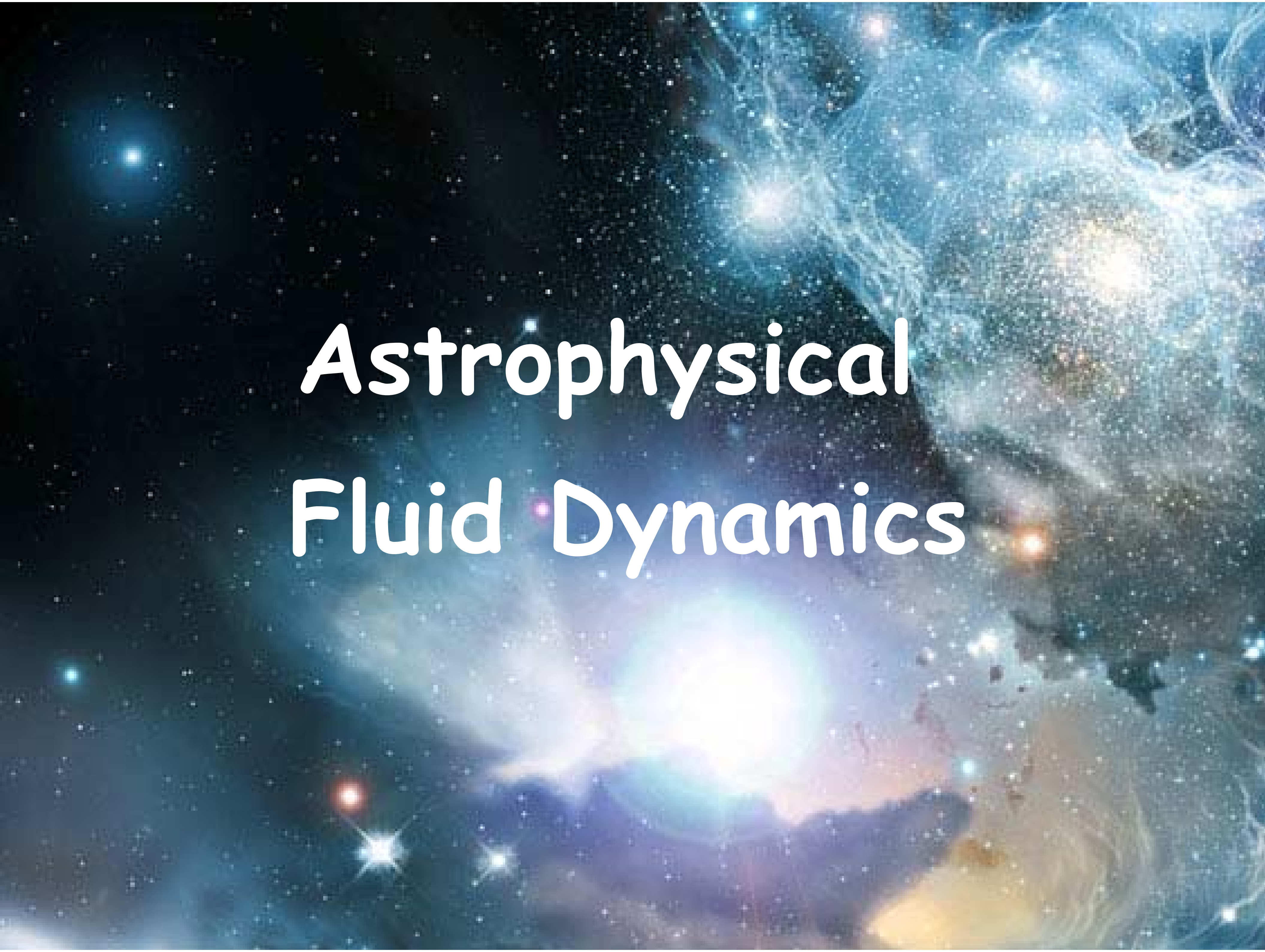
Astrophysical Fluid Dynamics



What is a Fluid ?

I. What is a fluid ?

I.1 The Fluid approximation:

The fluid is an idealized concept in which the matter is described as a continuous medium with certain macroscopic properties that vary as **continuous** function of position (e.g., density, pressure, velocity, entropy).

That is, one assumes that the scales l over which these quantities are defined is much larger than the mean free path λ of the individual particles that constitute the fluid,

$$l \gg \lambda; \qquad \lambda = \frac{1}{\sigma n}$$

Where n is the number density of particles in the fluid and σ is a typical interaction cross section.

I. What is a fluid ?

Furthermore, the concept of local fluid quantities is only useful if the scale *l* on which they are defined is much smaller than the typical macroscopic lengthscales *L* on which fluid properties vary. Thus to use the equations of fluid dynamics we require

$$L \gg l \gg \lambda$$

Astrophysical circumstances are often such that strictly speaking not all criteria are fulfilled.

I. What is a fluid ?

Astrophysical circumstances are often such that strictly speaking not all fluid criteria are fulfilled.

Mean free path astrophysical fluids (temperature T, density n):

$$\lambda \simeq 10^6 \left(T^2 / n\right) cm$$

1) Sun (centre): $T \simeq 10^7 K, \quad n \simeq 10^{24}\, cm^{-3} \quad \Rightarrow \quad \lambda \sim 10^{-4}\ cm$

$\lambda \ll R_\odot = 7 \times 10^{10}\ cm$ fluid approximation very good

2) Solar wind: $T \simeq 10^5 K, \quad n \simeq 10\ cm^{-3} \quad \Rightarrow \quad \lambda \sim 10^{15}\ cm$

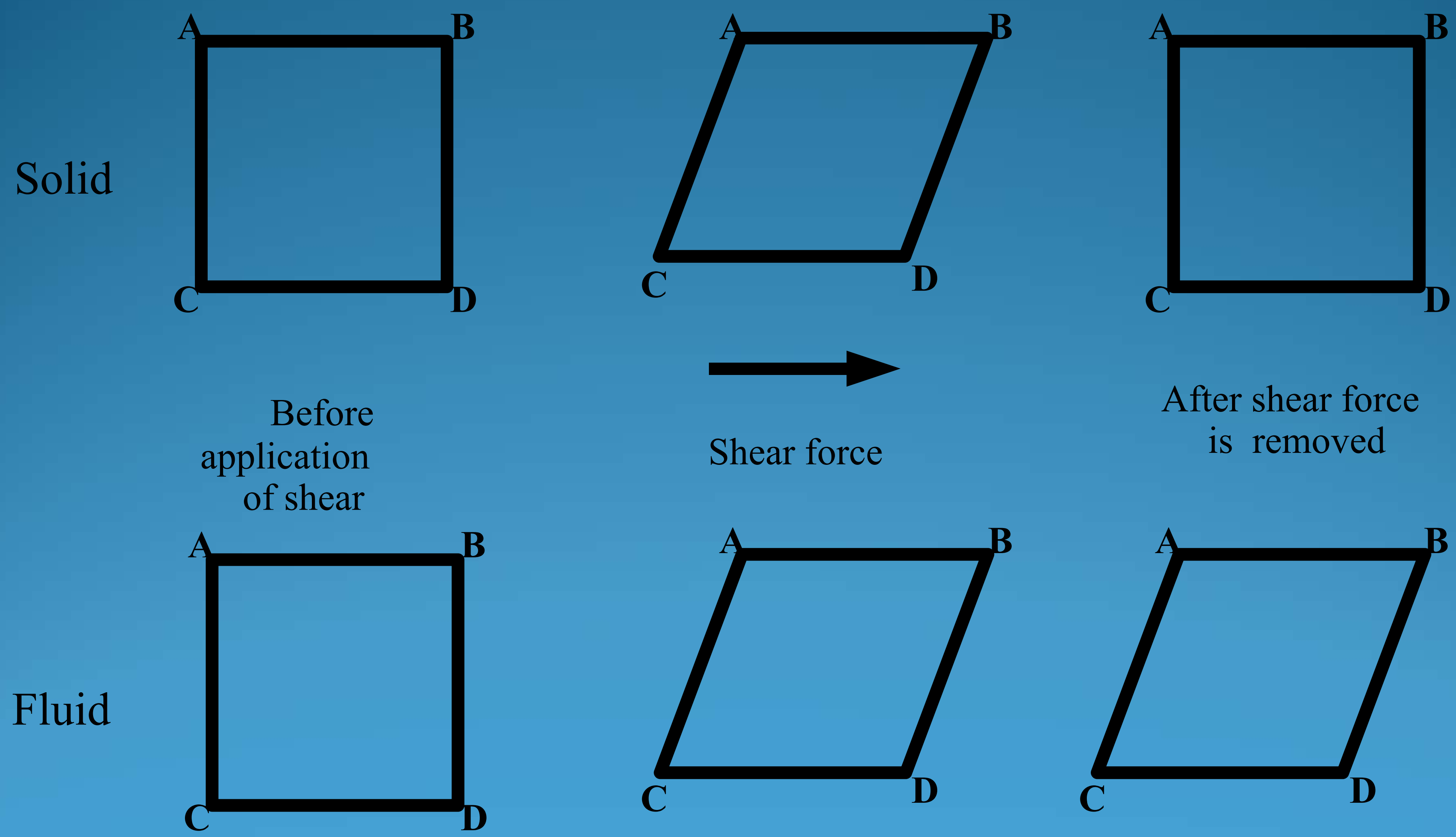
$\lambda \gg AU = 1.5 \times 10^{13}\ cm$ fluid approximation does not apply, plasma physics

3) Cluster: $T \simeq 3 \times 10^7 K, \quad n \simeq 10^{-3}\, cm^{-3} \quad \Rightarrow \quad \lambda \sim 10^{24}\ cm$

$\lambda \sim 1\ Mpc$ fluid approximation marginal

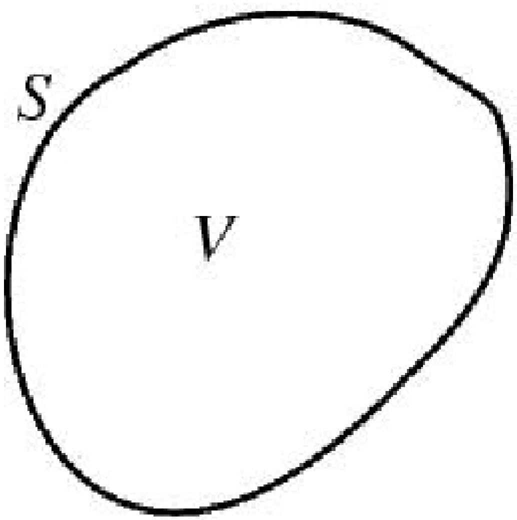
Solid vs. Fluid

By definition, a fluid cannot withstand any tendency for applied forces to deform it, (while volume remains unchanged). Such deformation may be resisted, but not prevented.



