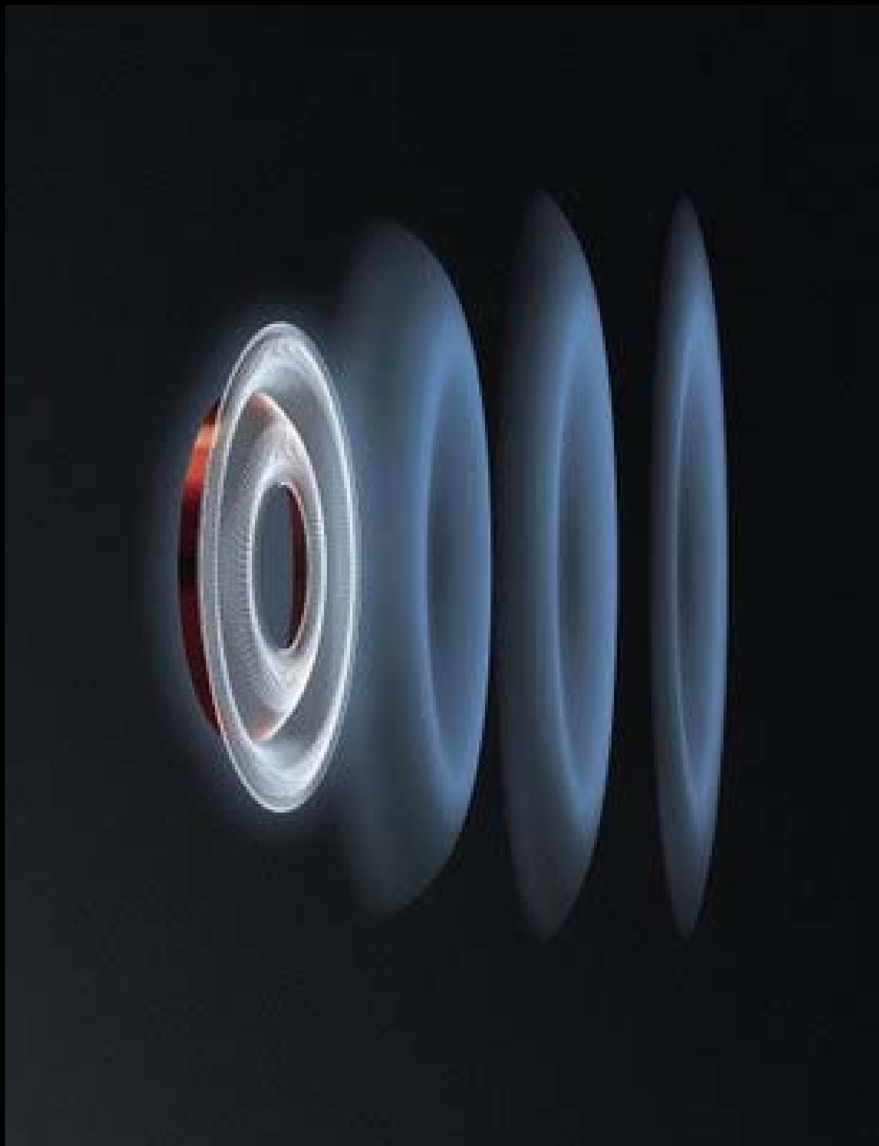
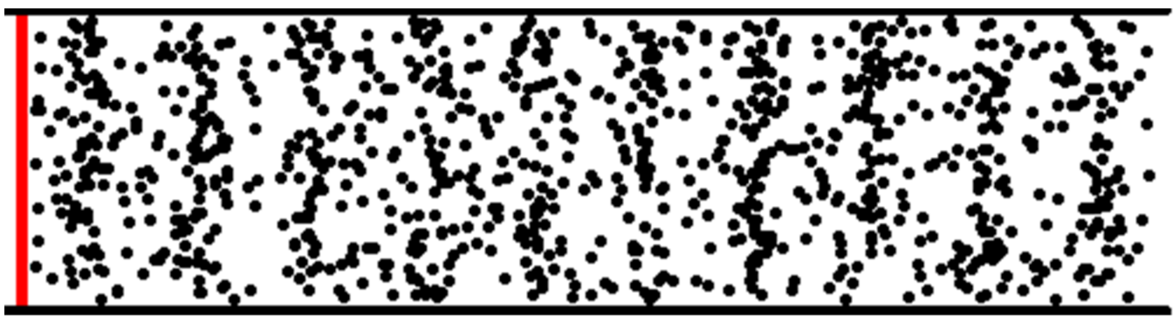
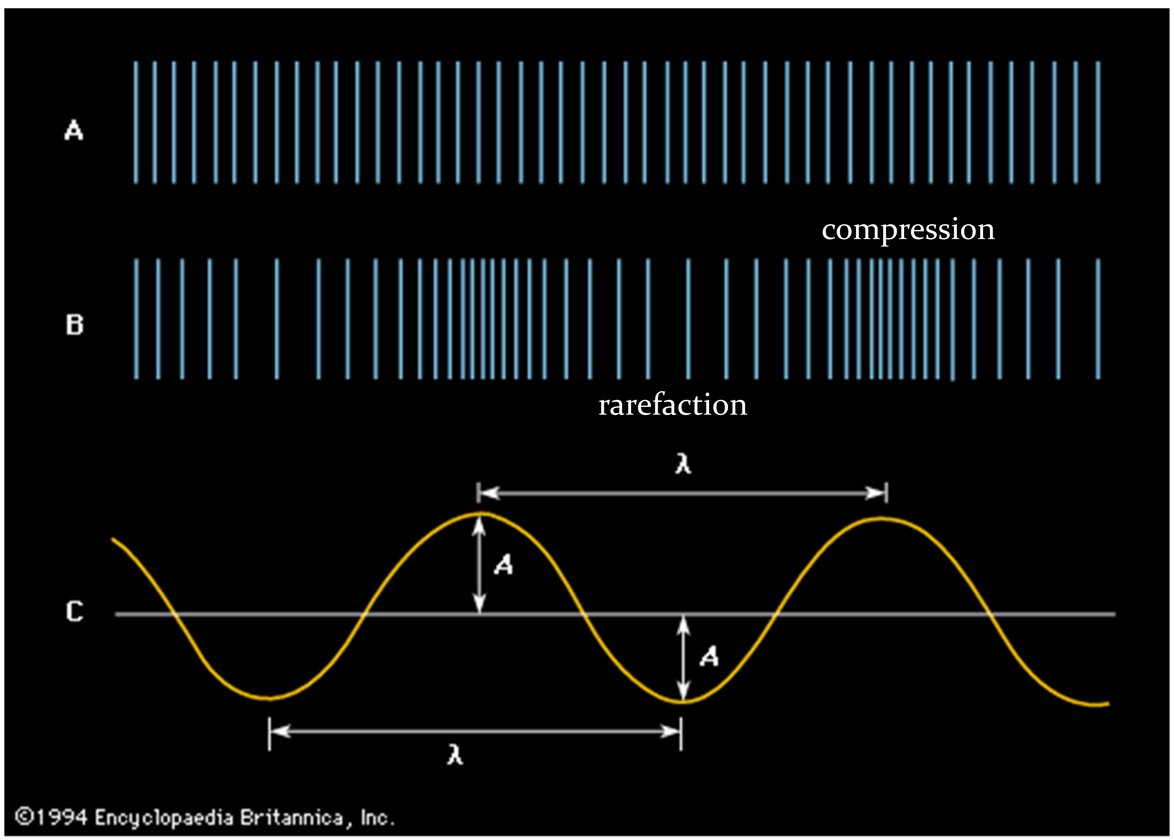
Sound Waves

Sound Waves:



