the Cosmic Web:

Lecture 4: Cosmic Web Pattern Analysis

Rien van de Weijgaert,
Cosmic Web, Caput Course, Oct. 2017
Cosmic Structure Analysis

The overwhelming complexity of the individual structures, as well as their connectivity, the lack of structural symmetries, the intrinsic multiscale nature and the wide range of densities that one finds in the cosmic matter distribution has prevented the use of simple and straightforward instruments.

To assess the key aspects of the nonlinear cosmic matter and galaxy distribution:

- multiscale character
- weblike network
- volume dominance voids
- hierarchical structure formation
- anisotropic collapse
- asymmetry overdense vs. underdense
Despite the multitude of elaborate qualitative descriptions it has remained a major challenge to characterize the structure, geometry and topology of the Cosmic Web.

Quantities as basic and general as the mass and volume content of clusters, filaments, walls and voids are still not well established or defined. Since there is not yet a common framework to objectively define filaments and walls, the comparison of results of different studies concerning properties of the filamentary network -- such as their internal structure and dynamics, evolution in time, and connectivity properties -- is usually rendered cumbersome and/or difficult.

The overwhelming complexity of the individual structures as well as their connectivity, the lack of structural symmetries, its intrinsic multiscale nature and the wide range of densities that one finds in the cosmic matter distribution has prevented the use of simple and straightforward toolbox.

Over the years, a variety of heuristic measures were forwarded to analyze specific aspects of the spatial patterns in the large scale Universe. Only in recent years these have lead to a more solid and well-defined machinery for the description and quantitative analysis of the intricate and complex spatial patterns of the Cosmic Web.

Nearly without exception, these methods borrow extensively from other branches of science such as image processing, mathematical morphology, computational geometry and medical imaging.
Structure Statistics:

Correlation Functions
Power Spectrum, et al.
If you want to know more...

**Standard Reference:**

Martinez & Saar
Ergodic Theorem

- Ensemble Averages $\Leftrightarrow$ Spatial Averages over one realization of random field

- Basis for statistical analysis of cosmological large scale structure

- In statistical mechanics, the Ergodic Hypothesis usually refers to time evolution of system, in cosmological applications to spatial distribution at one fixed time
Infinitesimal Definition Two-Point Correlation Function:

Joint probability that in each one of the two infinitesimal volumes $dV_1$ & $dV_2$, at distance $r$, lies a galaxy

\[ dP(r) = n^2 \left( 1 + \xi(r) \right) dV_1 dV_2 \]

mean density
Correlation Functions

In case of Homogeneous & Isotropic point process

then \( \xi(\vec{r}) \)

only dependent on \( |\vec{r}| = r \)

Infinitesimal Definition Two-Point Correlation Function:

\[
dP(r) = \bar{n}^2 (1 + \xi(r)) \, dV_1 \, dV_2
\]
Power-law Correlations

\[ \xi(r) = \left( \frac{r}{r_0} \right)^{-\gamma} \]

\[ \gamma \approx 1.8 \]
\[ r_0 \approx 5h^{-1}\text{Mpc} \]

Totsuji & Kihara 1969

Peebles 1975, 1980, …
Correlation Functions

\[ \xi_{cc}(r) = \left( \frac{r_o}{r} \right)^\gamma \]
\[ \xi(r_0) = 1 \]

Clustering length/
"Correlation" length

Coherence length

\[ \xi(r_a) = 0 \]
Angular & Spatial Clustering

\[ dP(\theta) = \bar{n}^2 \{1 + w(\theta)\} d\Omega_1 d\Omega_2 \]

Two-point angular correlation function is the "projection" of \( \xi (r) \)

Limber's Equation:

\[
w(\theta) = \frac{\iint p(\overline{x}_1) p(\overline{x}_2) x_1^2 x_2^2 dx_1 dx_2 \xi (| \overline{x}_1 - \overline{x}_2 |)}{\left[ \int_0^\infty x^2 p(x) dx \right]^2}
\]

\[ p(x): \text{ survey selection function} \]
Angular Clustering Scaling

Two-point correlation function:

- small angles: power-law
  \[ w(\theta) = \left( \frac{\theta_0}{\theta} \right)^\gamma \]
  \[ \gamma \approx 0.8 \]
- large angles \( \theta \to \infty \)
  ie. to homogeneity
Angular Clustering Scaling

Projection of more layers leads to decreasing amplitude $w(\theta)$

Angular size/structures smaller when more distant

APM survey
sky-redshift space

2-pt correlation function $\xi(\sigma, \pi)$

Correlation function determined in sky-redshift space:

$$\xi(\sigma, \pi)$$

sky position: $\sigma = (\alpha, \delta)$
redshift coordinate: $\pi = cz$

Close distances: distortion due to non-linear Finger of God
Large distances: distortions due to large-scale flows
Redshift Space Distortions
Correlation Function

On average, \( \xi_s(s) \) gets amplified wrt. \( \xi_r(r) \)

Linear perturbation theory (Kaiser 1987):

\[
\xi_s(s) = (1 + \frac{2}{3} \Omega^{0.6} + \frac{1}{5} \Omega^{1.2}) \xi_r(s)
\]

Large distances:
distortions due to large-scale flows
Structural Insensitivity

2-pt correlation function is highly insensitive to the geometry & morphology of weblike patterns:

\[ \langle r \rangle, \text{cq. } P(k), \]

but totally different phase distribution

In practice, some sensitivity in terms of distinction Field, Filamentary, Wall-like and Cluster-dominated distributions:

because of different fractal dimensions
Structural Sensitivity

Wall-dominated

Filamentary

Cluster-like
Power Spectrum
P(k) specifies the relative contribution of different scales to the density fluctuation field. It entails a wealth of cosmological information.

\[
\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k) \quad \Leftrightarrow \quad P(k) \propto \left\langle \hat{f}(\vec{k}) \hat{f}^*(\vec{k}) \right\rangle
\]

Formal definition:

\[
(2\pi)^3 P(k_1) \delta_D (\vec{k}_1 - \vec{k}_2) = \left\langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \right\rangle \quad \downarrow \quad P(k) \propto \left\langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \right\rangle
\]
**Power Spectrum – Correlation Function**

Gaussian random field fully described by 2\textsuperscript{nd} order moment:

- in Fourier space: power spectrum
- in Configuration (spatial) space: 2-pt correlation function

\[
(2\pi)^3 P(k_1) \delta_D(\vec{k}_1 - \vec{k}_2) = \left\langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \right\rangle
\]

\[
\xi(\vec{r}_1, \vec{r}_2) = \xi(|\vec{r}_1 - \vec{r}_2|) = \left\langle f(\vec{r}_1) f(\vec{r}_2) \right\rangle
\]

\[
P(k) = \int d^3r \, \xi(\vec{r}) \, e^{i\vec{k} \cdot \vec{r}}
\]

\[
\xi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} P(k) e^{-i\vec{k} \cdot \vec{r}}
\]
Random Field Phases

\[
f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \hat{f}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}
\]

\[
\hat{f}(\vec{k}) = \hat{f}_r(\vec{k}) + i \hat{f}_i(\vec{k}) = \left|\hat{f}(\vec{k})\right| e^{i\theta(k)}
\]

When a field is a Random Gaussian Field, its phases \(\phi(k)\) are uniformly distributed over the interval \([0, 2\pi]\):

\[
\theta(k) \in U[0, 2\pi]
\]

As a result of nonlinear gravitational evolution, we see the phases acquire a distinct non-uniform distribution.
DTFE:

Delaunay Tessellation Field Estimator

Points, Tessellations & Patterns

Schaap & van de Weygaert 2000
Van de Weygaert & Schaap 2007
Cautun & van de Weygaert 2012
DTFE
Delaunay Tessellation Field Estimator

- Density Estimate:
  Voronoi Tessellation (contiguous)

- multi-D field interpolation:
  Delaunay Tessellations
Voronoi Tessellations

\[ \Pi_i = \{ \vec{x} | d(\vec{x}, \vec{x}_i) < d(\vec{x}, \vec{x}_j) \quad \text{for all} \quad j \neq i \} \]
Dual Tessellations

