Bernoulli Applications

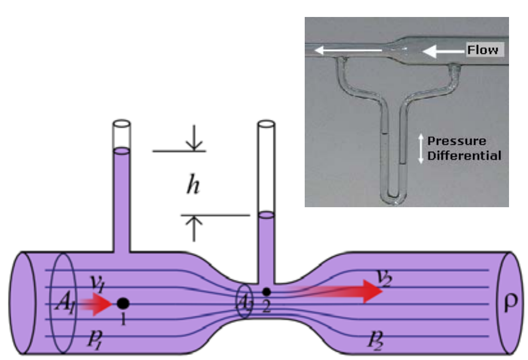
Venturi Meter

A Venturi meter is used to measure the flow rate through a tube.

It is based on the use of the Venturi effect, the reduction of fluid pressure that results when a fluid runs through a constricted section of pipe. It is called after Giovanni Battista Venturi (1746-1822), an Italian physicist.

Look at the construction in figure:

- we assume the flow is smooth and effectively inviscid, ie. friction is negligible.
- the fluid is incompressible, and has density ρ throughout the pipe.
- downstream we have a flow through a pipe section of area A_1, with a flow velocity v_1, and pressure p_1.
- in the narrow section with area A_2, the fluid flows with flow speed v_2, and has accompanying pressure p_2.
- as a result the two meters indicate the difference in pressure by means of a height difference h.



Venturi Meter

To find the pressure difference between the downstream flow and the pipe narrow, we invoke 1) the Bernoulli theorem and 2) the continuity equation. The latter assures that the rate of fluid flow through any section remains constant, ie. mass is preserved.

1) Bernoulli Theorem:

$$\frac{p_1}{\rho}+\frac{1}{2}v_1^{\,2}=\frac{p_2}{\rho}+\frac{1}{2}v_2^{\,2}$$

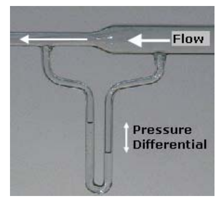
as the flow is horizontal, we do not have to take into account the gravity term.

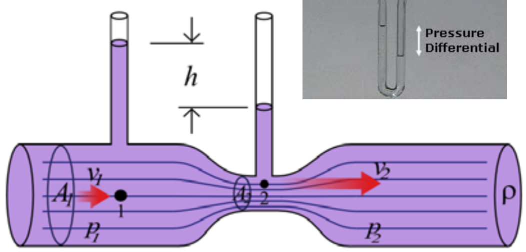
2) Continuity equation:

$$A_1 v_1 = A_2 v_2$$

Combining both equations, we find for the pressure difference in the two parts of the pipe:

$$p_1 - p_2 = \frac{1}{2}\rho v_1^{\,2}\left(\frac{A_1^{\,2}}{A_2^{\,2}}-1\right)$$





Venturi Meter

To read of the pressure difference between the two locations 1 and 2 in the fluid, we use the height difference h between the fluid level in the vertical tubes. To connect this height difference h to the pressure difference p1 and p2, we invoke the Euler equation:

From this we may then infer the flow velocity v_1 (as well as v_2):

$$v_1^{\,2}=\frac{2g\left(h_1-h_2\right)}{\left(A_1^{\,2}/A_2^{\,2}-1\right)}$$

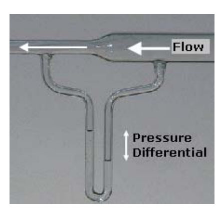
3) Euler equation (for a static medium):

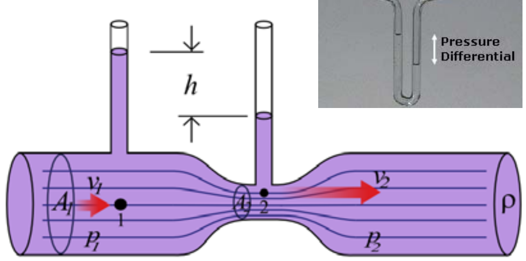
$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$

With the outside atmosphere pressure being P_{atm}, we then directly infer for the pressure P_1 and p_2,

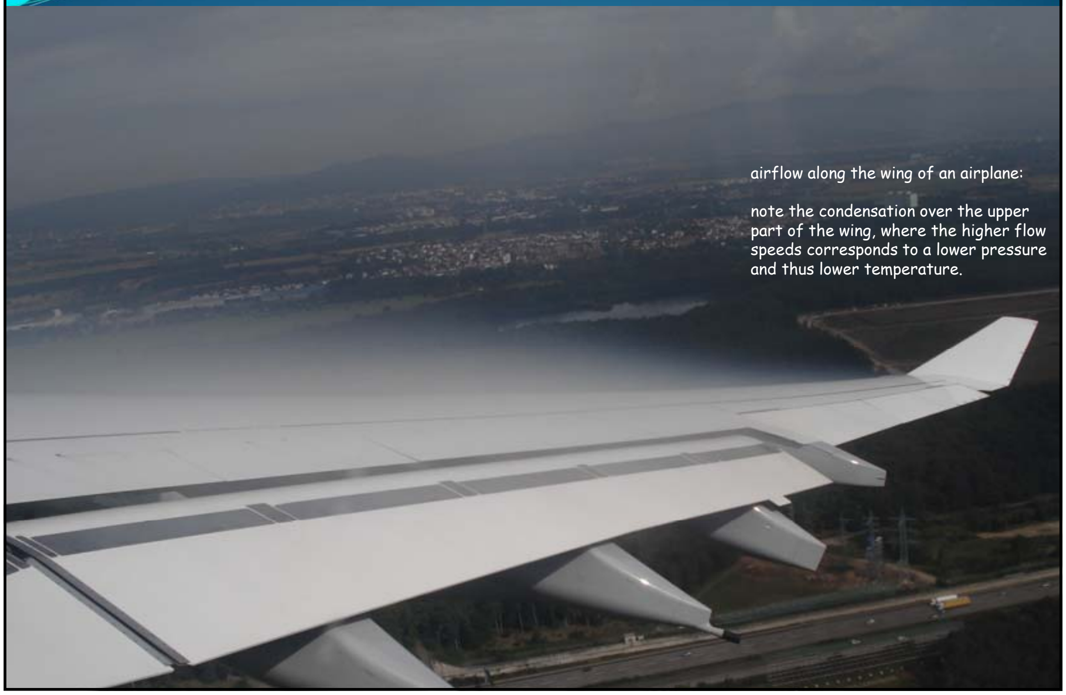
$$p_1 = p_{atm}+\rho g h_1$$

$$p_2 = p_{atm}+\rho g h_2$$





Air Flow along Wing

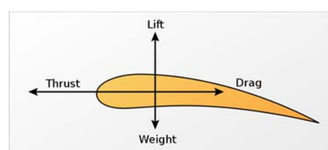


Air Flow along Wing

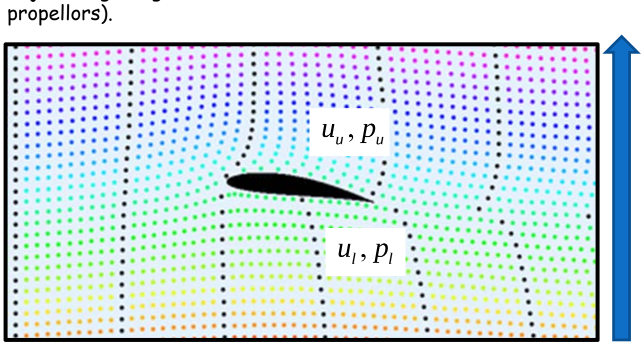
One of the most interesting applications of the Bernoulli equation, is the *flight of aeroplanes*.