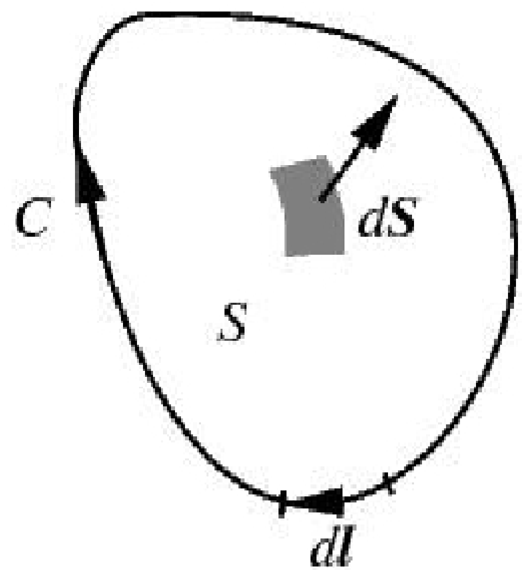
Mathematical Preliminaries

Mathematical preliminaries



$$\int_S \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F}\, dV$$

Gauss's Law



$$\int_C \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

Stoke's Theorem

Lagrangian vs. Eulerian View

There is a range of different ways in which we can follow the evolution of a fluid. The two most useful and best known ones are:

1) Eulerian view

Consider the system properties Q – density, flow velocity, temperature, pressure – at fixed locations. The temporal changes of these quantities is therefore followed by partial time derivative:

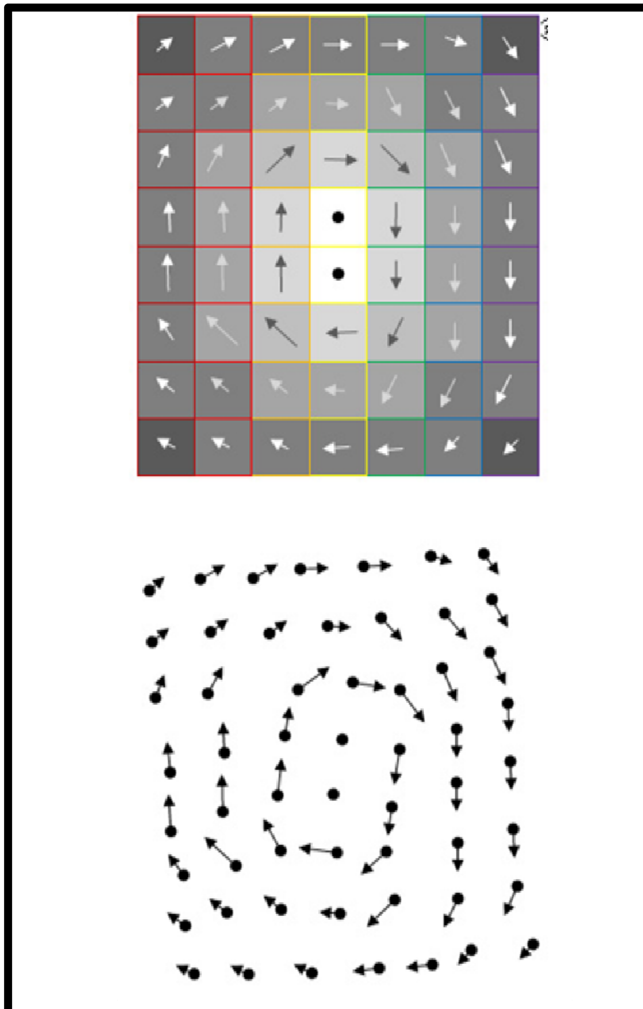
$$\frac{\partial Q}{\partial t}$$

2) Lagrangian view

Follow the changing system properties Q as you flow along with a fluid element. In a way, this "particle" approach is in the spirit of Newtonian dynamics, where you follow the body under the action of external force(s).

The temporal change of the quantities is followed by means of the "convective" or "Lagrangian" derivative

$$\frac{DQ}{Dt}$$



Lagrangian vs. Eulerian View

Consider the change of a fluid quantity $Q(\vec{r},t)$ at a location $\vec{r}$

1) Eulerian view:
   change in quantity Q in interval δt,
   at location $\vec{r}$:

$$\frac{\partial Q}{\partial t} = \frac{Q(\vec{r},t+\delta t) - Q(\vec{r},t)}{\delta t}$$

2) Lagrangian view:
   change in quantity Q in time interval δt,
   while fluid element moves from
   $\vec{r}$ to $\vec{r}+\delta\vec{r}$

$$\frac{DQ}{Dt} = \frac{Q(\vec{r}+\delta\vec{r},t+\delta t) - Q(\vec{r},t)}{\delta t}$$

$$= \frac{\partial Q}{\partial t} + \vec{v}\cdot\nabla Q$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$$

Convective/ Lagrangian Derivative

Einstein Summation Convention

In the practice of having to deal with equations involving a large number of vector and tensor quantities, we may quickly get overwhelmed by the large number of indices that we have to deal with.

Take for example the inproduct of two vectors $\vec{A}$ and $\vec{B}$, in 3-D space,

$$\vec{A}\cdot\vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \Sigma_{k=1}^{3} A_k B_k$$

The Einstein summation is a transparent means of writing this more succinctly,

$$\vec{A}\cdot\vec{B} = A_k B_k$$

By simply noting that in the case of an index occurring twice, it implies the summation over that index (k=1 to 3).

Another example that occurs many times in a fluid dynamical context is that of the divergence of a vector field $\vec{F}$,

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \frac{\partial F_k}{\partial x_k}$$

Basic

Fluid Equations

Conservation Equations

To describe a continuous fluid flow field, the first step is to evaluate the development of essential properties of the mean flow field . To this end we evaluate the first 3 moment of the phase space distribution function $f(\vec{r},\vec{v})$, corresponding to five quantities,

For a gas or fluid consisting of particles with mass m, these are

1) mass density
2) momentum density
3) (kinetic) energy density

$$\begin{pmatrix} \rho \\ \rho\vec{u} \\ \rho\varepsilon \end{pmatrix} = \int \begin{pmatrix} m \\ m\vec{v} \\ m\left|\vec{v}-\vec{u}\right|^2/2 \end{pmatrix} f\left(\vec{r},\vec{v},t\right) d\vec{v}$$

Note that we use $\vec{u}$ to denote the bulk velocity at location r, and $\vec{v}$ for the particle velocity. The velocity of a particle is therefore the sum of the bulk velocity and a "random" component $\vec{w}$,

$$\vec{v} = \vec{u} + \vec{w}$$

In principle, to follow the evolution of the (moment) quantities, we have to follow the evolution of the phase space density $f(\vec{r},\vec{v})$. The Boltzmann equation describes this evolution.

Boltzmann Equation

In principle, to follow the evolution of these (moment) quantities, we have to follow the evolution of the phase space density $f(\vec{r},\vec{v})$ This means we should solve the Boltzmann equation,

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f - \vec{\nabla}\Phi\cdot\vec{\nabla}_v f = \left(\frac{\delta f}{\delta t}\right)_c$$

The righthand collisional term is given by

$$\left(\frac{\delta f}{\delta t}\right)_c = \int \left|\vec{v}-\vec{v}_2\right| \sigma(\Omega)\left[f(\vec{v}')\, f(\vec{v}_2') - f(\vec{v})\, f(\vec{v}_2)\right] d\Omega\, d\vec{v}_2$$

in which

$$\sigma(\Omega) = \sigma\left(\vec{v}',\vec{v}_2' \,\middle|\, \vec{v},\vec{v}_2\right)$$

is the angle □-dependent elastic collision cross section.

On the lefthand side, we find the gravitational potential term, which according to the Poisson equation

$$\nabla^2\Phi = 4\pi G(\rho + \rho_{ext})$$

is generated by selfgravity as well as the external mass distribution $\rho_{ext}(\vec{x},t)$.

Boltzmann Equation

To follow the evolution of a fluid at a particular location x, we follow the evolution of a quantity χ(x,v) as described by the Boltzmann equation. To this end, we integrate over the full velocity range,

$$\int \left(\chi \frac{\partial f}{\partial t} + \chi v_k \frac{\partial f}{\partial x_k} - \chi \frac{\partial \Phi}{\partial x_k} \frac{\partial f}{\partial v_k} \right) d\vec{v} = \int \chi \left(\frac{\delta f}{\delta t} \right)_c d\vec{v}$$

If the quantity $\chi(\vec{x},\vec{v})$ is a conserved quantity in a collision, then the righthand side of the equation equals zero. For elastic collisions, these are mass, momentum and (kinetic) energy of a particle. Thus, for these quantities we have,

$$\int \chi \left(\frac{\delta f}{\delta t} \right)_c d\vec{v} = 0$$

The above result expresses mathematically the simple notion that collisions can not contribute to the time rate change of any quantity whose total is conserved in the collisional process.

For elastic collisions involving short-range forces in the nonrelativistic regime, there exist exactly five independent quantities which are conserved:

mass,	momentum	(kinetic) energy of a particle,
$\chi = m;$	$\chi = m v_i;$	$\chi = \frac{m}{2} \left| \vec{v}^2 \right|$

Boltzmann Moment Equations

When we define an average local quantity,

$$\langle Q \rangle = n^{-1} \int Q \, f \, d\vec{v}$$

for a quantity Q, then on the basis of the velocity integral of the Boltzmann equation, we get the following evolution equations for the conserved quantities ,

$$\frac{\partial}{\partial t}\left(n\langle \chi \rangle\right) + \frac{\partial}{\partial x_k}\left(n\langle v_k \chi \rangle\right) + n\frac{\partial \Phi}{\partial x_k}\left\langle \frac{\partial \chi}{\partial v_k} \right\rangle = 0$$

For the five quantities

$$\chi = m; \qquad \chi = m v_i; \qquad \chi = \frac{m}{2}\left|\vec{v}^2\right|$$

the resulting conservation equations are known as the

1) mass density — continuity equation
2) momentum density — Euler equation
3) energy density — energy equation

In the sequel we follow – for reasons of insight – a slightly more heuristic path towards inferring the continuity equation and the Euler equation.

Continuity equation

To infer the continuity equation, we consider the conservation of mass contained in a volume V which is fixed in space and enclosed by a surface S.