Sound Waves




A
compression
B
rarefaction
λ
A
C
A
λ
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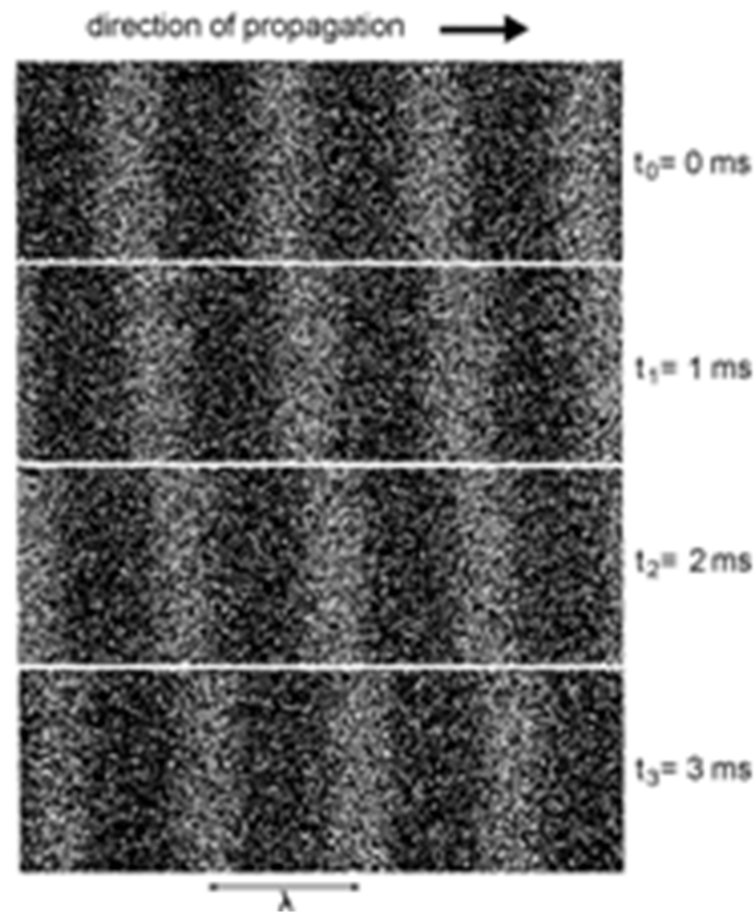
direction of propagation
t_0= 0 ms
t_1= 1 ms
t_2= 2ms
t_3= 3 ms
λ

Sound Waves

Inference of Sound Wave equation:

We look at small disturbances in a compressible medium
(note: compressible means that the density may change).

Rest configuration:

$$\rho_0, p_0, \vec{u}_0$$

Small (linear) disturbances, assume wrt. fluid at rest ($\vec{u}_0 = 0$)

$$\Delta\rho \ll \rho : \qquad \rho = \rho_0 + \Delta\rho$$

$$\Delta p \ll p : \qquad p = p_0 + \Delta p$$

$$\Delta u \ll u : \qquad \vec{u} = \vec{u}_0 + \Delta\vec{u} = \Delta\vec{u}$$

We are going to investigate the behaviour of the perturbations by assessing the 1st order version of the Continuity and Euler equation.

Sound Waves

Before looking into the details of fluid element displacements, and their expression in density, pressure and velocity perturbations, we first will look at the global effect of such first-order perturbations. We will see that the resulting equation for density and pressure perturbation is the well-known wave equation. In other words, the resulting perturbations result in a wave phenomenon.

To this end, first assess the consequence of the presence of small perturbations by analyzing the continuity and Euler equation (in a medium without external force):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho}$$

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Linearize these equations:
ie. ignore the 2nd order terms, ie. terms like

$$\Delta\rho \, \Delta\vec{u}$$

$$(\vec{u} \cdot \vec{\nabla})\vec{u}$$

only keep 1st order terms.

Sound Waves

- Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

$$\Downarrow$$

$$\frac{\partial \Delta \rho}{\partial t} + \Delta \vec{u} \cdot \nabla (\Delta \rho) + \rho \nabla \cdot \Delta \vec{u} = 0$$

$$\Downarrow (discard \;\; 2^{nd} \, order \, terms)$$

$$\frac{\partial \Delta \rho}{\partial t} + \rho_0 \nabla \cdot \Delta \vec{u} \approx 0$$

Sound Waves

- Euler Equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p$$

$$\Downarrow (ignore\ higher\ order\ terms)$$

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\vec{\nabla} p = -\frac{1}{\rho_0 + \Delta\rho}\vec{\nabla}\ (p_0 + \Delta p) \approx -\frac{1}{\rho_0}\vec{\nabla}\ (\Delta p)$$

- Subsequently take the following steps: $\rho_0\ \vec{\nabla} \cdot Euler\ eqn.$
  -

$$\frac{\partial}{\partial t}\ Continuity\ eqn$$

Sound Waves

- This results in the following 2 2nd order partial differential equations:

$$\rho_0 \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{u}) = -\nabla^2(\Delta\rho)$$

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) + \rho_0 \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{u}) = 0$$

- The combination of these 2 equations yields the following relation:

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - \nabla^2(\Delta p) = 0$$

Sound Waves

- We assume that the perturbations are adiabatic (ie. they are sufficiently fast that no hear or thermal conduction will occur). In this case we may use that p=p(ρ) and use the adiabatic gas law relating pressure and density,

$$p = cst. \times \rho^{\gamma} \quad \Rightarrow \quad \frac{\Delta p}{p_0} = \gamma \frac{\Delta \rho}{\rho_0} \quad \Rightarrow \quad \Delta p = \left(\gamma \frac{p_0}{\rho_0} \right) \Delta \rho$$

in which γ is the adiabatic gas index and

$$\left(\frac{\partial p}{\partial \rho} \right)_S = \left(\gamma \frac{p_0}{\rho_0} \right)$$

- Inserting this adiabatic relation into our 2nd order PDE, yields

$$\frac{\partial^2}{\partial t^2}(\Delta \rho) - \left(\frac{\partial p}{\partial \rho} \right)_S \nabla^2 (\Delta \rho) = 0$$

Sound Velocity

- We have arrived at a 2nd order partial differential equation that can be immediately recognized as a wave equation, through its combination of 2nd order gradient in time with the 2nd order spatial gradient (Laplacian).

- To make the resulting relation even more transparant, we should identify the connecting term as a velocity term:

It leads us to no less than the identification of the sound velocity of a fluid,

$$c_s{}^2 = \left(\frac{\partial p}{\partial \rho} \right)_S$$

- Note: for an adiabatic gas with temperature T the velocity of sound c_s is:

$$c_s{}^2 = \left(\frac{\partial p}{\partial \rho} \right)_S = \gamma \frac{p}{\rho}; \qquad p = \frac{R\rho T}{\mu} \quad \Rightarrow \quad c_s = \sqrt{\frac{RT}{\mu}}$$

Sound Velocity

- The definition for the sound velocity of a gas is,

$$c_s^{\,2} = \left(\frac{\partial p}{\partial \rho}\right)_S$$

- The sound velocity cs of a compressible fluid is a fundamental quantity characterizing the nature of the fluid.

- It fixes the maximum rate at which information about changes in pressure, density, velocity and temperature in a medium can pass through the fluid and modify its behaviour.

- Note:
  the sound velocity is a local quantity defined at each point of the fluid, and can vary with position and time.

Sound Wave Equation

- Using the definition for sound velocity, we thus arrived at the equation for the density (and hence pressure) perturbations in a gas, the

  (Sound) wave equation

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - c_s^{\ 2}\ \nabla^2(\Delta\rho) = 0$$

- The solution of this wave equation is that of harmonic motion, ie. of a temporal and spatial wave,

$$\Delta\rho(\vec{x},t) = A(\vec{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

  which describes a wave with temporal frequency ω and wavelength λ, travelling in the direction specified by the wavevector $\vec{k}:\quad |\vec{k}| = \frac{2\pi}{\lambda}$

Sound Wave Equation

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which describes a wave with temporal frequency ω and wavelength λ, travelling in the direction specified by the wavevector

$$\vec{k}: \quad \left|\vec{k}\right| = \frac{2\pi}{\lambda}$$

- The generic solution of the wave equation is therefore the superposition of such singular sound waves (expressed via the Fourier integral expression)

$$\Delta\rho(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\; A(\vec{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