Here we will provide a simplified explanation, based on the Bernoulli equation (reality is somewhat more complex).

Aeroplanes can fly because of the pressure difference between the flow below the wing and the flow over the wing*. This pressure difference results in a lift force that opposes the weight of the aeroplane (note that similar lifting forces work on many different objects, eg. wings of mills or wind turbines, sails on a sailboat, propellors).



Forces working on a wing – airfoil.



Flow around an airfoil: the dots move with the flow. Note that the velocities are much higher at the upper surface than at the lower surface. The black dots are on timelines, which split into two — an upper and lower part — at the leading edge. The part of a timeline below the airfoil does not catch up with the one above ! Colors of the dots indicate streamlines.

- Jet fighters often are not kept aflight by Bernoulli. Instead, they have the thrust of the jet motor, with vertical component, to keep them in the air.

Air Flow along Wing

According to Bernoulli,
the Bernoulli function B is constant along any *streamline.*
Thus, for a horizontally flying plane, we have that

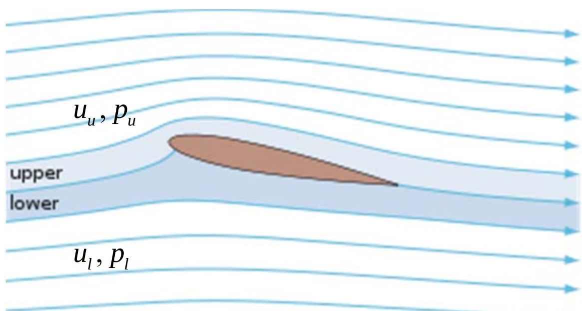
1) $$\frac{p_l}{\rho}+\frac{1}{2}{v_l}^2=\frac{p_u}{\rho}+\frac{1}{2}{v_u}^2=cst.$$

2) $$\rho A_l v_l=\rho A_u v_u \qquad \text{(Continuity eqn.)}$$

$\Rightarrow$

C_L is the lift coefficient, dependent on various factors, including the angle of wing wrt. air.

$$\boxed{p_2-p_1=\frac{1}{2}\rho v^2 C_L}$$



$$F_{lift}=\left(p_2-p_1\right)A$$

Along the wing, the flow over the upper (longer) edge of the wing has a (considerably) higher velocity u_u than below the wing, u_l *.

As a result (Bernoulli), the pressure p_l at the lower end of the wing is higher than the pressure p_u at the upper end.

The resulting pressure difference generates a lift force F_{lift}

The usual assumption of equal crosstime of upper and lower flow is not correct.

Air Flow along Wing

With the pressure difference between lower and upper wing being

$$p_2-p_1=\frac{1}{2}\rho v^2 C_L$$

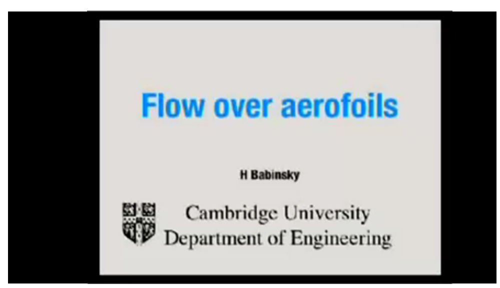
the total effective lift force

$$F_{lift}=\left(p_2-p_1\right)A$$

is, with A the "effective" planform area , and C_L the lift coefficient,

$$\boxed{F_{lift}=\frac{1}{2}\rho v^2 A C_L}$$

In other words, we need a certain speed V and wing area A to get sufficient lift force to lift a plane into the air … (see right) …

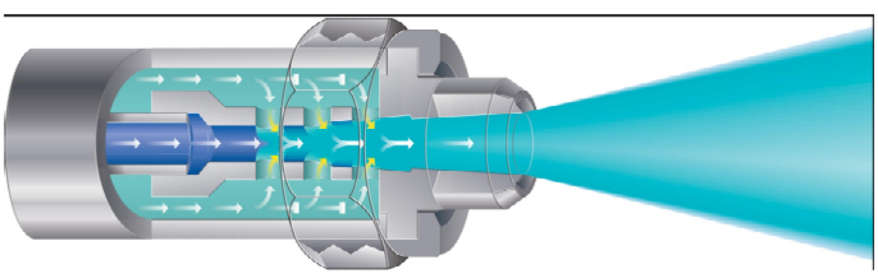


Air Flow along Wing

De Laval Nozzle

Bernoulli Equation: compressible fluids.

A very interesting application of the Bernoulli equation, for compressible fluids, concerns the *de Laval nozzle.*



A de Laval nozzle is a tube that is pinched in the middle, making a carefully balanced, asymmetric hourglass-shape. The nozzle was developed in 1888 by the Swedish inventor Gustaf de Laval for use on a steam turbine. The principle was first used for rocket engines by Robert Goddard.

The de Laval nozzle forms a nice platform to highlight the differences introduced by the compressibility of a gas when applying Bernoulli's theorem.

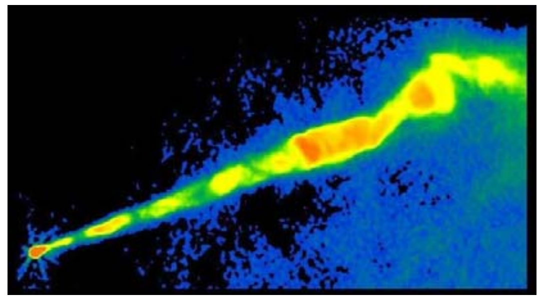
De Laval Nozzle

The de Laval nozzle is used to accelerate a hot, pressurised gas passing through it to a *supersonic speed*.

High-pressure gas coming from the combustion chamber enters the nozzle and flows into a region where the nozzle cross section decreases, $dA/dx < 0$. The thermal energy is converted into kinetic energy of the flow, and the flow goes through a sonic point at the critical point where the nozzle cross section narrows to its minimum ($dA/dx=0$). At that point the flow speed reaches the sound velocity. The cross section increases again after the critical point, and the gas is further accelerate to supersonic speeds.

The de Laval nozzle shapes the exhaust flow so that the heat energy propelling the flow is maximally converted into directed kinetic energy.

Because of its properties, the nozzle is widely used in some types of *steam turbine*, it is an essential part of the modern *rocket engine*, and it also sees use in *supersonic jet engines*.



Astrophysically, the flow properties of the de Laval nozzle have been applied towards understanding jet streams, such as observed in AGNs (see figure), the outflow from young stellar objects and likely occur in Gamma Ray Bursts (GRBs).

De Laval Nozzle

If we make the approximation of steady, quasi-1-D barotropic flow, we may write Bernoulli's theorem and the equation of continuity as

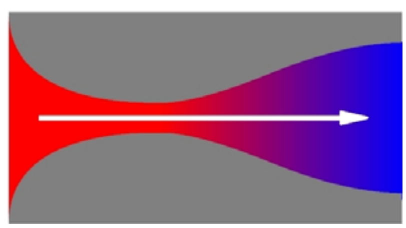
$$\frac{1}{2}u^2 + \int \frac{dP}{\rho} = cst.$$

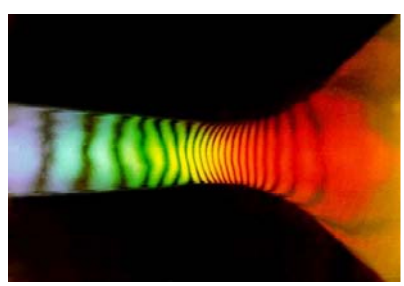
$$\rho u A = cst.$$

where A is the local sectional area of the nozzle.

