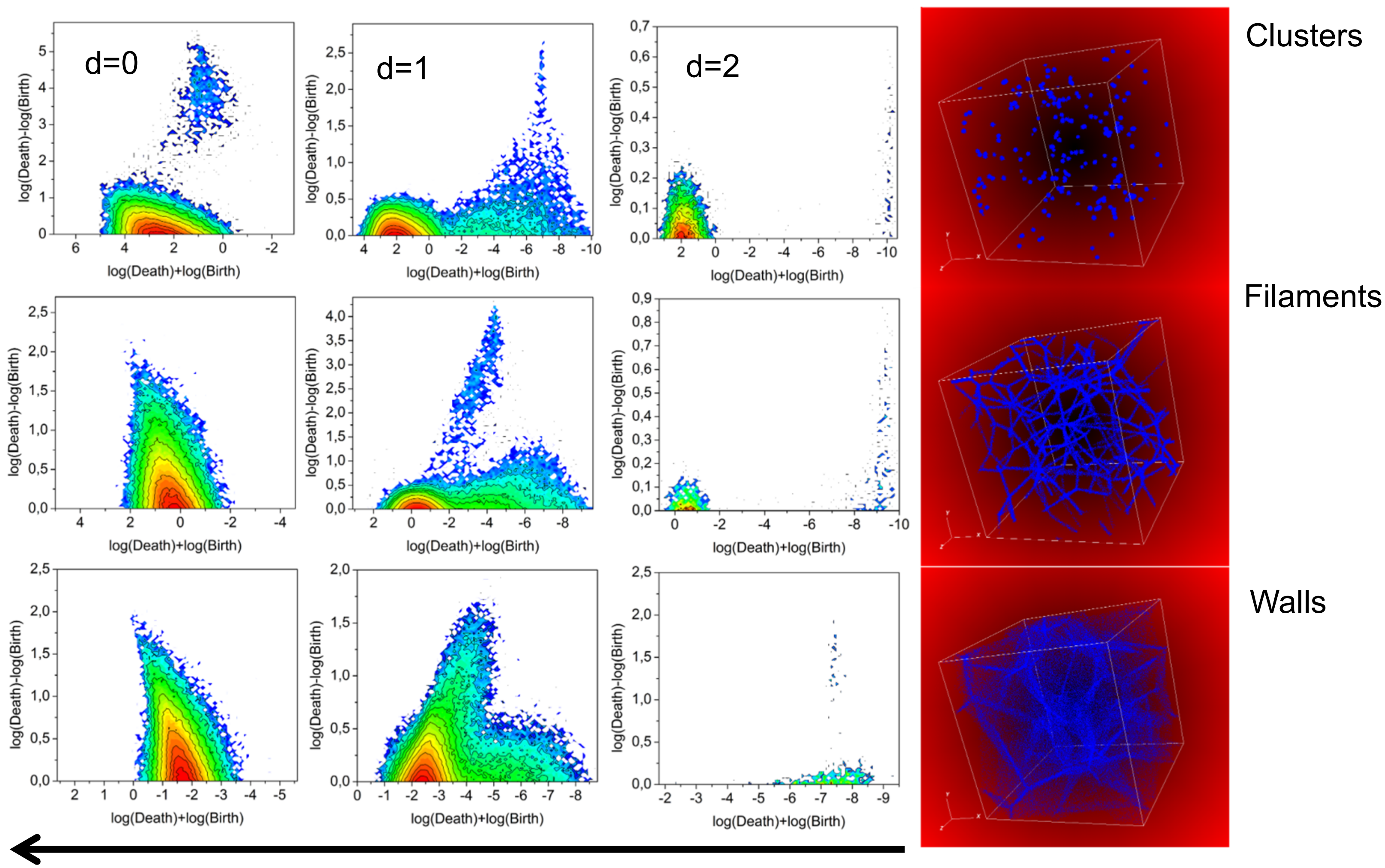


Filaments:

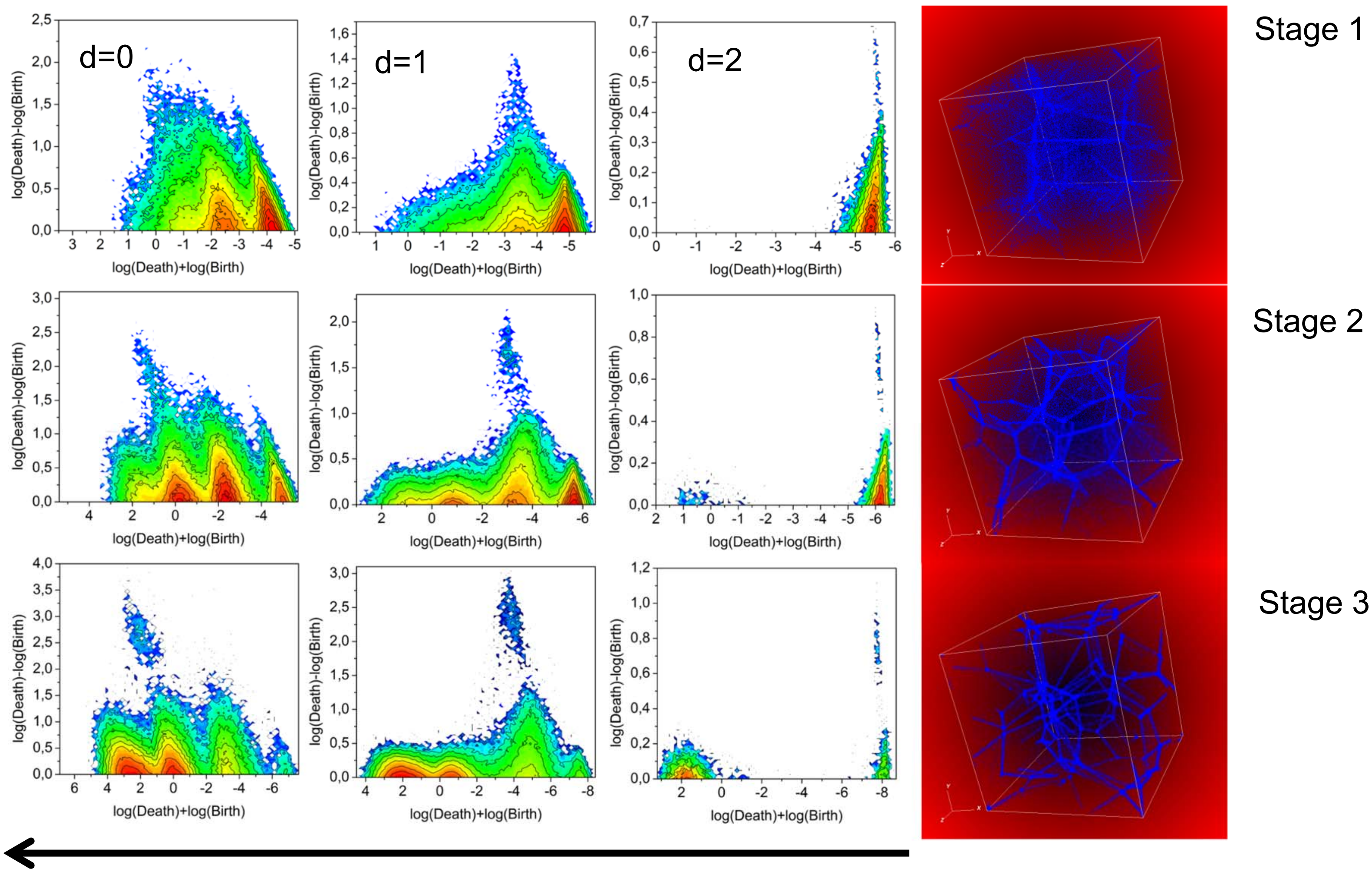
Pisces-Perseus chain



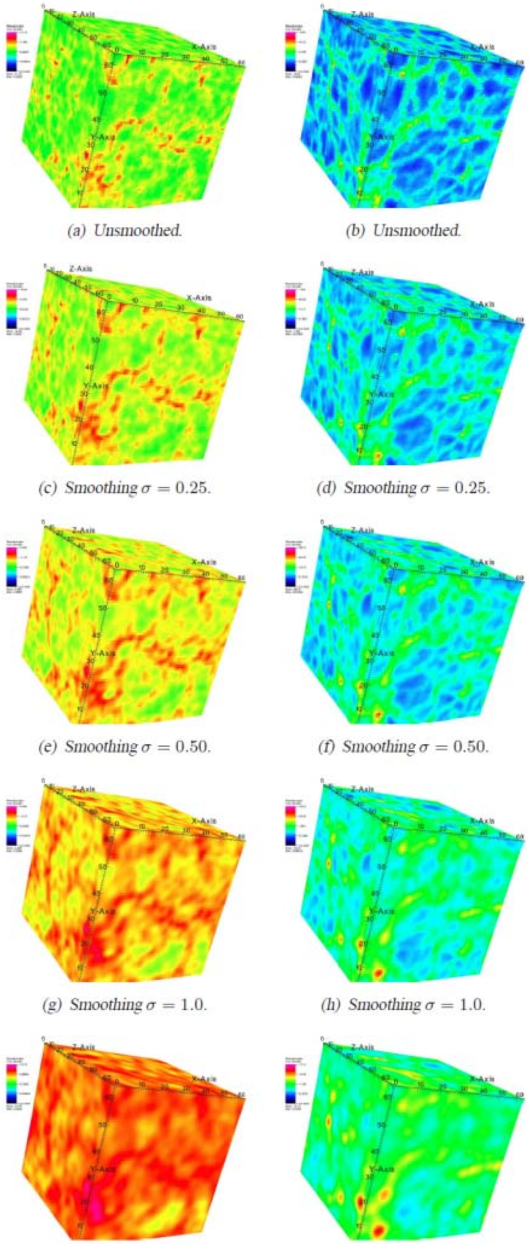
Voronoi Element Models: Persistence Diagrams