Dispersion Relation

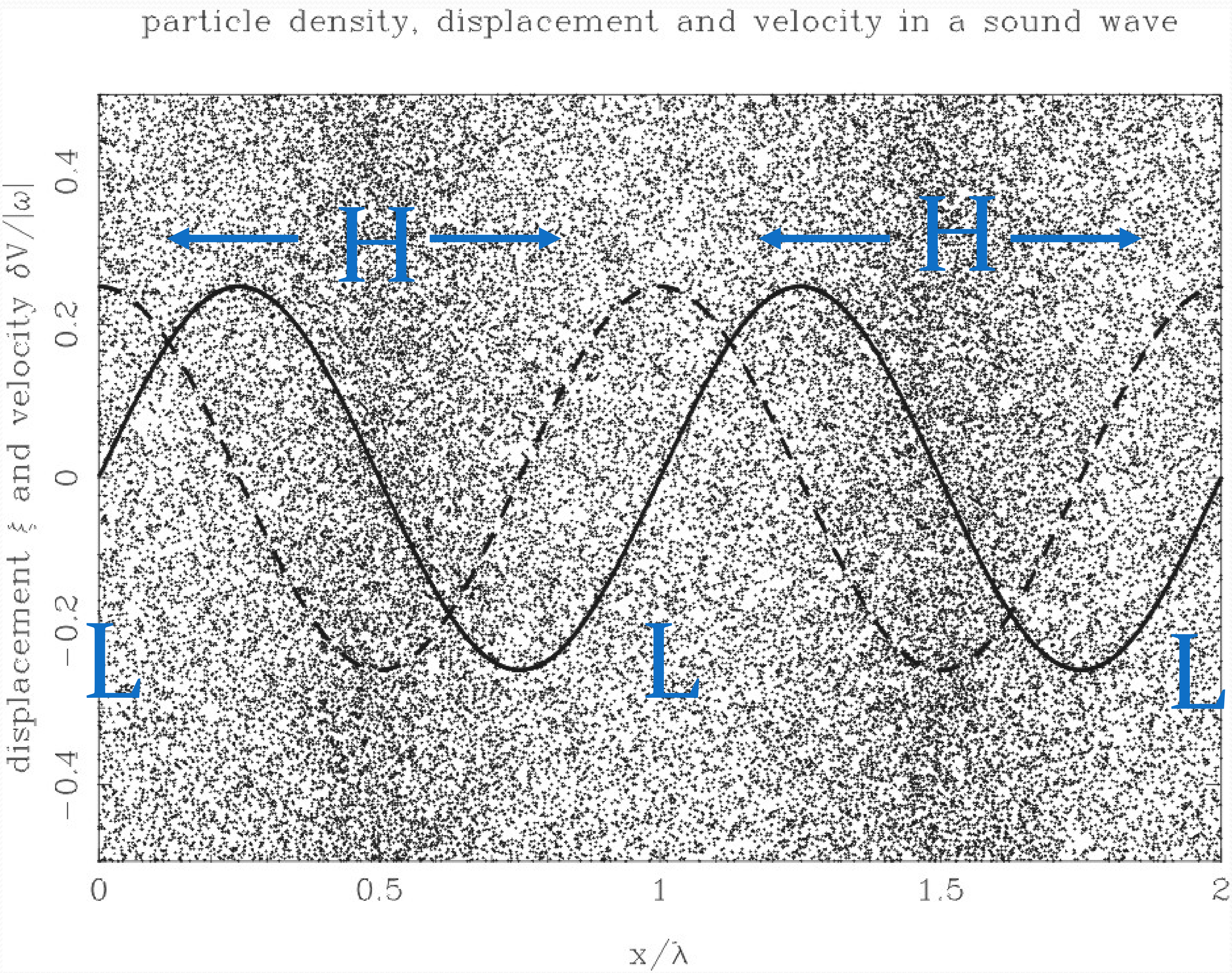
- Given the generic wave solution to the sound wave equation, we may find the physical behaviour of such waves from the

  DISPERSION RELATION

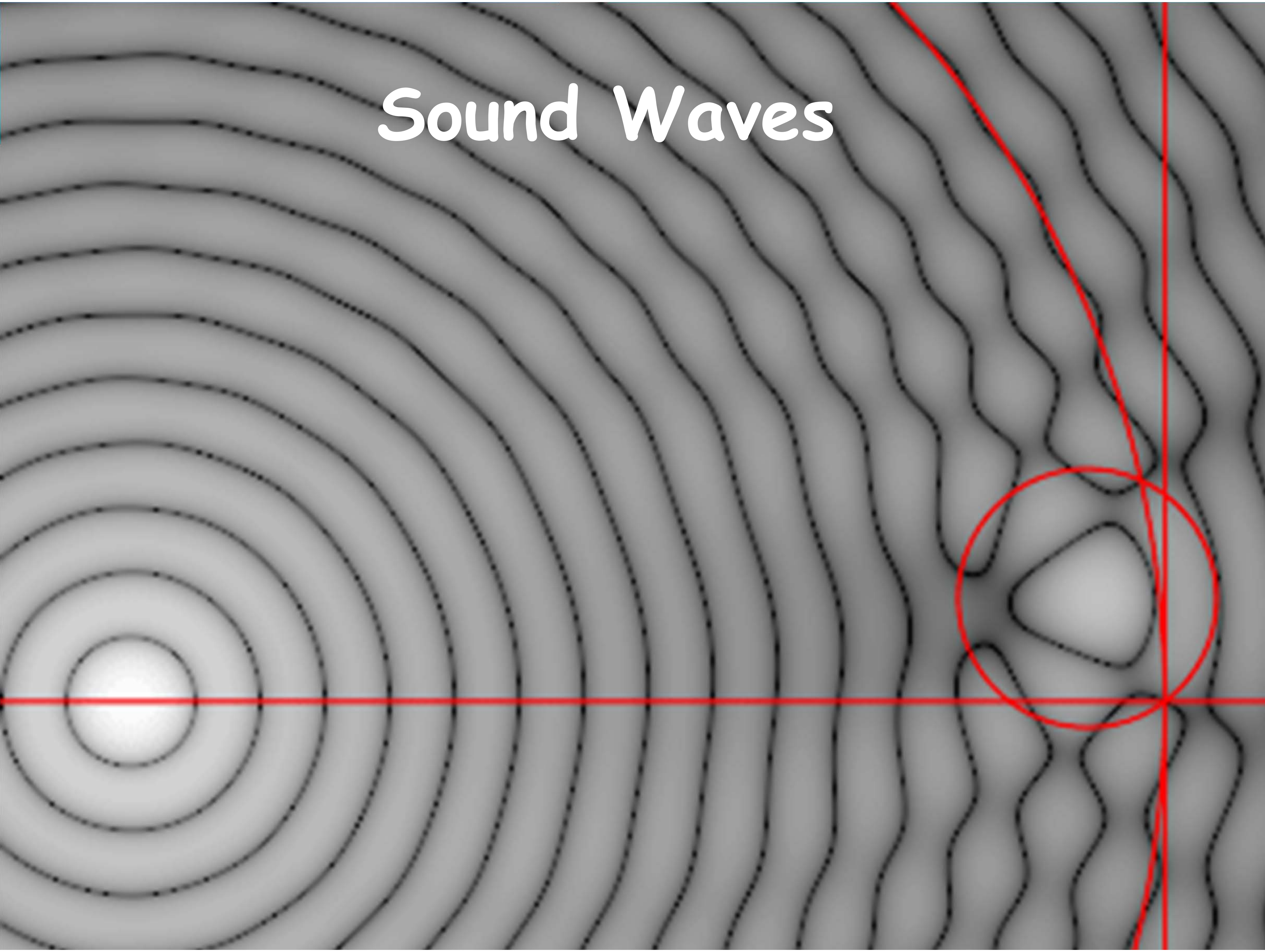
  which specifies the relation between the wavelength λ/wavenumber k and frequency ω of the wave

$$\omega^2(k) - c_s^{\,2}\, k^2 = 0$$

- Later we will see that the dispersion relation describes the effect of dispersion as the wave propagates through a medium: its refraction/refractive index is determined by the expressions for the
- Phase velocity $v_{ph} = \dfrac{\omega}{k}$ & Group velocity $v_{gr} = \dfrac{\partial \omega}{\partial k}$


particle density, displacement and velocity in a sound wave
H
H
L
L
L
displacement ξ and velocity δV/|ω|
0.4
0.2
0
−0.2
−0.4
0
0.5
1
1.5
2
x/λ

Sound Waves



Gravity Waves

Gravity Waves

Gravity Waves

While soundwaves are minor disturbances in a compressible medium, another interesting class is perturbations in an INCOMPRESSIBLE medium.

Such perturbations would involve the displacement and velocity of fluid elements around their equilibrium position. In particular interesting is those that are sustained by the presence of a gravitational field. The gravitational force is responsible for the tendency to restore the system towards its equilibrium configuration.