Voronoi Vertices

Centers Circumscribing Spheres 4 nuclei

Delaunay Tetrahedron
Sensitivity of Delaunay Tessellations to weblike geometry of particle distribution:

suggestion for exploiting this to explore the topology of the cosmic mass distribution

DTFE
Alpha Shapes
DTFE

• Delaunay Tessellation Field Estimator
• Piecewise Linear representation
density & other discretely sampled fields
• Exploits sample density & shape sensitivity of
Voronoi & Delaunay Tessellations
• Density Estimates from contiguous Voronoi cells
• Spatial piecewise linear interpolation by means of
Delaunay Tessellation

Schaap & vdW 2000
vdW & schaap 2009
Cautun & vdW 2012
DTFE Procedure

Summary

I. Construction
   Delaunay Tessellation
II. Point Sampling
III. Determination Field Values
IV. Calculation Field Gradient in Delaunay cell
V. - Interpolation to locations \( x \)
   - Image construction: interpolation to ordered locations
VI. Processing of field
DTFE website:

Tracing the Cosmic Web:

Pattern Classification
**Tracing the Cosmic Web**

<table>
<thead>
<tr>
<th>Classes</th>
<th>Identification &amp; Classification</th>
<th>procedures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Graph &amp; Percolation techniques</td>
<td>Minimal Spanning Tree</td>
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<tr>
<td></td>
<td>Stochastic Methods</td>
<td>Bisous Bayesian sampling geometric configurations</td>
</tr>
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<td></td>
<td>Geometric, Hessian-based methods</td>
<td>Vweb - velocity shear gradient velocity field Tweb - tidal field Hessian potential field</td>
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<td>Scale-space Multiscale Hessian-based methods</td>
<td>MMF/Nexus</td>
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<td>Topological Methods (Morse theory)</td>
<td>Watershed Void Finder / Voboz Disperse Spineweb</td>
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<tr>
<td></td>
<td>Phase-Space (multistream) structure:</td>
<td>Phase-space sheet &amp; flip-flop Origami Multistream</td>
</tr>
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</table>
Nexus/MMF:

Multiscale Morphology of the Cosmic Web

dec Formalism

Aragon-Calvo, Jones, vdW, van der Hulst 2007
Cautun, vdW & Jones 2013
Scale Space Analysis

Inspiration from Medical Imaging:
trace blood vessels, tumors, etc.

- Florack, Kuijper et al.; Lindeberg et al.
- Frangi et al. 1998 Multiscale vessel enhancement filter
Scale Space Pyramids

The ensemble of images is referred to as a scale space stack:
It is analysed as a single object.

Gaussian smoothing keeping same number of pixels.
Nexus Fields

- Nexus/Nexus+ fields relevant for cosmic web dynamics

- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:
  - Density
  - Log(Density)
  - Tidal field
  - Velocity Divergence
  - Velocity Shear
  - Nexus+ - log(density)

- from: Cautun et al. 2013
Scale Space Analysis

- Smooth the field over the range of relevant scales
  \[ f_n(\tilde{x}) = \int d\tilde{y} \ f_{DTFE}(\tilde{y}) \ W_n(\tilde{y}, \tilde{x}) \]
- with Gaussian filter
  \[ W_n(\tilde{y}, \tilde{x}) = \frac{1}{(2\pi R_n^2)^{3/2}} \exp\left(-\frac{||\tilde{y} - \tilde{x}||^2}{2R_n^2}\right) \]
- Scale space: stacking density maps \( f_n \)
  \[ \Phi = \bigcup_{\text{levels } n} f_n \]
Nexus/Nexus+ Scale Space

Input field

Gaussian smoothing

Log-Gaussian smoothing
Scale Space Analysis

- Smooth the field over the range of relevant scales
- Density field around location to 2nd order determined by Hessian:
  \[ f(x_0 + s) = f(x_0) + s^T \nabla f(x_0) + \frac{1}{2} s^T \mathcal{H}(x_0) s + ... \]
- Hessian filtered Density field:
  \[
  \frac{\partial^2}{\partial x_i \partial x_j} f_s(x) = f_{DTFE} \otimes \frac{\partial^2}{\partial x_i \partial x_j} W_G(R_s)
  = \int dy \ f(y) \ \frac{(x_i - y_i)(x_j - y_j) - \delta_{ij} R_s^2}{R_s^4} \ W_G(y, x)
  \]
Scale Space Analysis