Note that because of the compressibility of the gas we no longer assume a constant density, and thus have to keep ρ in the integral.

Gravitational potential variations are ignored, as for terrestrial applications the fast flow of jet gases is not relevant over the related limited spatial extent.





Two illustrations of the de Laval nozzle principle. The 2nd figure is a measurement of the flow speed in an experiment.

De Laval Nozzle

The variation of the area A along the axis of the nozzle will introduce spatial variations for each of the other quantities.

To consider the rate of such variations, take the differential of the Bernoulli equation,

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} = cst. \quad \Rightarrow \quad u\,du + \frac{1}{\rho}\frac{dp}{d\rho}d\rho = 0$$

Taking into account that the *sound velocity* c_s associated with the barotropic relation is

$$c_s{}^2 \equiv \frac{dp}{d\rho}$$

we find from the equations above that

$$u\,du + \frac{c_s{}^2}{\rho}d\rho = 0$$

We define the Mach number of the flow as the ratio of the flow velocity to the sound velocity,

$$M = \frac{u}{c_s}$$

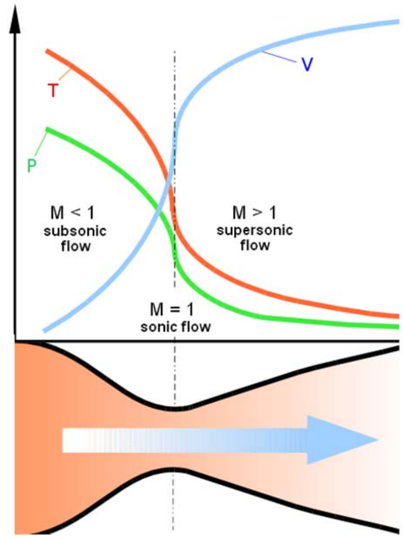


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

From the relation between velocity and density,

$$u\,du + \frac{c_s{}^2}{\rho}d\rho = 0$$

we find that the fractional change of density ρ is related to the fractional change of the fluid velocity u via the equation

$$\boxed{\frac{d\rho}{\rho} = -M^2\frac{du}{u}}$$

This equation states that the square of the Mach number provides a measure of the importance of compressibility.

In particular, flow of air at subsonic speeds past terrestrial obstacles can often be approximated as occurring at incompressibility, because the fractional change of density ρ is negligible in comparison with the fractional change of u if M<1.

In contrast, supersonic flight past obstacles necessarily involves substantial compressions and expansions.

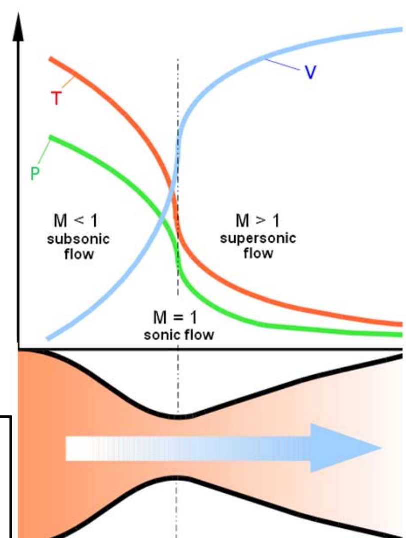


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

To relate the change of velocity u to the change of sectional area A in the nozzle, we take the logarithmic derivative of the continuity equation,

To consider the rate of such variations, take the differential of the Bernoulli equation,

$$\rho u A = cst.$$

$$\Downarrow$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

which, taking into account the relation between density ρ and flow speed u, yields the following relation between u and A:

$$\boxed{\left(1 - M^2\right)\frac{du}{u} = -\frac{dA}{A}}$$

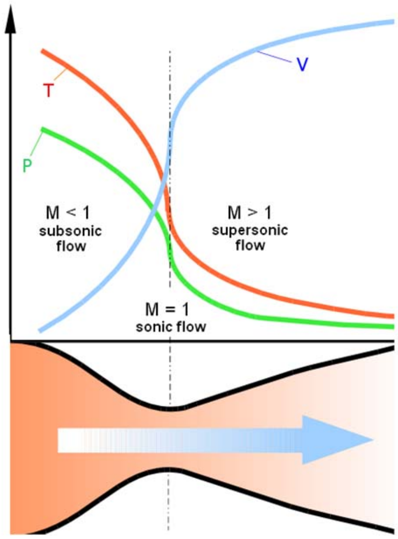


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

the relation

$$\boxed{\left(1 - M^2\right)\frac{du}{u} = -\frac{dA}{A}}$$

has the following implications:

1) *Subsonic speeds:*

$$\left(1 - M^2\right) > 0$$

$$\Downarrow$$

$$du > 0 \quad \Leftrightarrow \quad dA < 0$$

this corresponds to normal experience, eg. the speeding up of a river as the channel narrows.

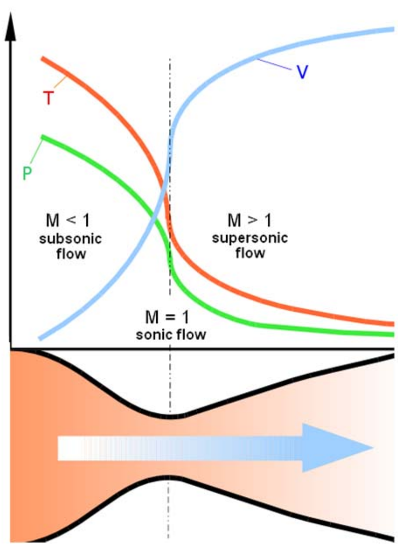


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

2) *Supersonic speeds*:

$$\boxed{\begin{gathered}\left(1-M^2\right)<0\\ \Downarrow\\ du>0 \iff dA>0\ !!!\end{gathered}}$$

In other words, an increase in the velocity requires an *increase* in the area of the nozzle, dA>0 !!!

This counterintuitive result has a simple explanation: for M>1, the density decreases faster than the area increases, so the velocity must increase to maintain a constant flux of mass.

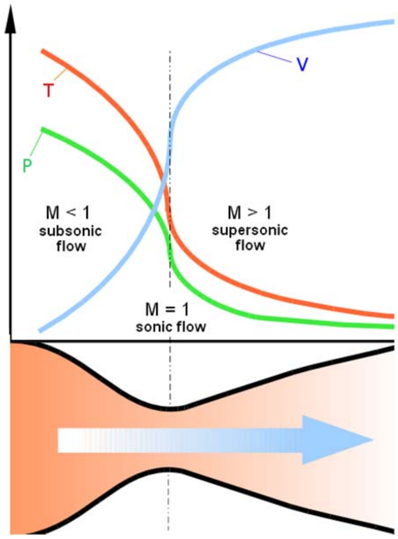


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

3) *A sonic transition* , ie.

$$\left(1-M^2\right)=0$$

can be made smoothly, ie. With

$$\frac{du}{dx}\ \ finite$$

only at the *throat of the nozzle* where

$$dA=0$$

To obtain supersonic exhaust, therefore, we must accelerate the reaction gases through a converging-diverging nozzle, a fundamental feature behind the design of jet engines and rockets.

4) Note that the converse does not necessarily hold:

M does not necessarily equal unity at the throat of the nozzle, where dA=0. If M<1, the fluid velocity reaches a local extremum when the area does, i.e. du=0 where dA=0.

Whether the extremum corresponds to a local maximum or minimum depends on whether we have subsonic or supersonic flow and whether the nozzle has a converging-diverging shape.