The implied result is that of a harmonic motion around an equilibrium position.

This wave phenomenon is known under the name GRAVITY WAVES (do not confuse this with the more exotic phenomenon of gravitational waves).

The most salient example of such gravity wave perturbations are sea and water surface waves.

Gravity Waves

Preliminaries:

- We assume small displacement and velocity perturbations of the fluid, with the fluid at rest in equilibrium ($\vec{u}_0 = 0$),

$$\vec{u} = \vec{u}_0 + \Delta\vec{u} = \Delta\vec{u} \qquad \Delta\vec{u} \ll \vec{u}$$

$$\vec{r} = \vec{r}_0 + \xi(\vec{r},t) \qquad \xi \ll \lambda$$

- As the velocity perturbations is small, we may discard 2nd and higher order terms in the fluid equations:

$$(\vec{u}\cdot\vec{\nabla})\vec{u} \ll \frac{\partial\vec{u}}{\partial t}$$

- Because of the incompressibility of the medium, we also have:

$$\vec{\nabla}\cdot\vec{u} = 0$$

Gravity Waves

Equation of motion: Euler equation

- We assume small displacement and velocity perturbation u:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\vec{\nabla} p}{\rho} + \vec{g}$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\Phi$$

- with p the pressure in the fluid and Φ the gravitational potential.

- From this relation we can immediately conclude that the velocity perturbation u itself must correspond to potential flow, ie. the velocity is the gradient of a velocity potential ψ:

$$\vec{u} = \vec{\nabla}\,\psi$$

Gravity Waves

- Processing the velocity potential ψ in the Euler equation for the incompressible fluid perturbation then leads to the following differential equation:

$$\vec{\nabla}\frac{\partial \psi}{\partial t} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}$$

- We subsequently chose a coordinate system such that the gravitational field vector g is oriented along the z-direction (with z increasing in vertical direction, with z=0 the equilibrium surface position, z>0 above the surface and z<0 into the water):

$$\vec{g} = (0 \quad 0 \quad g)$$

- We may then follow the Euler equation along its z-direction component and integrate this along the z-direction

$$\rho\frac{\partial}{\partial z}\frac{\partial \psi}{\partial t} = -\frac{\partial p}{\partial z} + \rho g \qquad \Rightarrow \qquad \rho\frac{\partial \psi}{\partial t} = -p + \rho g z$$

- where we also used the fact that we deal with an incompressible medium, for which ρ=cst.

Gravity Waves

- We may use this expression for the Euler equation to derive an important boundary condition for the generated wave motion:
- At the surface boundary of the wave, the pressure p must be equal to the atmospheric pressure p_0.
- While parameterizing the surface of the medium with its z-coordinate ζ, we find that the integrated Euler equation at the surface of the fluid attains the boundary relation,

$$\rho \frac{\partial \psi}{\partial t} = -p + \rho g z$$

$$\Downarrow$$

$$p_0 = -\rho g \zeta - \rho \frac{\partial \psi}{\partial t}$$

Gravity Waves

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$$\Downarrow$$

$$p_0 = -\rho g \zeta - \rho \frac{\partial \psi}{\partial t}$$

- Note that this enables a straightforward renormalization of the potential ψ, using the fact that atmospheric pressure p0 and density ρ are constants,

$$\psi' = \psi + \frac{p_0}{\rho} t \qquad \Rightarrow \qquad \vec{u} = \vec{\nabla} \psi'$$

- Which yields the following straightforward boundary relation for the perturbation:

$$g\zeta + \left.\frac{\partial \psi'}{\partial t}\right|_{z=\zeta} = 0$$

Gravity Waves

- We may convert the boundary condition into an expression solely in terms of the velocity potential ψ′ by using the fact that the z-component of the velocity u_z at the surface is equal to

$$u_z = \frac{\partial \zeta}{\partial t} = \left.\frac{\partial \psi'}{\partial z}\right|_{z=\zeta}$$

- Taking the time derivative of the Euler equation,

$$g\zeta + \left.\frac{\partial \psi'}{\partial t}\right|_{z=\zeta} = 0 \qquad \Rightarrow \qquad \frac{\partial \zeta}{\partial t} + \frac{1}{g}\left.\frac{\partial^2 \psi'}{\partial t^2}\right|_{z=\zeta} = 0$$

- and inserting the expression for u_z yields the following expression of the surface/boundary condition in terms of the potential ψ′

$$\left.\frac{\partial \psi'}{\partial t} + \frac{1}{g}\frac{\partial^2 \psi'}{\partial t^2}\right|_{z=\zeta} = 0$$

Gravity Waves

- In addition, we may infer the equation for the velocity potential ψ′ from the condition for incompressibility,

$$\vec{u} = \vec{\nabla}\psi' : \qquad \vec{\nabla}\cdot\vec{u} = \nabla^2\psi' = 0$$

in other words, the incompressibility of the fluid implies the Laplace equation for the velocity potential ψ′.

- In all, for the velocity potential ψ′ we obtain the following set of 2nd order partial differential equations, including that for the noundary condition

$$\nabla^2\psi' = 0$$

$$\left.\frac{\partial\psi'}{\partial z} + \frac{1}{g}\frac{\partial^2\psi'}{\partial t^2}\right|_{z=\zeta} = 0$$

Gravity Waves

- Generic solutions to the set of 2nd order partial differential equations, for the velocity potential ψ'

$$\nabla^2 \psi' = 0$$

$$\left. \frac{\partial \psi'}{\partial z} + \frac{1}{g} \frac{\partial^2 \psi'}{\partial t^2} \right|_{z=\zeta} = 0$$

- can be written as the product of a z-dependent function f(z) and a harmonic term for the motion along the x-direction (noting the 2nd order gradients in both x and t,

$$\psi'(\vec{r}, t) = f(z) \cos(kx - \omega t)$$

with ω the frequency of the wave along the x-direction, with wavenumber $k=2\pi/\lambda$ for wavelength λ.

Gravity Waves

- To establish the dependence of the z-direction function f(z) on the wavelength of the perturbation we insert the generic expression for ψ′ in the Laplace equation, to yield:

$$\frac{d^2 f}{dt^2} - k^2 f^2 = 0$$

- It is straightforward to see that the general solution for f(z) is the combination of 2 base functions

$$f(z) = a_1 f_1(z) + a_2 f_2(z)$$

$$f_1(z) = e^{kz}$$

$$f_2(z) = e^{-kz}$$

- Note: as z<0 concern locations into the fluid, and the perturbations by construction/assumption should be small, ie. f(z)<<1, the implication is that a_2=0.

Gravity Waves

- Hence, we continue with a general solution for the incompressible fluid displacement and velocity perturbation specified by the velocity potential

$$\psi'(x,z,t) = a_1 e^{kz} \cos(kx - \omega t)$$

- It is straightforward to infer the velocity components (u_x,u_z) by integrating the potential ψ′ along x and z,

$$\vec{u} = \vec{\nabla}\psi' \quad \Leftrightarrow \quad \psi'(x,z,t) = a_1 e^{kz} \cos(kx - \omega t)$$

$$u_x = -Ak\, e^{kz} \sin(kx - \omega t)$$

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$$u_x = -Ak\, e^{kz} \sin(kx - \omega t)$$
$$u_z = Ak\, e^{kz} \cos(kx - \omega t)$$

From this, we immediately see that the x- and y-component of the velocity display a harmonic dance, out of phase by π/2 wrt. each other: circular motion !

Gravity Waves

- As velocity by definition is the time derivative of the displacement of the fluid elements,

$$\vec{u} = \frac{d\vec{\xi}}{dt} = \frac{d}{dt}\begin{pmatrix} x - x_0 \\ z - z_0 \end{pmatrix}$$

- We infer the solution for the position of the fluid elements by integrating the velocity over time, to obtain

$$x - x_0 = -A\frac{k}{\omega}e^{kz}\ \cos(kx - \omega t)$$

$$z - z_0 = -A\frac{k}{\omega}e^{kz}\ \sin(kx - \omega t)$$

Gravity Waves

- the solution for the position of the fluid elements:

$$x - x_0 = -A\frac{k}{\omega}e^{kz}\ \cos(kx - \omega t)$$

$$z - z_0 = -A\frac{k}{\omega}e^{kz}\ \sin(kx - \omega t)$$

- Clearly, this is a harmonic motion in both
  x-direction (along the surface)
  z-direction (perpendicular to the surface)

- x- and z- motion are out of phase by π/2:
  this clearly reveals a circular motion of the fluid element around the equilibrium location (x_0, z_0).