Voronoi Kinematic Models: Persistence Diagrams



LCDM Persistence



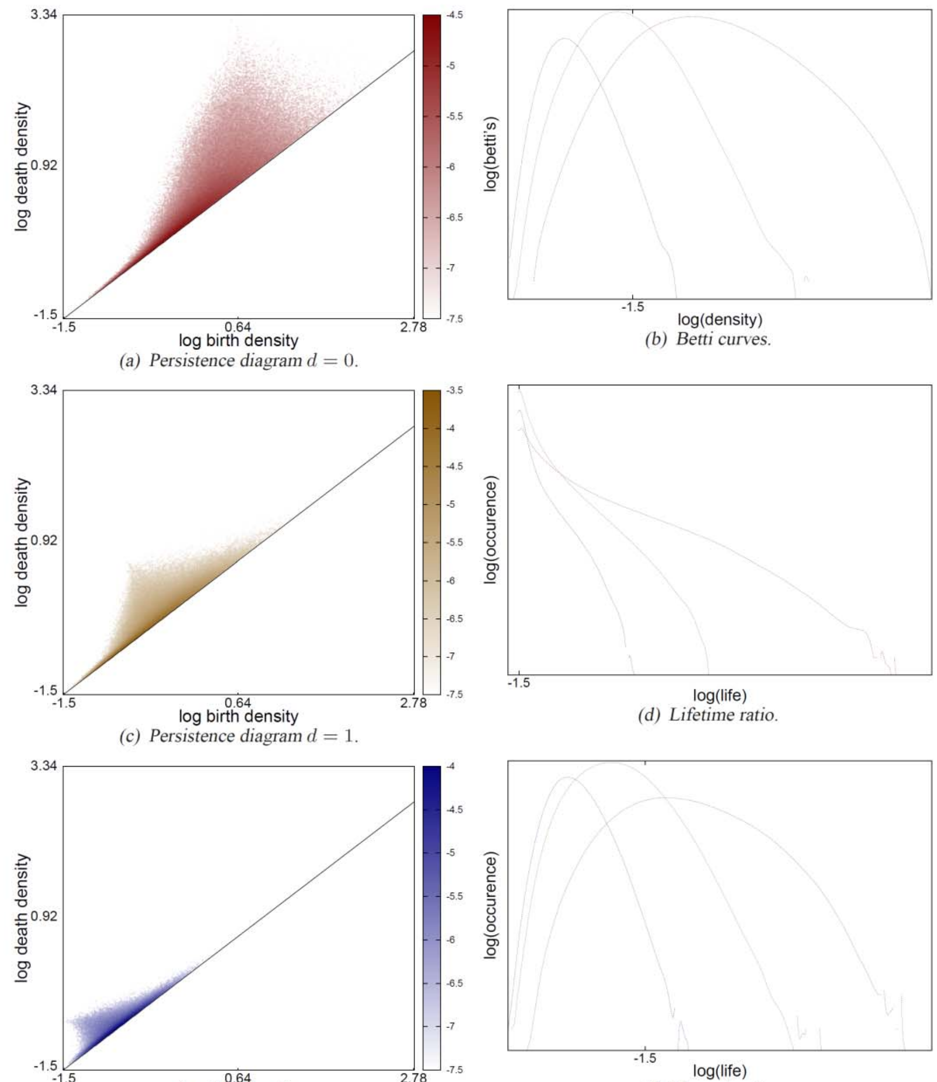
(a) Unsmoothed. (b) Unsmoothed.

