– ASTROPHYSICAL HYDRODYNAMICS –
Assignment 3

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March 02, 2018

The main goal of this assignment is to get used to the curvilinear co-ordinate systems. So, the first problem is for revising the properties of coordinate transformations, differential vector operators and so on. The second problem is a revision from the lecture. In the third problem we see how Euler's equation changes in the presence of magnetic field and we derive a fundamental equation of MHD. Then we consider hydrostatics and its application to stellar structure, and the problem of an immersed, moving sphere.

1 Vector analysis in curvilinear coordinates

In transforming from Cartesian coordinates (x, y, z) to a general coordinate (q_1, q_2, q_3), the square of the differential distance element can be written as:

$$ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = d\boldsymbol{r}^2 = \sum_{ij} \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} dq_i dq_j = \sum_{ij} g_{ij} dq_i dq_j \tag{1}$$

here g_{ij} is called *metric* which describes the transformation from Cartesian coordinates to another set of coordinates q_1, q_2 and q_3:

$$g_{ij}(q_1, q_2, q_3) = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j}\,. \tag{2}$$

We will only consider orthogonal systems, which means that the metric simplifies to a diagonal tensor g_{ii}:

$$g_{ii} = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 . \tag{3}$$

We further define the scale factors $h_i = \sqrt{g_{ii}}$. Now we can write down a number of results for general coordinate transformations of this form:

$$d\boldsymbol{r} = \sum_i h_i dq_i \hat{\boldsymbol{q}}_i \tag{4}$$

$$ds^2 = \sum_i (h_i dq_i)^2 \tag{5}$$

$$dS_{ij} = h_i h_j dq_i dq_j \tag{6}$$

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3\,. \tag{7}$$

In the transformed orthogonal curvilinear coordinate system the gradient, divergence, Laplacian and curl are written as:

$$
\begin{aligned}
\boldsymbol{\nabla}\phi &= \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial q_i}\hat{\boldsymbol{q}}_i && (8)\\
\boldsymbol{\nabla}\cdot\boldsymbol{V}(q_1,q_2,q_3) &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(V_1h_2h_3)+\frac{\partial}{\partial q_2}(V_2h_3h_1)+\frac{\partial}{\partial q_3}(V_3h_1h_2)\right] && (9)\\
\boldsymbol{\nabla}^2\phi(q_1,q_2,q_3) &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right] && (10)\\
\boldsymbol{\nabla}\times\boldsymbol{V}|_i &= \epsilon_{ijk}\frac{1}{h_jh_k}\left[\frac{\partial}{\partial q_j}(h_kV_k)-\frac{\partial}{\partial q_k}(h_jV_j)\right]\hat{\boldsymbol{q}}_i\,. && (11)
\end{aligned}
$$

Determine the scale factors and write down the line element, surface element, volume element and vector differential operators in these following two coordinate systems:

1. Circular cylindrical coordinates:

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} & x &= \rho\cos\phi && (12)\\
\phi &= \arctan\frac{y}{x} & y &= \rho\sin\phi && (13)\\
z &= z & z &= z\,, && (14)
\end{aligned}
$$

   where $(0\le\rho<\infty)$, $(0\le\phi\le 2\pi)$ and $(-\infty<z<\infty)$.

2. Spherical polar coordinate system:

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} & x &= r\sin\theta\cos\phi && (15)\\
\theta &= \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} & y &= r\sin\theta\sin\phi && (16)\\
\phi &= \arctan\frac{y}{x} & z &= r\cos\theta\,, && (17)
\end{aligned}
$$

   where $(0\le r<\infty)$, $(0\le\theta\le\pi)$ and $(0\le\phi\le 2\pi)$.

2 Kelvin's circulation theorem

The circulation Γ around a closed curve C is defined as follows:

$$\Gamma = \oint_C \boldsymbol{u}\cdot\boldsymbol{dr} \qquad (18)$$

1. If the curve C moves with the fluid (assumed to be inviscid and isentropic), show that Γ is a constant:

$$\frac{D\Gamma}{Dt} = 0 \qquad (19)$$

   This is known as "Kelvin's circulation theorem".