- The amplitude e^{kz} of the circular motion scales with depth:
  - larger depth, ie. lower z ($z<0$): smaller radius of circular motion

Gravity Waves

- To understand the dispersive character of the gravity waves (sea waves, atmospheric waves), we infer the dispersion relation of the waves – ie. dependence of frequency on wavelength – from the boundary condition (which followed from the Euler eqn.):

$$\left.\frac{\partial \psi'}{\partial z} + \frac{1}{g}\frac{\partial^2 \psi'}{\partial t^2}\right|_{z=0} = 0$$

$$\Downarrow$$

$$\omega^2 = kg$$

- Using this dispersion relation, it is interesting to investigate the phase and group velocity of gravity waves:

$$u_{ph} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \; ; \qquad u_{gr} = \frac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

This shows that both phase and group velocity are wavelength dependent, increasing with wavelength: larger waves travel faster. Entirely different than in the case of soundwaves.

Gravity Waves

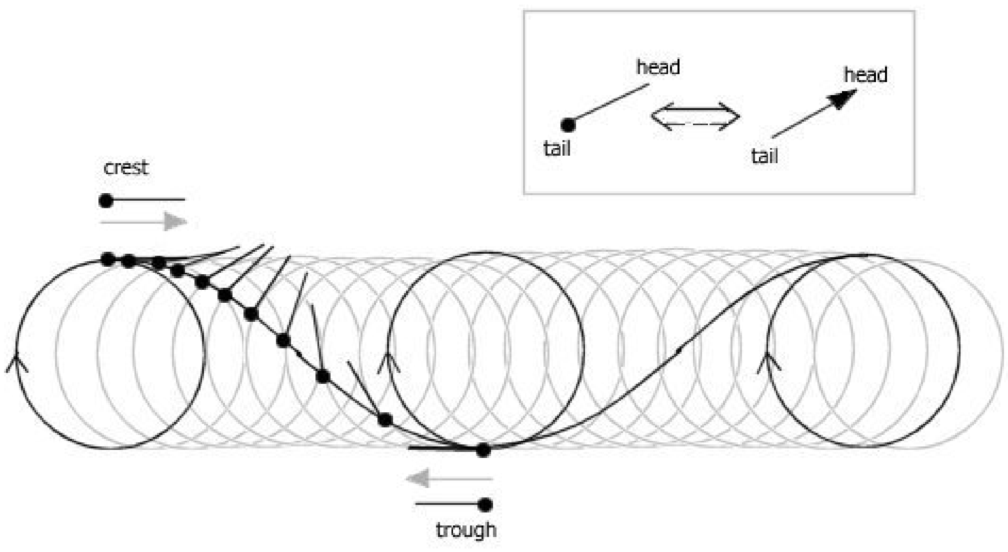
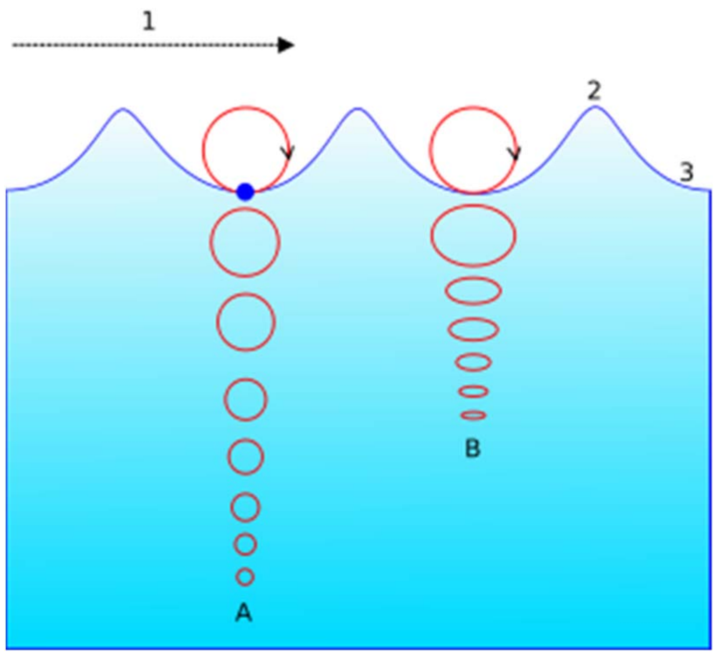
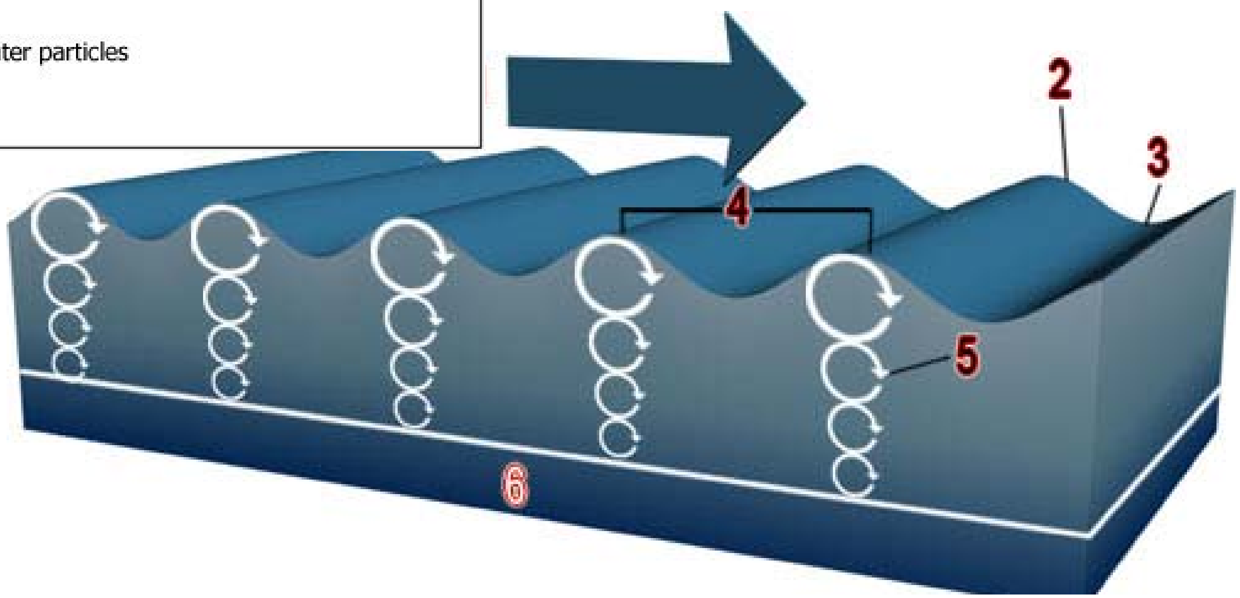