- Morphology determined by eigenvalues of Hessian:

\[
\frac{\partial^2 f_n(\bar{x})}{\partial x_i \partial x_j} - \lambda_a(\bar{x})\delta_{ij} = 0, \quad a = 1, 2, 3
\]

\[
\lambda_1 > \lambda_2 > \lambda_3
\]

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<tr>
<th>Structure</th>
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<td>Blob</td>
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Scale Space Analysis

- Smooth the field over the range of relevant scales
- Select the characteristic scale of a particular (local) morphological element

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**Nexus/MMF formalism:**

- Aragon-Calvo et al. 2007
- Aragon-Calvo et al. 2010
- Cautun et al. 2013
- Cautun et al. 2014
Scale Space Analysis

**Nexus/MMF Procedure**

- *Smooth* the field over the range of relevant scales
- *Hessian* filtered density field
- Morphological characterization in terms of eigenvalues Hessian
- Select the characteristic scale of a particular (local) morphological element
- **Nexus/MMF morphology Filter Bank**
Nexus: MMF Filter Bank
Scale Space Analysis

• *Scale Space Map Stack*
  \[ \Psi(\vec{x}) \]
  maximum morphology response across full range of scales

• To filter out morphology noise: *morphology dependent thresholds*,
  \[ \tau_c, \tau_f, \tau_w \]
  value dependent on dynamical and/or structural (percolation) considerations

• *Object Map*
  \[ O(\vec{x}) \]
Nexus
Morphological Signatures

- a) density field
- b) blob/cluster node signature
- c) filament signature
- d) wall signature

from: Cautun et al. 2013
Nexus Signature & Thresholds
Nexus:

Multiscale Morphology Identification

Filaments

Colouring:
Local scale filament
Stochastic Spatial Pattern

- Clusters,
- Filaments &
- Walls

around

- Voids

in which matter & galaxies have agglomerated through gravity

MMF/Nexus
Cautun et al. 2013, 2014
Nexus Filaments

- Nexus identification of filaments (blue)

- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:
  - Density
  - Velocity Divergence
  - Tidal field
  - Velocity Shear
  - Log(density)
  - Nexus+
Nexus Walls

- Nexus identification of walls (orange)

- Identification on the basis of 6 different physical characteristics of the cosmic mass distribution:
  - Density
  - Velocity Divergence
  - Tidal field
  - Velocity Shear
  - Log(density)
  - Nexus+

- from: Cautun et al. 2013
Nexus+ tracing of filaments:
inherent multiscale character of filamentary web

Hidding, Cautun, vdW 2016
Spine of the Cosmic Web
• Additional Analysis Step:
  projection of filament galaxies to Medial Axis of Nexus/MMF filaments
Nexus/MMF:

Cosmic Web Characteristics

Aragon-Calvo, vdW & Jones 2010
Cautun, vdW, Jones & Frenk 2014
Cosmic Web: Density-Morphology Connection

Mass & Volume content
Web morphologies

Density distribution
Individual morphologies

Cautun et al. 2014
Walls & Filaments

Internal Diameter & Density Distribution
Walls & Filaments

Density Profiles

Density $1+\delta(r)$ vs. distance $r$ (h$^{-1}$ Mpc) for different filament and wall categories.
Filament Segmentation
V-web:

velocity (shear) flow field of the Cosmic Web

Hoffmann et al. 2012
Libeskind et al. 2013, 2014
Large Scale Flows

Large-Scale Flows:

- Structure buildup accompanied by displacement of matter:
  - Cosmic flows

- On large (Mpc) scales, structure formation still in linear regime

- Directly related to cosmic matter distribution

- Note: redshift space distortion

\[ cz = H_r + v_{pec} \]

In principle possible to correct for this distortion, i.e. to invert the mapping from real to redshift space

- Condition: entire mass distribution within volume should be mapped

\[ \mathbf{v}(\mathbf{x}, t) = \frac{H}{4\pi} \frac{f(\Omega_m)}{b} a \int d\mathbf{x'} \delta_{gal}(\mathbf{x'}, t) \frac{(\mathbf{x'} - \mathbf{x})}{|\mathbf{x'} - \mathbf{x}|^3} \]
Flow in the Cosmic Web
Supergalactic Plane

mean KIGEN - adhesion reconstruction
Cosmic Web Flowlines:

<table>
<thead>
<tr>
<th>Stokes:</th>
<th>flow field components</th>
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</thead>
<tbody>
<tr>
<td>Divergence</td>
<td>dominant in voids</td>
</tr>
<tr>
<td>Shear</td>
<td>dominant along filaments</td>
</tr>
<tr>
<td>Vorticity:</td>
<td>only in high-density multistream regions</td>
</tr>
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</table>
Push of the Local Void

Tully et al. 2008: Local Void pushes with ~260 km/s against our local neighbourhood.
A velocity field can be decomposed into 3 basic components, according to the gradient of the flow field:

the **velocity divergence**, **shear** and **vorticity** in each tetrahedron.

\[
\theta = \frac{1}{H} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)
\]

\[
\sigma_{ij} = \frac{1}{2} \left\{ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\} - \frac{1}{3} (\nabla \cdot \mathbf{v}) \delta_{ij}
\]

\[
\omega_{ij} = \frac{1}{2} \left\{ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right\}
\]
A velocity field can be decomposed into 3 basic components, according to the gradient of the flow field:

the velocity divergence, shear and vorticity in each tetrahedron.

**Divergence:**
- Expansion/Contraction

**Shear:**
- Deformation
Shear Tensor: 
Eigenvalues & Deformation directions

\[\sigma_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) - \frac{1}{3} \left( \nabla \cdot \vec{u} \right) \delta_{ik}\]

\[\Rightarrow \sigma_1, \sigma_2, \sigma_3\]

Wall
- Inflow: 1 direction
- Outflow: 2 directions

Filament
- Inflow: 2 directions
- Outflow: 1 direction

Cluster node
- Inflow: 3 direction
PSCz
Divergence & Shear
Velocity Shear Field

Resolution:
\( R_G = 3.0 h^{-1} \) Mpc (left)
\( R_G = 10.0 h^{-1} \) Mpc (right)

Romano-Diaz & vdW 2007
Cosmic Web morphology: velocity shear based V-web identification flow pattern in cosmic web (Pomarede et al. 2017)
Watershed

Void Identification

Platen, vdW & Jones 2007
Definition of voids (Voronoi density & watershed)

WVF: Platen et al. 2007
ZOBOV: Neyrinck 2008
Sutter, Lavaux, Wandelt, Weinberg 2012
No exact definition of a void!
→ broad range and variety of void detection techniques

Our void finder:
• closely follows real geometry cosmic web
• no assumptions geometry void
• no user defined parameters

→ Watershed Void Finder by Platen et al., 2007.

Figure from Colberg et al., 2008
Watershed Void Identification
WATERSHEDS:
A cell is the union of points that are topological closer to a certain minimum

Topology Distance:
The path that connects two points via the steepest slope: the path a water-droplet would take, when running down a landscape

Segmentation:
A division of space in individual cells
Following the water-flow into the distinct catchment basins.

Each basin belonging to one individual minima defines one region
Pierce the local minima, and let the landscape sink slowly in a tub of water.
Every time two different flooding basins meet we draw a dividing wall.
Void Patches

Final segmentation lines
The Multiscale Watershed Void Finder

- local height $\rightarrow$ local density
- mountain ridges $\rightarrow$ walls and filaments
- watershed basins $\rightarrow$ voids
WVF: Watershed Void Finder
COSI Columbus Science Center:

Hands-On Voids by Watershed
Void persistence and merger trees

Adhesion model

Void evolution in idealized adhesion model:

- self gravity of walls and filaments modelled by artificial viscosity $\nu$
- discards nonlinear evolution on smaller scales
- models hierarchical evolution very good

$2$ adhesion models

- $P(k) \propto k^1$
- $P(k) \propto k^{-1}$

Zel’dovich, 1970
Gurbatov, Saichev and Shandarin, 1989
Hidding et al., 2012

Image courtesy: Johan Hidding
Void persistence and merger trees

- Merger tree is only based on one parent void!
- Combine information of all merger trees into
  **Persistence Diagram**
  (Edelsbrunner et al. 2000)
- Information w.r.t. formation and disappearance of voids due to hierarchical evolution
- Not only mathematical principle.
Spine of the Cosmic Web
Cosmic Spine