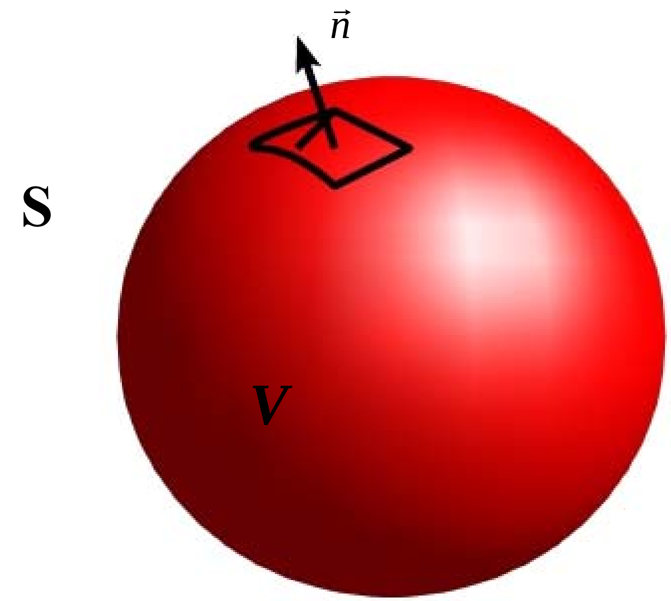
The mass M is

$$M = \int_V \rho \, dV$$

The change of mass M in the volume V is equal to the flux of mass through the surface S,

$$\frac{d}{dt}\int_V \rho \, dV = -\oint_S \rho \vec{u} \cdot \vec{n} \, dS$$

Where $\vec{n}$ is the outward pointing normal vector.



LHS:

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} dV$$

RHS,
using the divergence theorem (Green's formula):

$$\oint_S \rho \vec{u} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$

Continuity equation

Since this holds for every volume, this relation is equivalent to

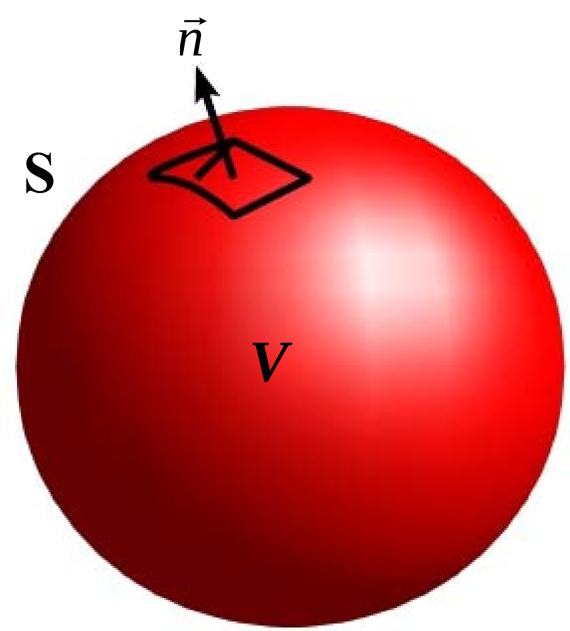
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{u} \right) = 0 \quad (I.1)$$

The continuity equation expresses

- mass conservation

AND

- fluid flow occurring in a continuous fashion !!!!!



One can also define the mass flux density as

$$\vec{j} = \rho \vec{u}$$

which shows that eqn. I.1 is actually a continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad (I.2)$$

Continuity Equation & Compressibility

From the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

we find directly that ,

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{u} = 0$$

Of course, the first two terms define the Lagrangian derivative, so that for a moving fluid element we find that its density changes according to

$$\boxed{\frac{1}{\rho} \frac{D\rho}{Dt} = -\vec{\nabla} \cdot \vec{u}}$$

In other words, the density of the fluid element changes as the divergence of the velocity flow. If the density of the fluid cannot change, we call it an *incompressible fluid* , for which $\vec{\nabla} \cdot \vec{u} = 0$

Momentum Conservation

When considering the fluid momentum, $\chi = mv_i$, via the Boltzmann moment equation,

$$\frac{\partial}{\partial t}\left(n\langle\chi\rangle\right)+\frac{\partial}{\partial x_k}\left(n\langle v_k\chi\rangle\right)+n\frac{\partial\Phi}{\partial x_k}\left\langle\frac{\partial\chi}{\partial v_k}\right\rangle=0$$

we obtain the equation of momentum conservation,

$$\frac{\partial}{\partial t}\left(\rho\langle v_i\rangle\right)+\frac{\partial}{\partial x_k}\left(\rho\langle v_i v_k\rangle\right)+\rho\frac{\partial\Phi}{\partial x_i}=0$$

Decomposing the velocity v_i into the bulk velocity u_i and the random component w_i, we have

$$\langle v_i v_k\rangle = u_i u_k + \langle w_i w_k\rangle$$

By separating out the trace of the symmetric dyadic $w_i w_k$, we write

$$\rho\langle w_i w_k\rangle = p\delta_{ik} - \pi_{ik}$$

Momentum Conservation

By separating out the trace of the symmetric dyadic $w_i w_k$, we write

$$\rho \langle w_i w_k \rangle = p\delta_{ik} - \pi_{ik}$$

where

P is the "*gas pressure*"

$$p \equiv \frac{1}{3}\rho \left\langle \left|\vec{w}\right|^2 \right\rangle$$

π_{ik} is the "*viscous stress tensor*"

$$\pi_{ik} \equiv \rho \left\langle \frac{1}{3}\left|\vec{w}\right|^2 \delta_{ik} - w_i w_k \right\rangle$$

we obtain the *momentum equation*, in its conservation form,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k + p\delta_{ik} - \pi_{ik}) = -\rho \frac{\partial \Phi}{\partial x_i}$$

Momentum Conservation

Momentum Equation

$$\frac{\partial}{\partial t}\left(\rho u_i\right)+\frac{\partial}{\partial x_k}\left(\rho u_i u_k + p\delta_{ik} - \pi_{ik}\right) = -\rho\frac{\partial \Phi}{\partial x_i}$$

Describes the change of the momentum density ρu_i in the i-direction:

The flux of the i-th component of momentum in the k-th direction consists of the sum of

1) a mean part: $\rho u_i u_k$
2) random part I, isotropic pressure part: $p\delta_{ik}$
3) random part II, nonisotropic viscous part: $-\pi_{ik}$

Force Equation

Momentum Equation

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_k}(\rho u_i u_k + p\delta_{ik} - \pi_{ik}) = -\rho\frac{\partial \Phi}{\partial x_i}$$

By invoking the continuity equation, we may also manipulate the momentum equation so that it becomes the *force equation*

$$\rho\frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Phi - \vec{\nabla}p + \vec{\nabla}\cdot\overleftrightarrow{\pi}$$

Viscous Stress

A note on the viscous stress term π_{ik} :

For Newtonian fluids:

Hooke's Law
states that the viscous stress π_{ik} is linearly proportional to the rate of strain $\partial u_i / \partial x_k$,

$$\pi_{ik} = 2\mu\Sigma_{ik} + \beta\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{ik}$$

where Σ_{ik} is the shear deformation tensor,

$$\Sigma_{ik} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right\} - \frac{1}{3}\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{ik}$$

The parameters ⍰ and ⍰ are called the *shear* and *bulk* coefficients of viscosity.

Euler equation

In the absence of viscous terms, we may easily derive the equation for the conservation of momentum on the basis of macroscopic considerations. This yields the *Euler equation.*

As in the case for mass conservation, consider an arbitrary volume V, fixed in space, and bounded by a surface S, with an outward normal $\vec{n}$.

Inside V, the total momentum for a fluid with density ρ and flow velocity $\vec{u}$ is

$$\int_V \rho \, \vec{u} \; dV$$

The momentum inside V changes as a result of three factors:

1) External (volume) force,
   a well known example is the gravitational force when V embedded in gravity field.

2) The pressure (surface) force over de surface ***S*** of the volume.
   (at this stage we'll ignore other stress tensor terms that can either be caused by viscosity, electromagnetic stress tensor, etc.):

3) The net transport of momentum by in- and outflow of fluid into and out of V

Euler equation

1) External (volume) forces,:

$$\int_V \rho \vec{f} \, dV$$

where $\vec{f}$ is the force per unit mass, known as the body force. An example is the gravitational force when the volume V is embeddded in a gravitational field.

2) The pressure (surface) force is the integral of the pressure (force per unit area) over the surface S

$$-\oint_S p \vec{n} \, dS$$

3) The momentum transport over the surface area can be inferred by considering at each surface point the slanted cylinder of fluid swept out by the area element δS in time δt, where δS starts on the surface S and moves with the fluid, ie. with velocity $\vec{u}$. The momentum transported through the slanted cylinder is

$$\delta(\rho \vec{u}) = -\rho \vec{u} (\vec{u} \cdot \vec{n}) \delta t \, \delta S$$

so that the total transported momentum through the surface S is:

$$\delta(\rho \vec{u}) = -\oint_S \rho \vec{u} (\vec{u} \cdot \vec{n}) \, dS$$

Euler equation

Taking into account all three factors, the total rate of change of momentum is given by

$$\frac{d}{dt}\int_V \rho \vec{u}\, dV = \int_V \rho\, \vec{f}\, dV - \oint_S p \vec{n}\, dS - \oint_S \rho \vec{u} (\vec{u} \cdot \vec{n}) dS$$

The most convenient way to evaluate this integral is by restricting oneself to the i-component of the velocity field,

$$\frac{d}{dt}\int_V \rho u_i\, dV = \int_V \rho\, f_i\, dV - \oint_S p n_i\, dS - \oint_S \rho u_i u_j n_j\, dS$$

Note that we use the Einstein summation convention for repeated indices.

Volume V is fixed, so that

$$\frac{d}{dt}\int_V \quad = \int_V \frac{\partial}{\partial t}$$

Furthermore, V is arbitrary. Hence,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho\, f_i$$

Euler equation

Reordering some terms of the lefthand side of the last equation,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho f_i$$

leads to the following equation:

$$\rho \left\{ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right\} + u_i \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right\} = -\frac{\partial p}{\partial x_i} + \rho f_i$$

From the *continuity equation*, we know that the second term on the LHS is zero. Subsequently, returning to vector notation, we find the usual exprssion for the *Euler equation*,

Returning to vector notation, and using the we find the usual expression for the Euler equation:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right) = -\vec{\nabla} p + \rho \vec{f} \qquad (I.4)$$

Euler equation

An slightly alternative expression for the Euler equation is

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\frac{\vec{\nabla} p}{\rho} + \vec{f} \qquad (I.5)$$

In this discussion we ignored energy dissipation processes which may occur as a result of internal friction within the medium and heat exchange between its parts (conduction). This type of fluids are called **ideal** fluids.