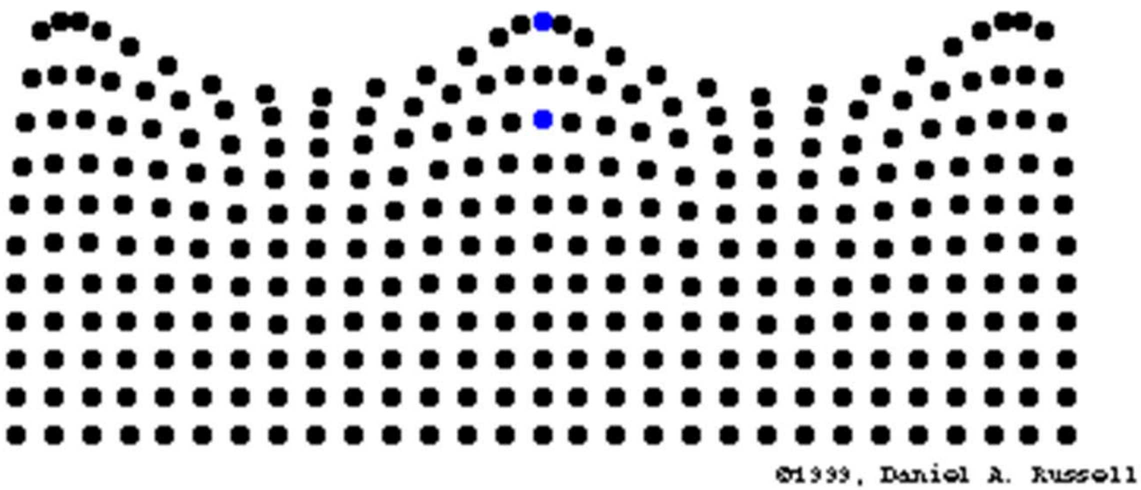
Fig 1

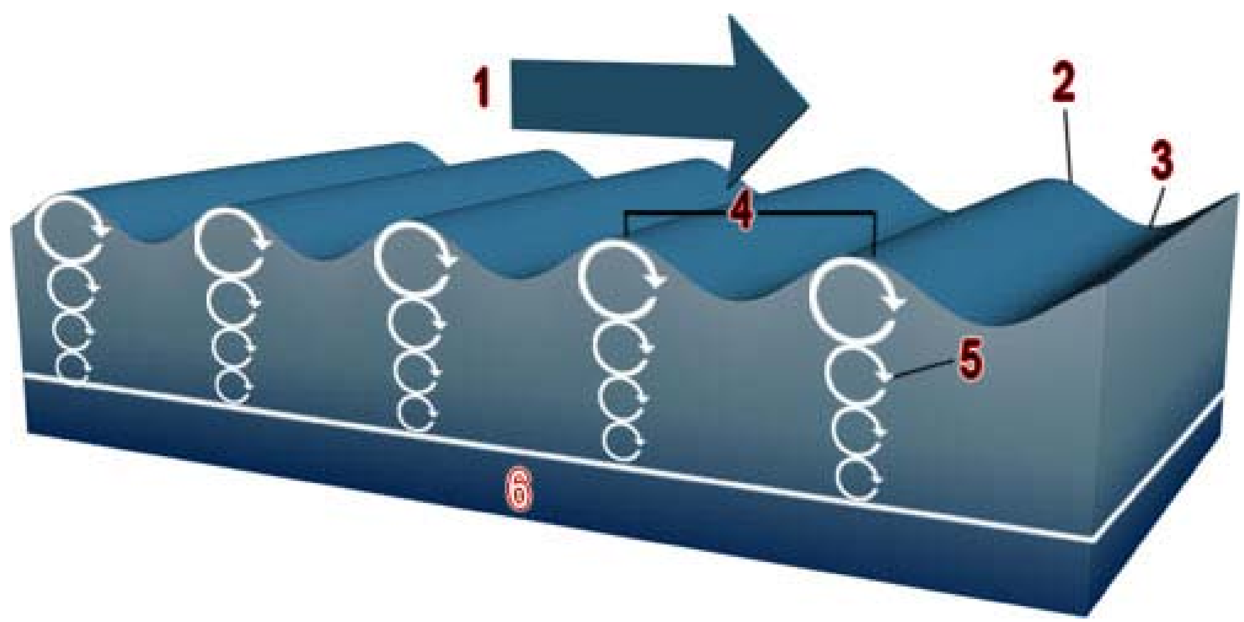
Selective velocity profiles for water particles
in deep water wave.