(c) Smoothing $\sigma = 0.25$. (d) Smoothing $\sigma = 0.25$.

(e) Smoothing $\sigma = 0.50$. (f) Smoothing $\sigma = 0.50$.

(g) Smoothing $\sigma = 1.0$. (h) Smoothing $\sigma = 1.0$.

(i) Smoothing $\sigma = 2.0$. (j) Smoothing $\sigma = 2.0$.



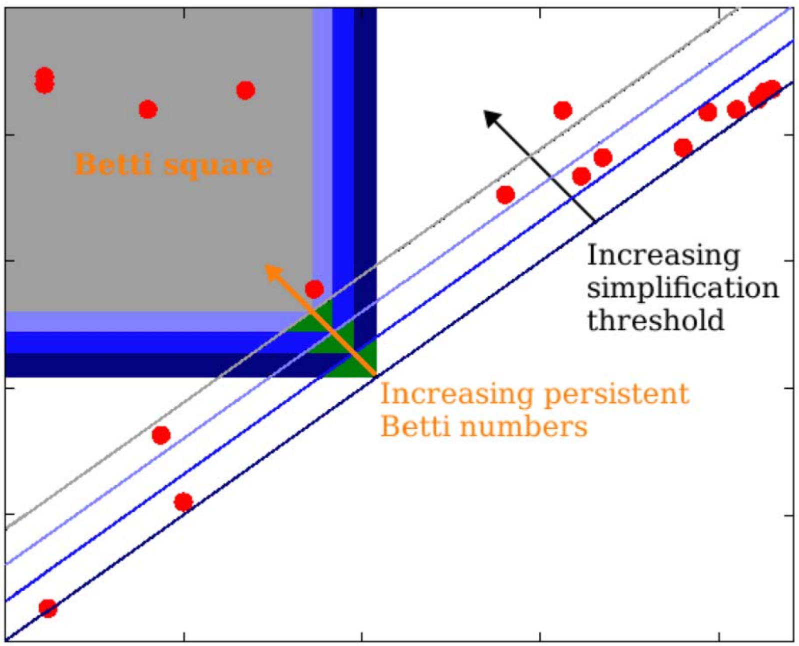
(a) Persistence diagram $d = 0$. (b) Betti curves.

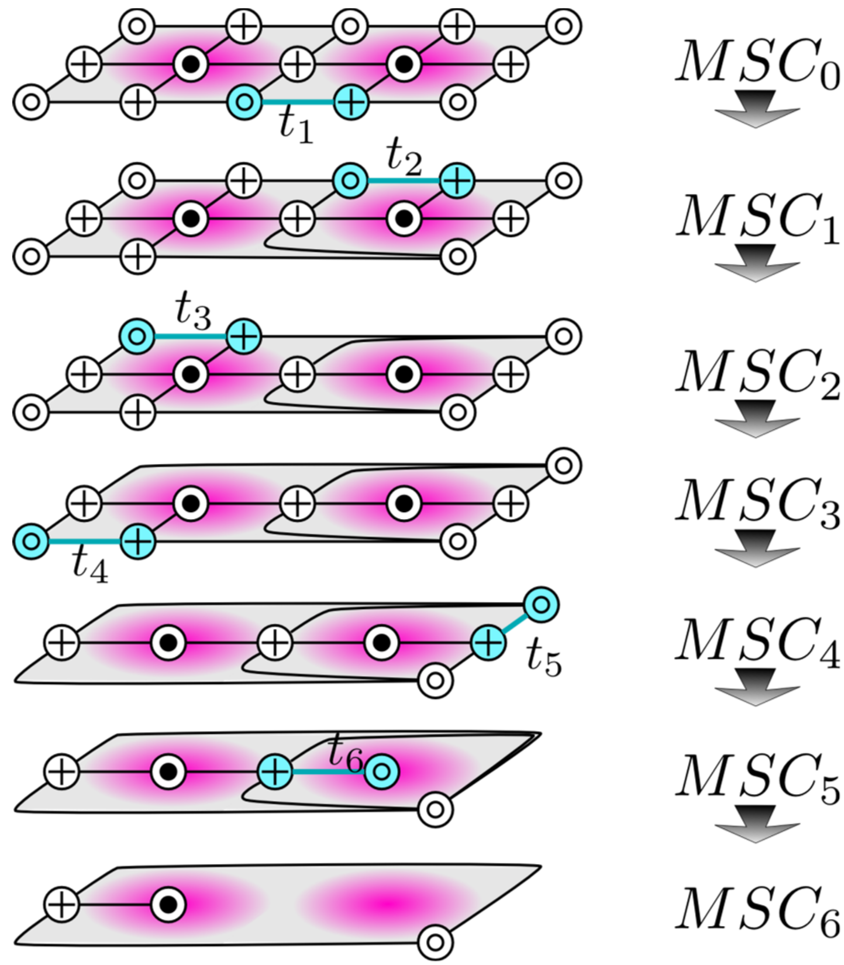
(c) Persistence diagram $d = 1$. (d) Lifetime ratio.