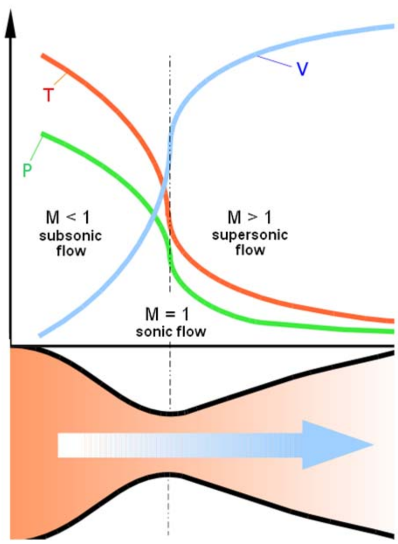


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

Final Notes:

5) whether supersonic exhaust is actually achieved in nozzle flow also depends on the boundary conditions. In particular, on the pressure of the ambient medium in comparison with the pressure of the reaction chamber.

   If a sonic transition does occur, the flow behaviour depends sensitively on the nozzle conditions, since the coefficient $1-M^2$ becomes arbitrarily small near the transition region.

6) When external body forces are present, we do not need to have a throat to achieve the smooth transition of subsonic to supersonic flow. The external forces can provide the requisite acceleration.

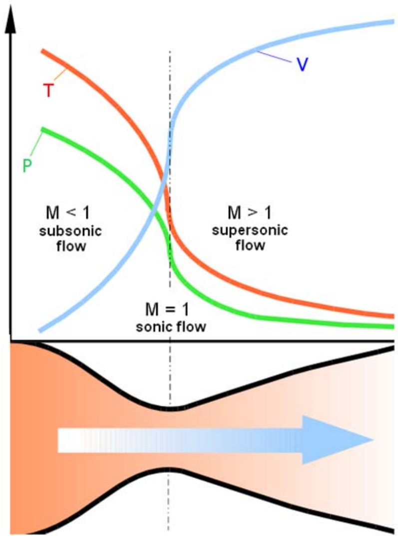


Illustration of the run of flow speed u, pressure p and temperature T, as the gas passes through the nozzle and its sonic point.

De Laval Nozzle

Werner von Braun:

father of the Saturn 5 moon rocket
(and the German V2 missiles WWII)

Standing near motor exhausts Saturn V

Apollo 11: mission to the moon

Launch Apollo 11

July 16, 1969

exhausts of motors
of the giant
Saturn 5 moon rocket

beautiful illustration
of workings
De Laval nozzle

Potential Flow

Incompressible Flow

Many problems of practical importance, involving a large number of engineering and terrestrial conditions concern incompressible flows.

For an incompressible flow, we have

$$\boxed{\vec{\nabla} \cdot \vec{u} = 0}$$

which follows directly from the continuity equation on the basis of the conditions

$$\frac{\partial \rho}{\partial t} = 0; \qquad \vec{\nabla} \rho = 0$$

In other words, for an incompressible fluid (a liquid) the variation of pressure p in the force (Euler) equation equals whatever it needs so that $\vec{\nabla} \cdot \vec{u} = 0$

Note that for an incompressible medium, Kelvin's circulation theorem is valid independent of the barotropic assumption: because $\vec{\nabla} \rho = 0$ the vorticity equation is true independent of

Incompressible Flow

Many problems in hydrodynamics involve the motion of a solid body (eg., a ship) through water that is stationary at infinity.

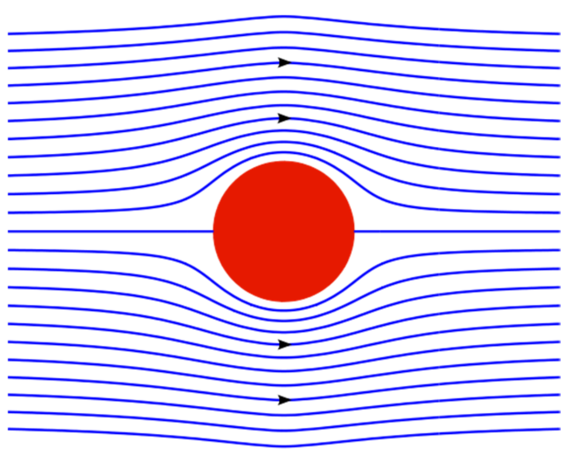
From the point of view of an observer fixed on the ship, the water flowing past the ship originates from a steady region of uniform conditions.

- Uniform flow has no vorticity: Kelvin's circulation then guarantees that no vorticity will be generated in the flow around the ship.

  Note: this is only true as long as the effect of viscosity can be ignored.

- If $\vec{\omega} = \vec{\nabla} \times \vec{u} = 0$ everywhere, the flow field can be derived from the gradient of a scalar, the velocity potential Φ_v

$$\vec{u} = \vec{\nabla} \Phi_v$$



Potential Flow

If we substitute the velocity potential definition

$$\vec{u} = \vec{\nabla}\Phi_v$$

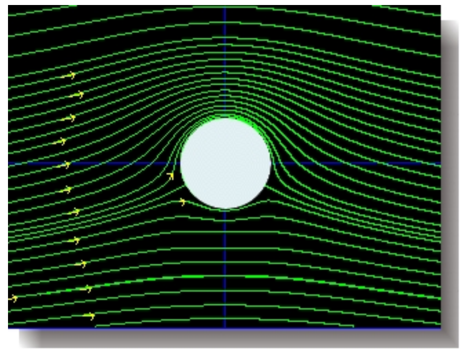
into the continuity equation

$$\vec{\nabla} \cdot \vec{u} = 0$$

we obtain the condition for *potential flow*,

$$\boxed{\nabla^2 \Phi_v = 0}$$

This is nothing else than the *Laplace equation.*



Thus, the solution of many problems in hydrodynamics boils down to a solution of the Laplace equation. The problem is *well-posed* , and there is a vast body of work on its solution.

To solution of the Laplace equation is dictated by the boundary (and initial) conditions that are imposed.

Boundary Conditions

To solve the Laplace equation, we need to specify the boundary conditions. There are a Variety of boundary conditions. Usually, these involve one (or more) of the following :

a) The value of Φ_v on the bounding surface of the fluid $\Longleftrightarrow$ Dirichlet boundary conditi

b) The value of its normal derivatives on the boundaries $\Longleftrightarrow$ Neumann boundary conditi

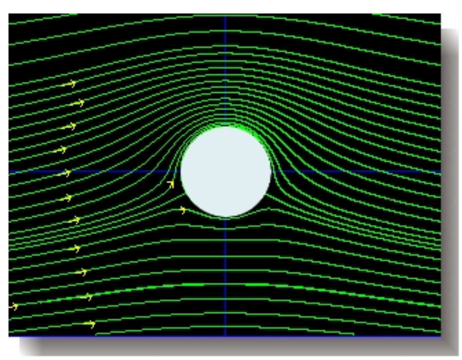
For the problem of a flow past a solid object like a ship or a sphere, we have the important condition that the water should not penetrate the object, ie. there should be no flow normal to its surface. This translates into the *Neumann boundary condition*:

a) for the object at rest:

$$\vec{u} \cdot \hat{n} = 0 \quad \Rightarrow \quad \vec{n} \cdot \vec{\nabla}\phi = \frac{\partial \phi}{\partial n} = 0$$

b) if the object moves with velocity $\vec{U}$, then

$$\vec{u} \cdot \hat{n} = \vec{U} \cdot \hat{n} \quad \Rightarrow \quad \hat{n} \cdot \vec{\nabla}\phi = \frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n}$$

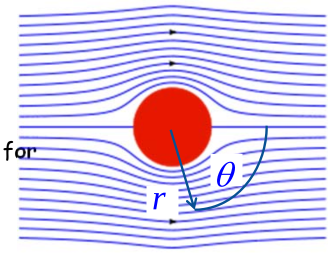


Potential Flow

To find the solutions to the Laplace equation,

$$\nabla^2 \Phi_v = 0$$

one can apply the mathematical machinery of potential theory.
To provide an idea of the solutions we concentrate on solutions for



Spherical geometry (of object) and Axisymmetric flow

The general axisymmetric solution of Laplace's equation

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi_v}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi_v}{\partial \theta}\right) = 0$$

is obtained by the separation-of-variables method in spherical polar coordinates, ie.

$$\Phi_v = R(r)\,\Theta(\theta)$$

which yields

$$\Phi_v = \sum_{n=0}^{\infty} \left\{ A_n r^n + B_n r^{-(n+1)} \right\} P_n(\cos\theta)$$

r is the spherical radius
 the co-latitude (see figure):

$$z = r\cos\theta$$

A_n and B_n are arbitrary constants, whose value is determined by the boundary conditions.