Gravity Waves

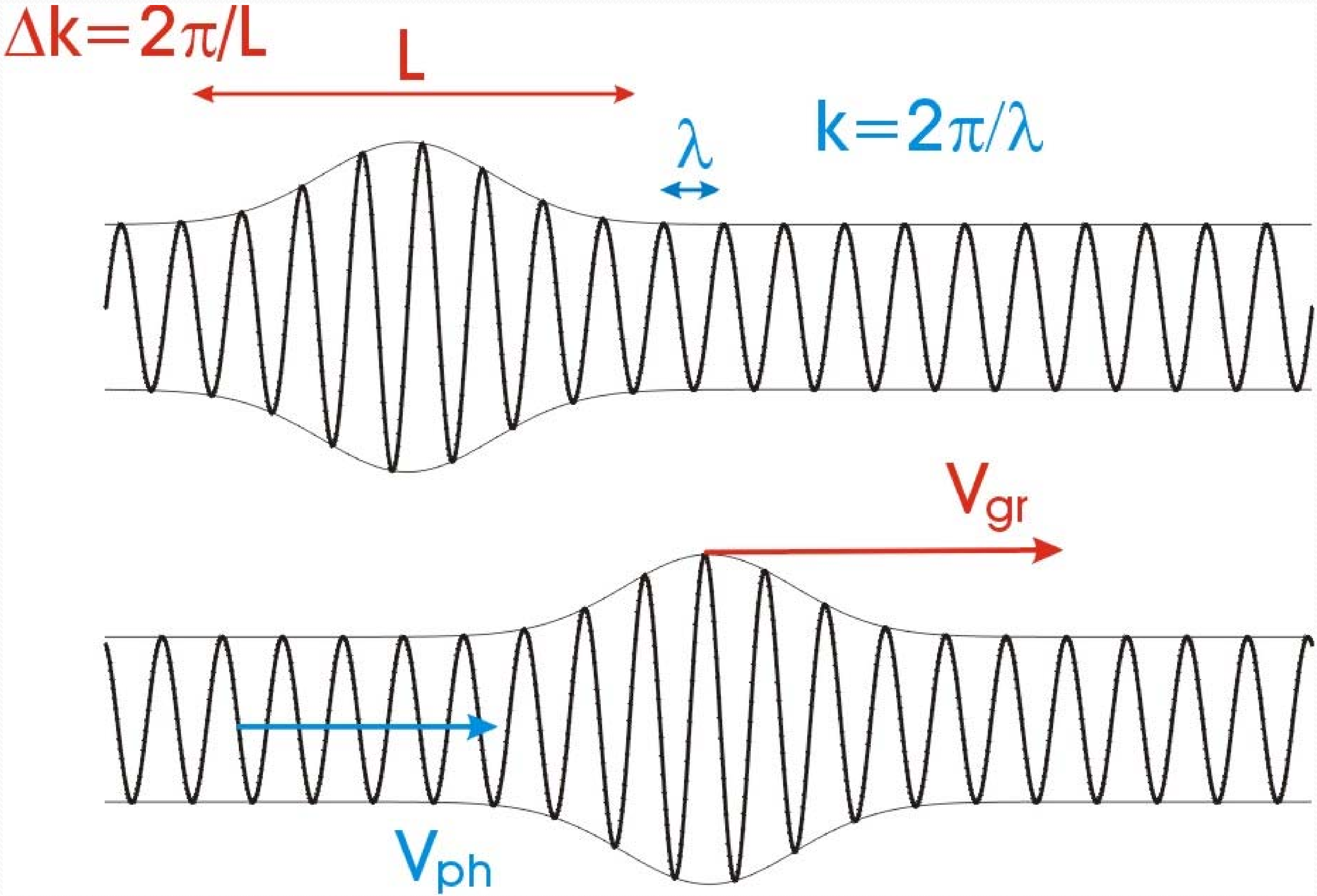




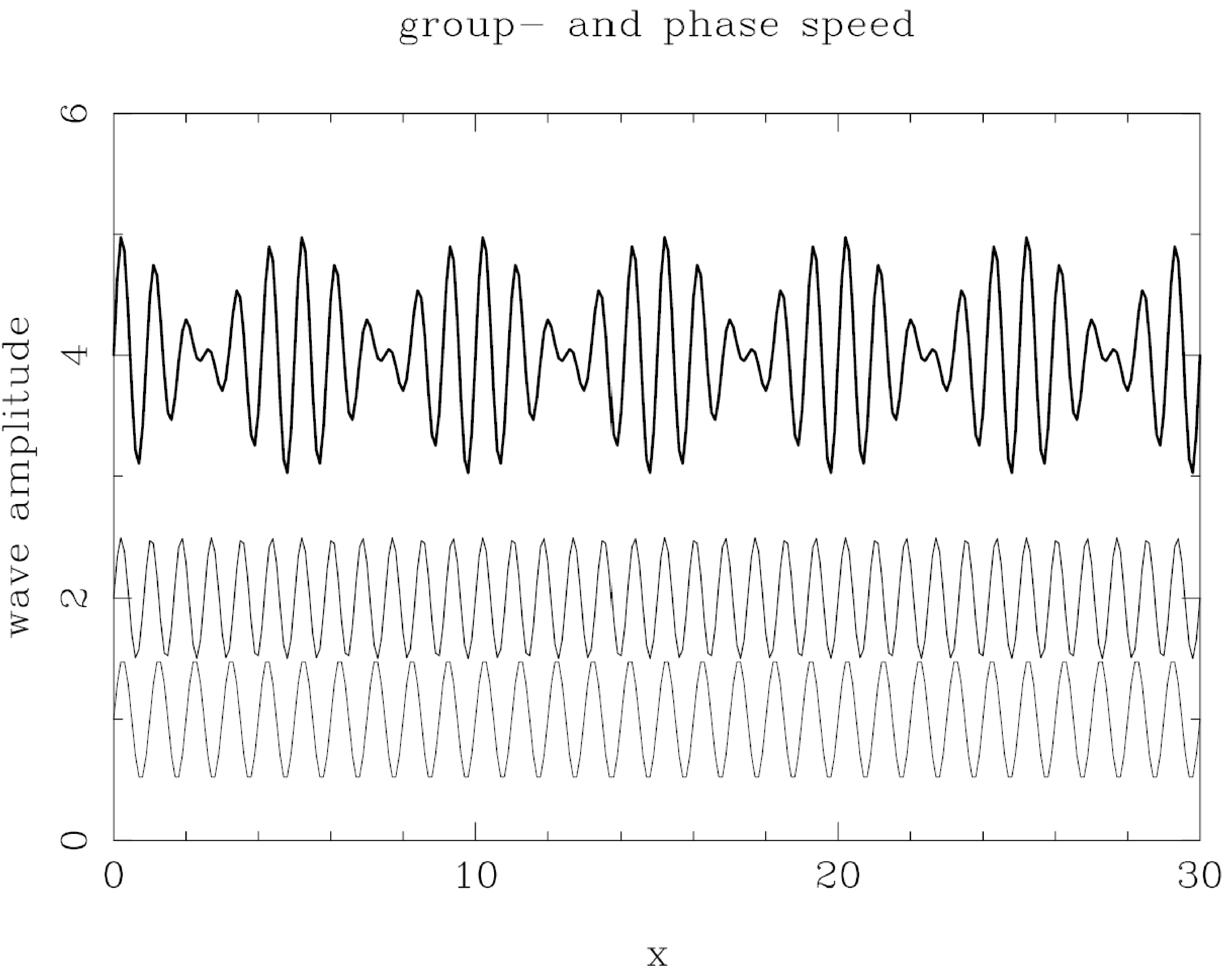
Waves:
sea & ocean waves

Kayak Surfing
on ocean gravity waves
Oregon Coast

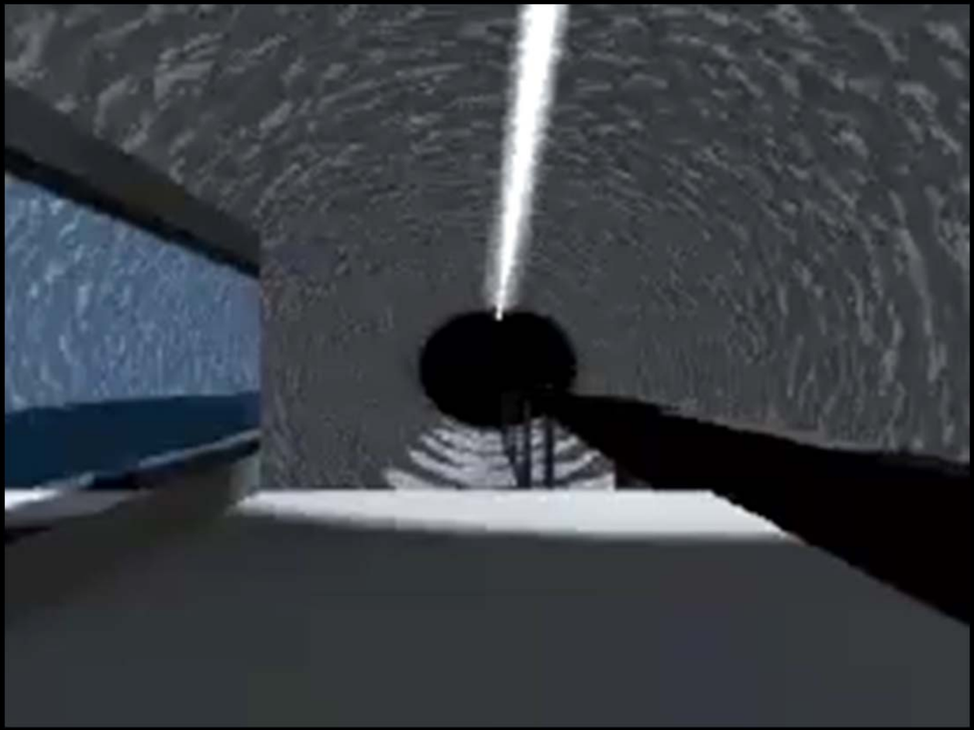
Phase & Group Velocity

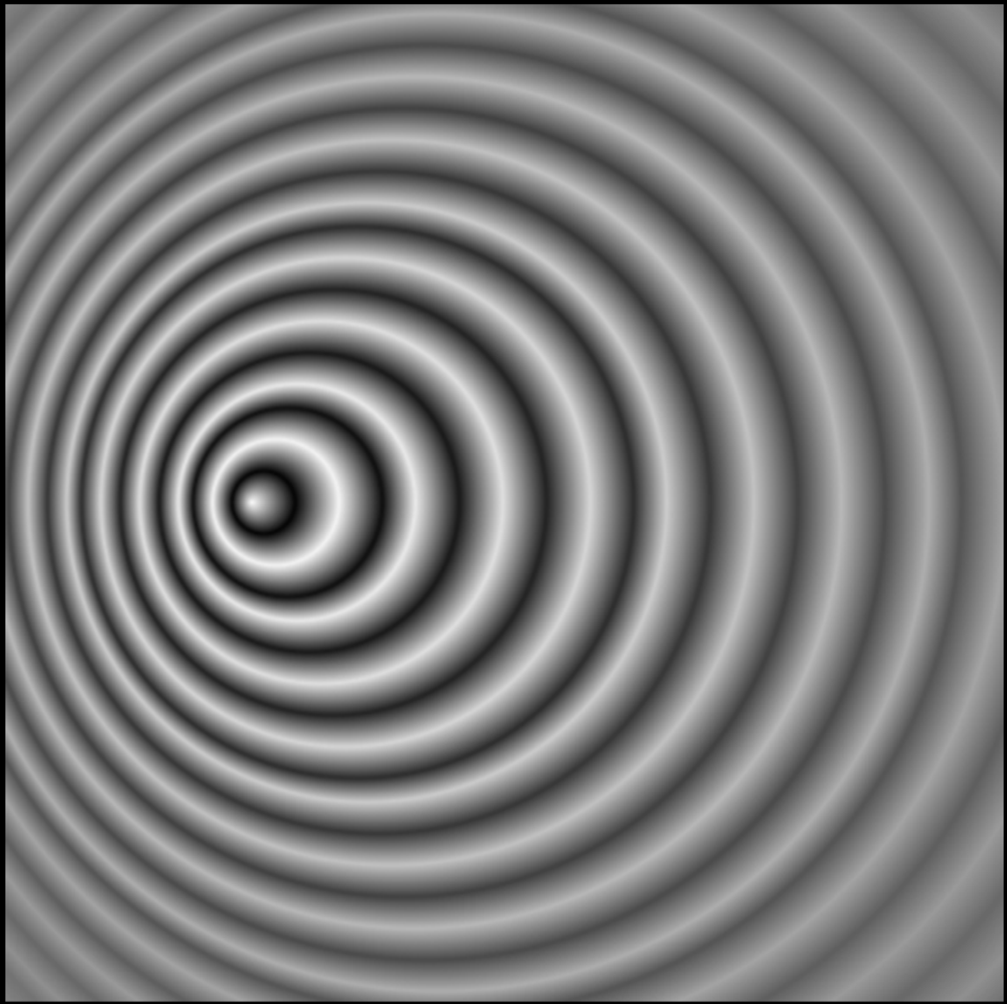


$\Delta k = 2\pi/L$
L
λ
$k = 2\pi/\lambda$
V_{gr}
V_{ph}

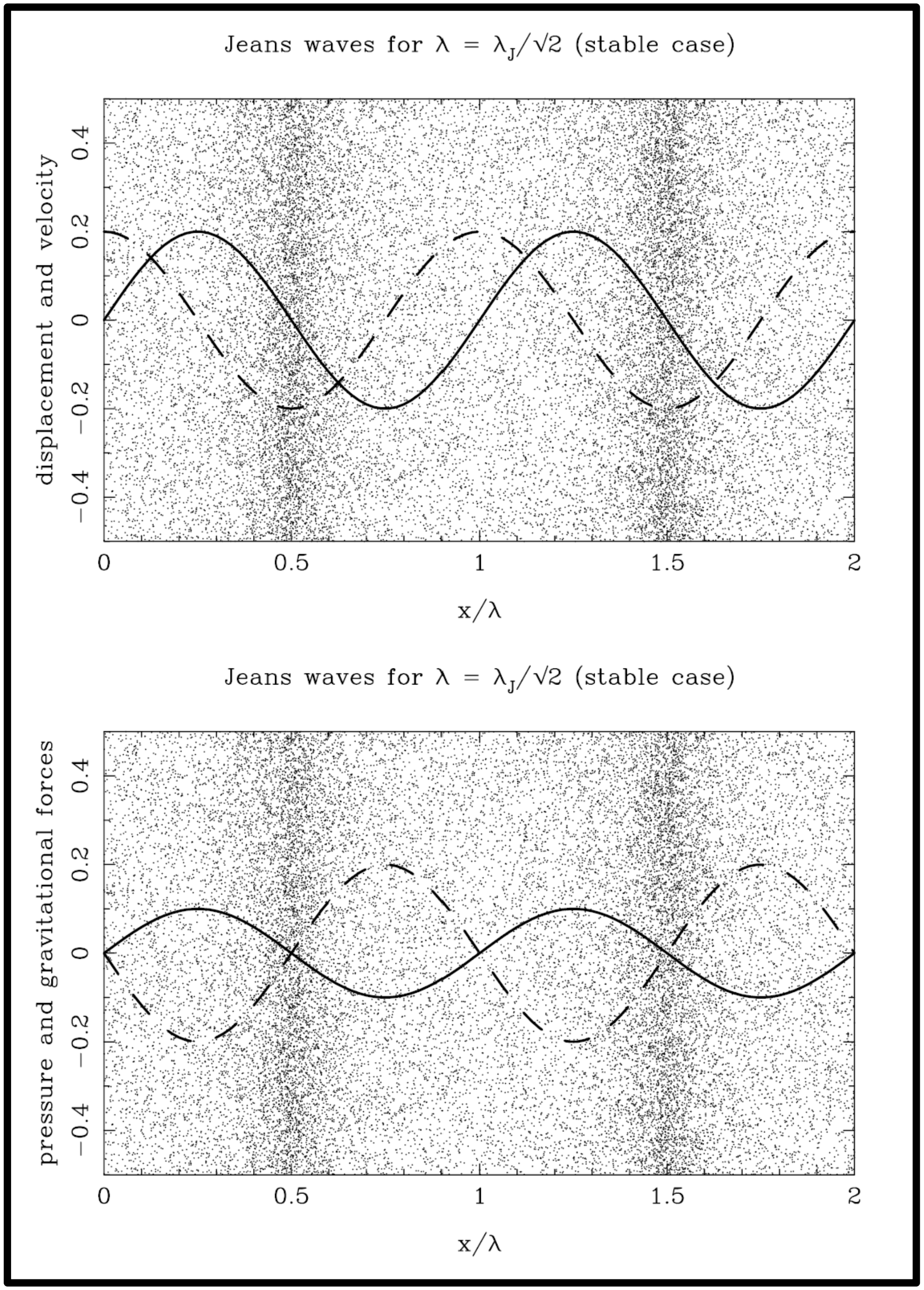

group– and phase speed