(e) Persistence diagram $d = 2$. (f) Mean density

Nevenzeel & vdW 2015

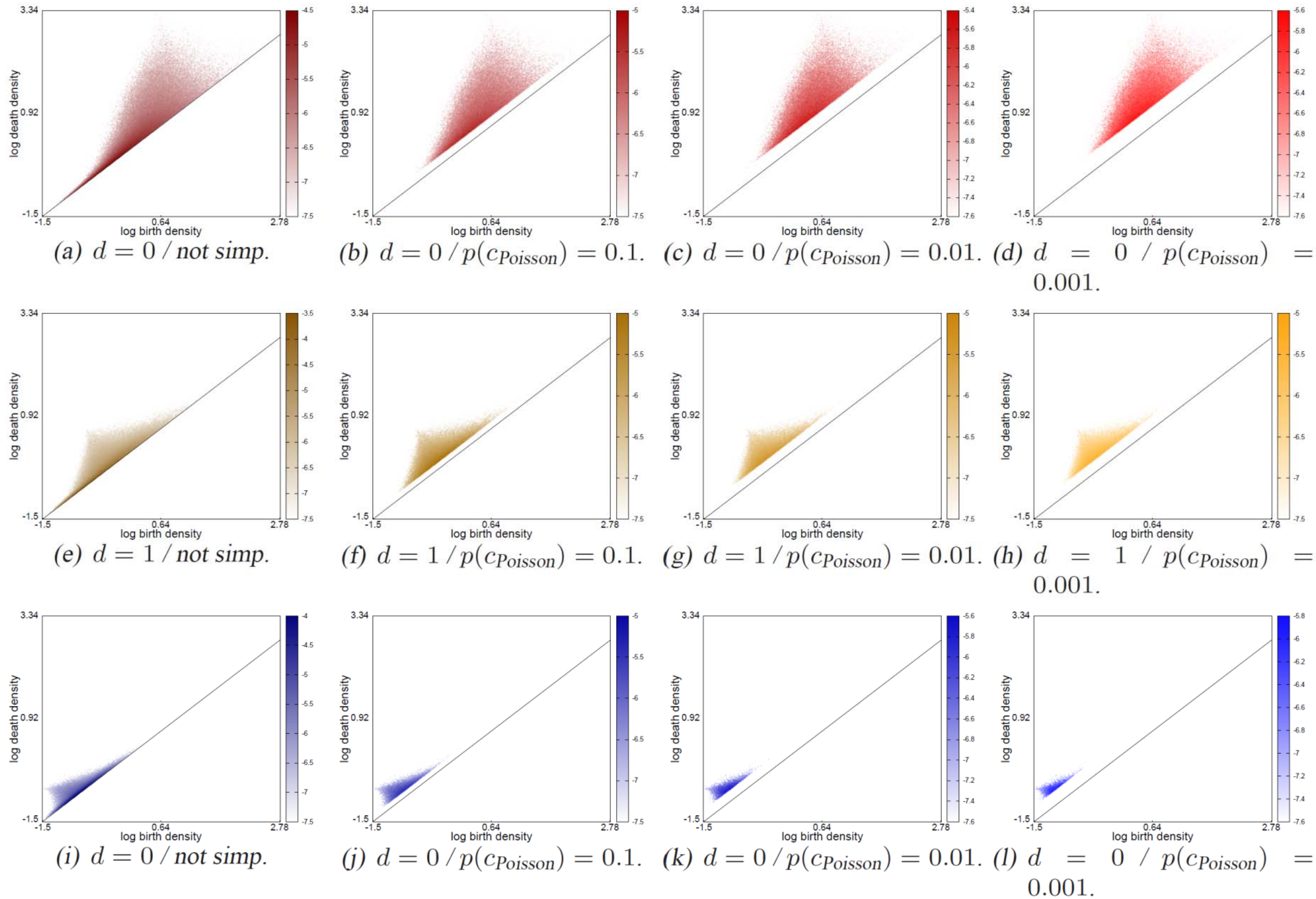
LCDM Persistence





Morse-Smale simplification

LCDM: Persistence & Morse Simplification



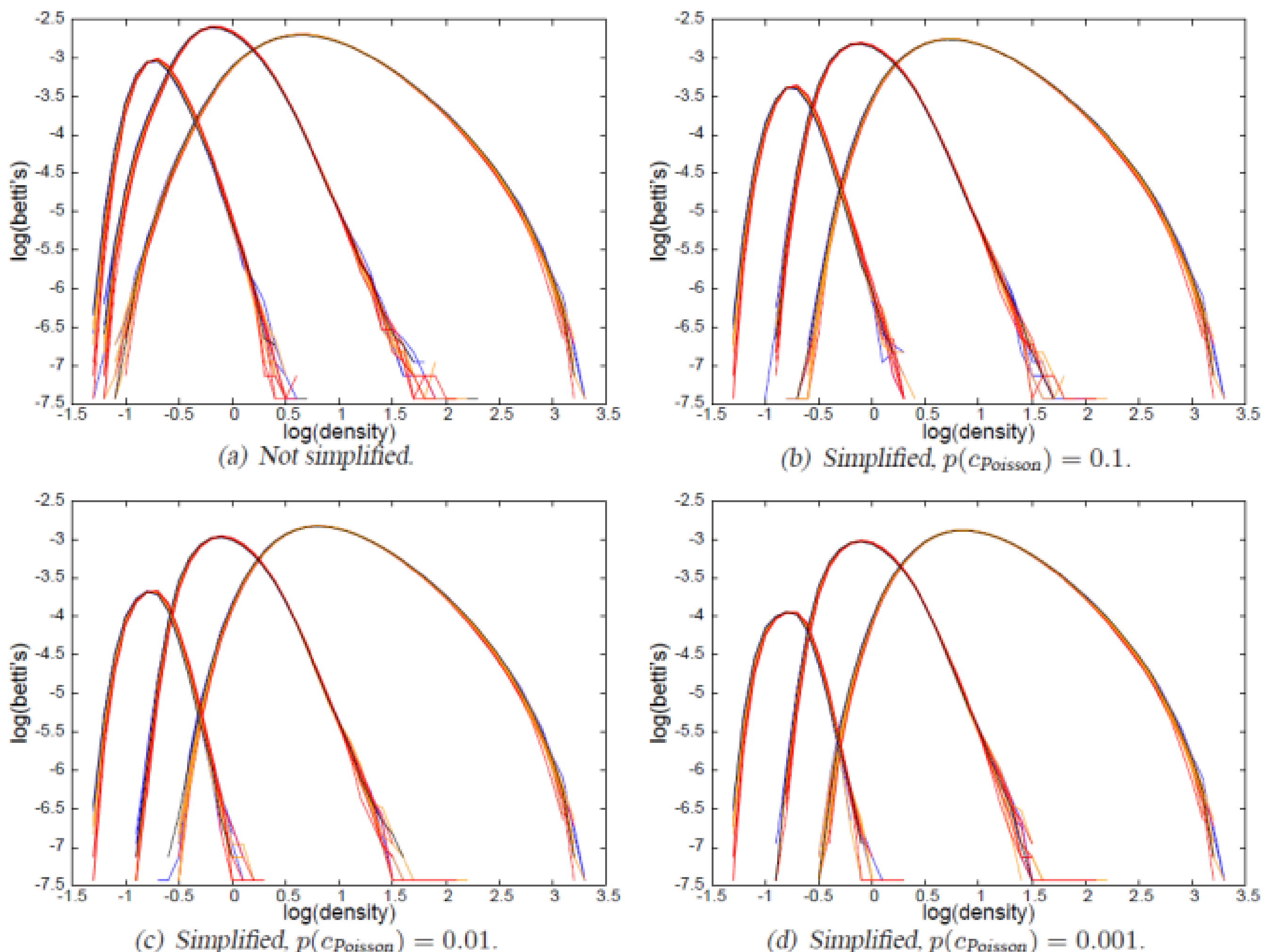
(a) $d = 0$ / not simp. (b) $d = 0 / p(c_{\text{Poisson}}) = 0.1$. (c) $d = 0 / p(c_{\text{Poisson}}) = 0.01$. (d) $d = 0 / p(c_{\text{Poisson}}) = 0.001$.

(e) $d = 1$ / not simp. (f) $d = 1 / p(c_{\text{Poisson}}) = 0.1$. (g) $d = 1 / p(c_{\text{Poisson}}) = 0.01$. (h) $d = 1 / p(c_{\text{Poisson}}) = 0.001$.

(i) $d = 0$ / not simp. (j) $d = 0 / p(c_{\text{Poisson}}) = 0.1$. (k) $d = 0 / p(c_{\text{Poisson}}) = 0.01$. (l) $d = 0 / p(c_{\text{Poisson}}) = 0.001$.