P_n is the Legendre Polynomial of degree n:

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$$

I.14 Euler & Potential flow

In the case of potential flow, we find from the fact that it is irrotational,

$$\omega = \vec{\nabla} \times \vec{u} = 0$$

and the velocity can be written as the gradient of a potential $_v$ that the Euler equation

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\vec{\nabla}\phi - \vec{\nabla} h$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) + \left(\vec{\nabla} \times \vec{u}\right) \times \vec{u} = -\vec{\nabla}\phi - \vec{\nabla} h$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) = -\vec{\nabla}\phi - \vec{\nabla} h$$

for barotropic flow and potential external forces can be written as

I.14 Euler & Potential flow

for barotropic flow and potential external forces can be written as

$$\vec{\nabla}\frac{\partial\Phi_v}{\partial t}+\vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right)=-\vec{\nabla}\phi-\vec{\nabla}h$$

$$\Downarrow$$

$$\vec{\nabla}\left(\frac{\partial\Phi_v}{\partial t}+\frac{1}{2}|\vec{u}|^2+\phi+h\right)=0$$

from which we can immediately infer that the Bernoulli function is a function of time:

$$\vec{\nabla}\left(\frac{\partial\Phi_v}{\partial t}+\frac{1}{2}|\vec{u}|^2+\phi+h\right)=0$$

$$\Downarrow$$

$$\frac{\partial\Phi_v}{\partial t}+\frac{1}{2}|\vec{u}|^2+\phi+h=\frac{\partial\Phi_v}{\partial t}+B(t)=T(t)$$

Hydrostatics

Hydrostatics

Systems where motion is absent altogether, or at least has no dynamic effects, are in *hydrostatic equilibrium* $\vec{u} = 0$

In those situations, the fluid equations reduce to simple equilibrium equations.

1) Continuity equation: $$\frac{\partial \rho}{\partial t} = 0$$
2) Euler equation: $$\frac{1}{\rho}\vec{\nabla} p = \vec{f} = -\vec{\nabla}\Phi$$

(the latter identity in the Euler equation is for the body force being the gravitational force).

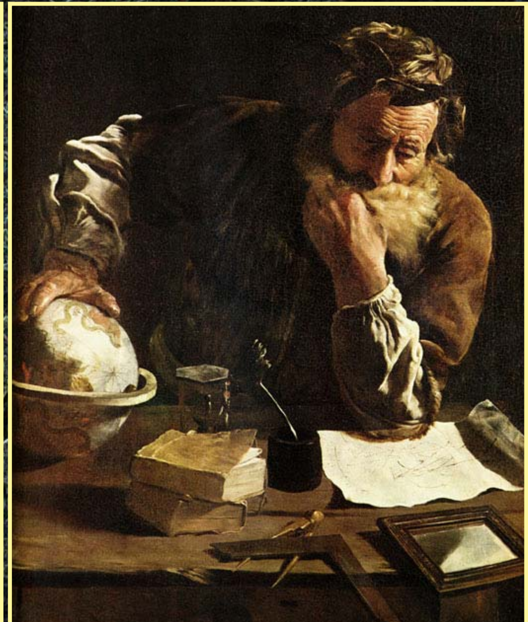
We will shortly address four typical examples of hydrostatic equilibrium, all of major astrophysical interest

1) Archimedes' Principle, bouyancy forces
2) Isothermal sphere
3) Stellar Structure equations
4) Mass determination of clusters from their X-ray emission.

Archimedes

Syracuse, 287-211/212 BC,
Greatest mathematician & scientist of antiquity (all time ?):

- Probably studied in Alexandria, under followers Euclides
- Killed by Roman soldier, upon Roman conquest Syracuse
- Family Hieron II, king Syracuse ?
- Inventions:
  - war machines ...
  - water screw
  - water organ (?)
  - burning mirrors (???)
  - planetarium !!!!!!



Archimedes' Principle

In the situation where an object is (partially) immersed in a fluid (see figure), Archimedes' principle states, shortly, that

Buoyancy = Weight of displaced fluid

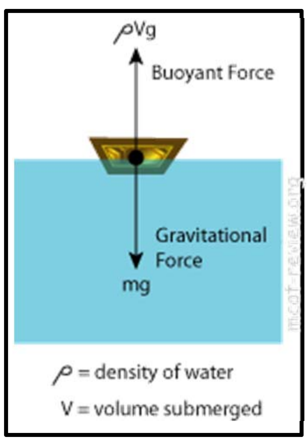
Pressure by water on displaced volume:

$$\frac{1}{\rho}\vec{\nabla}p = \vec{f} = \vec{g}$$

$$\Downarrow$$

$$\vec{F} = -\oiint_S p\, d\vec{S} = -\int_V \vec{\nabla}p\, dV = -\rho V\vec{g}$$

This is called the buoyancy force, and underlies a large amount of practical applications – starting from ships floating on water.

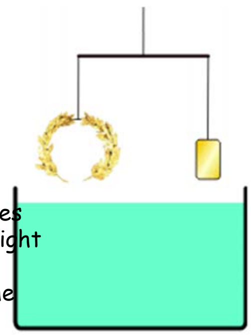


Archimedes Principle

'the principle is called after Archimedes of Syracuse (287-212 BC), Antiquities' greatest genius.

He got the idea when ordered by King Hieron II of Syracuse to investigate whether the

golden crown he had ordered to be manufactured contained the pure gold he had provided the goldsmith or whether the smith had been dishonest and included silver …

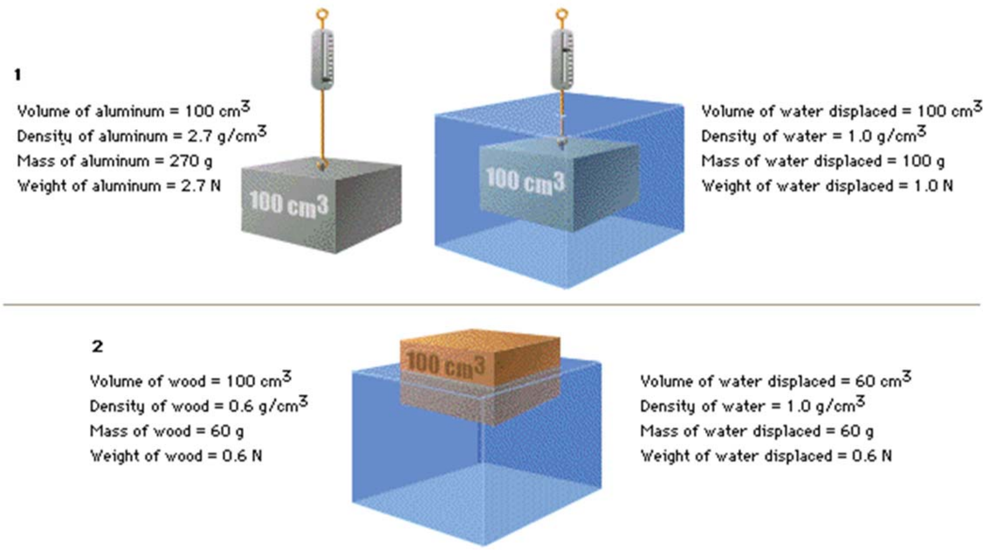


Immersing the crown in water, Archimedes determined the volume. Comparing its weight by a balance containing similar amount of pure gold, he found the density of the the crown … which turned out not to be pure