3 Magnetohydrodynamics: Cauchy momentum equation

Including the Lorentz force term $\boldsymbol{J} \times \boldsymbol{B}$, Euler's equation in the presence of magnetic field becomes,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{f} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} \tag{20}$$

1. Using Maxwell's equation show that (neglect displacement current) the above equation can be written in the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f} - \frac{1}{\rho}\nabla\Big(p + \frac{B^2}{2\mu_0}\Big) + \frac{(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}}{\mu_0 \rho} \tag{21}$$

The term $\frac{B^2}{2\mu_0}$ corresponds to magnetic pressure, introduced by the magnetic field. The last term above is related to the tension along the magnetic field lines, that is, the magnetic field is associated to an isotropic pressure and also to a tension along the field lines.

4 Hydrostatics: Stellar structure

This question gives some examples of static fluids (no velocities). In that case, Euler's equation becomes

$$\boldsymbol{\nabla} p = \rho \boldsymbol{g}\,, \tag{22}$$

where p is the pressure of the fluid and $\rho\boldsymbol{g}$ the force acting on the fluid.

1. Consider a very large, very massive fluid such that self-gravity becomes important. The Poisson equation – familiar from mechanics, stellar dynamics and/or stellar evolution courses – relates the gravitational potential ϕ to the density ρ

$$\nabla^2 \phi(\boldsymbol{r}) = 4\pi G \rho(\boldsymbol{r})\,, \tag{23}$$

where G is the gravitational constant. What is $\boldsymbol{g}$ in this scenario? Use the assumption of spherical symmetry to find (one of) the equations of stellar structure

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G \rho\,. \tag{24}$$

N.B. It's not only in stars that this equation is used. Another example are clusters of galaxies. Then ρ is the total matter density: baryonic (stellar, gas) and dark.

2. In order to solve this, we need a relation between p and ρ, called the *equation of state*. Often this is a polytropic relation

$$p \propto \rho^{1+1/n}\,, \tag{25}$$

where n is called the polytropic index. Low mass white dwarf stars are well approximated as $n = 1.5$ polytropes, red giants with $n = 3$, the giant planets Jupiter and Saturn with $n = 1$ and the small planets with a constant density (no relation between p and ρ). Assume a planet or a star has radius R and mass M. The pressure at the center is p_c, and the pressure at the surface is $p_R = 0$ (which in fact can be used as a definition of the surface).

Show that for the gaseous planets, $p = \alpha\rho^2$ the above differential equation turns into the following form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \rho}{\partial r}\right) + \beta^2 \rho = 0\,. \tag{26}$$

where $\beta^2 = \frac{2\pi G}{\alpha}$.

3. Derive an expression for the pressure $p(r)$ in Mercury as function of radius in terms of M and R (and G etc.). What is the central pressure?

5 The Immersed Moving Sphere

Consider a sphere with radius a which moves with a velocity u in an ideal, incompressible fluid, e.g. a protoplanet swiping through the planetary disk or a person wading through a crowded tunnel. We will estimate the velocity of the fluid, $\boldsymbol{v}(\boldsymbol{x})$, induced by the sphere. Ignore gravity (or take gravity constant in the volume). If we assume that the vorticity vanishes, $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, then fluid motion is a potential flow so $\boldsymbol{v}$ can be written in terms of a potential $\boldsymbol{v} = \boldsymbol{\nabla}\Psi$. The aim of this exercise is to find this potential by use of Laplace's equation

$$\nabla^2\Psi = 0\,, \tag{27}$$

combined with the boundary conditions

$$\lim_{r\to\infty} \Psi = 0\,, \tag{28}$$

$$v_N(a) = u_N\,, \tag{29}$$

with v_N the velocity component in the direction of the normal of the sphere.

1. Using spherical coordinates, write down the Laplace equation. Then use the *ansatz* $\Psi = R(r)\Theta(\theta)$ to split Laplace's equation into a radial and an angular part. Both sides of the equation you then end up with must be a constant of opposite sign i.e. the equation you will find is of the form

$$-f(\Theta;\theta) = g(R;r) = \text{constant}\,. \tag{30}$$

2. Focus on the angular side of the equation. Solutions for Θ can be found from the boundary condition on the sphere, since

$$v_r = \partial_r\Psi = \Theta(\theta)\partial_r R(r)\,. \tag{31}$$

   Thus, how should $v_r(a)$ vary with θ in order to satisfy the boundary conditions?

3. Focus on the radial part. Try a power law solution,

$$R(r) = Br^{-n}\,. \tag{32}$$

   Why should n be positive? Solve the radial equation and determine n.

4. Finally, write down the potential Ψ in terms of r and θ. Apply the boundary conditions to derive the proportionality constant(s).

References

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