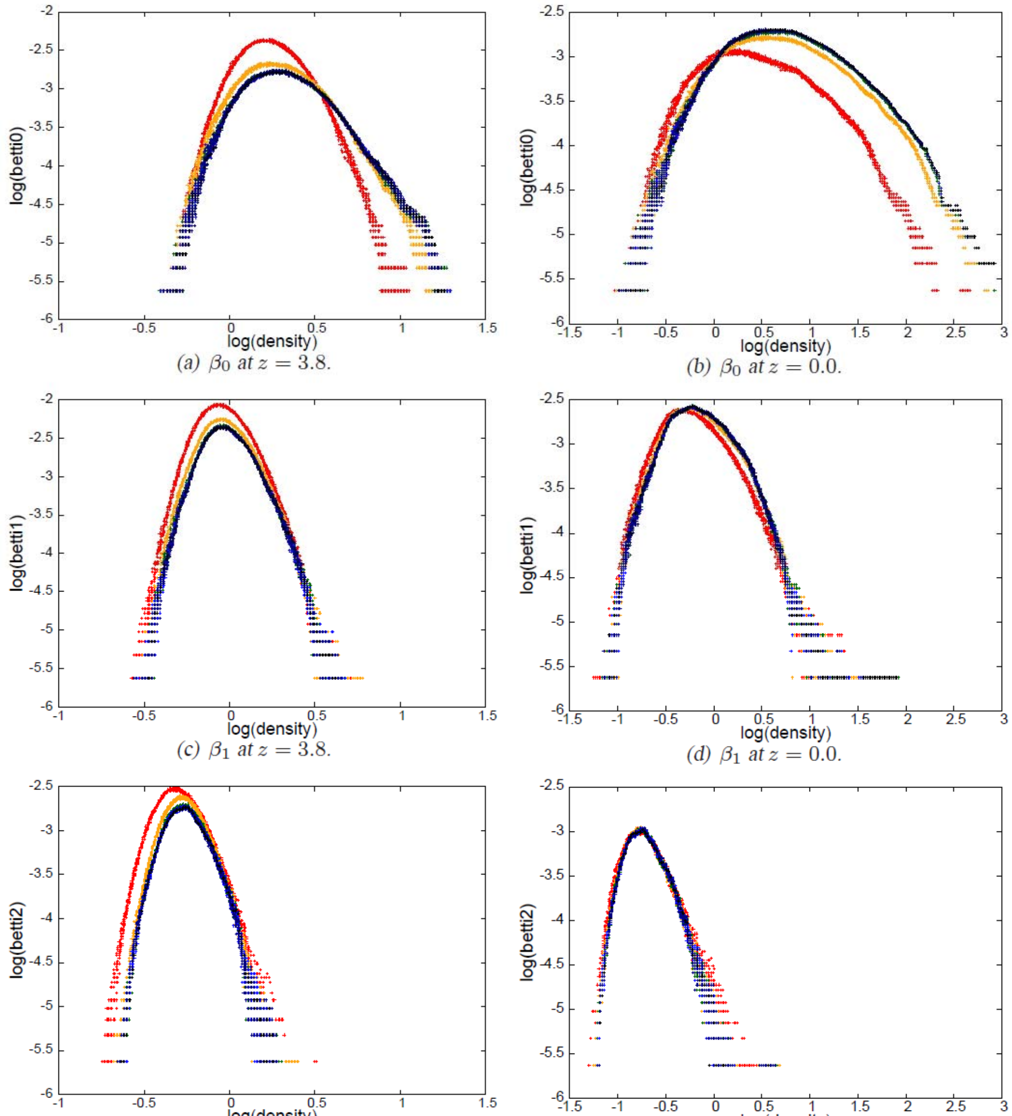
Nevenzeel & vdW 2017

LCDM:
Betti Curves & Morse Simplification



Nevenzeel & vdW 2017

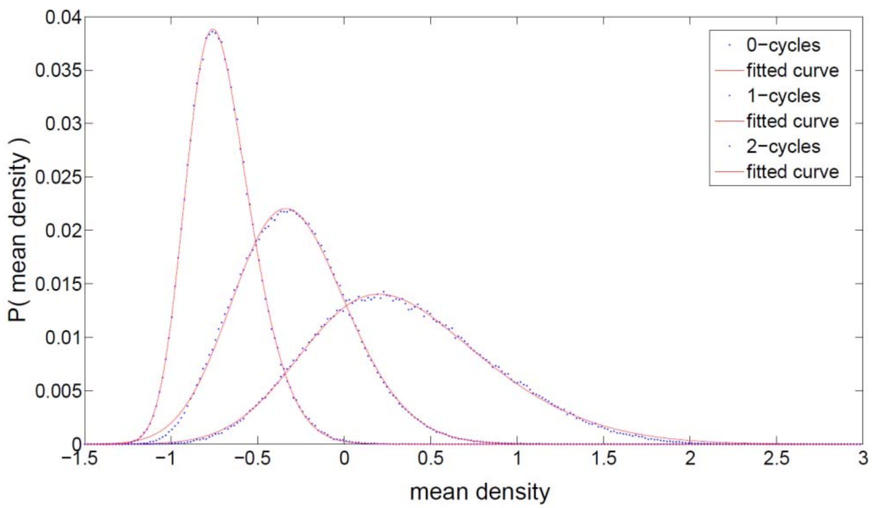
Betti Curve Stability



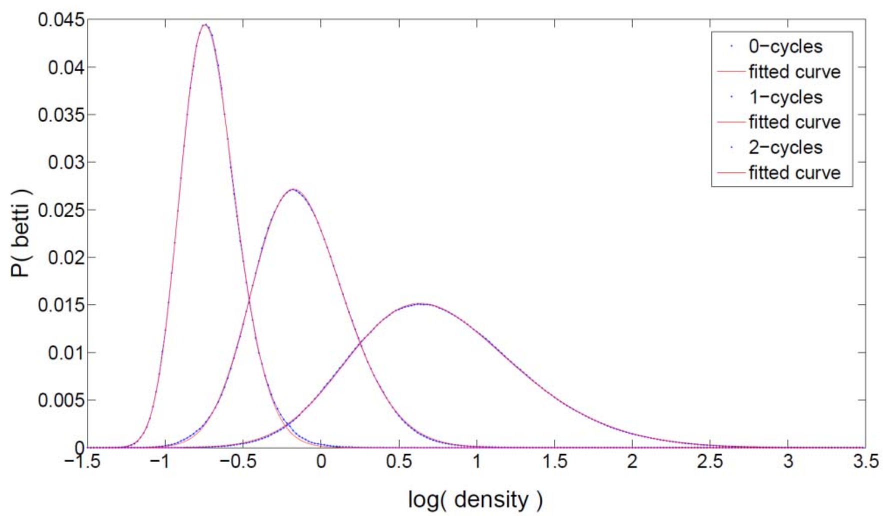
(a) β_0 at $z = 3.8$.
(b) β_0 at $z = 0.0$.
(c) β_1 at $z = 3.8$.
(d) β_1 at $z = 0.0$.
(e) β_2 at $z = 3.8$.
(f) β_2 at $z = 0.0$.