Gravity:

For gravity the force per unit mass is given by $\vec{f} = -\vec{\nabla}\phi$ where the Poisson equation relates the gravitational potential ϕ to the density ρ:

$$\vec{\nabla}^2 \phi = 4\pi G \rho$$

Euler equation

From eqn. (I.4)

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right) = -\vec{\nabla} p + \rho\,\vec{f} \qquad (I.4)$$

we see that the LHS involves the Lagrangian derivative, so that the *Euler equation* can be written as

$$\rho\,\frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \rho\,\vec{f} \qquad (I.6)$$

In this form it can be recognized as a statement of Newton's 2nd law for an inviscid (frictionless) fluid. It says that, for an infinitesimal volume of fluid,

mass times acceleration = total force on the same volume,

namely force due to pressure gradient plus whatever body forces are being exerted.

Energy Conservation

In terms of bulk velocity $\vec{u}$ and random velocity $\vec{w}$ the (kinetic) energy of a particle is,

$$\chi = \frac{m}{2}\vec{v}^2 = \frac{m}{2}\left|(\vec{u}+\vec{w})^2\right| = \frac{m\vec{u}^2}{2} + m\vec{w}\cdot\vec{u} + \frac{m\vec{w}^2}{2}$$

The Boltzmann moment equation for energy conservation

$$\frac{\partial}{\partial t}\left(n\langle\chi\rangle\right) + \frac{\partial}{\partial x_k}\left(n\langle v_k\chi\rangle\right) + n\frac{\partial\Phi}{\partial x_k}\left\langle\frac{\partial\chi}{\partial v_k}\right\rangle = 0$$

becomes

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}\left(\left|\vec{u}^2\right| + \left\langle\left|\vec{w}\right|^2\right\rangle\right)\right] + \frac{\partial}{\partial x_k}\left[\frac{\rho}{2}\left\langle(u_k+w_k)(u_i+w_i)^2\right\rangle\right] + \rho\frac{\partial\Phi}{\partial x_k}u_k = 0$$

Expanding the term inside the spatial divergence, we get

$$\left\langle(u_k+w_k)(u_i+w_i)^2\right\rangle = \left|\vec{u}\right|^2 u_k + 2u_i\left\langle w_i w_k\right\rangle + u_k\left\langle\left|\vec{w}\right|^2\right\rangle + \left\langle w_k\left|\vec{w}\right|^2\right\rangle$$

Energy Conservation

Defining the following energy-related quantities:

1) specific internal energy: $$\rho\varepsilon = \rho\left\langle \frac{1}{2}\left|\vec{w}\right|^2 \right\rangle = \frac{3}{2}P$$

2) "gas pressure" $$P = \frac{1}{3}\rho\left\langle \left|\vec{w}\right|^2 \right\rangle$$

3) conduction heat flux $$F_k = \rho\left\langle w_k \frac{1}{2}\left|\vec{w}\right|^2 \right\rangle$$

4) viscous stress tensor $$\pi_{ik} = \rho\left\langle \frac{1}{3}\left|\vec{w}\right|^2 \delta_{ik} - w_i w_k \right\rangle$$

Energy Conservation

The total energy equation for energy conservation in its conservation form is

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\vec{u}|^2 + \rho\varepsilon\right) + \frac{\partial}{\partial x_k}\left[\frac{\rho}{2}|\vec{u}|^2 u_k + u_i\left(P\delta_{ik} - \pi_{ik}\right) + \rho\varepsilon u_k + F_k\right] = -\rho u_k \frac{\partial\Phi}{\partial x_k}$$

This equation states that the total fluid energy density is the sum of a part due to bulk motion $\vec{u}$ and a part due to random motions $\vec{w}$.

The flux of fluid energy in the k-th direction consists of

1) the translation of the bulk kinetic energy at the k-th component of the mean velocity,

$$\left(\rho|\vec{u}|^2 / 2\right)u_k$$

2) plus the enthalpy – sum of internal energy and pressure – flux,

$$\left(\rho\varepsilon + P\right)u_k$$

3) plus the viscous contribution $-u_i\pi_{ik}$

4) plus the conductive flux F_k

Work Equation

Internal Energy Equation

For several purposes it is convenient to express energy conservation in a form that involves only the *internal energy* and a form that only involves the global *PdV work.*

The *work equation* follows from the full energy equation by using the Euler equation, by multiplying it by u_i and using the continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\vec{u}|^2\right) + \frac{\partial}{\partial x_k}\left(\frac{\rho}{2}|\vec{u}|^2 u_k\right) = -\rho u_i \frac{\partial \Phi}{\partial x_i} - u_i \frac{\partial P}{\partial x_i} + u_i \frac{\partial \pi_{ik}}{\partial x_k}$$

Subtracting the *work equation* from the full energy equation, yields the *internal energy equation* for the internal energy ε

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_k}(\rho\varepsilon u_k) = -P\frac{\partial u_k}{\partial x_k} - \frac{\partial F_k}{\partial x_k} + \Psi$$

where Ψ is the *rate of viscous dissipation* evoked by the viscosity stress π_{ik}

$$\Psi = \pi_{ik}\frac{\partial u_i}{\partial x_k}$$

Internal energy equation

If we use the continuity equation, we may also write the internal energy equation in the form of the first law of thermodynamics,

$$\rho \frac{D\varepsilon}{Dt} = -P\vec{\nabla}\cdot\vec{u} - \vec{\nabla}\cdot\vec{F}_{cond} + \Psi$$

in which we recognize

$$-P\vec{\nabla}\cdot\vec{u} = -P\left[\rho^{-1}\frac{D\rho}{Dt}\right]$$

as the rate of doing PdV work, and

$$-\vec{\nabla}\cdot\vec{F}_{cond} + \Psi$$

as the time rate of adding heat (through heat conduction and the viscous conversion of ordered energy in differential fluid motions to disordered energy in random particle motions).

Energy Equation

On the basis of the kinetic equation for energy conservation

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\vec{u}|^2+\rho\varepsilon\right)+\frac{\partial}{\partial x_k}\left[\frac{\rho}{2}|\vec{u}|^2 u_k+u_i\left(P\delta_{ik}-\pi_{ik}\right)+\rho\varepsilon u_k+F_k\right]=\rho u_k g_k$$

we may understand that the time rate of the change of the total fluid energy in a volume V (with surface area A), i.e. the kinetic energy of fluid motion plus internal energy, should equal the sum of

1) minus the surface integral of the energy flux (kinetic + internal)
2) plus surface integral of doing work by the internal stresses P_{ik}
3) volume integral of the rate of doing work by local body forces (e.g. gravitational)
4) minus the heat loss by conduction across the surface A
5) plus volumetric gain minus volumetric losses of energy due to local sources and sinks (e.g. radiation)

Energy Equation

The total expression for the time rate of total fluid energy is therefore

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho|\vec{u}|^2 + \rho\varepsilon\right)dV = -\oint_A \left[\left(\frac{1}{2}\rho|\vec{u}|^2 + \rho\varepsilon\right)\vec{u}\right]\cdot\hat{n}\,dA +$$
$$+\oint_A u_i P_{ik} n_k\, dA + \int_V \rho\vec{u}\cdot\vec{g}\,dV -$$
$$-\oint_A \vec{F}_{cond}\cdot\hat{n}\,dA + \int_V (\Gamma - \Lambda)\,dV$$

- P_{ik} is the force per unit area exerted by the outside on the inside in the i^{th} direction across a face whose normal is oriented in the k^{th} direction.
  For a dilute gas this is

$$P_{ik} = -\rho\langle w_i w_k\rangle = -p\delta_{ik} + \pi_{ik}$$

- Γ is the energy gain per volume, as a result of energy generating processes.
- Λ is the energy loss per volume due to local sinks (such as e.g. radiation)

Energy Equation

By applying the divergence theorem, we obtain the total energy equation:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}\left|\vec{u}\right|^2+\varepsilon\right)\right]+\frac{\partial}{\partial x_k}\left[\rho\left(\frac{1}{2}\left|\vec{u}\right|^2+\varepsilon\right)-u_i P_{ik}+F_k\right]=\rho\vec{g}\cdot\vec{u}+\Gamma-\Lambda$$

Heat Equation

Implicit to the fluid formulation, is the concept of *local thermal equilibrium*. This allows us to identify the trace of the stress tensor P_{ik} with the thermodynamic pressure p,

$$P_{ik} = -p\delta_{ik} + \pi_{ik}$$

Such that it is related to the internal energy per unit mass of the fluid, ε, and the specific entropy s, by the fundamental law of thermodynamics

$$d\varepsilon = Tds - pdV = Tds - pd\left(\rho^{-1}\right)$$

Applying this thermodynamic equation and subtracting the work equation, we obtain the *Heat Equation*,

$$\rho T \frac{Ds}{Dt} = -\vec{\nabla} \cdot \vec{F}_{cond} + \Psi + \Gamma - \Lambda$$

where ￼ equals the rate of viscous dissipation, $\Psi = \pi_{ik} \frac{\partial u_i}{\partial x_k}$

Fluid Flow Visualization

Flow Visualization: Streamlines, Pathlines & Streaklines

Fluid flow is characterized by a velocity vector field in 3-D space.
There are various distinct types of curves/lines commonly used when visualizing fluid motion: *streamlines*, *pathlines* and *streaklines*.

These only differ when the flow changes in time, ie. when the flow is not *steady*! If the flow is not steady, streamlines and streaklines will change.

1) Streamlines

Family of curves that are instantaneously tangent to the velocity vector $\vec{u}$. They show the direction a fluid element will travel at any point in time.