Cosmic Spine:

• Network of filamentary edges & sheetlike walls
• Connection of Cluster Nodes via filamentary bridges
Density Field Flow Lines

\[ \nabla f \]
Density  Field  Flow Lines

\[ \nabla f \]
Density Field Flow Lines

\[ \nabla f = 0 \]

Critical Points:
- Maxima
- Minima
- Saddle Points (of various signatures)
Density Field
Critical Points:
Ridges:
Connections
Saddles-
Maxima

Maximum
Saddle
Minimum
Topological structure well-behaved $C^2$ field:

- “flow” field
- singularities - minima, maxima, saddles
- critical integral lines: connection singularities
- saddles-maxima: spine of field - filaments, sheets
- basin minima: voids

Practical Computation:
- Watershed Transform
- Pseudo Morse complex !!!!
Density Field & Landscape
Segmentation & Flowlines
Watershed Segmentation
Watershed Segmentation
SpineWeb Formalism

Extension of Watershed Transform:

- determination boundary regions between the watershed basins (the “voids”).
- Identification of boundary pixels
- Topological Identity determined on the basis of # neighbouring voids/basins,
SpineWeb Procedure

Local Neighbourhood:

Counting Number Adjacent Voids:

\[ N_v = 2 \quad \text{wall} \]
\[ N_v = 3 \quad \text{filament} \]
SpineWeb Morphology Dissection
3-D SpineWeb Segmentation

- Colour: density contours
- White: watershed segmentation lines (cosmic spine)
Density Levels vs. Spine

Density field level sets

Spinal Components:
- walls
- filaments
Spinal Walls
Spinal Filaments
Spine Hierarchy
Spine of the Cosmic Web: directly on Delaunay grid
Cosmo Topology
Topology

Study of the

(multiscale) shapes, complexity and connectivity

of the Cosmic Web
Conventional Cosmological Topology Measure:

(Reduced) Genus

- # holes - # connected regions
- (Gott et al. 1986; Hamilton et al. 1986; Choi et al. 2010)

Complete quantitative characterization of local geometry in terms of

Minkowski Functionals

Minkowski Functionals:
- Volume
- Surface area
- Integrated mean curvature
- Genus/Euler Characteristic

- (Mecke, Buchert & Wagner 1994)
Minkowski functionals

- Weyl’s Tube formula:

Minkowski functionals $Q_k$ are the parameters specifying the contribution of volumes $r^k$ to the volume of a cube $M^r$ with rounded edges of radius $r$:

$$\text{Vol}(M^r) = Q_0 + Q_1r + Q_2r^2 + Q_3r^3$$
Topology:
Study of connectivity and spatial relations that remain invariant under homeomorphisms (= continuous mapping between two topological objects)

Homology:
Description of topology of a space in terms of the relationship between cycles and boundaries.

- p-chain: sum of p-simplices
- p-cycle: boundary of (p+1) chain

Torus:
- one 0-cycle: rank group $H_0$: 1
- two 1-cycles: rank group $H_1$: 2
- one 2-cycle: rank group $H_2$: 1

Group of p-cycles:
$\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$: on islands, tunnels & voids
on islands, tunnels & voids
$\tau_0, \tau_1, \tau_2$: on islands, tunnels & voids
Euler-Poincaré

3-manifold $\mathbb{M}$:

$$\chi(M) = \beta_0 - \beta_1 + \beta_2 + \beta_3$$

$$\approx \beta_0 - \beta_1 + \beta_2$$

Boundary 2-manifold $\partial \mathbb{M}$:

$$\chi(\partial M) = \beta_{0b} - \beta_{1b} + \beta_{2b}$$
the Rule of Euler

SIMPLICIAL TOPOLOGY
Simplices, complexes, cycles, numbers of simplices, Betti numbers

\[ \sum_{k} (-1)^k \# \{ k\text{-dimensional simplices} \} \]