Nevenzeel & vdW 2015

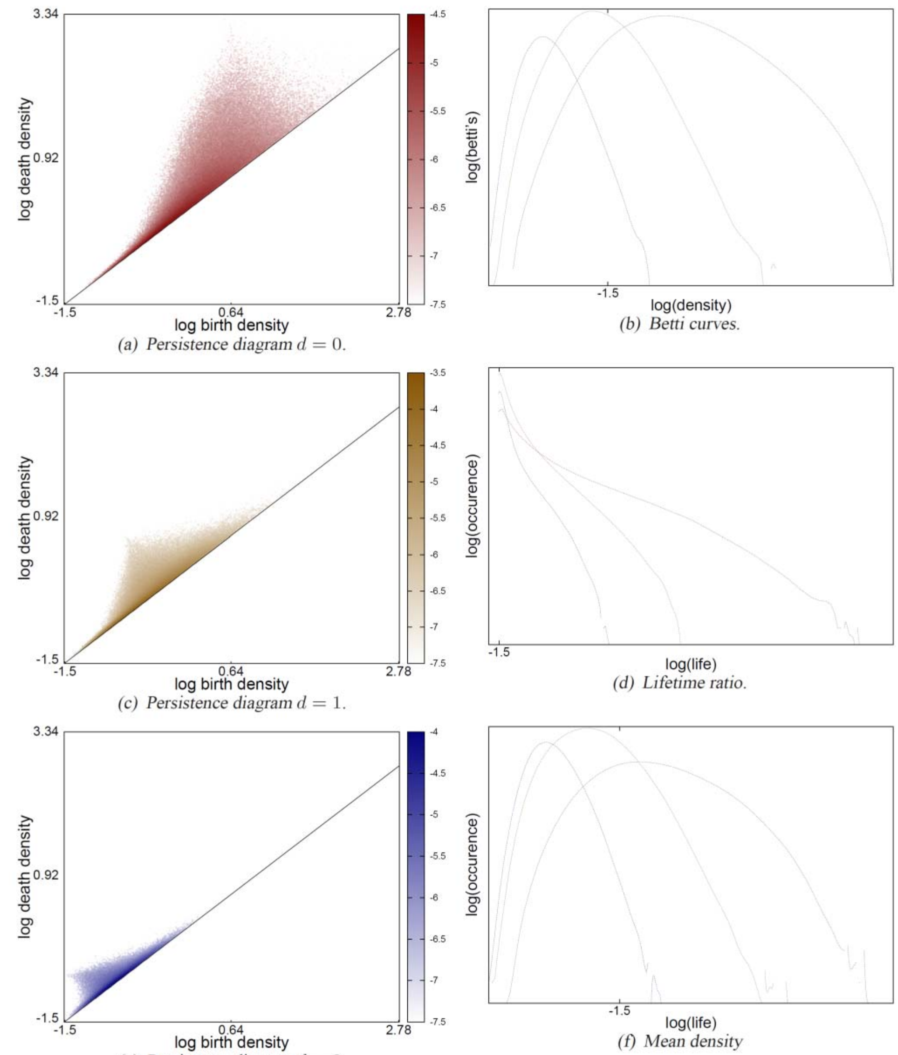
LCDM Persistence



(a) Skewed Gaussian fit of mean density curves.



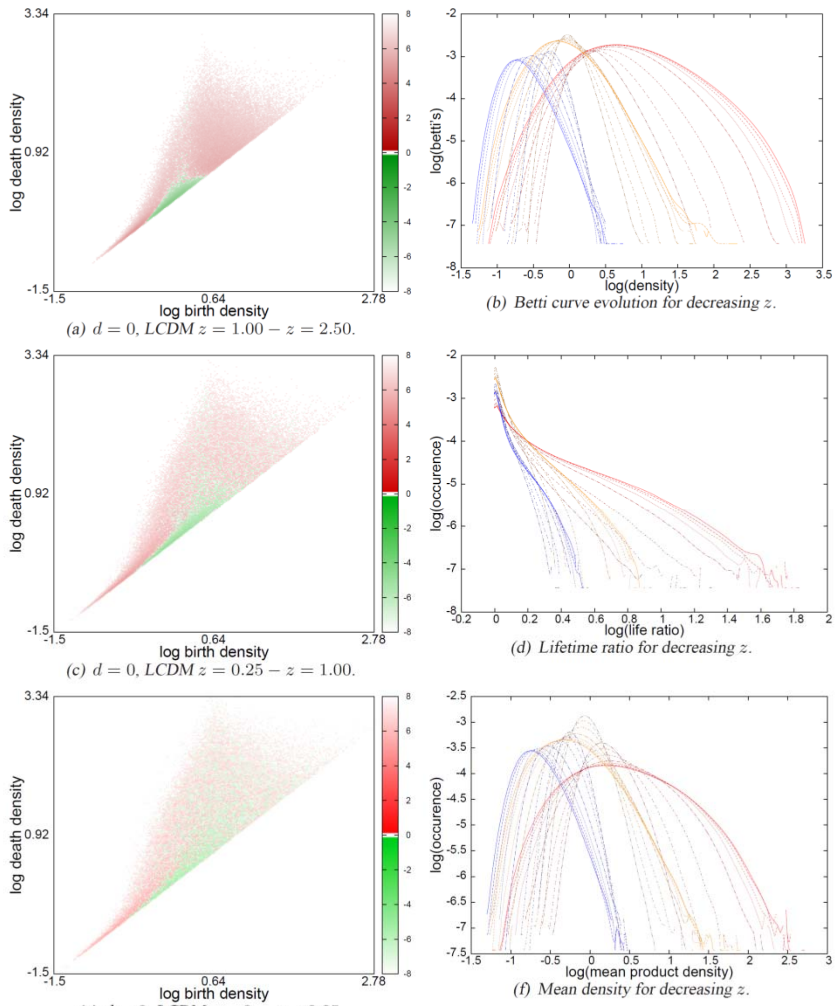
(b) Skewed Gaussian fit of Betti curves



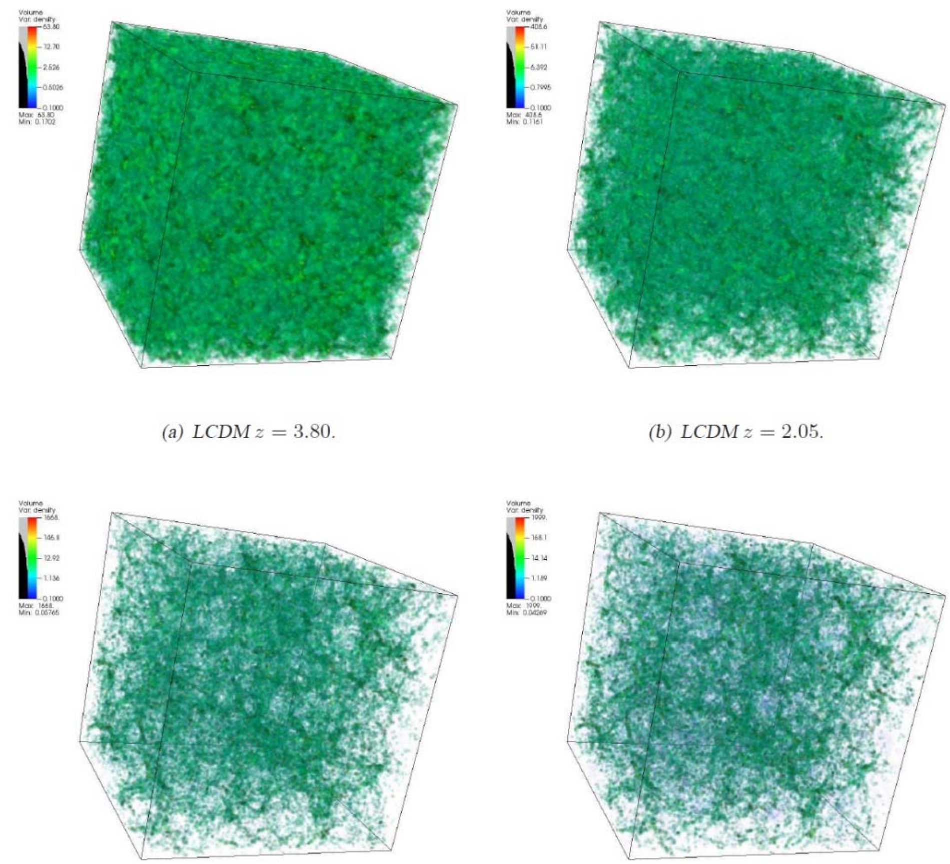
(a) Persistence diagram $d = 0$.
(b) Betti curves.
(c) Persistence diagram $d = 1$.
(d) Lifetime ratio.
(e) Persistence diagram $d = 2$.
(f) Mean density