If we parameterize one particular streamline $\vec{l}_S(s)$, with $\vec{l}_S(s=0)=\vec{x}_0$, then streamlines are defined as

$$\frac{d\vec{l}_S}{ds}\times\vec{u}(\vec{l}_S) = 0$$

Flow Visualization: Streamlines

Definition Streamlines:

$$\frac{d\vec{l}_S}{ds} \times \vec{u}(\vec{l}_S) = 0$$

If the components of the streamline can be written as

$$\vec{l}_S = (x, y, z)$$

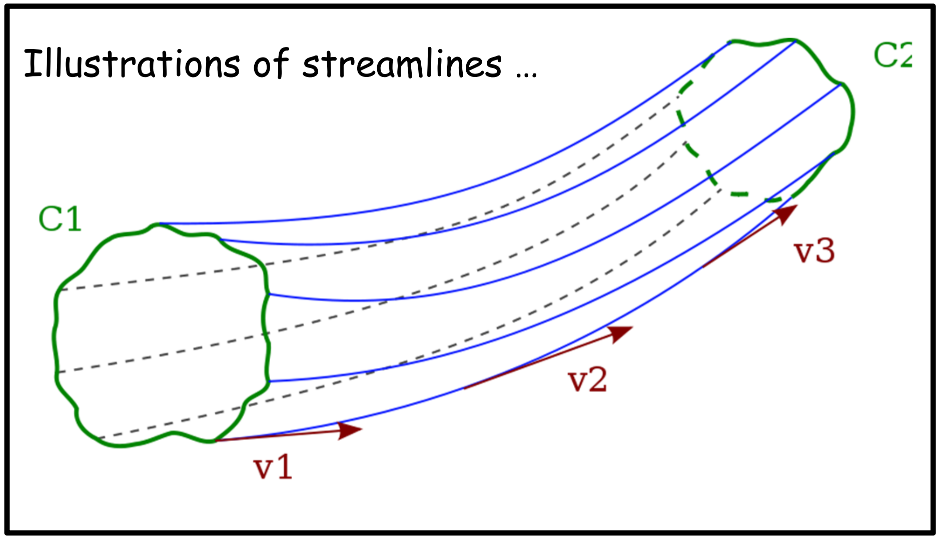
and

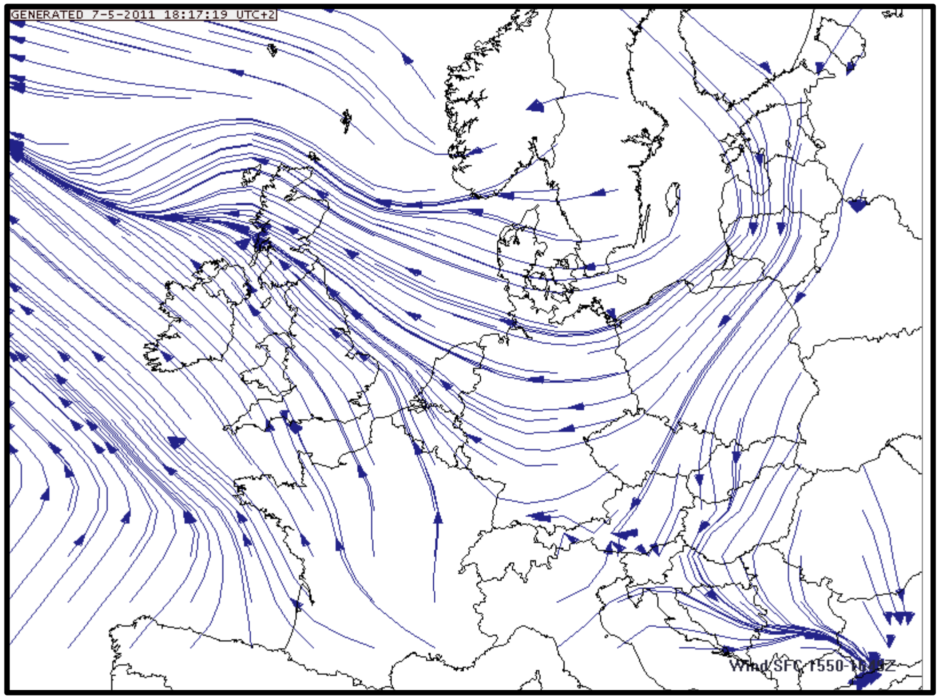
$$d\vec{l} = (d\vec{x}, d\vec{y}, d\vec{z})$$

$$\vec{u} = (u_x, u_y, u_z)$$

then

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$





Flow Visualization: Pathlines

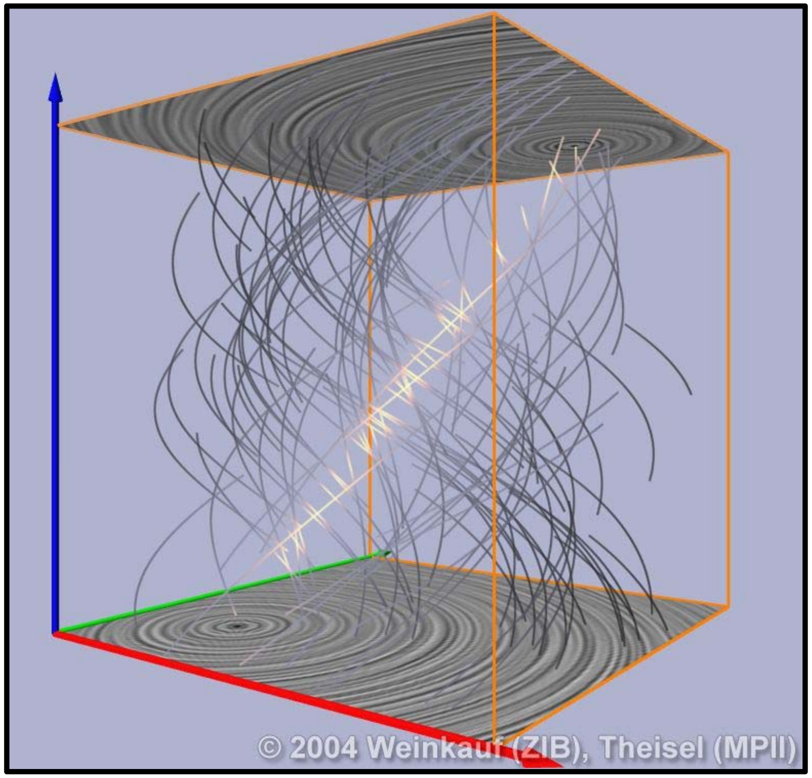
2) Pathlines

Pathlines are the trajectories that individual fluid particles follow. These can be thought of as a "recording" of the path a fluid element in the flow takes over a certain period.

The direction the path takes will be determined by the streamlines of the fluid at each moment in time.

Pathlines $\vec{l}_P(t)$ are defined by

$$\begin{cases} \dfrac{d\vec{l}_P}{dt} = \vec{u}(\vec{l}_P, t) \\ \vec{l}_P(t_0) = \vec{x}_{P0} \end{cases}$$



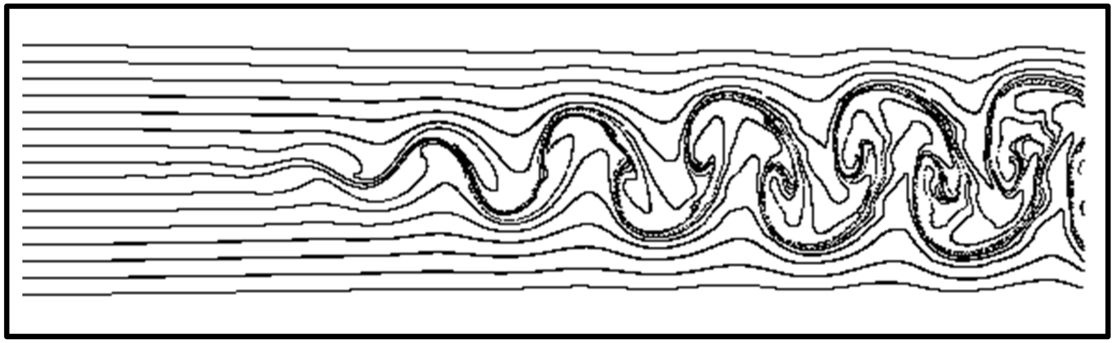
where the suffix P indicates we are following the path of particle P. Note that at location the curve is parallel to velocity vector $\vec{l}_P$, where the velocity vector $\vec{u}$ is evaluated at location $\vec{l}_P$ at time t.

Flow Visualization: Streaklines

3) Streaklines

Streaklines are are the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past.

Dye steadily injected into the fluid at a fixed point extends along a streakline. In other words, it is like the plume from a chimney.



Streaklines $\vec{l}_T$ can be expressed as

$$\begin{cases} \dfrac{d\vec{l}_T}{dt} = \vec{u}(\vec{l}_T, t) \\ \vec{l}_T(\tau_T) = \vec{x}_{T0} \end{cases}$$

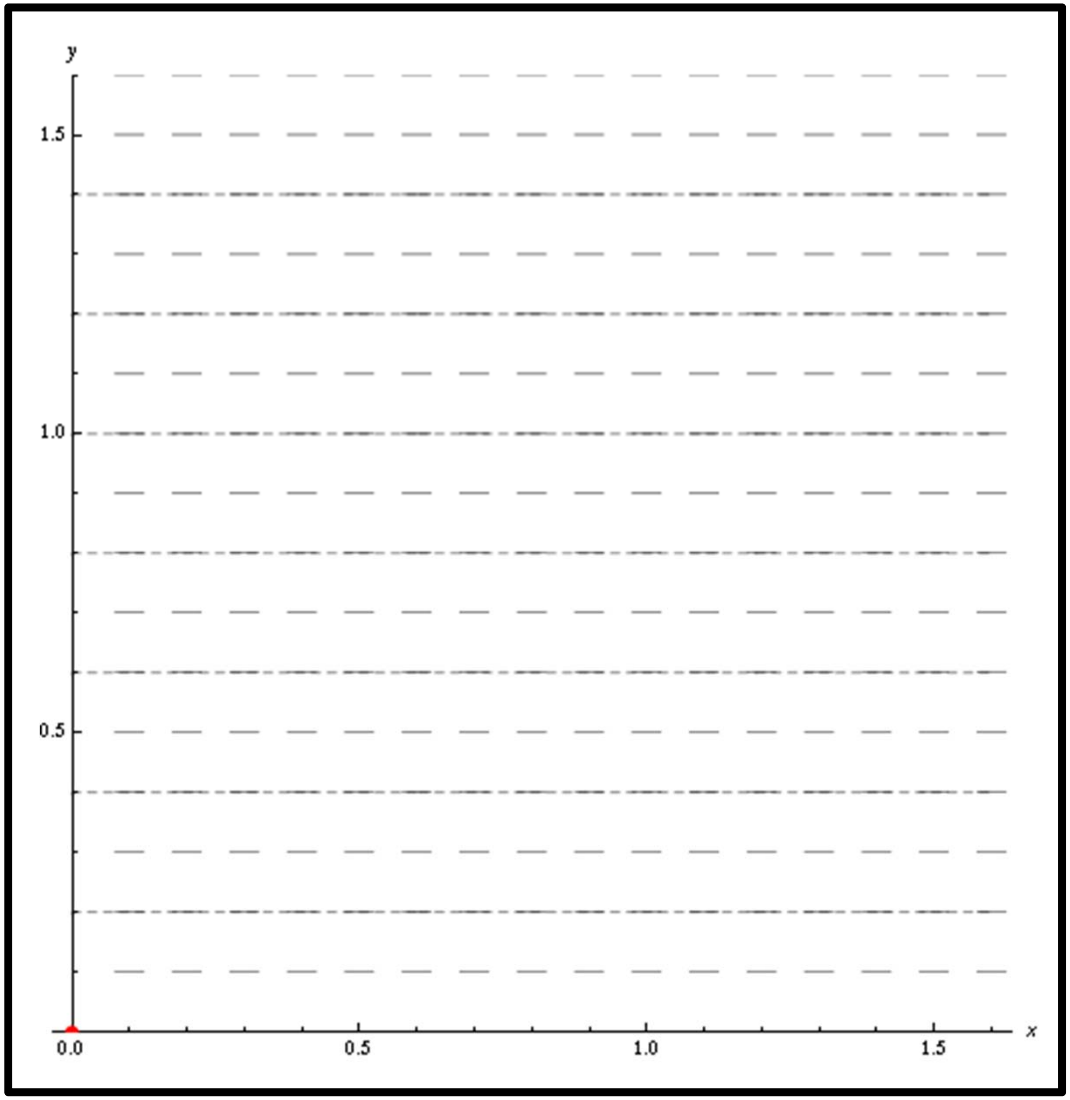
where $\vec{u}(\vec{l}_T, t)$ is the velocity at location $\vec{l}_T$ at time t. The parameter τ_T parameterizes the streakline $\vec{l}_T(t, \tau_T)$ and $0 \le \tau_T \le t_0$ with t_0 time of interest.