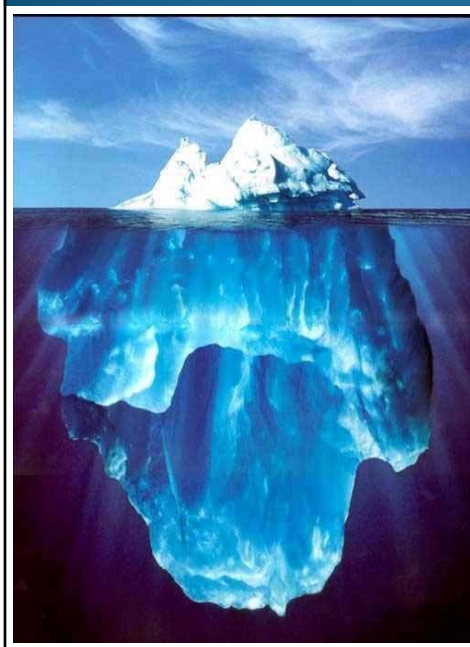
Buoyancy = Weight of displaced fluid

Archimedes & Hieron's Golden Crown

Archimedes Principle



Archimedes Principle: Iceberg



A telling example of how Archimedes' principle works is the floating of icebergs.

How much of an iceberg is visible over the water level depends on the density of ice wrt. the density of fluid water ?

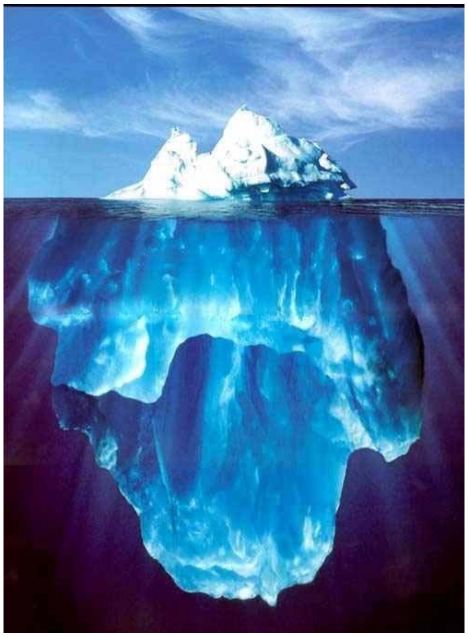
ρ_{ice} =0.9167 g/cm³ at T=0° C

ρ_{water}=0.9998 g/cm³ at T=0° C

With the volume of the iceberg =V_{ice}, and the volume of the iceberg immersed in the water V_{water} :

Determine the fraction of the iceberg's volume immersed in the water …

Archimedes Principle: Iceberg



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With the volume of the iceberg =V_{ice}, and the volume of the iceberg immersed in the water V_{water} :

$$\rho_{water} V_{water} g = \rho_{ice} V_{ice} g$$

$$\Downarrow$$

$$\frac{V_{water}}{V_{ice}} = \frac{\rho_{ice}}{\rho_{water}} \approx 0.92$$

Ie., only 8% of the iceberg is visible above the water, hence …

Archimedes Principle: Iceberg

Isothermal Sphere

What is the equilibrium configuration of a spherically symmetric gravitating body ?

The two equations governing the system are the hydrostatic equilibrium (Euler) equation and the Poisson equation:

$$\vec{\nabla} p = -\rho \vec{\nabla} \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

Because of spherical symmetry, we write the Laplacian in spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Therefore, in spherical coordinates the hydrostatic and Poisson equation become:

$$\boxed{\begin{gathered} \frac{dp}{dr} = -\rho \frac{d\phi}{dr} \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G \rho \end{gathered}}$$

Isothermal Sphere

Integration of the second equation gives:

$$r^2 \frac{d\phi}{dr} = Gm(r)$$

where m(r) is the mass contained within the shell of radius r,

$$m(r) = \int_0^r 4\pi x^2 \rho(x)\, dx$$

To solve this equation, we have to invoke the nature of the gas, ie. the equation of state p(?). We assume an ideal gas, for which

$$p = \frac{R}{\mu} \rho T = \rho c_s^{\,2}$$

We make the assumption that it concerns a gas with constant molecular weight ? and a constant temperature T (an isothermal sphere). This yields the following equation:

$$\frac{d}{dr}\left(\frac{r^2 c_s^{\,2}}{\rho} \frac{d\rho}{dr} \right) = -4\pi G r^2 \rho$$

Isothermal Sphere

The equation for isothermal sphere hydrostatic equilibrium,

$$\frac{d}{dr}\left(\frac{r^2 c_s^{\,2}}{\rho} \frac{d\rho}{dr} \right) = -4\pi G r^2 \rho$$

has the solution

$$\boxed{\rho = \frac{c_s^{\,2}}{2Gr^2} \; ; \qquad p = \frac{c_s^{\,4}}{2Gr^2}}$$

This is the well-known isothermal sphere solution.

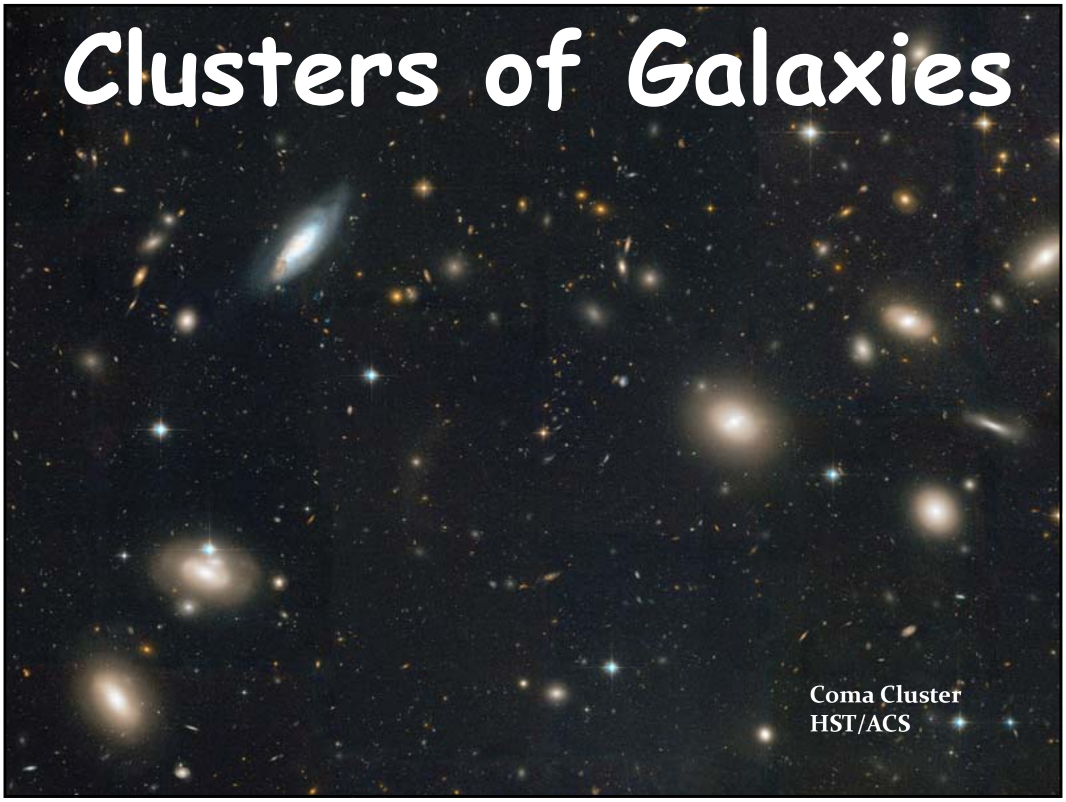
Notice that the isothermal sphere solution is singular at the center. Nevertheless, it provides a useful analytic approximation for various astronomical problems (sometimes a core is added).