Nevenzeel & vdW 2017

LCDM: Evolving Persistence



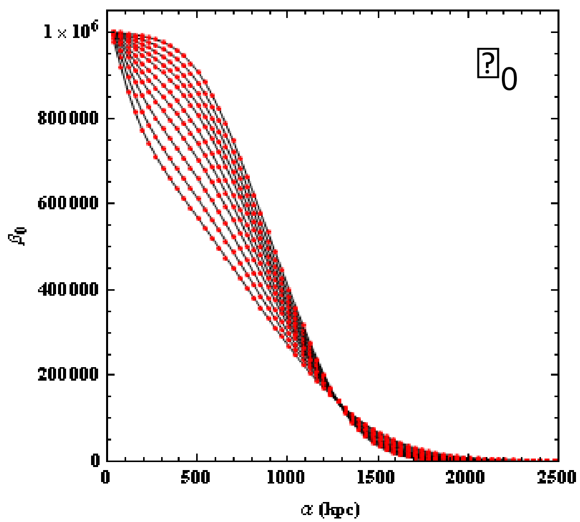
(a) $d = 0$, LCDM $z = 1.00 - z = 2.50$.
(b) Betti curve evolution for decreasing z.
(c) $d = 0$, LCDM $z = 0.25 - z = 1.00$.
(d) Lifetime ratio for decreasing z.
(e) $d = 0$, LCDM $z = 0 - z = 0.25$.
(f) Mean density for decreasing z.

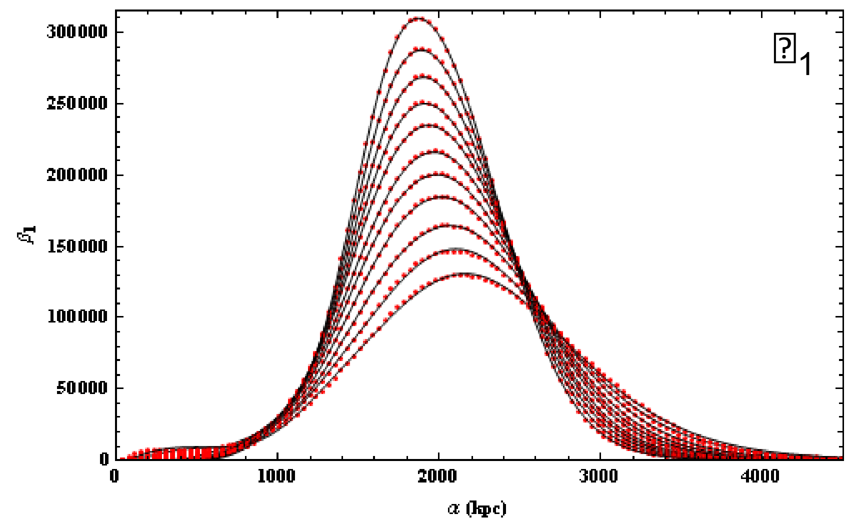


(a) LCDM $z = 3.80$.
(b) LCDM $z = 2.05$.
(c) LCDM $z = 0.507$.
(d) LCDM $z = 0.00$.

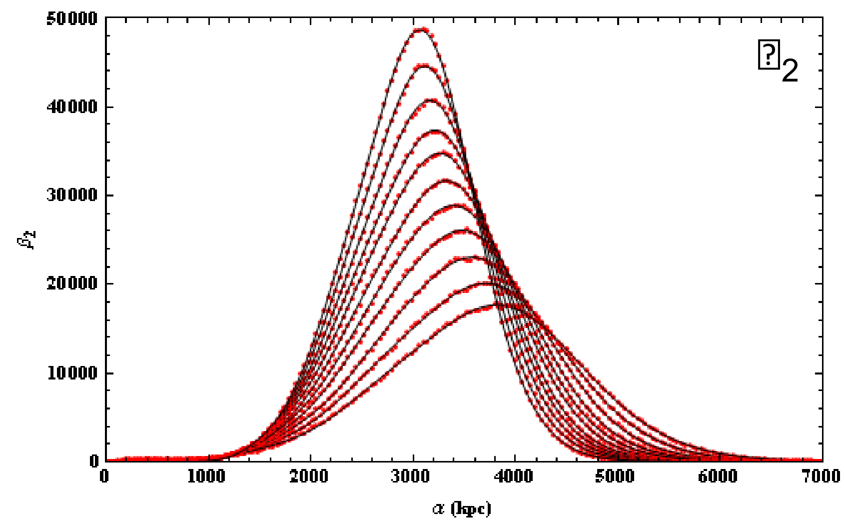
Nevenzeel & vdW 2017

Homology of evolving LCDM cosmology





□-shape homology



Betti$_2$:
evolving void populations