Flow Visualization: Streamlines, Pathlines, Streaklines

The following example illustrates the different concepts of streamlines, pathlines and streaklines:

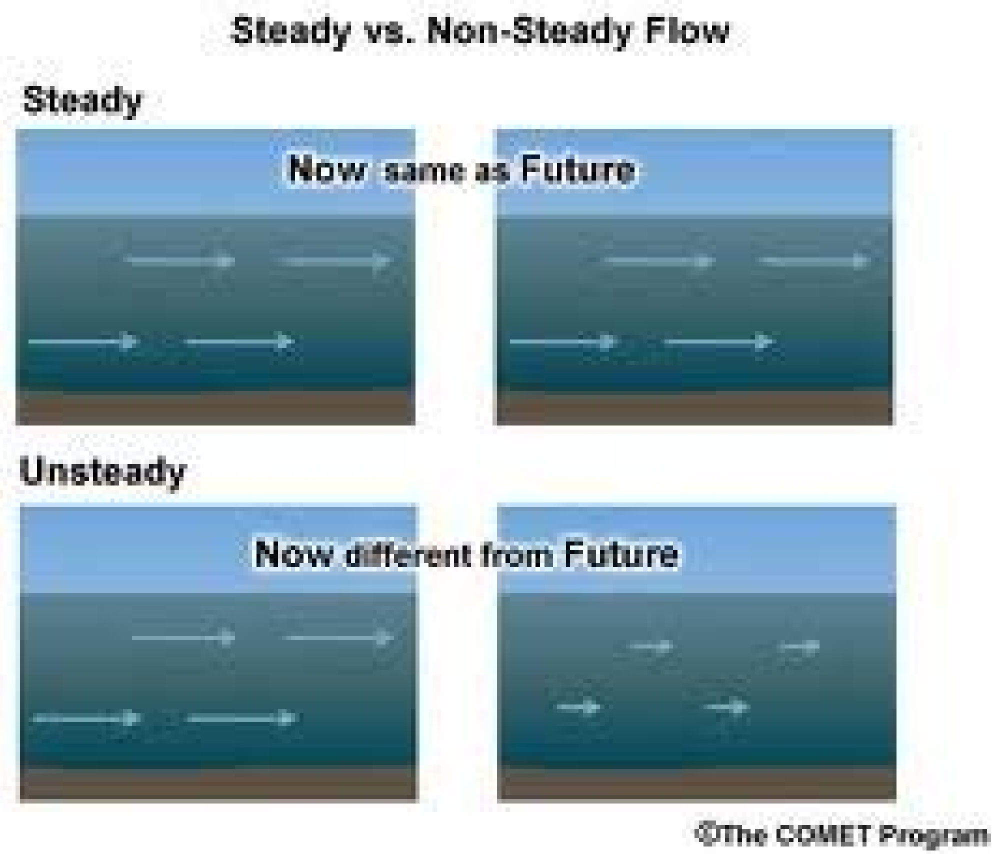
- red: pathline
- blue: streakline

- short-dashed: evolving streamlines



Steady flow

Steady flow is a flow in which the velocity, density and the other fields do not depend explicitly on time, namely $\partial / \partial t = 0$



In steady flow streamlines and streaklines do not vary with time and coincide with the pathlines.

Kinematics of Fluid Flow

Stokes' Flow Theorem

Stokes' flow theorem:

The most general differential motion of a fluid element corresponds to a

1) uniform translation
2) uniform expansion/contraction — divergence term
3) uniform rotation — vorticity term
4) distortion (without change volume) — shear term

The fluid velocity $\vec{u}\,(Q\,)$ at a point Q displaced by a small amount $\vec{R}$ from a point P will differ by a small amount, and includes the components listed above:

$$\vec{u}(Q) = \vec{u}(P) + H\vec{R} + \vec{\nabla}S + \vec{\Omega}\times\vec{R}$$

- $\vec{u}(P)$: uniform translation
- $H\vec{R}$: Divergence uniform expansion/contraction
- $\vec{\nabla}S$: Shear term distortion
- $\vec{\Omega}\times\vec{R}$: Vorticity uniform rotation

Stokes' Flow Theorem

Stokes' flow theorem:

the terms of the relative motion wrt. point P are:

2) Divergence term:

uniform expansion/contraction

$$H = \frac{1}{3} \vec{\nabla} \cdot \vec{u}$$

3) Shear term:

uniform distortion

S : shear deformation scalar

Σ_{ik} : shear tensor

$$S = \frac{1}{2} \Sigma_{ik} R_i R_k$$

$$\Sigma_{ik} = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right\} - \frac{1}{3} \left(\vec{\nabla} \cdot \vec{u} \right) \delta_{ik}$$

4) Vorticity Term:

uniform rotation

$$\Omega = \frac{1}{2} \vec{\nabla} \times \vec{u} = \frac{1}{2} \vec{\omega}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

Stokes' Flow Theorem

Stokes' flow theorem:

One may easily understand the components of the fluid flow around a point P by a simple Taylor expansion of the velocity field $\vec{u}(\vec{x})$ around the point P:

$$\delta u_i = u_i(\vec{x}+\vec{R},t) - u_i(\vec{x},t) = \frac{\partial u_i}{\partial x_k} R_k$$

Subsequently, it is insightful to write the rate-of-strain tensor $\partial u_i / \partial x_k$ in terms of its symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_k} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$$

The symmetric part of this tensor is the deformation tensor, and it is convenient -and insightful – to write it in terms of a diagonal trace part and the traceless shear tensor Σ_{ik},

$$\frac{\partial u_i}{\partial x_k} = \frac{1}{3}(\vec{\nabla}\cdot\vec{u})\delta_{ik} + \Sigma_{ik} + \omega_{ik}$$

Stokes' Flow Theorem

where

1) the symmetric (and traceless) shear tensor Σ_{ik} is defined as

$$\Sigma_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{ik}$$

2) the antisymmetric tensor ω_{ik} as

$$\omega_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$$

3) the trace of the rate-of-strain tensor is proportional to the velocity divergence term,

$$\frac{1}{3}\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{ik} = \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\delta_{ik}$$

Stokes' Flow Theorem

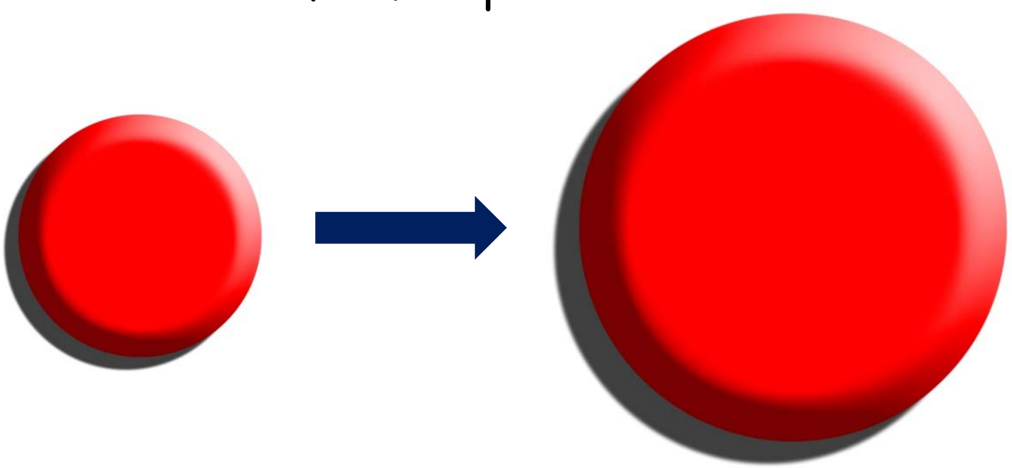
Divergence Term

$$\frac{1}{3}\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{ik} = \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\delta_{ik}$$

We know from the Lagrangian continuity equation,

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla}\cdot\vec{u}$$

that the term represents the uniform expansion or contraction of the fluid element.



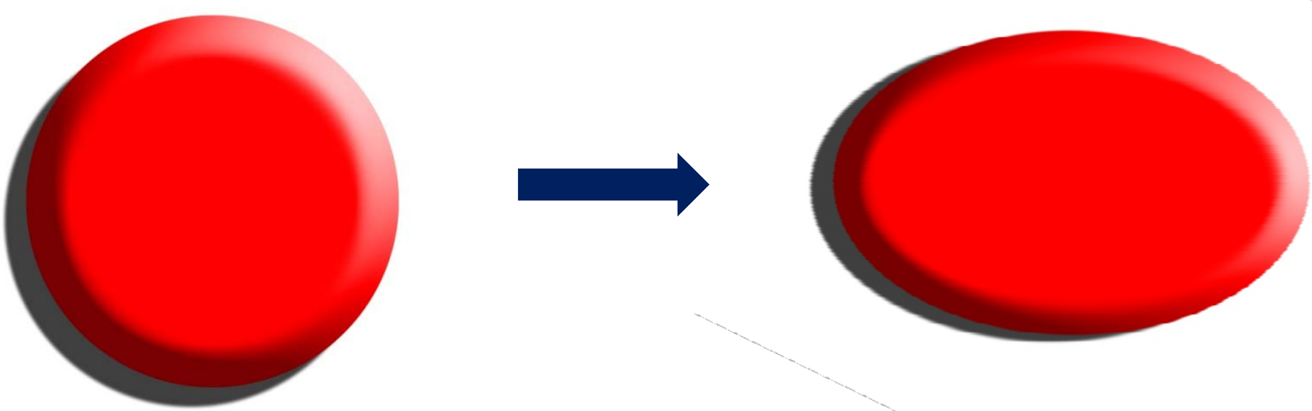
Stokes' Flow Theorem

Shear Term

The traceless symmetric shear term,

$$\Sigma_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}(\vec{\nabla} \cdot \vec{u})\delta_{ik}$$

represents the anisotropic deformation of the fluid element. As it concerns a traceless deformation, it preserves the volume of the fluid element (the volume-changing deformation is represented via the divergence term).