Note: in real stars the temperature and, with it, the pressure increases with depth, Which provide enough support against self collapse without the need for a singularity at r=0

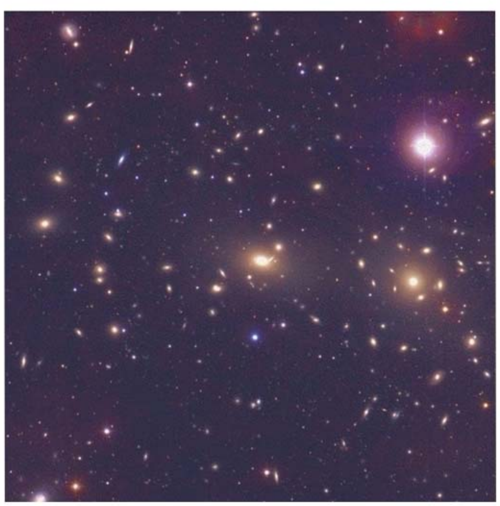
Clusters of Galaxies

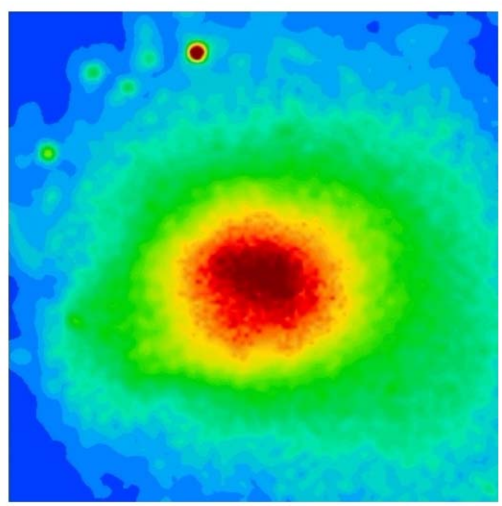
- Assemblies of up to 1000s of galaxies within a radius of only 1.5-2h^{-1} Mpc.
- Representing overdensities of δ~1000
- Galaxies move around with velocities ~ 1000 km/s
- They are the most massive, and most recently, fully collapsed structures in our Universe.

Clusters of Galaxies

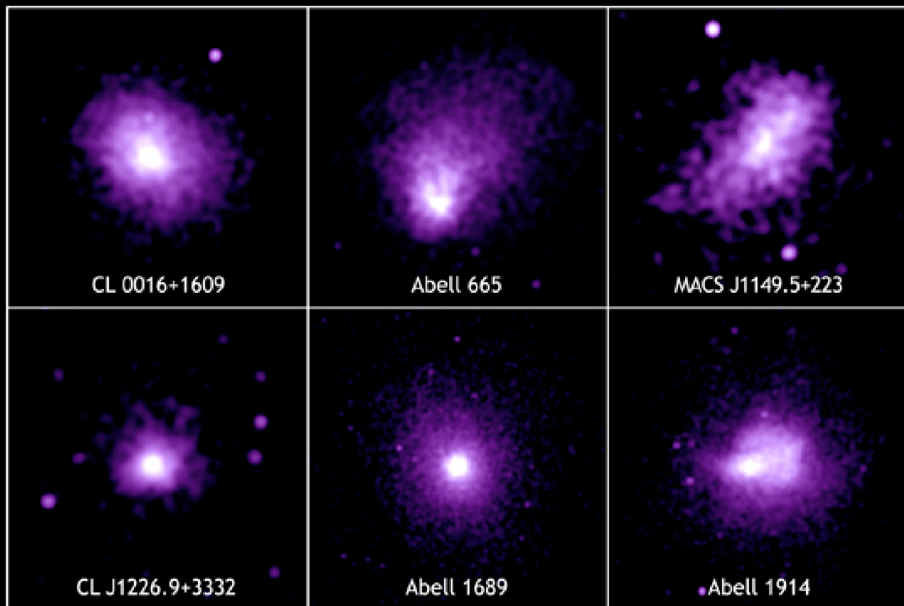


Cluster X-ray Emission





Cluster X-ray Emission

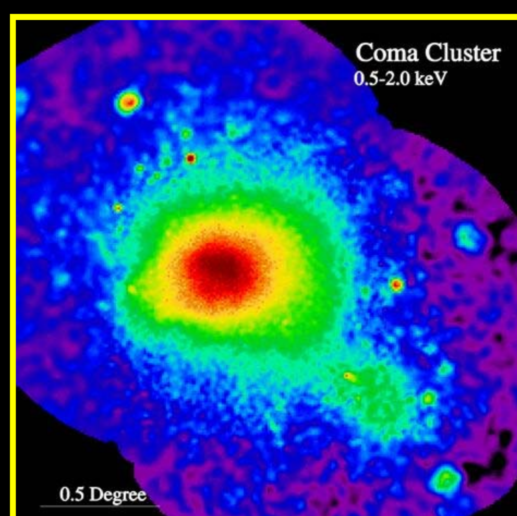


Clusters: X-ray emitting Hot Gas Spheres

- T ~ 10-100 million Kelvin !!!
- in Hydrostatic Equilibrium:

  Gravity = Pressure
- assume perfect spherically symmetric gas sphere:

$$F_{pressure} = F_{grav}$$
$$\Downarrow$$
$$4\pi r^2 dP = -\frac{GM(r)}{r^2}\rho(r)\,4\pi r^2\,dr$$



ROSAT X-ray image Coma Cluster

Clusters:
X-ray emitting Hot Gas Spheres

Hydrostatic Equilibrium:

Pressure = Gravity

$$F_{pressure} = F_{grav}$$
$$\Downarrow$$
$$4\pi r^2 dP = -\frac{GM(r)}{r^2}\rho(r)\,4\pi r^2\,dr$$

where M(r) is the mass within radius r:

$$M(r) = \int_0^r 4\pi x^2 \rho(x)dx$$

Hence,

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{GM(r)}{r^2}$$

Clusters:
X-ray emitting Hot Gas Spheres

For an ideal gas:

$$p = knT = \frac{\rho}{\mu m_p}kT$$

Hence,

$$\frac{k}{\mu m_p}\frac{1}{\rho}\frac{d}{dr}\{\rho(r)T(r)\} = -\frac{GM(r)}{r^2}$$

Which, after some algebraic manipulation, leads to …

Clusters:
X-ray emitting Hot Gas Spheres

$$\frac{k}{\mu m_p}\frac{1}{\rho}\frac{d}{dr}\{\rho(r)T(r)\} = -\frac{GM(r)}{r^2}$$