wave amplitude
x
0
2
4
6
0
10
20
30

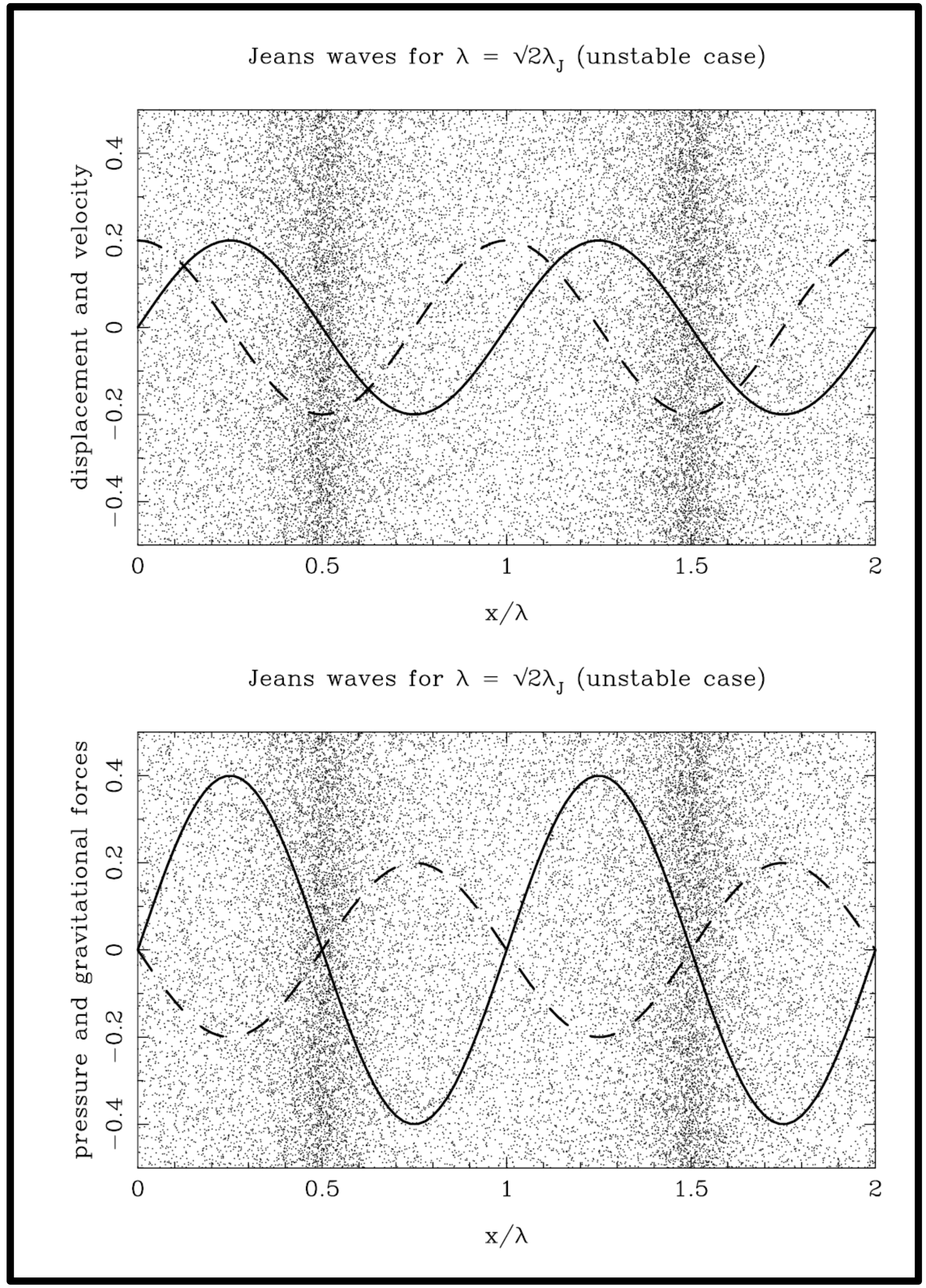
Doppler Effect





Jeans Instability


Jeans waves for λ = λJ/√2 (stable case)
displacement and velocity
x/λ
Jeans waves for λ = λJ/√2 (stable case)
pressure and gravitational forces
x/λ


Jeans waves for λ = √2λJ (unstable case)
displacement and velocity
x/λ
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x/λ