(intention of illustration is that the volume of the sphere and the ellipsoid to be equal)

Stokes' Flow Theorem

Shear Term

Note that we can associate a quadratic form – ie. an ellipsoid – with the shear tensor, the shear deformation scalar S,

$$S = \frac{1}{2}\Sigma_{ik} R_i R_k$$

such that the corresponding shear velocity contribution is given by

$$\delta u_{\Sigma,i} = \frac{\partial S}{\partial R_i} = \Sigma_{ik} R_k$$

We may also define a related quadratic form by incorporating the divergence term,

$$\Phi_v = \frac{1}{2} D_{mk} R_m R_k = \frac{1}{2}\left\{\Sigma_{mk} + \frac{1}{3}\left(\vec{\nabla}\cdot\vec{u}\right)\delta_{mk}\right\} R_m R_k$$

$$\frac{\partial \Phi_v}{\partial R_i} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right\} R_k$$

Evidently, this represents the *irrotational* part of the velocity field. For this reason, we call Φ_v the *velocity potential:*

$$\vec{u} = \vec{\nabla}\Phi_v \qquad \Rightarrow \qquad \vec{\nabla}\times\vec{u} = 0$$

Stokes' Flow Theorem

Vorticity Term

The antisymmetric term,

$$\omega_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$$

represents the rotational component of the fluid element's motion, the *vorticity*.

With the antisymmetric ω_{ik} we can associate a (pseudo)vector, the vorticity vector

$$\boxed{\vec{\omega} = \vec{\nabla} \times \vec{u}}$$

where the coordinates of the vorticity vector, $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, are related to the vorticity tensor via

$$\boxed{\omega_m = \varepsilon_{mik}\frac{\partial u_k}{\partial x_i} \quad \Leftrightarrow \quad 2\,\omega_{ik} = \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} = \varepsilon_{kim}\,\omega_m}$$

where ε_{kim} is the Levi-Cevita tensor, which fulfils the useful identity

$$\varepsilon_{kim}\,\varepsilon_{mps} = \delta_{kp}\delta_{is} - \delta_{ks}\delta_{ip}$$

Stokes' Flow Theorem

Vorticity Term

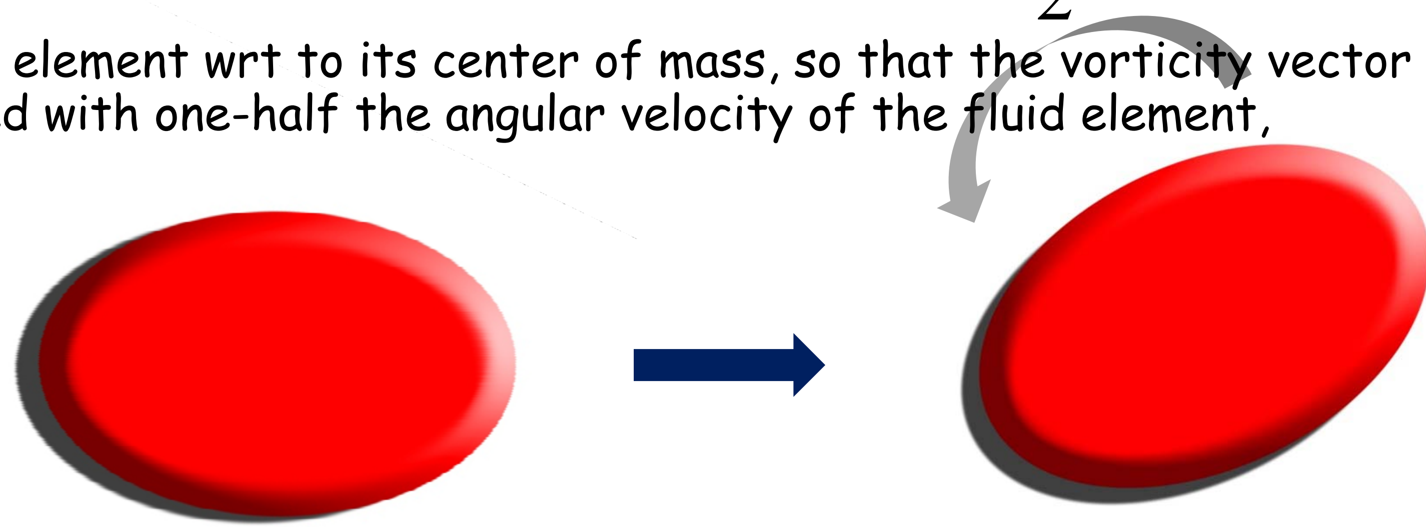
The contribution of the antisymmetric part of the differential velocity therefore reads,

$$\delta u_{\omega,i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right) R_k = \frac{1}{2}\varepsilon_{kim}\omega_m R_k = \varepsilon_{kim}\Omega_m R_k$$

The last expression in the eqn. above equals the i-th component of the rotational velocity

$$\vec{v}_{rot} = \vec{\Omega} \times \vec{R} \quad \Longleftrightarrow \quad \vec{\Omega} = \frac{1}{2}\vec{\nabla} \times \vec{u}$$

of the fluid element wrt to its center of mass, so that the vorticity vector can be identified with one-half the angular velocity of the fluid element,



Linear Momentum Fluid Element

The linear momentum $\vec{p}$ of a fluid element equal the fluid velocity $\vec{u}(Q)$ integrated over the mass of the element,

$$\vec{p} = \int \vec{u}(Q)\, dm$$

Substituting this into the equation for the fluid flow around P,

$$\vec{u}(Q) = \vec{u}(P) + H\vec{R} + \vec{\nabla} S + \vec{\Omega} \times \vec{R}$$

we obtain:

$$\vec{p} = \vec{u}(P) \int dm + \vec{\Omega} \times \int \vec{R}\, dm + H \int \vec{R}\, dm + \int \vec{\nabla} S\, dm$$

If P is the center of mass of the fluid element, then the 2nd and 3rd terms on the RHS vanish as

$$\int \vec{R}\, dm = \vec{0}$$

Moreover, for the 4th term we can also use this fact to arrive at,

$$\int \nabla_i S\, dm = \int \Sigma_{ik} R_k\, dm = \Sigma_{ik} \int R_k\, dm = 0$$

Linear Momentum Fluid Element

Hence, for a fluid element, the linear momentum equals the mass times the center-of-mass velocity,

$$\vec{p} = \int \vec{u}(Q)\, dm = m\, \vec{u}(P)$$

Angular Momentum Fluid Element

With respect to the center-of-mass P, the instantaneous angular momentum of a fluid element equals

$$\vec{J} \equiv \int \left[\vec{R} \times \vec{u}(Q) \right] dm$$

We rotate the coordinate axes to the eigenvector coordinate system of the deformation tensor D_{mk} (or, equivalently, the shear tensor Σ_{mk}), in which the symmetric deformation tensor is diagonal

$$\Phi_v = \frac{1}{2} D'_{mk} R'_m \; R'_k = \frac{1}{2}\left(D'_{11} R'^2_1 + D'_{22} R'^2_2 + D'_{33} R'^2_3 \right)$$

and all strains D_{mk} are extensional,

$$D'_{11} = \frac{\partial u'_1}{\partial x'_1}; \qquad D'_{22} = \frac{\partial u'_2}{\partial x'_2}; \qquad D'_{33} = \frac{\partial u'_3}{\partial x'_3}$$

Then

$$J'_1 = \int \left[R'_2 u'_3(Q) - R'_3 u'_2(Q) \right] dm$$

Angular Momentum Fluid Element

In the eigenvalue coordinate system, the angular momentum in the 1-direction is

$$J_1' = \int \left[R_2' u_3'(Q) - R_3' u_2'(Q) \right] dm$$

where

$$u_3'(Q) = u_3'(P) + \left(\Omega_1' R_2' - \Omega_2' R_1' \right) + D_{33}' R_3'$$

$$u_2'(Q) = u_2'(P) + \left(\Omega_3' R_1' - \Omega_1' R_3' \right) + D_{22}' R_2'$$

with $\vec{\Omega} = \vec{\nabla} \times \vec{u} / 2$ and D_{mk} evaluated at the center-of-mass P. After some algebra we obtain

$$J_1' = I_{11}' \Omega_1' + I_{22}' \Omega_2' + I_{33}' \Omega_3' + I_{23}' \left(D_{22}' - D_{33}' \right)$$

where I_{jl}' is the moment of inertia tensor

$$I_{jl}' \equiv \int \left(\left| \vec{R}' \right|^2 \delta_{jl} - R_j' R_l' \right) dm$$

Notice that I_{jl}' is not diagonal in the primed frame unless the principal axes of I_{jl} happen to coincide with those of D_{mk}.

Angular Momentum Fluid Element

Using the simple observation that the difference

$$D'_{22} - D'_{33} = \Sigma'_{22} - \Sigma'_{33}$$

since the isotropic part of I'_{jl} does not enter in the difference, we find for all 3 angular momentum components

$$\begin{aligned} J'_1 &= I'_{1l}\Omega'_l + I'_{23}\left(\Sigma'_{22} - \Sigma'_{33}\right) \\ J'_2 &= I'_{2l}\Omega'_l + I'_{31}\left(\Sigma'_{33} - \Sigma'_{11}\right) \\ J'_3 &= I'_{3l}\Omega'_l + I'_{12}\left(\Sigma'_{11} - \Sigma'_{22}\right) \end{aligned}$$

with a summation over the repeated l's.

Note that for a solid body we would have

$$J'_j = I'_{jl}\Omega'_l$$

For a fluid an extra contribution arises from the extensional strain if the principal axes of the moment-of-inertia tensor do not coincide with those of D_{ik}.