INTEGRAL GEOMETRY
Convexity, convex ring kinematic formulae Minkowski functionals

\[ M_k(M) = c_{dk} \int_{\text{Graff}(d,d-k)} \chi(M \cap V) d\mu^d_{d-k}(V) \]

\[ \sum_{k} (-1)^k \# \{ \text{critical points of index } k \} \]

\[ \int_M \text{Tr}(R^{m/2}) \text{Vol}_g \]

ALGEBRAIC TOPOLOGY
Homology, homotopy, dimensions of groups, Betti numbers, persistence

DIFFERENTIAL TOPOLOGY
Curvature, forms, Betti numbers, Morse theory, integration, Lipschitz-Killing curvatures

from: Robert Adler
Random Field Topology:
Morse Complex
Density Field Flow Lines

$\vec{\nabla} f$
Density Field Flow Lines

Critical Points:
- Maxima
- Minima
- Saddle Points (of various signatures)

\[ \nabla f = 0 \]
Betti & Morse

Relation to Morse Theory:

Topological Structure Continuous Field determined by **singularities**:

- maxima
- minima
- saddle points

(a) Minimum, 0, ⊙
(b) Saddle, 1, ⊕
(d) Monkey Saddle, ⊗
(c) Maximum, 2, ○
Betti & Morse

Number of singularities in field determines Euler characteristic:

\[ \zeta_0 : \text{minima} \]
\[ \zeta_1 : \text{saddle 1} \]
\[ \zeta_2 : \text{saddle 2} \]
\[ \zeta_3 : \text{maxima} \]

\[ \chi = \sum_{k=0}^{d} (-1)^k \zeta_k \]
Density Field & Landscape
Topological Hierarchy: Excursion Sets & Filtrations

Superlevel Sets

\[ M_v = \{ \bar{x} \in M \mid f_s(\bar{x}) \in [f_v, \infty) \} = f_s^{-1}[f_v, \infty) \]

Pranav et al. 2013a
Important source of information about topology of a field/point distribution:

Filtrations

Filtration provides view of topology as a function of scale.

Formally, given a space $\mathcal{E}$, a filtration is a nested sequence of subspaces

$$\emptyset = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots M_m = M$$

Nature of filtrations depends (amongst others) on representation of the mass distribution.
Topological Hierarchy

Persistent Homology:
“Cycling” over density filtration

Edelsbrunner & Harer 2010
field value filtration
tree hierarchy
Cosmic Web

Homology & Persistence
Voronoi Elements:
Filaments

Filaments:
Pisces-Perseus chain
Voronoi Element Models: Persistence Diagrams

d=0  d=1  d=2

Clusters  Filaments  Walls
Voronoi Kinematic Models: Persistence Diagrams

Stage 1

Stage 2

Stage 3
LCDM Persistence

Morse-Smale simplification

MSC₀
MSC₁
MSC₂
MSC₃
MSC₄
MSC₅
MSC₆
LCDM: Persistence & Morse Simplification

(a) $d = 0 / \text{not simp.}$
(b) $d = 0 / p(c_{\text{Poisson}}) = 0.1.$
(c) $d = 0 / p(c_{\text{Poisson}}) = 0.01.$
(d) $d = 0 / p(c_{\text{Poisson}}) = 0.001.$

(e) $d = 1 / \text{not simp.}$
(f) $d = 1 / p(c_{\text{Poisson}}) = 0.1.$
(g) $d = 1 / p(c_{\text{Poisson}}) = 0.01.$
(h) $d = 1 / p(c_{\text{Poisson}}) = 0.001.$

(i) $d = 0 / \text{not simp.}$
(j) $d = 0 / p(c_{\text{Poisson}}) = 0.1.$
(k) $d = 0 / p(c_{\text{Poisson}}) = 0.01.$
(l) $d = 0 / p(c_{\text{Poisson}}) = 0.001.$

Nevenzeel & vdW 2017
LCDM: Betti Curves & Morse Simplification

Nevenzeel & vdW 2017
LCDM Persistence

(a) Skewed Gaussian fit of mean density curves.

(b) Skewed Gaussian fit of Betti curves
Homology of evolving LCDM cosmology

$H_0$-shape homology
Betti$_2$: evolving void populations