Which, after some algebraic manipulation, leads to …

$$\begin{aligned}
M(r) &= -\frac{k}{\mu m_p}T(r)r\frac{r}{\rho T}\frac{d}{dr}\left(\rho T\right) \\
&= -\frac{k}{\mu m_p}T(r)r\frac{d\ln\left(\rho T\right)}{d\ln r} \\
&= -\frac{k}{\mu m_p}T(r)r\left\{\frac{d\ln\rho}{d\ln r}+\frac{d\ln T}{d\ln r}\right\}
\end{aligned}$$

Cluster Mass:
X-ray emitting Hot Gas Spheres

Hydrostatic Equilibrium:

$$\frac{GM(r)}{r^2} = -\frac{k_B T}{\mu m_H}\left[\frac{d\log\rho}{dr}+\frac{d\log T}{dr}\right]$$

Determination Mass from X-ray observations:

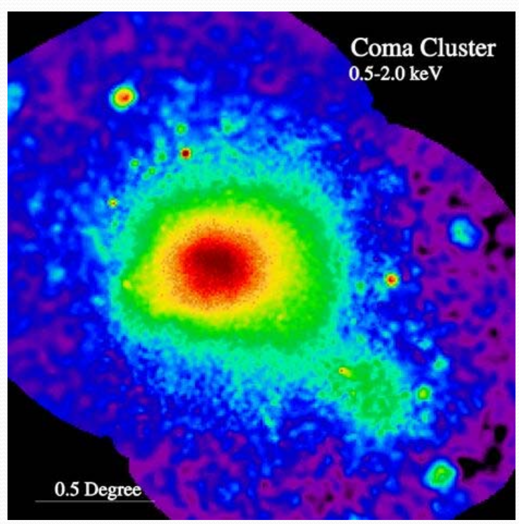
-assumption:

Isothermal: $T(r)=T_o$

- temperature: T ~10-100 million Kelvin

-density profile:

X-ray emission Bremsstrahlung: $L(r) \sim \rho(r)^2$



ROSAT X-ray image Coma Cluster

Clusters of Galaxies

- Clusters not only contain galaxies:
- in fact, galaxies & stars are a minor component:

I. Clusters are Halos of Dark Matter:

M_{DM}/M_{total} ~ 82%

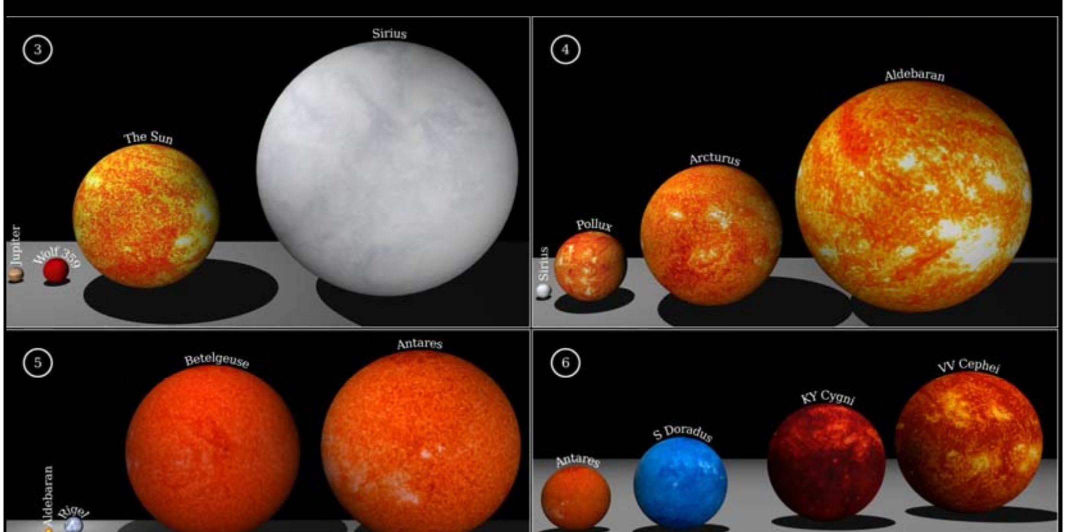
II. Clusters are Hot Balls of (highly ionized) Gas

M_{ICM}/M_{total} ~ 16-17 %

III. Galaxies are mainly raisins in a sea of dark matter & hot gas

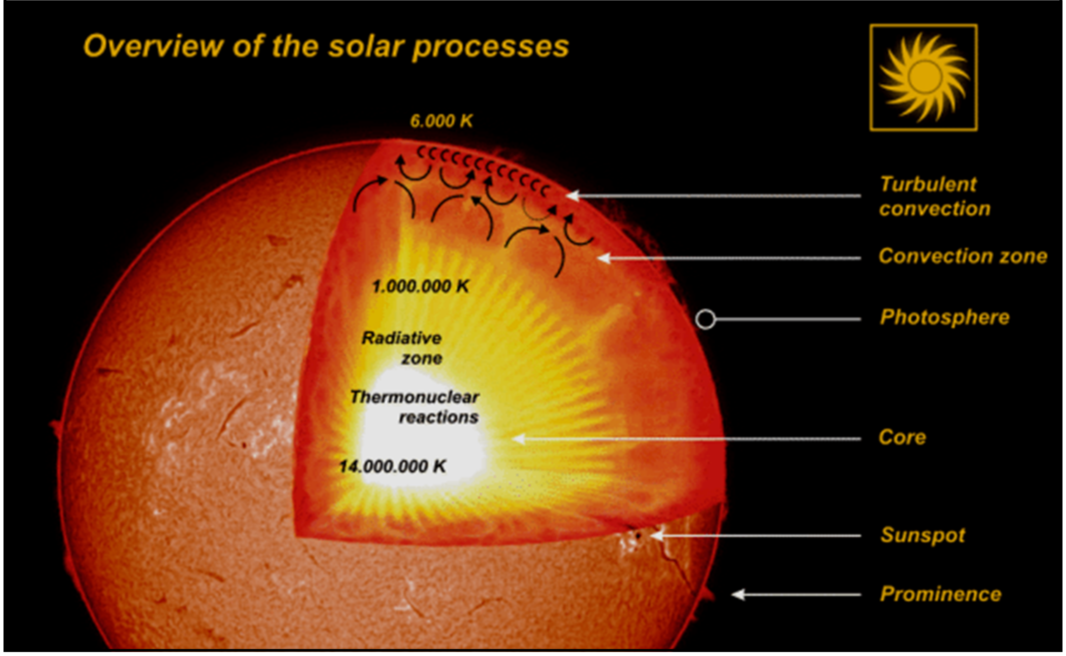
M_{stars}/M_{total} ~ 2%

Stellar Structure:



Hydrostatic Equilibrium

Stellar Structure



Stellar Structure Equations

Continuity equation:

conservation of mass
in shell (r,r+dr)

$$\frac{dr}{dm_r} = \frac{1}{4\pi\rho r^2}$$

Hydrostatic Equilibrium:

Pressure = Gravity

dP = pressure difference over
shell mass dm_r

$$\frac{dP}{dm_r} = -\frac{Gm_r}{4\pi r^2}$$

Energy conservation & generation:

Energy generated by shell dm_r:
- nuclear energy ε_n
- thermodynamic energy ε_g
- energy loss neutrinos ε_ν

$$\frac{dL}{dm_r} = \varepsilon_n + \varepsilon_g - \varepsilon_\nu$$

Energy transport

radiative & conductive energy
transport, shell opacity κ

$$\frac{dT}{dm_r} = \frac{3\kappa}{64\pi^2 ac}\frac{L_r}{r^4 T^3}$$

Stellar Structure