Notice, in particular, that a fluid element can have angular momentum wrt. its center of mass without possesing spinning motion, ie. even if $\vec{\Omega} = \vec{\nabla} \times \vec{u} / 2 = 0$!

Inviscid Barotropic Flow

Inviscid Barotropic Flow

In this chapter we are going to study the flow of fluids in which we ignore the effects of *viscosity* .

In addition, we suppose that the energetics of the flow processes are such that we have a barotropic equation of state

$$P = P(\rho, S) = P(\rho)$$

Such a replacement considerably simplifies many dynamical discussions, and its formal justification can arise in many ways.

One specific example is when heat transport can be ignored, so that we have adiabatic flow,

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) s = 0$$

with s the specific entropy per mass unit. Such a flow is called an *isentropic flow.* However, barotropic flow is more general than *isentropic flow.* There are also various other thermodynamic circumstances where the barotropic hypothesis is valid.

Inviscid Barotropic Flow

For a barotropic flow, the *specific enthalpy* h

$$dh = T\,ds + V\,dp$$

becomes simply

$$dh = V\,dp = \frac{dp}{\rho}$$

and

$$\boxed{h = \int \frac{dp}{\rho}}$$

Kelvin Circulation Theorem

Assume a fluid embedded in a uniform gravitational field, i.e. with an external force

$$\vec{f} = \vec{g}$$

so that – ignoring the influence of viscous stresses and radiative forces - the flow proceeds according to the Euler equation,

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \vec{g} - \frac{\vec{\nabla} p}{\rho}$$

To proceed, we use a relevant vector identity

$$\left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \left(\vec{\nabla} \times \vec{u}\right) \times \vec{u} + \vec{\nabla}\left(\frac{1}{2}\left|\vec{u}\right|^2\right)$$

which you can most easily check by working out the expressions for each of the 3 components.

The resulting expression for the Euler equation is then

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}\left|\vec{u}\right|^2\right) + \left(\vec{\nabla} \times \vec{u}\right) \times \vec{u} = \vec{g} - \frac{\vec{\nabla} p}{\rho}$$

Kelvin Circulation Theorem

If we take the curl of equation

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}|\vec{u}|^2\right) + (\vec{\nabla} \times \vec{u}) \times \vec{u} = \vec{g} - \frac{\vec{\nabla} p}{\rho}$$

we obtain

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \vec{\nabla} \times \vec{g} + \frac{\vec{\nabla} \rho}{\rho^2} \times \vec{\nabla} p$$

where $\vec{\omega}$ is the vorticity vector,

$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

and we have used the fact that the curl of the gradient of any function equals zero,

$$\vec{\nabla} \times \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) = 0; \qquad \vec{\nabla} \times \vec{\nabla}(p) = 0$$

Also, a classical gravitational field $\vec{g} = -\vec{\nabla}\phi$ satisfies this property,

$$\vec{\nabla} \times \vec{g} = 0$$

so that gravitational fields cannot contribute to the generation or destruction of vorticity.

Vorticity Equation

In the case of barotropic flow, ie. if

$$p = p(\rho) \qquad \Rightarrow \qquad \nabla p = \left(\frac{\partial p}{\partial \rho}\right) \nabla \rho$$

so that also the 2nd term on the RHS of the vorticity equation disappears,

$$\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p = \frac{1}{\rho^2} \left(\frac{\partial p}{\partial \rho}\right) \vec{\nabla} \rho \times \vec{\nabla} \rho = 0$$

The resulting expression for the vorticity equation for barotropic flow in a conservative gravitational field is therefore,

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = 0}$$

which we know as the *Vorticity Equation.*

Kelvin Circulation Theorem

Interpretation of the vorticity equation:

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = 0$$

Compare to magnetostatics, where we may associate the value of $\vec{B}$ with a certain number of magnetic field lines per unit area.

With such a picture, we may give the following geometric interpretation of the vorticity equation, which will be the physical essence of the

Kelvin Circulation Theorem

The number of vortex lines that thread any element of area, that moves with the fluid , remains unchanged in time for inviscid barotropic flow.

Kelvin Circulation Theorem

To prove Kelvin's circulation theorem, we define the circulation Γ around a circuit *C* by the line integral,

$$\Gamma = \oint_C \vec{u} \cdot d\vec{l}$$

Transforming the line integral to a surface integral over the enclosed area *A* by Stokes' theorem,

$$\Gamma = \int_A \left(\vec{\nabla} \times \vec{u}\right) \cdot \vec{n}\, dA$$

we obtain

$$\Gamma = \int_A \vec{\omega} \cdot \vec{n}\, dA$$

This equation states that the circulation Γ of the circuit *C* can be calculated as the number of vortex lines that thread the enclosed area *A*.

Kelvin Circulation Theorem

Time rate of change of Γ

Subsequently, we investigate the time rate of change of Γ if every point on C moves at the local fluid velocity $\vec{u}$.

Take the time derivative of the surface integral in the last equation.
It has 2 contributions:

$$\frac{d\Gamma}{dt} = \int_A \frac{\partial \vec{\omega}}{\partial t} \cdot \vec{n}\, dA + \int \vec{\omega} \cdot (time\ rate\ of\ change\ of\ area)$$

where $\hat{n}$ is the unit normal vector to the surface area.

Kelvin Circulation Theorem

Time rate of change of □

$$\frac{d\,\Gamma}{d\,t} = \int_A \frac{\partial\,\vec{\omega}}{\partial\,t} \cdot \vec{n}\; d\,A \;+\; \int \vec{\omega} \cdot (\textit{time rate of change of area})$$

The time rate of change of area can be expressed mathematically with the help of the figure illustrating the change of an area A moving locally with fluid velocity .

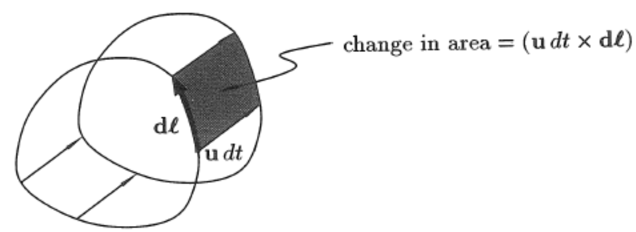


FIGURE 6.2
The time-rate change of an area A moving locally with the fluid velocity $\mathbf{u}$ comes from the parts swept up or left behind by the motion of the boundary C. Each line element $\mathbf{d}\ell$ on C contributes $\mathbf{u} \times \mathbf{d}\ell$ to the time rate of change of the area, an expression that contains both the magnitude of the effect as well as the direction of the local normal to the additional area element.

On the basis of this, we may write,

$$\frac{d\,\Gamma}{d\,t} = \int_A \frac{\partial\,\vec{\omega}}{\partial\,t} \cdot \hat{n}\; d\,A \;+\; \oint_C \vec{\omega} \cdot \left(\vec{u} \times d\,\vec{l}\,\right)$$

We then interchange the cross and dot in the triple scalar product

$$\vec{\omega} \cdot \left(\vec{u} \times d\,\vec{l}\,\right) = \left(\vec{\omega} \times \vec{u}\,\right) \cdot d\,\vec{l}$$

Kelvin Circulation Theorem

Time rate of change of ?

Using Stokes' theorem to convert the resulting line integral

$$\frac{d\Gamma}{dt} = \int_A \frac{\partial \vec{\omega}}{\partial t} \cdot \hat{n}\, dA + \oint_C (\vec{\omega} \times \vec{u}) \cdot d\vec{l}$$

to a surface integral, we obtain:

$$\frac{d\Gamma}{dt} = \int_A \left[\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) \right] \cdot \hat{n}\, dA$$

The vorticity equation tells us that the integrand on the right-hand side equals zero, so that we have the geometric interpretation of Kelvin's circulation theorem,

$$\boxed{\frac{d\Gamma}{dt} = 0}$$

Kelvin Circulation Theorem

Time rate of change of ?

Using Stokes' theorem to convert the resulting line integral

$$\frac{d\Gamma}{dt} = \int_A \frac{\partial \vec{\omega}}{\partial t} \cdot \hat{n}\, dA + \oint_C (\vec{\omega} \times \vec{u}) \cdot d\vec{l}$$

to a surface integral, we obtain:

$$\frac{d\Gamma}{dt} = \int_A \left[\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) \right] \cdot \hat{n}\, dA$$

The vorticity equation tells us that the integrand on the right-hand side equals zero, so that we have the geometric interpretation of Kelvin's circulation theorem,

$$\boxed{\frac{d\Gamma}{dt} = 0}$$

the Bernoulli Theorem

Closely related to Kelvin's circulation theorem we find Bernoulli's theorem.

It concerns a flow which is *steady* and *barotropic*, i.e.

$$\frac{\partial \vec{u}}{\partial t} = 0$$

and

$$p = p(\rho)$$

Again, using the vector identity,

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = (\vec{u} \times \vec{\nabla}) \times \vec{u} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right)$$

we may write the Euler equation for a steady flow in a gravitational field

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla}\phi - \frac{\vec{\nabla} p}{\rho}$$

$$\Downarrow$$

$$\vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) + (\vec{\nabla} \times \vec{u}) \times \vec{u} = -\vec{\nabla}\phi - \frac{\vec{\nabla} p}{\rho}$$

the Bernoulli Theorem

The Euler equation thus implies that

$$\vec{\omega} \times \vec{u} = -\vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) - \vec{\nabla}\phi - \vec{\nabla}h$$

where h is the specific enthalpy, equal to

$$h = \int \frac{dp}{\rho}$$

for which

$$-\vec{\nabla}h = -\frac{\vec{\nabla}p}{\rho}$$

We thus find that the Euler equation implies that

$$\vec{\omega} \times \vec{u} = -\vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2 + h + \phi\right)$$

the Bernoulli Theorem

Defining the Bernoulli function B

$$B = \frac{1}{2}|\vec{u}|^2 + \phi + h$$

which has dimensions of energy per unit mass. The Euler equation thus becomes

$$\boxed{\vec{\omega} \times \vec{u} + \vec{\nabla} B = 0}$$

Now we consider two situations, the scalar product of the equation with and $\vec{u}$ and $\vec{\omega}$,

1) $$(\vec{u} \cdot \vec{\nabla}) B = 0$$

B is constant along streamlines this is *Bernoulli's streamline theorem*

2) $$(\vec{\omega} \cdot \vec{\nabla}) B = 0$$

B is constant along vortex lines ie. along integral curves $\vec{\omega}(\vec{x})$

* vortex lines are curves tangent to the vector field $\vec{\omega}(\vec{x})$